Appeared in *Letters of Math. Phys.*, **53** No. 4 2000, 313–320.

Perron-Frobenius Theory and Symmetry of Solutions to Non-linear PDE's*

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Abstract. In this paper we suggest a simple but general method of establishing symmetry properties of stable solutions of nonlinear elliptic equations. The method relies on characterization of symmetry breaking with a help of zero modes and on a generalization of the Perron-Frobenius theory.

1. Introduction

In this paper we suggest a new method of establishing symmetry properties of systems of non-linear partial differential equations of the form

$$F(u, \partial u, \partial^2 u, x) \;=\; 0 \tag{1.1}$$

on a domain $\Omega \subset \mathrm{R}^n$ with a smooth boundary and with the Dirichlet boundary condition. Here $F(u, \xi, \eta, x)$ is a twice differentiable function from $\mathrm{R}^m \times \mathrm{R}^{mn} \times \mathrm{R}^{mn^2} \times \Omega$ to R^m, u is a vector-function from Ω to R^m, ∂u and $\partial^2 u$ are the collections of the first and second derivatives of u defined as $(\partial_1 u, \dots, \partial_n u)$ and $(\partial_{ij} u,\ i,j = 1, \dots, n)$, respectively, where ∂_i is the partial derivative w.r. to x_i, and $\partial_{ij} = \partial_i \partial_j$. We let $u = (u^1, \dots, u^m)$

* Research on this paper was supported by NSERC under Grant NA7901.

and $F = (F^1, \ldots, F^m)$. In what follows partial derivatives are signified by assigning the variable over which they are taken as a subindex, e.g., $u_{x_i} \equiv \partial_i u \equiv \frac{\partial u}{\partial x_i}$.

We assume that the function F satisfies the following conditions:

$$\delta^{-1}|\xi|^2 \geq -\sum_{i,j} F^{\alpha}_{u^{\alpha}_{x_i x_j}}(u, \partial u, \partial^2 u, x)\xi_i \xi_j \geq \delta|\xi|^2 \tag{1.2}$$

for each α and for some $\delta > 0$, i.e. the matrices $(F^{\alpha}_{u^{\alpha}_{x_i x_j}})$ are uniformly elliptic. In the vector case, $m \geq 2$, we assume, in addition, that

$$F^{\alpha}_{u^{\beta}_{x_i x_j}} = F^{\alpha}_{u^{\alpha}_{x_i x_j}} \delta_{\alpha,\beta} \quad \text{and} \quad F^{\alpha}_{u^{\beta}_{x_i}} = F^{\alpha}_{u^{\alpha}_{x_i}} \delta_{\alpha,\beta} \ . \tag{1.3}$$

If Ω is unbounded, then to minimize technicalities we assume also that

$$\partial_i F^{\alpha}_{u^{\alpha}_{x_i x_j}} = F^{\alpha}_{u^{\alpha}_{x_j}} \quad \text{and} \quad F^{\alpha}_{u^{\beta}} = F^{\beta}_{u^{\alpha}} \tag{1.4}$$

We believe that condition (1.3) can be relaxed and condition (1.4) can be removed. See the conjecture and the discussion following it at the end of the next section.

We say that a solution u_0 to an equation $\phi(u) = 0$ is *stable* iff $\operatorname{spec} D\phi(u_0) \subset \{z \in \mathrm{C} \mid \operatorname{Re} z \geq 0\}$. Here $D\phi(u)$ is the Fréchet derivative of $\phi(u)$ at u.

If F is linear in $\partial^2 u$, then condition (1.4) guarantees that there is a function ϕ: $\mathrm{R}^m \times \mathrm{R}^{mn} \times \Omega \to \mathrm{R}$ s.t. $F(u, \partial u, \partial^2 u, x)$ is the Fréchet derivative of $u \to \phi(u, \partial u, x)$ in u. The solutions to Eqn (1.1) are critical points of the energy functional

$$\mathcal{E}(u) = \int_A \phi\big(u(x), \partial u(x), x\big) d^n x \ , \tag{1.5}$$

and stable solutions are local (or relative) minima of this functional. This observation is not used in our paper.

