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GINZBURG-LANDAU EQUATION I. STATIC VORTICES *

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Abstract. We consider radially symmetric solutions of the Ginzburg-Landau equation (without magnetic field) in dimension 2. Such solutions are called vortices and are specified by their winding number at infinity (vorticity). For a given vorticity n we prove existence and uniqueness (modulo symmetry transformations) of an n-vortex and show that for $n = 0, \pm 1$ such vortices are stable while for $|n| \geq 2$, unstable. We introduce the renormalized Ginzburg-Landau energy and use it for the existence and uniqueness proof. Our stability proof is novel and uses the concept of symmetry breaking and its consequence in the form of zero modes of the linearized equation.
Keywords: Ginzburg-Landau equation, energy functional, vortex, stability, symmetry breaking.

Contents

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1. Introduction

This paper starts our study of the Ginzburg-Landau and related equations. By the former we mean the equation

$$-\Delta\psi + (|\psi|^2 - 1)\psi \;=\; 0 \tag{1.1}$$

for ψ: $\mathbb{R}^2 \to \mathbb{C}$ subject to the boundary condition

$$|\psi| \to 1 \text{ as } |x| \to \infty \text{ uniformly in } \hat{x} = \frac{x}{|x|} \;. \tag{1.2}$$

The origin of this boundary condition is dictated by the physical interpretation of this equation (see below).* This boundary condition allows one to introduce the degree of ψ as the index of ψ, considered as a vector field on $\mathbb{R}^2$, at ∞:

$$\deg\psi \;=\; \frac{1}{2\pi}\int_{|x|=R} d(\arg\psi) \tag{1.3}$$

for R sufficiently large (so that $|\psi| \neq 0$ on the circle $|x| = R$). The mathematical problem here is to study solutions to Eqn (1.1) for a given $\deg\psi = n$.

In Physics Eqn (1.1) describes a superfluid and $\deg\psi \neq 0$ corresponds to rotation of this superfluid. A similar equation, namely the one in which the rôle of $\deg\psi$ is played by the magnetic flux, is used to study superconductors. More generally, such equations arise as the equation for critical points of the free energy functional (Ginzburg-Landau energy functional) describing the matter near phase transitions (see e.g. [ZJ]).

In this paper we present some general mathematical facts about this equation, the most important of which is analysis of the symmetry breaking and introduction of the renormalized Ginzburg-Landau energy functional, and study the simplest solutions of (1.1). The latter are solutions of the form

$$\psi^{(n)}(x) \;=\; f^{(n)}(r)e^{in\theta} \;, \tag{1.4}$$

* The maximum principle implies that $|\psi| < 1$ (see Appendix A).

where (r, θ) are the polar coordinates of x (w.r.t. the origin $x = 0$). These are the most symmetric solutions and following [JT] we call them radially symmetric solutions (or n-vortices). Clearly $\deg \psi^{(n)} = n$. We prove existence of such solutions and their uniqueness in the class of functions of the form (1.4) (which is, probably, a known fact, but the one we could not locate in the literature) and classify them as minima and saddle points of the renormalized energy functional. Namely we show that

(a) For $|n| = 1$, $\psi^{(n)}$ are strict local minima.

(b) For $|n| \geq 2$, $\psi^{(n)}$ are saddle points.

Besides, it follows from the definition that for $n = 0$, a strict absolute minimum is given by $\psi^{(0)} \equiv z$ for any $z \in \mathbb{C}$ with $|z| = 1$.

In order to keep the arguments as simple as possible we develop them for Eqn (1.1) only. In fact, these arguments are rather general and applicable to a large class of non-linear equations. For instance, they hold for equations of the form

$$\Delta\psi = p(|\psi|^2)\psi$$

under various assumptions on the function p on $[0, \infty)$. To begin with we assume that p is continuously differentiable, has a single zero at $s = 1$ (or any other positive point) and $(s-1)p(s) > 0$ if $s \neq 1$. The existence result holds if $P(s) \geq c(1-s)_+^\alpha$ for some $c, \alpha > 0$. Here $P(s)$ is defined by $P'(s) = p(s)$ and $P(1) = 0$. The partial convexity result, and, consequently, uniqueness result hold if

$$p'(s) \geq 0 .$$

The latter condition suffices also for stability result (a), above. Instability result (b) requires more precise information about the function $p(s)$.

After this paper was completed we found that some of the statements it contains were proven elsewhere. In particular, the result that $|\psi| \leq 1$ was obtained in [BMR]. Theorem 4.1 on relation between the standard Ginzburg-Landau energy and the degree was proven in [C, BMR]. The existence of radially symmetric solutions and their uniqueness for each n in the class of functions of the form (1.4) was proven in [H] (see also Appendix in [BBH] and references therein). The fact that only radially symmetric solutions with $n = 0$, ± 1 are local minimizers was established in [LL, M] for the problem in a ball and in [S], as in our case, for $\mathbb{R}^2$. In fact, [S] has shown more: that any minimizer (in the sense of [S]) must have a single zero. The methods in the listed papers are different from ours. In addition, it was shown in [BMR] that for any solution, ψ, to (1.1) of degree n, $\int(|\psi|^2 - 1)^2 = 2\pi n^2$. Existence of radial vortices and their properties for the Ginzburg-Landau equation coupled to a magnetic field were proven in [JT].

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2. Symmetry breaking

The purpose of this section is to set up a stage for utilizing some concepts, which first appeared in quantum field theory, for non-linear differential equations. First we note that the *symmetry group*, G_{sym}, of Eqn (1.1), i.e. the maximal group of transformations, g, of ψ s.t., if ψ is a solution to (1.1), then so is $g\psi$, is

$$G_{\text{sym}} \;=\; \mathbb{R}^2 \times O(2) \times U(1) \times \text{Charge}\ , \tag{2.1}$$

where $\mathbb{R}^2$ acts as translations of the spatial variable x, $\psi(x) \to \psi(x-h)$, $h \in \mathbb{R}^2$, $O(2)$, as rotations of the spatial variable x, $\psi(x) \to \psi(R^{-1}x)$, $R \in O(2)$, $U(1)$, as gauge transformations, $\psi(x) \to \lambda\psi(x)$, $\lambda \in U(1)$ (i.e. $\lambda \in \mathbb{C}$ and $|\lambda| = 1$) and, finally, the discrete group Charge (of charge transformations) acts as $\psi \to \bar{\psi}$. Again the first two groups $\mathbb{R}^2$ and $O(2)$, which constitute the group of rigid motions of $\mathbb{R}^2$, act on the underlying domain space of map ψ, while the other two groups, on the target space of ψ.

By the *symmetry group* G_ψ of a *solution* ψ we understand the largest subgroup of G_{sym} which leaves ψ fixed i.e. $G_\psi = \{g \in G_{\text{sym}} \mid g\psi = \psi\}$. Then the part of G_{sym} broken by ψ is G_{sym}/G_ψ, which, in general, is not a subgroup of G_{sym}, but can be identified with one in our situation. We will also talk about (one parameter) subgroup $H \subset G_{\text{sym}}$ *preserved* (or *broken*) by ψ meaning by this that $h\psi = \psi \; \forall h \in H$ (or $h\psi \neq \psi \; \forall h \in H$, $h \neq \text{id}$, in the latter case we might choose to ignore a discrete subgroup of H preserved by ψ).

As an example, the subgroup of translations, $\mathbb{R}^2$, is preserved iff ψ is independent of x. This happens only if $\deg\psi = 0$ and the solution ψ in this case is $\psi = e^{i\alpha}$, $\alpha \in \mathbb{R}$. This solution preserves also the subgroup of rotations but breaks the gauge and Charge subgroups. As a result we have,in fact, a whole continuum of solutions. The next simplest class of solutions are radially symmetric solutions, i.e. solutions of the form

$$\psi^{(n)}(x) \;=\; f^{(n)}(r)e^{in\theta} \;, \tag{2.2}$$

where, recall, (r, θ) are polar coordinates of x (w.r.t. the origin) and $f^{(n)}(r)$ satisfies the ordinary differential equation

$$\Big(-\Delta_r + \frac{n^2}{r^2}\Big)f + (f^2 - 1)f \;=\; 0 \;, \tag{2.3}$$

where Δ_r is the radial laplacian in dimension 2, i.e. $\Delta_r f = \frac{1}{r}\partial_r(r\partial_r f)$. Clearly $\deg\psi^{(n)} = n$. The symmetry group of $\psi^{(n)}$, $n \neq 0$, is

$$\Gamma \times U(1)^{-n}O(2) \;, \tag{2.4}$$

where Γ is the discrete subgroup of $O(2)$ of rotations by the angles $\frac{2\pi k}{n}$, $k \in \mathbb{Z}$, and

$$U(1)^n O(2) \;=\; \{u(\varphi)^n r(\varphi) \mid \varphi \in [0, 2\pi]\} \;, \tag{2.5}$$

where $r(\varphi)\psi(x)=\psi\big(R(\varphi)^{-1}x\big)$ with $R(\varphi)$, the rotation by the angle φ, i.e $\begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$, and $u(\varphi)\psi(x)=e^{i\varphi}\psi(x)$. Thus $\psi^{(n)}$ breaks the translation subgroup, $\mathbb{R}^2$, the rotation subgroup $O(2)/\Gamma$ (or the corresponding gauge subgroup) and the charge subgroup.

