Algorithms in Computational Algebra and Algebraic Geometry

Quest University Canada Block 3, Spring 2016

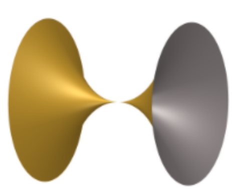
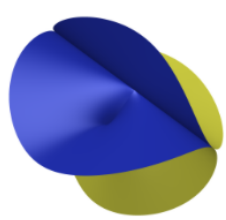
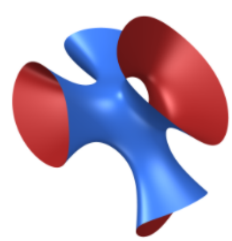
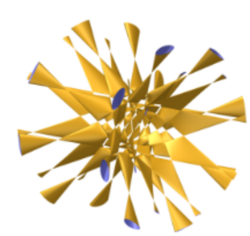
Today, algebraic geometry touches nearly every branch of mathematics.
– Karen E. Smith

Geometric objects are often described as the zero-sets of polynomial equations, or sets of polynomial equations. For example, in high school you learned that the set of coordinates (x, y) satisfying an equation of the form $ax^2 + bx + c = y = 0$ will always form a parabola. Things get much more interesting if you look at polynomial equations in three variables (corresponding to objects in 3-dimensional space), multiple polynomial equations at once, or expand your view to larger and larger number of variables. Some examples of varieties are shown below

This is the starting point of algebraic geometry: a bridge between the worlds of algebra and geometry. In this course, you will investigate questions such as:

- How can one compute the equations for the intersection or union of two zero-sets of polynomials?
- When do two systems of algebraic equations describe the same or overlapping geometric objects?
- Is it necessary to use an infinite number of polynomials to describe certain geometric problems?
- How can we find polynomials to describe a geometric object?
- How can we translate geometric problems into algebraic ones, and vice versa?

Even though polynomials are nice compared to other functions, they can get very complicated and answering these questions remained impractical through the mid-twentieth century. The answer to simplifying these questions comes from reducing sets of polynomials to sets of monomials. The field was revolutionized in 1976, when Bruno Buchberger published an algorithm for working with large numbers of polynomials in an efficient way. The field of computational commutative algebra was born, and it has remained one of the most influential areas of mathematics in the twenty-first century.

By the end of this course you should be able to:

- Understand key problems in commutative algebra and algebraic geometry, and algorithms for solving them
- Read complex mathematical texts
- Improve your ability to learn mathematics independently
- Write and critique proofs and arguments
- Use a computer utility to execute mathematical algorithms

Course Text and Computer Programs

The required text for this class is *Ideals, Varieties, and Algorithms*, 3rd ed. by David A. Cox, John B. Little, and Don O'Shea. It has *not* been ordered to the University Bookstore, but it should be available through online retailers.

You can do most of the computations that you will need for this course on Maple. If you would like to try out a more specialized program, feel free to explore other options! In North America, most computational commutative algebraists use a specialized program called *Macaulay 2*.[1]

Software

As a *computational* algebra course, you will need some software. You are encouraged to check calculations in the book with software whenever possible, and to run additional examples on your own beyond what is included on the syllabus. The Appendix of the text explains basic syntax for both Maple and Macaulay2. I have found Macaulay2 to be more reliable with certain calculations, but there may be more of a learning curve with it, as most of the documentation is written for specialists in the field. This learning curve has recently been made much easier with the advent of an online platform for Macaulay2: `http://habanero.math.cornell.edu:3690`.

Meetings and Office Hours

I will hold approximately 2 hours of office hours for this course per week. You should expect to attend each of these, as I will use this time to highlight key points as well as answering questions. If you are struggling with parts of the material and reaching a roadblock as a class, I may be able to add additional meetings upon request.

Course Requirements

To pass this course you will need to do the following.

- Study the assigned material, and complete suggested practice problems.
- Perform well on daily quizzes

[1] Similar programs that are more popular in Europe include CoCoA and Singular.

- Write mathematical proofs
- Work collaboratively with peers

Reading and Problems

The reading schedule for the course is given at the end of the syllabus. It is generally one to two sections a day. There are also some problems to go along with each reading. You will notice that there are not as many problems provided as in a typical math course, as I would like you to select problems that will aid in your own learning, or that intrigue you.

I expect that you will discuss the problems together, and work on them collaboratively. I will ask you to submit the problems as a portfolio at the end of the course.

I may occasionally post some supplemental readings that might help you to understand the material, but I expect you to do further research on your own, and seek out additional resources that you might need or find interesting.

Solutions to many of the are available from me.

