RESTRICTIONS OF MIXED HODGE MODULES USING GENERALIZED V-FILTRATIONS

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Abstract. We study generalized V-filtrations, defined by Sabbah, on $\mathcal{D}$-modules underlying mixed Hodge modules on $X \times \mathbf{A}^r$. Using cyclic covers, we compare these filtrations to the usual V-filtration, which is better understood. The main result shows that these filtrations can be used to compute $\sigma^!$, where $\sigma \colon X \times \{0\} \to X \times \mathbf{A}^r$ is the inclusion of the zero section.

As an application, we use the restriction result to study singularities of complete intersection subvarieties. These filtrations can be used to study the local cohomology mixed Hodge module. In particular, we classify when weighted homogeneous isolated complete intersection singularities in $\mathbf{A}^n$ are k-Du Bois and k-rational.

1. Introduction

Over the complex numbers, singularities of local complete intersection subvarieties have recently been studied using Saito's theory of mixed Hodge modules [MP22, CDMO24, CDM22]. One of the cornerstones of these applications is the extension, beyond the hypersurface case, of the relation between the Hodge module structure on local cohomology with V-filtrations and Bernstein-Sato polynomials.

A key technical tool in extending this relationship is an understanding of the V-filtration of mixed Hodge modules along higher codimension smooth subvarieties. This V-filtration and its relation to Bernstein-Sato polynomials was first introduced in [BMS06]. The Hodge module theoretic properties were further studied in [CD23, CDS23]. For a brief review of V-filtrations and mixed Hodge modules, see Section 2 below.

For a smooth complex algebraic variety X, we consider $T = X \times \mathbf{A}^r_t$ with coordinates $t_1, \ldots, t_r$ on $\mathbf{A}^r_t$. Kashiwara and Malgrange showed that any $\mathcal{D}_T$-module $\mathcal{M}$ underlying a mixed Hodge module M on T admits a V-filtration $(V^\lambda \mathcal{M})_{\lambda \in \mathbf{Q}}$ by $\mathcal{D}_X$ (not $\mathcal{D}_T$)-submodules.

Three important properties of this filtration are that it is discretely indexed, we have

$$t_i V^\lambda \mathcal{M} \subseteq V^{\lambda+1}\mathcal{M}, \quad \partial_{t_i} V^\lambda \mathcal{M} \subseteq V^{\lambda+1}\mathcal{M},$$

and the shifted Euler operator $\sum_{i=1}^r t_i \partial_{t_i} - \lambda + r$ is nilpotent on $\mathrm{Gr}_V^\lambda(\mathcal{M})$.

We can define Koszul-like complexes

$$A^\lambda(\mathcal{M}) = \left[V^\lambda \mathcal{M} \xrightarrow{t_i} \bigoplus_{i=1}^r V^{\lambda+1}\mathcal{M} \xrightarrow{t_i} \ldots \xrightarrow{t_i} V^{\lambda+r}\mathcal{M} \right],$$

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and

$$B^\lambda(\mathcal{M}) = \left[\mathrm{Gr}_V^\lambda(\mathcal{M}) \xrightarrow{t_i} \bigoplus_{i=1}^r \mathrm{Gr}_V^{\lambda+1}(\mathcal{M}) \xrightarrow{t_i} \dots \xrightarrow{t_i} \mathrm{Gr}_V^{\lambda+r}(\mathcal{M})\right]$$

placed in degree $0, 1, \dots, r$.

It is not hard to see (as shown in Proposition 2.10 below) that most of these complexes are acyclic. The most important one (and essentially the only one) that is not necessarily acyclic is the complex for $\lambda = 0$. In fact, the cohomologies of this complex are shown to compute the $\mathcal{D}$-module theoretic restriction of the module $\mathcal{M}$ to the zero section $X \times \{0\} \subseteq X \times \mathbf{A}_t^r$.

The main results of [CD23, CDS23] are extensions of the acyclicity and restriction results to the setting of mixed Hodge modules. Careful statements of these results are found in Theorem 2.11 below.

The main objective of this paper is to extend these results to more general V-filtrations, defined by Sabbah [Sab87]. The proper definition is given in Section 2 below. For now, we just mention that associated to a tuple $(a_1, a_2, \dots, a_r) \in \mathbf{Z}_{\geq 0}$, we can define a linear form $L = \sum_{i=1}^r a_i s_i$, which is called a *slope*. Such a slope is *non-degenerate* if $a_i > 0$ for all i. We set $|L| = \sum_{i=1}^r a_i$ and $L(t\partial_t) = \sum_{i=1}^r a_i t_i \partial_{t_i}$.