3. Linearized equation

Given an equation $F(\psi)=0$ and its solution ψ_0, the *linearization* of this equation around ψ_0 is the equation

$$DF(\psi_0)\xi \;=\; 0\;, \tag{3.1}$$

where $DF(\psi_0)$ is the tangent map (or differential, or Fréchet derivative, or gradient map) of F at ψ_0. The latter is defined as

$$DF(\psi)\xi \;=\; \frac{\partial}{\partial\lambda}F(\psi_\lambda)\Big|_{\lambda=0}\;,$$

where ψ_λ is a path starting at ψ, with the velocity ξ, i.e. $\psi_0=\psi$ and $\frac{\partial}{\partial\lambda}\psi_\lambda\Big|_{\lambda=0}=\xi$. If F maps a topological space X into a topological space Y, then $DF(\psi_0)$ maps $T_{\psi_0}X$ into $T_{F(\psi_0)}Y$ where $T_{\psi_0}X$ is the tangent space for X at ψ_0 i.e.

$$T_{\psi_0}X \;=\; \Big\{\xi \mid \exists \text{ path } \psi_\lambda \text{ in } X \text{ s.t. } \psi_0=\psi \text{ and } \frac{\partial}{\partial\lambda}\psi_\lambda\Big|_{\lambda=0}=\xi\Big\}$$

and similarly for $T_{F(\psi_0)}Y$. In our case, $X=\{\psi\in C^2(\mathbb{R}^2)\mid \psi \text{ satisfies (1.2)}\}$ and

$$F(\psi) \;=\; -\Delta\psi+(|\psi|^2-1)\psi \tag{3.3}$$

and consequently

$$DF(\psi)\xi \;=\; -\Delta\xi+(2|\psi|^2-1)\xi+\psi^2\bar{\xi}\;. \tag{3.4}$$

As we see here, for complex ψ, $DF(\psi)$ is only a *real* linear operator. The linearization of Eqn (1.1) around ψ_0 is

$$-\Delta\xi+(2|\psi_0|^2-1)\xi+\psi_0^2\bar{\xi} \;=\; 0\;. \tag{3.5}$$

We have the following elementary but crucial for our application result:

Theorem 3.1. *Let ψ_0 be a solution to the equation $F(\psi) = 0$ breaking an one parameter subgroup $g(s) \in G_{\rm sym}$ (the symmetry group of this equation). Let T be the generator of $g(s)$. Then $T\psi_0$ solves the linearized equation $DF(\psi_0)\xi = 0$.*

Proof. By the definition of $G_{\rm sym}$, $\psi_s = g(s)\psi_0$ solves $F(\psi) = 0$. Therefore, by the definition of $DF(\psi_0)$ and T,

$$0 \;=\; \frac{\partial}{\partial s} F(\psi_s)\Big|_{s=0} \;=\; DF(\psi_0)T\psi_0 \ . \qquad \square$$

Applying this result to our situation and observing that the generators of translations, rotations and gauge tranformations are ∇_x, $x_1\partial_{x_2} - x_2\partial_x = \partial_\theta$ and i, respectively, we arrive at the following important observation.

Corollary 3.2. *The functions $\partial_{x_1}\psi^{(n)}$, $\partial_{x_2}\psi^{(n)}$ and $i\psi^{(n)}$ solve the linearized equation*

$$L_{\psi^{(n)}}(\xi) \;=\; 0 \ , \tag{3.6}$$

where L_ψ is the tangent map to (3.3) at ψ:

$$L_\psi(\xi) \;=\; -\Delta\xi + (2|\psi|^2 - 1)\xi + \psi^2\bar{\xi} \ . \tag{3.7}$$

For future needs we present

Lemma 3.3. *We have*

$$\begin{aligned} \partial_{x_1}\psi^{(n)} \;=\; & \frac{1}{2}\Big(f^{(n)\prime} - \frac{n}{r}f^{(n)}\Big)e^{i(n+1)\theta} \\ & + \frac{1}{2}\Big(f^{(n)\prime} + \frac{n}{r}f^{(n)}\Big)e^{i(n-1)\theta} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \partial_{x_2}\psi^{(n)} \;=\; & -\frac{i}{2}\Big(f^{(n)\prime} - \frac{n}{r}f^{(n)}\Big)e^{i(n+1)\theta} \\ & + \frac{i}{2}\Big(f^{(n)\prime} + \frac{n}{r}f^{(n)}\Big)e^{i(n-1)\theta} \ . \end{aligned} \tag{3.9}$$

Proof. Using the representation $\psi^{(n)} = f^{(n)} e^{in\theta}$ and the relation

$$\nabla\theta(x) \;=\; \frac{Jx}{r^2}\ , \text{ where } J \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ , \tag{3.10}$$

we obtain

$$\nabla\psi^{(n)} \;=\; f^{(n)\prime}\frac{x}{r}e^{in\theta} + f^{(n)}\frac{in}{r}\frac{Jx}{r}e^{in\theta}\ .$$

Taking into account that

$$x_1 \;=\; r\cos\theta \;=\; \frac{1}{2}r(e^{i\theta} + e^{-i\theta})$$

and

$$x_2 \;=\; r\sin\theta \;=\; \frac{1}{2i}r(e^{i\theta} - e^{-i\theta})\ ,$$

we derive easily (3.8) and (3.9). □

4. Renormalized energy functional

It is a straightforward observation that Eqn (1.1) is the equation for critical points of the following functional

$$\mathcal{E}(\psi) \;=\; \frac{1}{2}\int\Big(|\nabla\psi|^2 + \frac{1}{2}(|\psi|^2-1)^2\Big)\ . \tag{4.1}$$

Indeed, if we define the variational derivative, $\partial_\psi\mathcal{E}(\psi)$, of $\mathcal{E}$ by

$$\mathrm{Re}\int \xi\partial_\psi\mathcal{E}(\psi) \;=\; \frac{\partial}{\partial\lambda}\mathcal{E}(\psi_\lambda)\Big|_{\lambda=0} \tag{4.2}$$

for any path ψ_λ s.t. $\psi_0 = \psi$ and $\frac{\partial}{\partial\lambda}\psi_\lambda\Big|_{\lambda=0} = \xi$, then the l.h.s. of Eqn (1.1) is equal to $\overline{\partial_\psi\mathcal{E}(\psi)} = \partial_{\bar\psi}\mathcal{E}(\psi)$ for $\mathcal{E}(\psi)$ given in (4.1).

(4.1) is the celebrated Ginzburg-Landau (free) energy. However, there is a problem with it in our context.

Theorem 4.1. *Let ψ be an arbitrary C^1 vector field on $\mathbb{R}^2$ s.t. $|\psi| \to 1$ as $|x| \to \infty$ uniformly in $\hat{x} = \frac{x}{|x|}$ and $\deg\psi \neq 0$. Then $\mathcal{E}(\psi) = \infty$.*

Proof. Let $\psi = fe^{i\varphi}$ with $f = |\psi|$. Then

$$|\nabla\psi|^2 \;=\; |\nabla f|^2 + f^2|\nabla\varphi|^2$$

and therefore

$$\int |\nabla\psi|^2 \;\geq\; \int f^2|\nabla\psi|^2 \;.$$

Moreover, by the condition on $f = |\psi|$, there is R s.t. $f \geq \frac{1}{\sqrt{2}}$ for all $|x| \geq R$. Hence

$$\int |\nabla\psi|^2 \;\geq\; \frac{1}{2}\int_{|x|\geq R} |\nabla\varphi|^2 \;.$$

On the other hand, the relation $\int\limits_{|x|=r} d\varphi = 2\pi\deg\psi$ implies that

$$\begin{aligned} 2\pi|\deg\psi| \;&\leq\; r\int_0^{2\pi} |\nabla\psi| d\theta \\ &\leq\; r\Big(2\pi\int_0^{2\pi} |\nabla\varphi|^2 d\theta\Big)^{\frac{1}{2}} \;, \end{aligned}$$

by the Schwarz inequality. Thus

$$\int_0^{2\pi} |\nabla\varphi|^2 d\theta \;\geq\; \frac{2\pi(\deg\psi)^2}{r^2} \;, \tag{4.3}$$

which yields

$$\int |\nabla\psi|^2 \;\geq\; \pi(\deg\psi)^2\int_R^{\infty} \frac{1}{r^2} r dr \;=\; \infty \;. \qquad \square$$

The proof above actually shows how to save the energy functional, which is an important variational tool. We renormalize it as follows. Let $\chi(x)$ be a smooth real function on $\mathbb{R}^2$ s.t.

$$\chi(x) \;=\; \begin{matrix} 1 & \text{for } |x| \geq 2 \\ 0 & \text{for } |x| \leq 1 \end{matrix} \tag{4.4}$$

Define

$$\mathcal{E}_{\text{ren}}(\psi) \;=\; \frac{1}{2}\int \Big(|\nabla\psi|^2 - \frac{(\deg\psi)^2}{r^2}\chi + P(|\psi|^2)\Big) d^2x \tag{4.5}$$

where

$$P(u) \;=\; \frac{1}{2}(u-1)^2 \;. \tag{4.6}$$

We list here the most important properties of $\mathcal{E}_{\rm ren}(\psi)$. Below $\varphi = \arg\psi$.

(a) $\partial_{\bar\psi}\mathcal{E}_{\rm ren}(\psi) = -\Delta\psi + P'(|\psi|^2)\psi$.

(b) $\forall n$ let $M_n = \Big\{\psi = fe^{i\varphi} \mid \int\limits_{|x|\geq 2} \frac{1}{r^2}|1-f^2| < \infty$, f is continuous and $f(0)=0$, $\int|\nabla(\varphi - n\theta)|r^{-1} < \infty$ and $\int|\nabla(\varphi-n\theta)|^2 < \infty\Big\}$. Then $\mathcal{E}_{\rm ren}(\psi) < \infty \;\forall\psi\in M_n$.