Let G_{sym} be the *symmetry group* of the equation $\phi(u) = 0$, i.e. the set of all transformations, g, of u s.t. $\phi(u) = 0 \Rightarrow \phi(gu) = 0$. Now we are ready to state the main result of this paper.

Theorem 1.1. *Assume conditions (1.2)–(1.4) hold and assume that the group, $O(n)$, of rotations of x, is a symmetry subgroup of Eqn (1.1). If u is a stable solution to (1.1), s.t. (a) $F^\alpha_{u^\beta_{x_ix_j}} \in C^1(\Omega)$ and $F^\alpha_{u^\beta_{x_i}}$, $F^\alpha_{u^\beta} \in L^\infty(\Omega)$, (b) $F^\alpha_{u^\beta} \le 0$ and $\not\equiv 0\ \forall\, \alpha \ne \beta$ and (c) $\partial^\nu u \in \langle x\rangle^{-|\nu|}L^2(\Omega)$, $\forall\ |\nu| \le 1$, then u is spherically symmetric.*

In general, the reverse implication is not true. However, sometimes the method developed in this paper does yield inverse implication, namely that certain type of spherically symmetric solutions are stable. This was done in [OS] for the Ginzburg-Landau and related equations.

There is nothing particularly special about the spherical symmetry. A similar result holds for any other continuous symmetry possessed by the equation in question. Thus for Eqn (1.1) on R^n with F not explicitly depending on x (i.e. translationally invariant), stable solutions are translationally invariant, i.e. are constant. An example of other symmetry groups is given in

Theorem 1.2. *Assume, besides conditions (1.2)–(1.4), that $m \ge 3$, and the group $O(m)$ of rotations of u is a symmetry subgroup of (1.1). The only stable solution, u, to (1.1) s.t. (a) $F^\alpha_{u^\beta_{x_ix_j}} \in C^1(\Omega)$ and $F^\alpha_{u^\beta_{x_i}}$, $F^\alpha_{u^\beta} \in L^\infty(\Omega)$ and (b) $F^\alpha_{u^\beta} \le 0$ and $\not\equiv 0\ \forall\, \alpha \ne \beta$, $\forall\, x \in \Omega$, is a trivial one, $u \equiv 0$ (if such exists).*

The class of equations covered by Theorems 1.1 and 1.2 is rather wide. Below we give two examples of such equations which arize in applications.

Let B_R be the ball in R^n of radius R centered at the origin. The first equation is

$$\mathrm{div}(|\nabla u^\alpha|^p \nabla u^\alpha) = f^\alpha(|x|, u)\ ,$$

on $\Omega = B_R$, where $\alpha = 1, \dots, m$, $u = (u^1, \dots, u^m)$, $p \ge 0$ and f^α are real functions satisfying the inequalities

$$\frac{\partial f^\alpha}{\partial u^\beta}(|x|, u) < 0 \quad \text{for} \quad \alpha \ne \beta\ .$$

The second equation – also on B_R – is

$$\tanh(\beta\Delta u - \beta u + h) = u \ ,$$

where h is a smooth function of $|x|$.

The first of these equations appears in the case of potential flows (electrodynamics, magnetodynamics, diffusion, etc.) in nonlinear media with a response coefficient (dialectric constant, permeability, diffusion coefficient, etc.) which is a function of the field strength; in fluid dynamics; in gravity and in computer vision. Furthermore, equations of these type were considered in quantum chromodynamics [AP] (as an effective action approximation), in modified Newtonian gravity [BM,M] (as a way to avoid the dark-matter hypothesis for galatic systems) and in the theory of nonlinear composite media [BB]. The second of the equations above appears as the mean-field approximation in the problems of phase transition for the Ising model in statistical mechanics and in mathematical epidemiology (e.g. the travelling wave problem for the Kermack-McKendrick-Kendall model for spatial spread of an epidemic (see e.g. [AR, DK] and references therein).