Daily Quizzes

Every day, I will post 1 to 3 basic questions about the material covered in the class so far. You may be asked to give a definition, a theorem, or an example. The details about administration of these quizzes (where you will find them, when they are due), will be announced.

Collaboration

The success of this Independent Study rests on how you collaborate and work with others. You do not need to work together continually throughout the day, but you should be a good team player, and work to aid in the learning of others while improving on your own.

Prior to the first day of the course, I will ask you to get together to lay out a basic plan for how you would like to organize your work during the class. For example, where and at what times will the entire class meet? When should each member of the class be expected to complete the reading? How will you organize your critiques of each other's writing for submission on the formal assignments? I will also ask you to lay out a rubric for participation and collaboration that you will use to assess each other throughout the block. You may modify your plans throughout the course, but having clear expectations for yourself and others will help the independent study to be successful.

Formal Mathematical Writing

Each Monday, you will submit a single written assignment *as a class.* I will assign one problem per member of the class from the previous week's material. I suggest that you randomly assign one person to write a first draft, and then follow a feedback-revision loop until everyone in the class is happy with the draft. If there are disagreements that cannot be settled by the due date, include a description of these in your solution.

Here are the problems due each week:

- Due π Day: §1.1 - 6b; §1.2 - 6ab; §1.5 - 10 (bonus for programmers: write Maple code to implement the algorithm!); §2.2 - 7, 11
- Due March 21: §2.4 - 1; §2.5 - 5; §2.6 - 1; §3.3 - 9, 11a
- Due March 28: §3.4 - 10; §4.1 - 1; §4.2 - 5; §4.3 - 11abc; §4.5 - 12

Academic Integrity

While googling a homework problem or trading solutions with a classmate may seem like good strategies for doing well in this class, these actions will prevent you from learning material, refining your problem-solving skills, and developing self-sufficiency and self-esteem.

The consequences for cheating are severe. *Any* blatant academic dishonesty will result in failure of the course and immediate reporting to the Chief Academic Officer. If you have questions about what counts as academic dishonesty in this course, please come and speak with me.

Grading

Your final grade will be calculated as follows.

40%	Team Assignments
30%	Quizzes
30%	Final Exam

The course grading scale is:

A	93-100%	B	83-86%	C	73-76%
A-	90-92%	B-	80-82%	C-	70-72%
B+	87-89%	C+	77-79%	D	60-69%

Disability Accommodations

If you have a disability for which you seek accommodation, please make sure to have registered with the Learning Commons, as specified in the Student Accommodation Policy (`http://www.questu.ca/pdfs/_uploads/content/student_accommodation_policy.pdf`), and provide us with your Memorandum by the second day of class.

Schedule

Notice that each day lists the work to be done on that day, not due on that day. For example, a quiz on Tuesday, March 8 would only cover material from §1.1 and Appendix A, not §1.2 and §1.3.

Date	**Reading**	**Problems**
March 7 (M)	App. A (§1), §1.1	§1.1: 1, 2, 6b
March 8 (T)	§1.2, §1.3	§1.2: 1a, 2, 4d, 6, 8, 9; §1.3: 1, 4, 8
March 9 (W)	§1.4	§1.4: 2, 3b, 6, 9, 15a
March 10 (Th)	§1.5, §2.1	§1.5: 9, 10, 12; §2.1: 5a
March 11 (F)	§2.2, §2.3	§2.2: 1, 2, 11; §2.3: 1a, 2a, 3, 5, 9
March 14 (M)	§2.4, §2.5	§2.4: 1, 3, 9, §2.5: 1, 2a, 5,7, 17
March 15 (T)	§2.6, §2.7	§2.6: 1, 5bd, 6, 9, §2.7: 2ab, 3ab
March 16 (W)	§2.8	§2.8: 1, 3, 5, 8
March 17 (Th)	§3.1, §3.2	§3.1: 1, 2, 3, 5 §3.2: 3, 5
March 18 (F)	§3.3, §3.4	§3.3: 6, 9, 11a §3.4: 4, 10
March 21 (M)	§3.5, §3.6	§3.5: 5, 7, 11, §3.6: 2, 3, 8a
March 22 (T)	§4.1, §4.2	§4.1: 1, 2, 4, §4.2: 2, 3, 5, 7a
March 23 (W)	§4.3, §4.4	§4.3: 1, 11abc; §4.4: 1, 2
March 24 (Th)	§4.5, 4.6	§4.5: 6, 7, 12; §4.6: 1,4, 7
March 25 (F)	Good Friday	Identify Topics from Ch. 5, 6, or 8
March 28 (M)	Ch. 5, 6, 8	
March 29 (T)	Ch. 5, 6, 8	
March 30 (W)	Ch. 5, 6, 8	