(c) We have the following bound from below

$$\mathcal{E}_{\rm ren}(\psi) \;\geq\; \mathcal{E}_{B(0,2)}(\psi) + \frac{1}{2}\int_{|x|\geq 2}\Big(\big|\nabla|\psi|\big|^2 - \frac{1}{2}|\nabla\varphi|^4\Big)d^2x \;, \tag{4.7}$$

where for $\Omega\subset\mathbb{R}^2$,

$$\mathcal{E}_\Omega(\psi) \;=\; \frac{1}{2}\int_\Omega\Big(|\nabla\psi|^2 - \frac{(\deg\psi)^2}{r^2}\chi + P(|\psi|^2)\Big)d^2x \;. \tag{4.8}$$

Properties (a) and (b) are straightforward, while to prove (c), we minimize

$$W(\psi) \;=\; |\psi|^2|\nabla\varphi|^2 + P(|\psi|^2)$$

w.r.t. $|\psi|$ to obtain that

$$W(\psi) \;\geq\; \Big(1 - \frac{1}{2}|\nabla\varphi|^2\Big)|\nabla\varphi|^2 \;.$$

This together with (4.3) yields (4.7). □

We call $\mathcal{E}_{\rm ren}$ the *renormalized Ginzburg-Landau* (GL) energy functional.

5. Partial convexity of $\mathcal{E}_{\rm ren}(\psi)$

The main result of this section is the following

Theorem 5.1. *Assume ψ is a critical point of $\mathcal{E}_{\rm ren}$ and η is s.t. $\eta\psi^{-1}$ is C^1 and is either real or imaginary or has proportional real and imaginary parts. Then* $d^2_\psi\mathcal{E}(\eta) := \frac{\partial^2}{\partial\lambda^2}\mathcal{E}(\psi+\lambda\eta)\Big|_{\lambda=0} > 0$.

Proof. We present the proof only for the case of $\eta\psi^{-1} \equiv h$ real, which is of the interest for us. The proof of the other cases is similar. We show that in this case

$$d_\psi^2 \mathcal{E}(\eta) \; = \; \int (|\nabla h|^2 |\psi|^2 + 2|h|^2|\psi|^4) \; > \; 0 \; . \tag{5.1}$$

Observe that

$$d_\psi^2 \mathcal{E}(\eta) \; = \; \mathrm{Re} \int \bar{\eta} L_\psi(\eta) \; , \tag{5.2}$$

where L_ψ is given by (3.7):

$$L_\psi(\eta) \; = \; -\Delta\eta + (2|\psi|^2 - 1)\eta + \psi^2\bar{\eta} \; . \tag{5.3}$$

We compute the r.h.s. of the expression above. We start with

$$\begin{aligned} \mathrm{Re}\langle h\psi, \Delta(h\psi)\rangle \; &= \mathrm{Re}\langle \psi, h[\Delta, h]\psi\rangle \\ &\quad + \mathrm{Re}\langle \psi, h^2\Delta\psi\rangle \; . \end{aligned}$$

Next, since h is real,

$$\mathrm{Re}\big\langle \psi, h[\Delta, h]\psi\big\rangle \; = \; \frac{1}{2}\big\langle \psi, \big[h, [\Delta, h]\big]\psi\big\rangle \; .$$

Since

$$[\Delta, h] \; = \; 2\nabla h \cdot \nabla + \Delta h \; ,$$

we obtain

$$\big[h, [\Delta, h]\big] \; = \; -2|\nabla h|^2 \; .$$

Hence

$$\mathrm{Re}\langle h\psi, [\Delta, h]\psi\rangle \; = \; -\int |\nabla h|^2 |\psi|^2 \; . \tag{5.4}$$

Now Eqn (1.1) yields

$$\begin{aligned} \mathrm{Re}\langle \psi, h^2\Delta\psi\rangle \; &= \; \langle \psi, h^2(|\psi|^2 - 1)\psi\rangle \\ &= \; \int h^2(|\psi|^4 - |\psi|^2) \; . \end{aligned}$$

Hence

$$-\mathrm{Re}\langle h\psi, \Delta(h\psi)\rangle \; = \; \int (|\nabla h|^2|\psi|^2 + h^2|\psi|^2 - h^2|\psi|^4) \; .$$

Recalling (5.3), we obtain

$$\mathrm{Re}\int \bar{\eta} L_\psi(\eta) \;=\mathrm{Re}\int \big(|\nabla h|^2|\psi|^2 + h^2|\psi|^2 - h^2|\psi|^4 \\ + (2|\psi|^2-1)|\eta|^2 + \psi^2\bar{\eta}^2\big)\;.$$

Remembering that $\eta = h\psi$, we arrive at (5.1). □

6. Existence of radially symmetric vortices

The main result of this section is

Theorem 6.1. *For any n, there is a unique radially symmetric vortex of degree n.*

This theorem follows directly from the following stronger result and a simple argument showing that any radially symmetric vortex of degree n can be written, up to a symmetry transformation, in the form (1.4) with $f^{(n)}$ real.

Theorem 6.2. *For any n, the functional $\mathcal{E}(f) = \mathcal{E}_{\mathrm{ren}}(fe^{in\theta})$ has a unique minimizer among real f s.t. $\mathcal{E}(f) < \infty$. This minimizer $f^{(n)}$, is radially symmetric and monotonically increasing and $f^{(n)}e^{in\theta}$ is an n-vortex, i.e. solution to Eqn (1.1) of degree n.*

Proof. Let $\psi = fe^{in\theta}$ with f real. Then

$$|\nabla\psi|^2 \;=\; |\nabla f|^2 + n^2 f^2|\nabla\theta|^2$$

(this is true, in general, only if f is either real or radially symmetric or both). Hence

$$\mathcal{E}(f) \;=\; \frac{1}{2}\int \Big\{|\nabla f|^2 + \frac{n^2}{r^2}(f^2-\chi) + \frac{1}{2}(f^2-1)^2\Big\}d^2x\;. \tag{6.1}$$

First of all we note that $\mathcal{E}(f)$ is bounded from below. Indeed

$$\frac{n^2}{r^2}(f^2-1) + \frac{1}{2}(f^2-1)^2 \;\geq\; -\frac{n^4}{2r^4} \tag{6.2}$$

and therefore, since $\chi \equiv 1$ for $|x| \geq 2$, we have

$$\begin{aligned}\mathcal{E}(f) \;&=\; \frac{1}{2}\int_{|x|\leq 2}\{\cdots\}d^2x + \frac{1}{2}\int_{|x|\geq 2}\{\cdots\}d^2x \\ &\geq\; \frac{1}{2}\int_{|x|\leq 2}\{\cdots\}d^2x + \frac{1}{2}\int_{|x|\geq 2}\Big\{|\nabla f|^2 - \frac{n^4}{2r^4}\Big\}d^2x \\ &\geq\; -\frac{1}{2}\int_{1\leq r\leq 2}\frac{n^2}{r^2}d^2x - \frac{1}{4}\int_{r\geq 2}\frac{n^4}{r^4} \;>\; -\infty\;.\end{aligned}$$

Next, we show that, if $0 \le f \le 1$, then

$$\mathcal{E}(f) \ \ge \ \frac{1}{2} \int \left\{ |\nabla \xi|^2 - \frac{2n^2}{r^2} \chi_{r \ge 1} \xi + \frac{1}{2} \xi^2 \right\} , \tag{6.3}$$

where $\xi = 1 - f$. Indeed, $|\nabla f| = |\nabla \xi|$, $(1 - f^2)^2 = \xi^2 (2 - \xi)^2 \ge \xi^2$ if $\xi \le 1$ and, finally

$$\frac{n^2}{r^2}(f^2 - \chi) \ = \ \frac{n^2}{r^2} \chi_{r \le 1} f^2 + \frac{n^2}{r^2} \chi_{r \ge 1} (f^2 - \chi)$$

$$\ge \ \frac{n^2}{r^2} \chi_{r \ge 1} (f^2 - 1) \ = \ -\frac{n^2}{r^2} \chi_{r \ge 1} \xi (2 - \xi) \ \ge \ -\frac{2n^2}{r^2} \chi_{\ge r} \xi \ .$$

Let $M = \{ f \text{ real} \mid \mathcal{E}(f) < \infty \}$. Since $\mathcal{E}(f)$ is bounded from below on M, there is a minimizing sequence $f_m \in M$ for $\mathcal{E}$:

$$\lim_{m \to \infty} \mathcal{E}(f_m) \ = \ \inf_{v \in M} \mathcal{E}(v) \ . \tag{6.4}$$

Without a loss of generality we can take $0 \le f_m \le 1$. Otherwise we pass from f_m to $f'_m(x) = \min\big(|f(x)|, 1\big)$. Since

$$\mathcal{E}(f'_m) \ \le \ \mathcal{E}(f_m) \ ,$$

$\{f'_m\}$ would be also a minimizing sequence.