Theorems 1.1 and 1.2 seem to be new, but results in this spirit have appeared in the literature before. In the special scalar case (i.e. $m = 1$ and u is a real function) with the function F linear in $\partial^2 u$, i.e. when Eqn (1.1) is an Euler-Lagrange equation of certain functional (see the paragraph containing Eqn (1.5)), the symmetric rearrangement technique gives the result of Theorem 1.1 (see [LL]). Furthermore, pioneering works [GNN, O] have shown that if a scalar version of Eqn (1.1) is spherically symmetric in x, then positive solutions of this equation are, also, spherically symmetric. These results were significantly extended in [ChK, Li, LiNi1,2,3]. They seem to be complementary to ours: the works mentioned above deal with unstable but positive solutions of the equations in question, while our paper, with stable ones but with no positivity required. The works above use the method of moving planes which is, at least formally, rather different from our method. Also, in the case of $\Omega = \{x \in \mathrm{R}^n \mid |x| \geq R\}$ for some R these papers require detailed information about asymptotic behaviour of the solutions in question, something which is irrelevant for our techniques. Note that neither the symmetric rearrangement method nor the method of moving planes are applicable to global gauge symmetries treated in Theorem 1.2.

2. Proof of Theorem 1.1

To exhibit the ideas of our approach we first consider the scalar case of $m = 1$. Consider the linearized operator for Eqn (1.1): $L = -\sum a_{ij}(x)\partial_{ij} + \sum b_i(x)\partial_i + c(x)$, where $a_{ij}(x) = -F_{u_{x_i x_j}}\big(u(x), \partial u(x), \partial^2 u(x), x\big)$, $b_i(x) = F_{u_{x_i}}\big(u(x), \partial u(x), \partial^2 u(x), x\big)$ and $c(x) = F_u\big(u(x), \partial u(x), \partial^2 u(x), x\big)$ (the operator L is the Fréchet derivative of the map $u \to F(u, \partial u, \partial^2 u, x)$ at u). By the conditions of the theorem, $a_{ij} \in C^1(\Omega)$ and $b_i, c \in L^\infty(\Omega)$. Hence if Ω is bounded, then the number $\lambda_1 = \inf \mathrm{Re}\, \mathrm{spec}\, L$ is an eigenvalue of L (the principal eigenvalue), it is algebraically simple and the corresponding eigenfunction is strictly positive, modulo a multiplicative constant (see e.g. [NP] and [BNV]). If Ω is unbounded, then by conditions (1.4) and (a), L is self-adjoint, so the previous conclusions still hold. (In that case one argues as follows. Approximating Ω by bounded domains, one shows that $(L-\lambda)^{-1}$ has a positive in Ω integral kernel, provided $\lambda < \inf \mathrm{spec}\, L$. Hence $(L-\lambda)^{-1}$ is positivity improving (i.e. $(L-\lambda)^{-1}u > 0$ whenever $u \geq 0$, $u \not\equiv 0$) for $\lambda < \inf \mathrm{spec}\, L$. On the other hand, by the variational principle for self-adjoint operators, if ψ_1 is an eigenfunction corresponding to $\lambda_1 = \inf \mathrm{spec}\, L$, then so are $\mathrm{Re}\psi_1$, (or $\mathrm{Im}\psi_1$) and $|\psi_1|$. So we can take ψ_1 to be real. By the positivity improving property established above $|\psi_1| > 0$ and therefore $\psi_1 = \pm|\psi_1|$, which implies that λ_1 is a simple eigenvalue. Hence the lowest eigenvalue of L is nondegenerate and the corresponding eigenfunction is strictly positive. (Cf. Theorem XIII.43 of [RSIV]).)

Now let us fix an axis in R^n and let θ be the angle of rotation around this axis. Let u be a solution to Eqn (1.1) which is not invariant under the rotations around this axis. Then $\frac{\partial u}{\partial \theta}$ satisfies the equation $L\xi = 0$, which is the linearization of Eqn (1.1) around u.