Sabbah defines the unique ${}^L V$-filtration on a $\mathcal{D}_T$-module $\mathcal{M}$ by similar properties to the V-filtration described above, except one requires $L(t\partial_t) - \lambda + |L|$ to be nilpotent on $\mathrm{Gr}_{{}^L V}^\lambda(\mathcal{M})$, which for the remainder of the paper we will write as $\mathrm{Gr}_L^\lambda(\mathcal{M})$. This imposes the other conditions $t_i {}^L V^\lambda \mathcal{M} \subseteq {}^L V^{\lambda+a_i}\mathcal{M}$ and $\partial_{t_i} {}^L V^\lambda \mathcal{M} \subseteq {}^L V^{\lambda - a_i}\mathcal{M}$. This filtration heavily depends on the ordered choice of coordinates $t_1, \dots, t_r$: for example, even reordering the coordinates gives a different filtration, corresponding to the slope with permuted coefficients.

If $L = \sum_{i=1}^r s_i$, then ${}^L V^\bullet \mathcal{M} = V^\bullet \mathcal{M}$ from above. In fact, if $L = \sum_{i \in I} s_i$ for some $I \subseteq \{1, \dots, r\}$, the filtration ${}^L V^\bullet \mathcal{M}$ is the V-filtration along $\{t_i \mid i \in I\}$.

In analogy with the above, we can define Koszul-like complexes

$$A_L^\lambda(\mathcal{M}) = \left[{}^L V^\lambda \mathcal{M} \xrightarrow{t_i} \bigoplus_{i=1}^r {}^L V^{\lambda+a_i}\mathcal{M} \xrightarrow{t_i} \dots \xrightarrow{t_i} {}^L V^{\lambda+|L|}\mathcal{M}\right],$$

$$B_L^\lambda(\mathcal{M}) = \left[\mathrm{Gr}_L^\lambda(\mathcal{M}) \xrightarrow{t_i} \bigoplus_{i=1}^r \mathrm{Gr}_L^{\lambda+a_i}(\mathcal{M}) \xrightarrow{t_i} \dots \xrightarrow{t_i} \mathrm{Gr}_L^{\lambda+|L|}(\mathcal{M})\right]$$

placed in degree $0, 1, \dots r$.

Proposition 3.9 below shows that, at the $\mathcal{D}$-module level, we have the acyclicity and restriction results for these complexes. The main result of this note is the extension to the mixed Hodge module setting.

If $(\mathcal{M}, F, W)$ is a bi-filtered $\mathcal{D}_T$-module underlying a mixed Hodge module M on T, we define filtrations

$$F_p A_L^\lambda(\mathcal{M}) = \left[F_{p+r} {}^L V^\lambda \mathcal{M} \xrightarrow{t_i} \bigoplus_{i=1}^r F_{p+r} {}^L V^{\lambda+a_i}\mathcal{M} \xrightarrow{t_i} \dots \xrightarrow{t_i} F_{p+r} {}^L V^{\lambda+|L|}\mathcal{M}\right],$$

$$F_p B_L^\lambda(\mathcal{M}) = \left[F_{p+r}\mathrm{Gr}_L^\lambda(\mathcal{M}) \xrightarrow{t_i} \bigoplus_{i=1}^r F_{p+r}\mathrm{Gr}_L^{\lambda+a_i}(\mathcal{M}) \xrightarrow{t_i} \dots \xrightarrow{t_i} F_{p+r}\mathrm{Gr}_L^{\lambda+|L|}(\mathcal{M})\right].$$

Moreover, the nilpotent operator $L(t\partial_t) - \lambda + |L|$ on $\mathrm{Gr}_L^\lambda(\mathcal{M})$ and the induced filtration $M_\bullet \mathrm{Gr}_L^\lambda(\mathcal{M}) = \mathrm{Gr}_L^\lambda(W_\bullet\mathcal{M})$ give rise to the *relative monodromy filtration* $W_\bullet \mathrm{Gr}_L^\lambda(\mathcal{M})$. This is recalled in more detail in Section 2 below. The complex $B_L^0(\mathcal{M})$ admits a filtration $W_\bullet B_L^0(\mathcal{M})$ by taking the relative monodromy filtration on each piece.

With this notation in place, we have the following:

Theorem A. *Let $L = \sum_{i=1}^r a_i s_i$ be a non-degenerate slope and let $(\mathcal{M}, F, W)$ be a bi-filtered $\mathcal{D}_T$-module underlying a mixed Hodge module. The complexes*

$$A_L^\chi(\mathcal{M}, F) \text{ and } B_L^\chi(\mathcal{M}, F) \text{ are filtered acyclic for all } \chi > 0.$$

Moreover, we have filtered quasi-isomorphisms

$$\sigma^!(\mathcal{M}, F) \cong A_L^0(\mathcal{M}, F) \cong B_L^0(\mathcal{M}, F),$$

and the latter two complexes are strictly filtered.

Finally, the filtration $W_\bullet \mathcal{H}^i B_L^0(\mathcal{M}, F)$ induced by $W_\bullet B_L^0(\mathcal{M}, F)$ satisfies

$$\mathrm{Gr}_k^W \mathcal{H}^i B_L^0(\mathcal{M}, F) \cong \mathrm{Gr}_{k+i}^W \mathcal{H}^i \sigma^!(\mathcal{M}, F),$$

as filtered $\mathcal{D}_X$-modules underlying polarizable Hodge modules of weight $k+i$.