Since $0 \le f_m \le 1$, we have due to (6.3) that

$$\int (|\nabla \xi_m|^2 + |\xi_m|^2) d^2x \ \le \ K \ ,$$

where $\xi_m = 1 - f_m$ for some fixed $K < \infty$. Hence by the Banach-Alaoglu theorem $\{\xi_m\}$ is weakly comapct in $H_1(\mathbb{R}^2)$ and by the Kondrashov-Sobolev embedding theorem $\{\xi_m\}$ is compact in $L^2(\Omega)$ for any compact $\Omega \subset \mathbb{R}^2$. Hence using a diagonalization procedure we can find a subsequence $\{\xi_{m'}\}$ s.t.

$$\xi_{m'} \ \to \ \xi_0 \qquad \text{weakly in } H_1(\mathbb{R}^2)$$

$$\xi_{m'} \ \to \ \xi_0 \qquad \text{a.e. in } \mathbb{R}^2$$

as $m' \to \infty$. Let $f_0 = 1 - \xi_0$. Since $\int |\nabla \xi|^2$ is weakly lower semicontinuous in $H_1(\mathbb{R}^2)$ and since by the Fatou lemma

$$\begin{aligned} \varliminf_{m' \to \infty} \int V(\xi_{m'}) &\geq \int \varliminf_{m' \to \infty} V(\xi_{m'}) \\ &= \int V(\xi_0) \end{aligned}$$

where $V(\xi) = \frac{1}{2}\left((1-\xi)^2 - 1\right)^2 \geq 0$, we have that

$$\varliminf_{m' \to \infty} \mathcal{E}(f_{m'}) \geq \mathcal{E}(f_0) .$$

On the other hand since f_0 is real and $\mathcal{E}(f_0) < \infty$, we have that $f_0 \in M$ and therefore $\mathcal{E}(f_0) \geq \inf\limits_{v \in M} \mathcal{E}(v)$. The last two inequalities yield

$$\inf_{v \in M} \mathcal{E}(v) = \mathcal{E}(f_0) . \tag{6.5}$$

Next, we want to show that f_0 is radially symmetric and monotonically increasing. The method of symmetric rearrangement, applied to the function $1 - |f|$, seems to yield only that there exists a radially symmetric monotonically increasing minimizer. So we do the problem from scratch. Let $u = \sqrt{\overline{f_0^2}}$, where $\bar{g}(r) = \frac{1}{2\pi} \int\limits_0^{2\pi} g(r, \theta) d\theta$. Then $\overline{f_0^2} = u^2$ and

$$\overline{f_0^4} \geq u^4$$

and

$$\overline{|\nabla_r f_0|^2} \geq |\nabla_r u|^2 .$$

The last inequality implies that $\int |\nabla f_0|^2 d^2x \geq \int |\nabla_r u|^2 r dr$ with the inequality taking place only if f_0 is independent of θ. Hence

$$\mathcal{E}(f_0) \geq \mathcal{E}(u)$$

and the equality holds only if f_0 is radially symmetric. Since f_0 is a minimizer we conclude that it must be radially symmetric.

Now we show that $f_0' > 0$. Differentiating the equation $\mathcal{E}'(f_0) = 0$ w.r. to r, we obtain

$$\Big(\mathcal{E}''(f_0) + \frac{1}{r^2}\Big) f_0' \;=\; \frac{3n^2}{r^3} f_0 \;, \tag{6.6}$$

where $f_0' = \frac{\partial}{\partial r} f_0$ and

$$\mathcal{E}''(f_0) \;=\; -\Delta_r + \frac{n^2}{r^2} + 3f_0^2 - 1 \;.$$

Since f_0 is the minimizer we have that $\mathcal{E}''(f_0) \geq 0$. Since $f_0 > 0$, the maximum principle (see [GT, Str, Thm B.4]) applied to Eqn (6.6) (take $h = f_0$ in Thm B.4 of [Str]) yields that $f_0' > 0$.

Next, f_0 satisfies the Euler-Lagrange equation

$$\Big(- \Delta + \frac{n^2}{r^2}\Big) f + (f^2 - 1) f \;=\; 0 \;.$$

Since f_0 is radially symmetric, we have that $\nabla f_0 \cdot \nabla \theta = 0$. Thus

$$\Delta(f_0 e^{in\theta}) \;=\; \Big(\Delta f_0 - \frac{n^2}{r^2} f_0\Big) e^{in\theta}$$

and therefore $f_0 e^{in\theta}$ satisfies the equation

$$-\Delta \psi + (|\psi|^2 - 1)\psi \;=\; 0 \;.$$

Since $\deg(f_0 e^{in\theta}) = n$, we have proven that (i) $\mathcal{E}_{\rm ren}(\psi)$ has a minimizer on the set $\{ f e^{in\theta} \mid f$ real and $\mathcal{E}_{\rm ren}(f e^{in\theta}) < \infty \}$ and (ii) any minimizer of $\mathcal{E}_{\rm ren}(\psi)$ on this set has f radially symmetric and monotonically increasing. A partial convexity result of Section 5 implies that $\mathcal{E}(f)$, for f real, is strictly convex. Hence $\mathcal{E}$ has a unique minimizer. □

7. Hessian

Before proceeding to the last and key section of this paper, which classifies the critical points of the renormalized Ginzburg-Landau functional, $\mathcal{E}_{\rm ren}(\psi)$, we present a few generalities about Hessians of functionals, used there. Let $\psi_{\lambda,\mu}$ be a two parameter variation of ψ along ξ and η, i.e. $\psi_{0,0} = \psi$,

$$\frac{\partial}{\partial \lambda} \psi_{\lambda,\mu}\Big|_{\lambda=\mu=0} \;=\; \xi,\; \frac{\partial}{\partial \mu} \psi_{\lambda,\mu}\Big|_{\lambda=\mu=0} \;=\; \eta \text{ and } \frac{\partial^2 \psi_{\lambda,\mu}}{\partial \lambda \partial \mu}\Big|_{\lambda=\mu=0} \;=\; 0 \;.$$

Then the second variation of $\mathcal{E}$ at ψ along ξ and η is computed as

$$\frac{\partial^2 \mathcal{E}(\psi_{\lambda,\mu})}{\partial\lambda\partial\mu}\Big|_{\lambda=\mu=0} = \mathrm{Re}\int \bar{\eta} L_\psi(\xi) \ ,$$

where L_ψ is the tangent map for the map $\psi \to \partial_{\bar{\psi}}\mathcal{E}(\psi)$, i.e. in the case of (4.5), it is given by (3.4).

Now it is natural to define $X = \{\psi \mid \mathcal{E}_{\mathrm{ren}}(\psi) < \infty\}$. Then $T_\psi X$ can be identified with the linear space

$$\{\xi \in L^4(\mathbb{R}^4) \mid \int |\mathrm{Re}(e^{-i\varphi}\xi)|^2 < \infty \text{ and } \int |\nabla\xi|^2 < \infty\} \ , \tag{7.1}$$

where $\varphi = \arg\psi$. Note that the quadratic form $\mathrm{Re}\int \bar{\eta} L_\psi(\xi)$ is well defined on this space. Indeed, we have

$$\begin{aligned}\mathrm{Re}\int \bar{\eta} L_\psi(\xi) &= \mathrm{Re}\int \nabla\bar{\eta}\cdot\nabla\xi \\ &\quad + 2\int |\psi|^2 \mathrm{Re}(e^{-i\varphi}\eta)\mathrm{Re}(e^{-i\varphi}\xi) \ < \ \infty \ .\end{aligned}$$

Note that if for ξ we define $\vec{\xi} = \binom{\xi}{\bar{\xi}}$, then

$$\mathrm{Re}\int \bar{\eta}\xi = \frac{1}{2}\langle \vec{\eta}, \vec{\xi}\rangle \ , \tag{7.2}$$

where the inner product on the r.h.s. is a standard one:

$$\begin{aligned}\langle \vec{\eta}, \vec{\xi}\rangle &\equiv \int \overline{(\eta, \bar{\eta})}\cdot\begin{pmatrix}\xi \\ \bar{\xi}\end{pmatrix} \\ &= \int (\eta_1, \eta_2)\cdot\begin{pmatrix}\xi_1 \\ \xi_2\end{pmatrix} \ ,\end{aligned} \tag{7.3}$$

with $\xi_1 = \mathrm{Re}\,\xi$ and $\xi_2 = \mathrm{Im}\,\xi$, etc. Let $\vec{T}_\psi X = \{\vec{\xi} \mid \xi \in T_\psi X\}$ and define $\mathrm{Hess}\,\mathcal{E}(\psi)$, the Hessian of $\mathcal{E}$ at ψ, as a map on $\vec{T}_\psi X$ given through the relation

$$\begin{aligned}\langle \vec{\eta}, \mathrm{Hess}\,\mathcal{E}(\psi)\vec{\xi}\rangle &= \frac{\partial^2}{\partial\lambda\partial\mu}\mathcal{E}(\psi_{\lambda,\mu})\Big|_{\lambda=\mu=0} \\ &= \mathrm{Re}\int \bar{\eta} L_\psi(\xi) \ ,\end{aligned} \tag{7.4}$$

where $\psi_{\lambda,\mu}$ is a two parameter variation of ψ specified above. Otherwise it can be written as

$$\operatorname{Hess}\mathcal{E}(\psi) = \begin{pmatrix} \partial^2_{\psi\bar\psi}\mathcal{E}(\psi) & \partial^2_{\bar\psi\bar\psi}\mathcal{E}(\psi) \\ \partial^2_{\psi\psi}\mathcal{E}(\psi) & \partial^2_{\psi\bar\psi}\mathcal{E}(\psi) \end{pmatrix} . \tag{7.5}$$

Observe also that (7.2) and (7.4) imply that

$$\operatorname{Hess}\mathcal{E}(\psi)\vec{\xi} = \overrightarrow{L_\psi \xi} . \tag{7.6}$$

In the case of $\mathcal{E}(\psi)$ given by (4.5) we have

$$\operatorname{Hess}\mathcal{E}(\psi) = \begin{pmatrix} -\Delta + 2|\psi|^2 - 1 & \psi^2 \\ \bar\psi^2 & -\Delta + 2|\psi|^2 - 1 \end{pmatrix} . \tag{7.7}$$

Finally we note an important symmetry relation

$$\operatorname{Re}\int \bar\eta L_\psi(\xi) = \operatorname{Re}\int \overline{L_\psi(\eta)}\,\xi , \tag{7.8}$$

which implies that $\operatorname{Hess}\mathcal{E}(\psi)$ is symmetric w.r.t. inner product (7.3).