Besides $\frac{\partial u}{\partial \theta}$ satisfies the Dirichlet boundary conditions. Moreover $\int_0^{2\pi} \frac{\partial u}{\partial \theta} d\theta = 0$, so $\frac{\partial u}{\partial \theta}$ changes the sign. Hence by the result of the previous paragraph, 0, the eigenvalue of L corresponding to the eigenfunction $\frac{\partial u}{\partial \theta}$, is not the lowest point of $\mathrm{Re}\,\mathrm{spec}\, L$. Thus u is not a stable solution. This proves Theorem 1.1 in the scalar case. To prove this theorem in the vector case one observes that due to the condition (1.3), the part $-\sum a_{ij}(x)\partial_{ij} + \sum b_i(x)\partial_i$ of the linearized operator L is diagonal. This and the condition $F^{\alpha}_{u^\beta} \leq 0$ and $\not\equiv 0$ for $\alpha \neq \beta$ imply that L satisfies the conditions of Corollary 4.3 of Section 4. This corollary then can be used instead of the conclusion of the previous paragraph. Λ

Conjecture. Let L be a uniformly elliptic operator, as in the beginning of this section, in a domain Ω with the Dirichlet boundary conditions on $\partial\Omega$ ($\cup\{\infty\}$, if Ω is unbounded). Let $\lambda_1 = \inf \mathrm{Re}\,\mathrm{spec}\, L$, where spec stands for the L^2-spectrum, is an eigenvalue of L. Then λ_1 is algebraically simple and the corresponding eigenfunction is positive, modulo a multiplicative constant.

As was mentioned at the beginning of this section this conjecture is proven in the case when Ω is bounded or L is self-adjoint. Thus the statement remains open for Ω unbounded and L non-self-adjoint. If the conjecture is valid, then condition (1.4) is superfluous and condition (c) of Theorem 1.1 can be weakened to the condition

(c′) there is $a \geq 0$ s.t. $\partial^\alpha u \in \langle x \rangle^a L^2(\Omega)$ for $|\alpha| \leq 1$.

Indeed, if condition (c′) is satisfied, pick $b > a$. Then the operator $L' = \langle x \rangle^{-b} L \langle x \rangle^b$, where L is the linearized operator for Eqn (1.1) given at the beginning of this section, has the desired properties pointed out above. Moreover, it satisfies $\mathrm{spec}\, L' = \mathrm{spec}\, L$ and can be used instead of L in the proof of Theorem 1.1.

3. Proof of Theorem 1.2

Let u be a stable solution to (1.1) and let ℓ_j, $j = 1, \ldots, \frac{1}{2}m(m-1)$, be the generators of the unitary representation of $SO(m)$ on the space R^m. Due to the conditions of the theorem, $\ell_j u$, $j = 1, \dots, \frac{1}{2}m(m-1)$, satisfy the linearized equation $L\xi = 0$. Hence by the argument presented in the proof of Theorem 1.1, $\mathrm{Span}\left\{\ell_j u \mid j = 1, \dots, \frac{1}{2}m(m-1)\right\}$ is a one dimensional space. For $m \geq 3$, this contradicts the fact that ℓ_j, $i = 1, \dots, \frac{1}{2}m(m-1)$, generate the algebra of $SO(m)$. Hence u must be the identical zero. Λ

4. Perron-Frobenius Theory

In this section we adapt the theory, which goes back to O. Perron and G. Frobenius and was developed by M.G. Krein, M. Rutman, J. Glimm, A. Jaffe, B. Simon and others (see [GJ, RSIV] and [Z] for reviews) to the situation at hand. We begin with some definitions. The statements below are made modulo sets of zero measure.

Let Ω be a domain in R^n. For $\varphi\colon \Omega \to \mathrm{R}^m$ we denote $\varphi \geq 0$ iff $\varphi_i \geq 0\ \forall\ i$ and $\varphi \not\equiv 0$. If, in addition, $\varphi_i \not\equiv 0$ or $\varphi_i > 0\ \forall i$, then we use the notation $\varphi \dot{\geq} 0$ or $\varphi > 0$, respectively. Note that in the scalar case the notions $\varphi \geq 0$ and $\varphi \dot{\geq} 0$ coincide. An operator T acting on $L^2(\Omega, \mathrm{R}^m)$ is said to be positivity *preserving* (resp. *improving*) iff $T\varphi \geq 0$ (resp. $T\varphi > 0$), whenever $\varphi \geq 0$.

Elementary examples of positivity preserving operators are $T = \mathrm{diag}(T_i)$, the diagonal matrix-operator with the diagonal entries $T_1, \dots, T_m$, where the T_i's are positivity preserving in $L^2(\Omega, \mathrm{R})$, and an operator of multiplication by $m \times m$ matrix whose matrix elements are non-negative. We combine these two examples to obtain a less obvious one. Let

$$L = -\sum a_{ij}(x)\partial_{ij} + \sum b_i(x)\partial_i + c(x) \tag{4.1}$$

on $L^2(\Omega, \mathrm{R}^m)$ with the Dirichlet boundary conditions, where the first two terms on the r.h.s. are diagonal w.r. to R^m and $c(x) = \big(c_{\alpha\beta}(x)\big)$ is a real, symmetric $m \times m$ matrix. We assume that $b_i(x)$ and $c_{\alpha\beta}(x)$ are bounded, and $a_{ij}(x)$ is uniformly elliptic: $\delta^{-1}|\xi|^2 \geq \sum a_{ij}(x)\xi_i\xi_j \geq \delta|\xi|^2$ for some $\delta > 0$. Of course, L is bounded from below. Let $\lambda_1 =$

$\inf \operatorname{Re} \operatorname{spec} L$. For $\lambda < \lambda_1$ we consider the operator

$$T = (L - \lambda)^{-1} .$$

This operator is bounded. In fact, we have

Proposition 4.1. *(i) If $c_{\alpha\beta}(x) \leq 0 \ \forall \alpha \neq \beta$, then T is positivity preserving. (ii) If $c_{\alpha\beta}(x) \leq 0$ and $c_{\alpha\beta}(x) \not\equiv 0 \ \forall \alpha \neq \beta$, then T is positivity improving.*

Proof. We rewrite L as $L = L_0 - U(x)$, where $U(x)$ is defined by $U_{\alpha\beta}(x) = -c_{\alpha\beta}(x)$ $\forall \alpha \neq \beta$ and $U_{\alpha\alpha}(x) < 0$ and bounded but otherwise arbitrary $\forall \alpha$ and

$$L_0 = \operatorname{diag}(H_\alpha) \quad \text{with} \quad H_\alpha = -\sum a_{ij}(x)\partial_{ij} + \sum b_i(x)\partial_i + c_\alpha(x) ,$$

$c_\alpha(x) = c_{\alpha\alpha}(x) + U_{\alpha\alpha}(x)$. By Theorems XIII.44 and XIII.45 of [RSIV] $(H_\alpha - \lambda)^{-1}$ is positivity improving on $L^2(\Omega, \mathrm{R})$ for $\lambda < \inf \operatorname{Re} \operatorname{spec} H_\alpha$. Consequently, $(L_0 - \lambda)^{-1}\varphi > 0$ whenever $\varphi \dot{\geq} 0$, provided $\lambda < \inf \operatorname{Re} \operatorname{spec} L_0$. On the other hand obviously, $U\varphi \geq 0$ (respectively, $U\varphi \dot{\geq} 0$) for $\varphi \geq 0$ if $U_{\alpha\beta} \geq 0$ (respectively, if $U_{\alpha\beta} \geq 0$ and $U_{\alpha\beta} \not\equiv 0$) $\forall \alpha, \beta$, i.e. in the case (i) (respectively, case (ii)). Hence the operator $K \equiv (L_0 - \lambda)^{-1} U$ is positivity preserving in the case (i) and positivity improving in the case (ii). Hence so is the operator T due to the expansion

$$T = \sum_{n=0}^{\infty} K^n (L_0 - \lambda)^{-1} ,$$

which converges in norm for λ sufficiently negative. Λ

Now an elementary extension of Theorem XIII.44 of [RSIV] (see also the proof of Krein-Rutman theorem [Z]) yields the following result.