Eqn (7.6) shows that

$$\operatorname{Null}\operatorname{Hess}\mathcal{E}(\psi) = \overrightarrow{\operatorname{Null} L_\psi} .$$

Denote by $\operatorname{Sym}\operatorname{Null}\operatorname{Hess}\mathcal{E}(\psi)$ the maximal null space of $\operatorname{Hess}\mathcal{E}(\psi)$ due to the symmetry breaking, i.e. $\operatorname{Sym}\operatorname{Null}\operatorname{Hess}\mathcal{E}(\psi) = \{\overrightarrow{L\psi} \mid L$ generator of a one parameter subgroup of G_{sym} broken by $\psi\} \cap \vec{T}_\psi X$. Then

$$\psi \text{ is a local minimum of } \mathcal{E} \leftrightarrow \operatorname{Hess}\mathcal{E}(\psi) \geq 0 \text{ and}$$

$$\operatorname{Null}\operatorname{Hess}\mathcal{E}(\psi) = \operatorname{Sym}\operatorname{Null}\operatorname{Hess}\mathcal{E}(\psi)$$

and

$$\psi \text{ is a saddle point of } \mathcal{E} \leftrightarrow \operatorname{Hess}\mathcal{E}(\psi) \text{ has a negative eigenvalue.}$$

Due to (7.6) these statements can be reformulated in terms of L_ψ.

8. Classification of radially symmetric vortices

Now we want to take a closer look at radially symmetric solutions to (1.1) as critical points of the renormalized GL energy functional. The main result of this section is

Theorem 8.1. *$\psi^{(n)}$ are local minima of $\mathcal{E}_{\rm ren}(\psi)$ for $n=0,\pm 1$ and are saddle points for $|n|\geq 2$.*

The problem is to understand the spectrum of the Hessian of $\mathcal{E}_{\rm ren}(\psi)$, $\mathrm{Hess}\,\mathcal{E}_{\rm ren}(\psi)$. To fix ideas we assume that $n\geq 0$. The case $n\leq 0$ is obtained by observing that $\overline{\psi^{(n)}}=\psi^{(-n)}$.

We begin with an elementary harmonic analysis of the linearization operator $L_{\psi^{(n)}}$ closely related to $\mathrm{Hess}\,\mathcal{E}_{\rm ren}(\psi)$. We consider $\xi(x)$ as a function of the polar coordinates r and θ, i.e. a function on $\overline{\mathbb{R}^+}\times S^1$, and, abusing notation, keep the same notation for the new function, $\xi(r,\theta)=\xi(x)$. This function can be expanded in the Fourier series in θ, i.e.

$$\xi(r,\theta) \;=\; \sum_{k=-\infty}^{\infty}\xi_k(r)e^{ik\theta}\ , \tag{8.1}$$

where the Fourier coefficients are given by

$$\xi_k(r) \;=\; (2\pi)^{-1}\int_0^{2\pi}\xi(r,\theta)e^{-ik\theta}d\theta\ .$$

Consider a map π of measurable functions ξ: $\mathbb{R}^2\to\mathbb{C}$ into measurable functions $\hat\xi=\bigoplus_{k\geq n}\binom{\xi_k}{\bar\xi_{2n-k}}$. If ξ's are endowed with inner product (7.2), then π is unitary, provided $\hat\xi$'s are endowed with the inner product

$$\langle\hat\eta,\hat\xi\rangle \;=\; \mathrm{Re}\langle\eta_n,\xi_n\rangle+\sum_{k>n}\mathrm{Re}\Big\langle\begin{pmatrix}\xi_k\\ \bar\xi_{2n-k}\end{pmatrix},\begin{pmatrix}\xi_k\\ \bar\xi_{2n-k}\end{pmatrix}\Big\rangle\ .$$

We define the real linear operator $\hat L_{\psi^{(n)}}$ on functions $\hat\xi$ by

$$\hat L_{\psi^{(n)}}\pi\xi \;=\; \pi L_{\psi^{(n)}}\xi\ . \tag{8.2}$$

Theorem 8.2. *The operator $\hat L_{\psi^{(n)}}$ is block diagonal of the form*

$$\hat L_{\psi^{(n)}}\xi \;=\; \bigoplus_{k\geq n}L^{(k)}_{\psi^{(n)}}\begin{pmatrix}\xi_k\\ \bar\xi_{2n-k}\end{pmatrix}\ , \tag{8.3}$$

where the operators $L^{(k)}_{\psi^{(n)}}$ are given by

$$L^{(k)}_{\psi^{(n)}} \;=\; \begin{pmatrix}-\Delta_r+\frac{k^2}{r^2}+2|\psi^{(n)}|^2-1 & |\psi^{(n)}|^2\\ |\psi^{(n)}|^2 & -\Delta_r+\frac{(2n-k)^2}{r^2}+2|\psi^{(n)}|^2-1\end{pmatrix}\ . \tag{8.4}$$

Here Δ_r is the radial Laplacian : $\Delta_r f = \frac{1}{r}\partial_r(r\partial_r f)$.

Proof. First of all we claim that

$$\left(L_{\psi^{(n)}}(\xi)\right)_k = -\Delta_r \xi_k + \frac{k^2}{r^2}\xi_k + (2|\psi^{(n)}|^2 - 1)^2 \xi_k + |\psi^{(n)}|^2 \overline{\xi}_{2n-k} . \tag{8.5}$$

Indeed, since $\Delta = \Delta_r + r^{-2}\partial_\theta^2$, we have that

$$(-\Delta\xi)_k = -\Delta_r \xi_k + \frac{k^2}{r^2}\xi_k . \tag{8.6}$$

Moreover, we have

$$\begin{aligned}\left(\psi^{(n)^2}\bar{\xi}\right)_k &= |\psi^{(n)}|^2 (2\pi)^{-\frac{1}{2}} \int_0^{2\pi} e^{2in\theta}\bar{\xi} e^{-ik\theta} d\theta \\ &= |\psi^{(n)}|^2 (2\pi)^{-\frac{1}{2}} \int_0^{2\pi} \xi e^{-i(2n-k)\theta} d\theta = |\psi^{(n)}|^2 \overline{\xi}_{2n-k} .\end{aligned}$$

Eqn (8.5) implies

$$\begin{pmatrix} \left(L_{\psi^{(n)}}\xi\right)_k \\ \overline{\left(L_{\psi^{(n)}}\xi\right)}_{2n-k} \end{pmatrix} = L^{(k)}_{\psi^{(n)}} \begin{pmatrix} \xi_k \\ \overline{\xi}_{2n-k} \end{pmatrix},$$

which, due to (8.2), yields (8.3). □

Note that $\hat{L}_{\psi(n)}$ is defined on the space $\bigoplus_{k \geq n} \left(L^4(\mathbb{R}^+, rdr) \oplus L^4(\mathbb{R}^4, rdr)\right) \supset \pi T_\psi X$. To keep the exposition as simple as possible we consider here only the L^2-spectral theory. How to extend our conclusions to the L^4-spaces can be gleaned from Appendix B.

Theorem 8.3. *(a)* $L^{(n)}_{\psi^{(n)}} \geq 0$ *and* 0 *is not an eigenvalue.*

(b) $L^{(n+1)}_{\psi^{(n)}} \geq 0$ and 0 is its non-degenerate eigenvalue due to breaking of the translational symmetry.

(c) $L^{(k)}_{\psi^{(n)}} \geq 0$ for $k \geq 3n$ and 0 is not an eigenvalue.

(d) $L^{(k)}_{\psi^{(n)}}$, for $n+2 \leq k \leq 2n$, has a negative eigenvalue.

(e) $\operatorname{cont\,spec} L^{(k)}_{\psi^{(n)}} = [0, \infty)$ for any k.

Proof. We omit the subindex $\psi^{(n)}$ at $L^{(k)}_{\psi^{(n)}}$ and write $L^{(k)}$ for $L^{(k)}_{\psi^{(n)}}$.

(a) Recall that the fact that $\psi^{(n)}$ breaks up the rotational (= gauge) symmetry implies that

$$L_{\psi^{(n)}}(i\psi^{(n)}) \;=\; 0 \;.$$

After separating out the angular variable this implies

$$\Big(-\Delta_r + \frac{n^2}{r^2} + {f^{(n)}}^2 - 1\Big) f^{(n)} \;=\; 0 \;, \tag{8.7}$$

where, recall, $f^{(n)} = |\psi^{(n)}|$. Since $f^{(n)} > 0$, bounded and $\notin L^2(\mathbb{R}^+, rdr)$, we conclude that the ordinary differential operator

$$-\Delta_r + \frac{n^2}{r^2} + {f^{(n)}}^2 - 1$$

is non-negative and 0 is not its eigenvalue (see Appendix B). This implies the same properties for the operator $-\Delta_r + \frac{n^2}{r^2} + 3{f^{(n)}}^2 - 1$. On the other hand using that $2n - k = n$ for $k = n$ we compute for $R = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$

$$RL^{(n)}R^{-1} \;=\; L \;, \tag{8.8}$$

where

$$L \;=\; \begin{pmatrix} -\Delta_r + \frac{n^2}{r^2} + 3{f^{(n)}}^2 - 1 & 0 \\ 0 & -\Delta_r + \frac{n^2}{r^2} + {f^{(n)}}^2 - 1 \end{pmatrix} . \tag{8.9}$$

By a result above the operator L is non-negative and 0 is not its eigenvalue. This implies statement (a).