Theorem 4.2. *Let $T = (L - \mu)^{-1}$, $\mu < \inf \operatorname{Re} \operatorname{spec} L$, be positivity improving and let $\sup \operatorname{Re} \operatorname{spec} T$ be an eigenvalue. Let λ be an eigenvalue of T with an eigenfunction ψ. Then*

$$\psi > 0 \quad \leftrightarrow \quad \lambda = \sup \operatorname{Re} \operatorname{spec} T .$$

Moreover, $\sup \operatorname{Re} \operatorname{spec} T$ *is a simple eigenvalue.*

Combining this theorem is Proposition 4.1, we arrive at

Corollary 4.3. *Let L be the operator defined in (4.1) and let* $\inf \operatorname{Re} \operatorname{spec} L$ *be an eigenvalue (which is always true if Ω is bounded). Assume L satisfies the condition of Proposition 4.1(ii). If λ is an eigenvalue of L and ψ its corresponding eigenfunction, then*

$$\psi > 0 \quad \leftrightarrow \quad \lambda = \inf \operatorname{Re} \operatorname{spec} L \ .$$

Moreover, $\inf \operatorname{Re} \operatorname{spec} T$ *is a simple eigenvalue.*

References

[AP] S.L. Adler and T. Piran, *Rev. Mod. Phys.* **56** (1984), 1.

[AR] C. Atkinson and G.E.H. Reuter, Deterministic epidemic waves, *Math. Proc. Camb. Phil. Soc.* **80** (1976), 315–330.

[BB] R. Blumenfeld and D.J. Bergman, *Physica A* **157** (1989), 428.

[BM] J. Bekenstein and M. Milgrom, *Astrophys. J.* **286** (1984), 7.

[BNV] H. Berestycki, L. Nirenberg and S.R.S. Varadhan, The principal eigenvalue and maximum principle for second-order elliptic operators on general domains, *Comm. Pure Appl. Math.* **47** (1994), 47–92.

[CK] S. Chanillo and Kiessling, M., Rotational symmetry of solutions of some nonlinear problems in statistical mechanics and in geometry, *Comm. Math. Phys.* **160** (1994), 217–238.

[DK] O. Diekmann and H.G. Kaper, On the bounded solutions of a nonlinear convolution equation, *Nonlinear Analysis, Theory, Methods & Applications* **2** (1978) 721–737.

[GNN] B. Gidas, W.-M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68** (1979), 209–243.

[GJ] J. Glimm and A. Jaffe, The $\lambda(\varphi^4)$ quantum field theory without cut-offs: The field operators and the approximate vacuum, *Annals of Math* **91** (1970), 362–401.

[L] C.-M. Li, Monotonicity and symmetry of solutions of fully nonlinear elliptic equations on unbounded domains, *Commun. PDE* **16** (1991), 585–615.

[LN1] Y. Li and W.M. Ni, On the existence and symmetry properties of finite total mass solutions of the Matukama equation, the Eddington equation, and their generalizations, *Arch. Rat. Mech. Anal.* **108** (1989), 175–194.

[LN2] Y. Li and W.M. Ni, On the asymptotic behaviour and radial symmetry of positive solutions of semilinear elliptic equations in R^n, I, *Arch. Rat. Mech. Anal.* **118** (1992), 195–222.

[LN3] Y. Li and W.M. Ni, On the asymptotic behaviour and radial symmetry of positive solutions of semilinear elliptic equations in R^n, II, *Arch. Rat. Mech. Anal.* **118** (1992), 223–243.

[LL] L.H. Lieb and M. Loss, Analysis, *AMS*, 1997.

[M] M. Milgrom, *Astrophys. J.* **302** (1986), 617.

[NP] R.D. Nussbaum and Y. Pinchover, On variational principles for the generalized principal eigenvalue of second order elliptic operators and some applications, *J. Anal. Math.* **59** (1992), 161–177.

[O] M. Obata, The conjectures on conformal transformations of Riemannian manifolds, *J. Diff. Geom.* **6** (1971), 247–258.

[OSI] Yu.N. Ovchinnikov and I.M. Sigal, Ginzburg-Landau equation I: Static vortices, preprint, 1995.

[PW] M.H. Protter and H.F. Weinberger, *Maximum Principles in Differential Equations,* Springer-Verlag, 1984.

[Z] E. Zeidler, *Nonlinear Functional Analysis and its Applications I,* Springer-Verlag, 1988.