Note parenthetically that, due to (8.7), there is a positive solution, $\binom{0}{f^{(n)}}$, to the equation

$$L\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \;=\; 0 \;. \tag{8.10}$$

A different way to arrive at this conclusion is by using that

$$\widehat{i\psi^{(n)}} \;=\; \bigoplus_{k \geq n} \begin{pmatrix} if^{(n)} \\ -if^{(n)} \end{pmatrix} \delta_{k,n} \;. \tag{8.11}$$

(b) Statement (b) is proven similarly to (a), but instead of the zero mode due to breaking the gauge group we use the zero mode due to breaking the translation group. The zero mode due to breaking the translational symmetry is $\nabla\psi^{(n)}$. Due to Lemma 3.3 and since $n-1=2n-k$ for $k=n+1$, $\widehat{\partial_{x_j}\psi^{(n)}}$ contain only the $k=n+1$ block, $(n+1,n-1)$:

$$\widehat{\partial_{x_1}\psi^{(n)}} \;=\; \bigoplus_{k\geq n} g^{(n)}\delta_{k,n+1} \tag{8.12}$$

and

$$\widehat{\partial_{x_2}\psi^{(n)}} \;=\; \bigoplus_{k\geq n} -ig^{(n)}\delta_{k,n+1} \tag{8.13}$$

where $g^{(n)}=\frac{1}{2}\binom{f^{(n)\prime}-\frac{n}{r}f^{(n)}}{f^{(n)\prime}+\frac{n}{r}f^{(n)}}$. Hence

$$0 \;=\; \hat{L}(\widehat{\partial_{x_1}\psi^{(n)}}) \;=\; \bigoplus_{k\geq n}(L^{(n+1)}g^{(n)})\delta_{k,n+1}$$

and therefore

$$L^{(n+1)}g^{(n)} \;=\; 0\ . \tag{8.14}$$

The zero mode $\widehat{\partial_{x_2}\psi^{(n)}}$ leads to the same equation.

Now observe that $Rg^{(n)}=\binom{\frac{n}{r}f^{(n)}}{f^{(n)\prime}}$, where R is the orthogonal transformation given by $R=\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}$. Since $f^{(n)}>0$ and $f^{(n)\prime}>0$, $r>0$, the vector $Rg^{(n)}$ has positive entries. It is shown in Appendix B that the latter fact and Eqn (8.14) imply that 0 is the lowest eigenvalue of $L^{(n+1)}$ and it is non-degenerate. This and statement (e) imply statement (b).

(c) Statement (c) follows from (b) and the inequality

$$L^{(k)}-L^{(n+1)} \;=\; \begin{pmatrix}\frac{k^2-(n+1)^2}{r^2} & 0\\ 0 & \frac{(k-2n)^2-(n-1)^2}{r^2}\end{pmatrix} \;>\; 0\ , \tag{8.15}$$

valid for $k\geq 3n$.

(d) Note that for $n=0,1$, the set of k's s.t. $n+2\leq k\leq 3n-1$ is empty. So we let $n\geq 2$. We consider only the easiest case $k=2n$, the $(2n,0)$ block. This case is already enough in order to draw the desirable conclusions from the present theorem. In this case

$$L^{(2n)} \;=\; \begin{pmatrix} -\Delta_r + \frac{4n^2}{r^2} + 2f^{(n)^2} - 1 & f^{(n)^2} \\ f^{(n)^2} & -\Delta_r + 2f^{(n)^2} - 1 \end{pmatrix} . \tag{8.16}$$

We use the variational criterion in order to prove existence of a negative eigenvalue for $L^{(2n)}$. In other words we find $\eta = \binom{\eta_{2n}}{\eta_0}$ s.t.

$$\langle L^{(2n)}\eta, \eta\rangle \;<\; 0 \;. \tag{8.17}$$

In order to capitalize on the fact that the operator standing in the right bottom corner of matrix (8.16) is smaller than the one standing in the left-top corner, we take η of the form $\binom{0}{\eta_0}$. In this case we have

$$\langle L^{(2n)}\eta, \eta\rangle \;=\; \langle(-\Delta_r + 2f^{(n)^2} - 1)\eta_0, \eta_0\rangle \;. \tag{8.18}$$

Next, it can be shown numerically that for $n \geq 2$

$$f^{(n)} \;\leq\; \frac{r^2}{r^2 + \frac{n^2}{2}} \;. \tag{8.19}$$

Using this we compute for $\eta_0 = e^{-\mu r^2/2}$ with $\mu = 0.3169$,

$$\langle(-\Delta_r + 2f^{(n)^2} - 1)\eta_0, \eta_0\rangle \;=\; -0.27911 \;. \tag{8.20}$$

This implies (8.17) for $\eta = \binom{0}{e^{-\mu r^2/2}}$, which yields that $L^{(2n)}$ has a negative eigenvalue.

(e) As $|x| \to \infty$, we have that

$$L^{(k)} \;\to\; \begin{pmatrix} -\Delta + 1 & 1 \\ 1 & -\Delta + 1 \end{pmatrix} \;\equiv\; L_0 \;. \tag{8.21}$$

Since the continuous spectrum is determined by a neighbourhood of ∞, we have that

$$\text{cont spec}\, L^{(k)} \;=\; \text{spec}\, L_0 \;. \tag{8.22}$$

Now we diagonalize L_0 by finding the eigenvalues of its symbol matrix:

$$\begin{vmatrix} p^2 + 1 - \lambda & 1 \\ 1 & p^2 + 1 - \lambda \end{vmatrix} \;=\; (p^2 + 1 - \lambda)^2 - 1 \;=\; 0 \;.$$

This implies $\lambda_1 = p^2 + 2$ and $\lambda_2 = p^2$. Hence L_0 can be transformed by a rotation in $\mathbb{R}^2$ to the form

$$\begin{pmatrix} -\Delta + 2 & 0 \\ 0 & -\Delta \end{pmatrix} .$$

Thus $\operatorname{spec} L_0 = \operatorname{spec} \begin{pmatrix} -\Delta + 2 & 0 \\ 0 & -\Delta \end{pmatrix} = [2, \infty) \cup [0, \infty)$, which together with (8.22) yields (e). $\square$

Proof of Theorem 8.1. For $n = 0, 1$, statement (d) of Theorem 8.3 is vacuous, since there is no k's satisfying $n + 2 \leq k \leq 3n - 1$. Consequently, due to Theorem 8.2, for $n = 1$, $\hat{L}_{\psi^{(n)}} \geq 0$ with the zero modes due to the symmetry breaking. For $n \geq 2$, statement (d) and Theorem 8.2 imply that the operator $\hat{L}_{\psi^{(n)}}$ has a negative eigenvalue. $\square$

Appendix A. $|\psi| < 1$

The following theorem is the main result of this appendix (cf. [JT, BGP, BMR, Z]).

Theorem. *If ψ is a solution to (1.1)–(1.2), then $|\psi| < 1$.*

Proof. Let $\psi = f e^{i\varphi}$, where $f = |\psi|$. Then multiplying Eqn (1.1) by $e^{-i\varphi}$ and taking the real part of the result, we obtain

$$-\Delta f + |\nabla \varphi|^2 f + (f^2 - 1) f \ = \ 0 \ . \qquad (A.1)$$

Let $D = \{x \in \mathbb{R}^2 \mid f > 1\}$. Then on D

$$\Delta f \ = \ |\nabla \varphi|^2 f + (f^2 - 1) f \ > \ 0 \ .$$

Hence f is a subharmonic on the open set D. Therefore it reaches its maximum either on ∂D or at ∞. Since $f = 1$ on ∂D and at ∞ we conclude that $f < 1$ on D which leads to a conflict with the definition of D.

Let $u = 1 - f$ and rewrite equation (A.1) as an equation for u:

$$(-\Delta + f^2 + f) u \ = \ |\nabla \varphi|^2 f \ \geq \ 0 \ .$$

Since $0 \leq u \leq 1$ and $u \to 0$ as $|x| \to \infty$, we have by a maximum principle (see [GT]) that $u > 0$, i.e. $f < 1$. $\square$

Appendix B. Perron-Frobenius argument

In this appendix we investigate the relation between positive solutions of (systems of) ordinary differential equations and their spectra. Results of this sort belong to the realm of the Perron-Frobenius theory. Our results extend somewhat the existing theory (see [GJ, RSIV]). Below $r = |x|$.

Theorem B.1. *Let $V(x)$ be a positive bounded function on $\mathbb{R}^2$, s.t. $V(x) = O(r^{-2})$ as $r \to \infty$ and let $\alpha > 0$. If the equation*

$$\Big(-\Delta + \frac{\alpha}{r^2} - V(x) \Big)\psi \;=\; 0$$

has a positive, bounded solution, ψ_1, then the spectrum of the operator $-\Delta + \frac{\alpha}{r^2} - V(x)$ on $L^p(\mathbb{R}^2)$, $2 \leq p < \infty$, is $[0, \infty)$ and any other solution of the above equation is of the form $\mathrm{const} \cdot \psi_1$.

Before proceeding to the proof we describe our machinery. Consider the Birman-Schwinger-type operator function

$$K(\lambda) \;=\; V^{\frac{1}{2}} R_0(\lambda) V^{\frac{1}{2}} \;, \qquad (B.1)$$

where $R_0(\lambda) = (L_0 - \lambda)^{-1}$, $L_0 = -\Delta + \frac{\alpha}{r^2}$, for $\lambda \leq 0$. Then ψ solves the equation

$$(L - \lambda)\psi \;=\; 0 \;,$$

where $L = -\Delta + \frac{\alpha}{r^2} - V(x)$, iff $\varphi = V^{\frac{1}{2}}\psi$ solves the equation

$$K(\lambda)\varphi \;=\; \varphi$$

and conversely, if φ solves the latter equation, then

$$\psi \;=\; R_0(\lambda) V^{\frac{1}{2}} \varphi \qquad (B.2)$$

solves the former. Moreover, since $V^{\frac{1}{2}} \le C\langle r\rangle^{-1}$, we conclude that if $\psi \in L^p(\mathbb{R}^2)$, then $\varphi = V^{\frac{1}{2}}\psi \in L^2(\mathbb{R}^2)$ and if $\varphi \in L^2(\mathbb{R}^2)$, then for $\lambda < 0$, $|\psi(x)| \le Ce^{-\sqrt{|\lambda|}r}$.

We will consider $K(\lambda)$ on $L^2(\mathbb{R}^+, rdr)$. Then $K(\lambda) \ge 0$, bounded and monotonically decreasing as $\lambda < 0$ decreases. One has the following elementary

Lemma B.2. *(i) λ is an eigenvalue of L iff 1 is an eigenvalue of $K(\lambda)$. (ii) λ is the lowest eigenvalue of L iff 1 is the largest eigenvalue of $K(\lambda)$.*

Proof. Statement (i) was already proven above. We prove (ii). If $K(\lambda)$ has an eigenvalue $\mu > 1$, then, by the monotonicity property of $K(\lambda)$, there is a number $\lambda_0 < \lambda$ s.t. $K(\lambda_0)$ has an eigenvalue 1 and therefore λ_0 is an eigenvalue of L. Thus λ cannot be the lowest eigenvalue of L. In the opposite direction, if L has an eigenvalue $\lambda_0 < \lambda$, then $K(\lambda_0)$ has an eigenvalue 1 and therefore $K(\lambda)$ has an eigenvalue > 1. □

Next, by standard results (see [RSIV, Theorems XIII.44 and XIII.45]) $K(\lambda)$ with $\lambda < 0$ is positivity improving, i.e. $K(\lambda)\varphi > 0$ (modulo a set of zero measure), whenever $\varphi \ge 0$. With a little more work one can show the same for $\lambda = 0$. This implies (see [RSIV, Theorem XIII.43]) that the eigenfunction of $K(\lambda)$, corresponding to the largest eigenvalue, is positive (and, which is not used here, that the largest eigenvalue itself is non-degenerate).

Now we are ready to proceed to the proof of Theorem B.1.

Proof of Theorem B.1. Since $\operatorname{ess\,spec} L = [0, \infty)$, L can have only isolated eigenvalues of finite multiplicities on $\mathbb{R}^-$. Assume L has negative eigenvalues. Let $\lambda_0 < 0$ be its lowest eigenvalue. Then 1 is the largest eigenvalue of $K(\lambda_0)$. Let $\varphi_0 > 0$ be the corresponding eigenfunction. Then $\psi_0 = R_0(\lambda_0)V^{\frac{1}{2}}\varphi_0 > 0$ is an eigenfunction of L, it is exponentially bounded together with its derivatives to the second order. Let ψ_1 be the solution to $L\psi = 0$ mentioned in the theorem. Then

$$\begin{aligned} 0 \; > \; \lambda_0 \int \psi_0\psi_1 \; &= \; \int L\psi_0\psi_1 \\ &= \int \psi_0 L\psi_1 \; = \; 0 \, , \end{aligned} \tag{B.3}$$

which is a contradiction. Hence L does not have negative eigenvalues.

Assume now that L has a zero eigenvalue. Then 1 is the largest eigenvalue of $K(0)$. Let $\varphi_0 > 0$ be the corresponding eigenfunction. Then the function $\psi_0 := L_0^{-1} V^{\frac{1}{2}} \varphi_0$ is positive and solves the equation $L\psi_0 = 0$. Let Ω_ε be a bounded domain in $\mathbb{R}^2$ s.t. $\psi_0 \geq \varepsilon$ on Ω_ε, $\varepsilon > 0$. Let ψ_1 be the solution to the equation $L\psi = 0$ mentioned in the theorem. Then $\psi_a = \psi_0 - a\psi_1 > 0$ in Ω_ε for $a > 0$ sufficiently small and there is $a_0 > 0$ s.t. $\psi_{a_0} \geq 0$ and ψ_{a_0} is not strictly positive in Ω_ε ($a_0 = \sup\{\psi_a | \psi_a > 0 \text{ in } \Omega_\varepsilon\}$). On the other hand by the strong maximum principle for $L + b$, where $b \geq \sup V + 1$, $\psi_{a_0} = \frac{1}{b}(L + b)\psi_{a_0} > 0$. Thus we arrived at the contradiction. $\square$

Now consider matrix-operators (or operator-valued matrices). Let L be a (2×2)-matrix operator on $X_{p,q} = L^p(\mathbb{R}^2) \oplus L^q(\mathbb{R}^2)$ for some $2 \leq p, q < \infty$, of the form $L = L_0 - V$, where

$$L_0 \;=\; \begin{pmatrix} -\Delta + W_1(x) & 0 \\ 0 & -\Delta + W_2(x) \end{pmatrix}, \tag{B.4}$$

with $W_j(x)$ bounded for $r \neq 0$ and such that for some $\alpha_j \geq 0$, β_j and $\gamma_j \geq 0$, $W_j(x) = \frac{\alpha_j}{r^2} + O\Big(\frac{1}{r}\Big)$ as $r \to 0$ and $W_j(x) = \gamma_j + \frac{\beta_j}{r^2} + O\Big(\frac{1}{r^3}\Big)$ as $r \to \infty$, and

$$V \;=\; \frac{\alpha}{r^2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{B.5}$$

with $\frac{1}{4}(\alpha_1 + \alpha_2) \geq \alpha > 0$. Moreover, we assume that $\beta_j > 0$ if $\gamma_j = 0$. This operator generalizes slightly the operator $RL^{(n+1)}R^{-1}$, where, recall, $L^{(k)} = L^{(k)}_{\psi^{(n)}}$ is given by (8.4) and $R = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$. We just dropped some irrelevant details from the definition of the latter. There is one important property of the operator L which we want to mimic. We assume that

$$-\Delta + W_j(x) \;\geq\; 0$$

and 0 is not its eigenvalue, $j = 1, 2$. Indeed, compute

$$RL^{(n+1)}R^{-1} \;=\; \begin{pmatrix} -\Delta_r + \frac{(n+1)^2}{r^2} + {f^{(n)}}^2 - 1 & 0 \\ 0 & -\Delta_r + \frac{(n+1)^2}{r^2} + 3{f^{(n)}}^2 - 1 \end{pmatrix} - \frac{2n}{r^2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

We have shown in the proof of Theorem 8.3(a) that $-\Delta_r + \frac{n^2}{r^2} + f^{(n)^2} - 1 \geq 0$. Consequently, $-\Delta_r + \frac{(n+1)^2}{r^2} + f^{(n)^2} - 1 \geq 0$ and $-\Delta_r + \frac{(n+1)^2}{r^2} + 3f^{(n)^2} - 1 > 0$ and 0 is not an eigenvalue of these operators. These are the properties of $RL^{(n+1)}R^{-1}$ which are encoded in the above assumption. In what follows the inequality $\psi > 0$ for $\psi = \binom{\psi_1}{\psi_2} \in \mathbb{R}^2$ means that $\psi_1, \psi_2 > 0$ (modulo a set of zero measure). We say in this case that ψ is positive. Similarly for $\psi \geq 0$. Recall that the operator $RL^{(n+1)}R^{-1}$ has an eigenfunction $Rg^{(n)} = \binom{\frac{n}{r}f^{(n)}}{f^{(n)\prime}} > 0$, corresponding to the eigenvalue 0. The next theorem shows that this fact implies that 0 is the lowest point of the spectrum of the operator $RL^{(n+1)}R^{-1}$ (and therefore of $L^{(n+1)}$) and is a non-degenerate eigenvalue of this operator, which completes the proof of part (b) of Theorem 8.3.

Theorem B.3. *Let the equation $Lg = 0$, where $L = L_0 - V$, with L_0 and V, given by (B.4) and (B.5), have a positive solution in $X_{p,q}$. Then 0 is the lowest point of the spectrum of L in $X_{p,q}$ and it is a non-degenerate eigenvalue. If the above-mentioned positive solution is not in $X_{p,q}$, then 0 is not an eigenvalue.*

Proof. There is a direct proof of this theorem which uses the (self-adjoint) variational principle in combination with the strong maximum principle, but we prefer to give a proof which is close to the one of Theorem B.1. First we introduce the Birman-Schwinger type operator family,

$$K(\lambda) \;=\; V^{\frac{1}{2}} R(\lambda) V^{\frac{1}{2}} \;, \tag{B.6}$$

where $R(\lambda) = (L_0 - \lambda)^{-1}$. We consider these operators on $Y \equiv L^2(\mathbb{R}^2) \oplus L^2(\mathbb{R}^2)$. By the assumption on $-\Delta + W_j(x)$, $L_0 \geq 0$ on Y and 0 is not its eigenvalue. Thus $K(\lambda)$ is well defined for $\lambda \leq 0$ on a dense set in Y, as a family of non-negative operators, monotonically decreasing as $\lambda < 0$ decreases. We show at the end of this proof that $K(\lambda)$, with $\lambda \leq 0$, is bounded (note that it is not compact as required in other applications of the Birman-Schwinger operators). It is straightforward to show that it has all the properties stated in Lemma B.2. Thus Lemma B.2 holds for the operators L and $K(\lambda)$.

Again it is a standard result (see e.g. [RSIV]) that for $\lambda < 0$, $R(\lambda)$ is positivity improving in each entry (modulo a set of zero measure), i.e. $\big(R(\lambda)\varphi\big)_i > 0$, provided $\varphi_i \geq 0$ and $\varphi_i \not\equiv 0$, $i = 1, 2$. Since

$$V^{\frac{1}{2}} \;=\; \frac{\sqrt{\alpha}}{\sqrt{2}r}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{B.7}$$

has positive entries, this implies that $K(\lambda)$ is positivity improving, i.e. $K(\lambda)\varphi > 0$ if $\varphi \geq 0$ and $\varphi \not\equiv 0$. With some extra work one shows that $K(0)$ is positivity improving.

To conclude our agument we follow the proof of Theorem XIII.43 from [RSIV] and Theorem 2.3.2 from [GJ]. We adopt the notation $|f| = \binom{|f_1|}{|f_2|}$ for $f = \binom{f_1}{f_2}$. Let φ be an eigenfunction of $K(\lambda)$ corresponding to the highest eigenvalue, μ, of $K(\lambda)$. Without loss of generality it can be taken to be real. We claim that $|\varphi|$ is also an eigenfunction of $K(\lambda)$ corresponding to the same eigenvalue. Let $\varphi = \binom{\varphi_1}{\varphi_2}$. Then $\left|\binom{\varphi_1}{0}\right| \pm \binom{\varphi_1}{0} \geq 0$ and $\left|\binom{0}{\varphi_2}\right| \pm \binom{0}{\varphi_2} \geq 0$ and therefore $K(\lambda)\left|\binom{\varphi_1}{0}\right| \geq \left|K(\lambda)\binom{\varphi_1}{0}\right|$ and $K(\lambda)\left|\binom{0}{\varphi_2}\right| \geq \left|K(\lambda)\binom{0}{\varphi_2}\right|$. This implies $K(\lambda)|\varphi| \geq |\varphi|$, which yields that

$$\begin{aligned} \langle|\varphi|, K(\lambda)|\varphi|\rangle \;&\geq\; \langle|\varphi|, |K(\lambda)\varphi|\rangle \\ &\geq\; \langle\varphi, K(\lambda)\varphi\rangle \;=\; \mu\|\varphi\|^2 \;. \end{aligned}$$

Since μ is the maximal eigenvalue of $K(\lambda)$, $|\varphi|$ must be an eigenfunction of $K(\lambda)$ with the eigenvalue μ, as claimed. Now we show that φ must have a definite sign. Indeed, write $\varphi = \varphi_+ - \varphi_-$, where $\varphi_\pm \geq 0$. Then the equation

$$\langle|\varphi|, K(\lambda)|\varphi|\rangle \;=\; \langle\varphi, K(\lambda)\varphi\rangle$$

implies that

$$\langle\varphi_+, K(\lambda)\varphi_-\rangle + \langle\varphi_-, K(\lambda)\varphi_+\rangle \;=\; 0 \;.$$

Since $K(\lambda)\varphi_\pm > 0$ if $\varphi_\pm \geq 0$ and $\varphi_\pm \not\equiv 0$, we conclude that either $\varphi_+ \equiv 0$ or $\varphi_- \equiv 0$. Thus every eigenfunction of $K(\lambda)$ corresponding to an eigenvalue μ is, modulo a constant multiplier, positive. Since linear combinations of eigenfunctions with the same eigenvalue is an eigenfunction itself, we conclude that the eigenvalue μ must be simple. Finally using Lemma 6.2, we pass from $K(\lambda)$ to L to obtain that the lowest eigenvalue of L is non-degenerate and the corresponding eigenfunction is positive. This eigenvalue must be 0. Indeed, if it were negative, the corresponding eigenfunction would have an exponential decay and therefore would be orthogonal to the positive solution of $Lg = 0$, mentioned in the theorem. The latter is clearly impossible (cf. Eqn (B.3)). This contradiction proves the theorem.

In conclusion we prove that the operators $K(\lambda)$ defined in (B.6) are bounded for $\lambda \leq 0$. Due to (B.7), it suffices to show that $\frac{1}{r}(-\Delta + W_j - \lambda)^{-1}\frac{1}{r}$ extends to a bounded operator for $j = 1, 2$ and $\lambda \leq 0$. We drop the subindex j for a moment. This will not cause a confusion since α entering (B.5) and (B.7) does not appear in the rest of the proof. To fix ideas we assume $\gamma = 0$ and therefore $\beta > 0$. The case of $\gamma > 0$ and arbitrary β is even simpler. We also denote $T := -\Delta + W - \lambda$, $\lambda \leq 0$. We use the representation

$$\left\| \frac{1}{r} T^{-\frac{1}{2}} \varphi \right\|^2 = \left\langle T^{-\frac{1}{2}} \frac{1}{r^2} T^{-\frac{1}{2}} \right\rangle_{\varphi} .$$

Using a smooth partition of unity $\chi_1^2 + \chi_2^2 = 1$, where χ_1 is supported in $r \leq 2$ and χ_2 is supported in $r \geq 1$, we transform the r.h.s. as

$$\begin{aligned} \left\langle \frac{1}{r^2} \right\rangle_u &= \left\langle \frac{1}{r^2}\chi_1^2 + \frac{1}{r^2}\chi_2^2 \right\rangle_u = \frac{1}{\alpha}\langle \chi_1 T \chi_2 \rangle_u + \frac{1}{\beta}\langle \chi_2 T \chi_2 \rangle_u \\ &\quad - \frac{1}{\alpha}\big\langle \chi_1(-\Delta - \lambda + U_1)\chi_1 \big\rangle_u - \frac{1}{\beta}\big\langle \chi_2(-\Delta - \lambda + U_2)\chi_2 \big\rangle_u , \end{aligned}$$

where $u = T^{-\frac{1}{2}}\varphi$,

$$U_1 := W - \frac{\alpha}{r^2} = O\Big(\frac{1}{r}\Big) \qquad \text{as } r \to 0$$

and

$$U_2 = W - \frac{\beta}{r^2} = O\Big(\frac{1}{r^3}\Big) \qquad \text{as } r \to \infty .$$

This equation implies

$$\begin{aligned} \left\| \frac{1}{r} u \right\|^2 &\leq \max\Big(\frac{1}{\alpha}, \frac{1}{\beta}\Big) \sum_{i=1}^{2} \langle \chi_i T \chi_i \rangle_u \\ &\quad - \frac{1}{\alpha}\langle \chi_1^2 U_1 \rangle_u - \frac{1}{\beta}\langle \chi_2^2 U_2 \rangle_u . \end{aligned}$$

The IMS formula (see [CFKS]) yields then that

$$\left\|\frac{1}{r}u\right\|^2 \leq \max\left(\frac{1}{\alpha},\frac{1}{\beta}\right)\Big\{\langle T\rangle_u + 2\big\||\nabla\chi_1|u\big\|^2 - \sum_{i=1}^{2}\big\|\chi_i|U_i|^{\frac{1}{2}}u\big\|^2\Big\}\ .$$

Recall now that $u = T^{-\frac{1}{2}}\varphi$. A fairly standard analysis shows that $|\nabla\chi_1|^2T^{-\frac{1}{2}}$ and $\chi_i|U_i|^{\frac{1}{2}}T^{-\frac{1}{2}}$ are bounded operators. Moreover, $\langle T\rangle_u = \|\varphi\|^2$. Thus the r.h.s. is bounded by const $\cdot\ \|\varphi\|^2$. Hence $\frac{1}{r}T^{-\frac{1}{2}}$ is a bounded operator and therefore so is $\frac{1}{r}T^{-1}\frac{1}{r}$. $\square$

Remark. One can show directly that e^{-tL} is positivity improving. Indeed, since L_0 is diagonal, standard arguments (see [RSIV, Theorem XIII.45) are applicable and show that e^{-tL_0} is positivity improving entrywise. Next, since all the entries of V are positive, then so are the entries of e^{tV} (in fact an explicit computation yields that

$$e^{tV} = \frac{1}{2}\big(e^{\frac{2\alpha t}{r^2}} - 1\big)\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + id \quad)\ . \tag{$B.8$}$$

Therefore e^{tV} is positivity preserving and, moreover, $e^{-tL} = e^{-tL_0+tV}$ is positivity improving, as can be shown using the Trotter product formula.

References

[BBH] F. Bethuel, H. Brezis, and F. Hélein, *Ginzburg-Landau Vortices,* Birkhäuser, Basel, 1994.

[BGP] A.M. Boutet de Monvel-Berthier, G. Georgescu and R. Purice, A boundary value problem for the Ginzburg-Landau Model, *Comm. Math. Phys.* **142** (1991), 1–23.

[BMR] H. Brezis, F. Merle, and T. Rivière, Quantization effects for $-\Delta u = u(1-|u|^2)$ in $\mathbb{R}^2$, *Arch. Rational Mech. Anal.* **126** (1994), 35–58.

[BN] H. Brezis and L. Nirenberg, Degree theory and BMO; Part I: Compact manifolds without boundaries, preprint.

[C] Th. Cazenave, unpublished.

[CFKS] H.L. Cycon, R.G. Froese, W. Kirsch and B. Simon, *Schrödinger Operators,* Springer, 1987.

[FS] J. Fröhlich and M. Struwe, Variational problems on vector bundles, *Comm. Math. Phys.* **131** (1990), 431–464.

[GJ] J. Glimm and A. Jaffe, The $\lambda(\varphi^4)_2$ quantum field theory without cut-offs: II. The field operators and the approximate vacuum, *Annals of Math.* **91** (1970), 362–401.

[GT] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order,* 2nd Edition, Springer-Verlag, New York, 1983.

[H] P. Hagan, Spiral waves in reaction diffusion equations, *SIAM J. Applied Math.* **42** (1982), 762–786.

[JT] A. Jaffe and C. Taubes, *Vortices and Monopolies,* Birkhäuser, 1980.

[LL] E.H. Lieb and M. Loss, Symmetry of the Ginzburg-Landau minimizer in a disc, *Math. Res. Lett.* **1** (1994), 701–715.

[M] P. Mironescu, On the stability of radial solutions of the Ginzburg-Landau equation, *Journal of Functional Analysis* **130** (1995), 334–344.

[RS] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV,* Academic Press.

[S] I. Shafrir, Remarks on solutions of $-\Delta u = (1-|u|^2)u$ in $\mathbb{R}^2$, *C.R. Acad. Sci. Paris,* t. 318, Série I (1994), 327–331.

[Str] M. Struwe, *Variational Methods,* Springer-Verlag, 1990.

[Z] G.M. Zhislin, Private communication.

[ZJ] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena,* Oxford Sci. Publ., 1993.