Remark 1.1. The old ideas in [CD23] are not sufficient to prove Theorem A because the ${}^LV$-filtration depends on the choice of coordinates.

The ideas in [CDS23] are used below; however, they do not automatically give us the strictness of the complex $B_L^0(\mathcal{M}, F)$. Indeed, the same problem arises in this situation: the ${}^LV$-filtration depends on the choice of coordinates $t_1, \dots, t_r$, and to prove strictness as in [CDS23], one takes t_r to be a general linear combination of $t_1, \dots, t_r$.

Remark 1.2. A few results from [CD23] are missing in this paper. Namely, we would like to understand the Koszul-like complexes

$$C_L^\lambda(\mathcal{M}) = \left[\mathrm{Gr}_L^{\lambda+|L|}(\mathcal{M}) \xrightarrow{\partial_{t_i}} \bigoplus_{i=1}^r \mathrm{Gr}_L^{\lambda+|L|-a_i}(\mathcal{M}) \xrightarrow{\partial_{t_i}} \dots \xrightarrow{\partial_{t_i}} \mathrm{Gr}_L^\lambda(\mathcal{M}) \right]$$

in degree $-r, \dots, -1, 0$.

It is not hard to see that, at the $\mathcal{D}$-module level, $C_L^\chi(\mathcal{M})$ is acyclic for all $\chi \neq 0$. For $L = \sum_{i=1}^r s_i$, however, we know that $C^0(\mathcal{M})$ is quasi-isomorphic to $\sigma^*(\mathcal{M})$. At the moment, we do not see a way to prove this even in the $\mathcal{D}$-module setting. Naturally, one would also want to study the corresponding filtered complexes and compute σ^* for mixed Hodge modules using the complex $C_L^0(\mathcal{M}, F)$.

Another missing result is the understanding of the Fourier-Laplace transform of an L-monodromic mixed Hodge module. The notion of L-monodromic modules is reviewed below. If one had an understanding of the Fourier-Laplace transform of such a module, then applying it to $\mathrm{Sp}_L(M)$ (as constructed below), and using Theorem A, one would obtain the results concerning the complex $C^\chi(\mathcal{M})$ mentioned at the beginning of this remark.

The paper ends with an application of Theorem A to the study of singularities of local complete intersection subvarieties. If $Z \subseteq X$ is defined by a regular sequence $f_1, \dots, f_r$ in $\mathcal{O}_X(X)$, we can define a pure Hodge module B_f on T of weight $n = \dim(X)$. Its underlying $\mathcal{D}_T$-module is

$$\mathcal{B}_f = \bigoplus_{\alpha \in \mathbf{N}^r} \mathcal{O}_X \partial_t^\alpha \delta_f,$$

whose $\mathcal{D}$-module structure is explained in Section 5 below.

If $\sigma\colon X \times \{0\} \to T$ is the inclusion of the zero section, we have a natural isomorphism

$$\mathcal{H}^r \sigma^! B_f \cong (\mathcal{H}^r_Z(\mathcal{O}_X), F, W),$$

where the right hand side is the bi-filtered local cohomology $\mathcal{D}_X$-module along Z, which is traditionally computed using the Čech complex for $f_1, \dots, f_r$.

By [MP22, CDMO24, CDM22], the Hodge module structure of $\mathcal{H}^r_Z(\mathcal{O}_X)$ is related to higher classes of singularities. These are the classes of k-Du Bois and k-rational singularities, where $k \in \mathbf{Z}_{\geq 0}$, whose definitions are reviewed in Section 5 below. For $k = 0$, these agree with the classical notions of Du Bois and rational singularities, hence the terminology.

In [CDMO24], the authors and Mustaţă define a numerical invariant of Z, the *minimal exponent* $\widetilde{\alpha}(Z)$, in terms of $V^\bullet \mathcal{B}_f$. The minimal exponent satisfies the following implications:

$$\widetilde{\alpha}(Z) \geq r + k \iff Z \text{ has } k\text{-Du Bois singularities},$$

$$\widetilde{\alpha}(Z) > r + k \iff Z \text{ has } k\text{-rational singularities}.$$

From this, we see that Z has k-rational singularities if it has $(k+1)$-Du Bois singularities, and it has k-Du Bois singularities if it has k-rational singularities. We say Z has k-liminal (or strictly k-Du Bois) singularities if $\widetilde{\alpha}(Z) = r + k$.

Our first main result in the study of singularities is the following description of the mixed Hodge module structure on local cohomology:

Theorem B. *Let $Z = V(f_1, \dots, f_r) \subseteq X$ be a complete intersection of pure codimension r. Let L be a non-degenerate slope. Then for all $p, \ell \in \mathbf{Z}_{\geq 0}$, we have*

$$F_{p+r} W_{n+r+\ell} \mathcal{H}^r_Z(\mathcal{O}_X)$$

$$= \left\{ \sum_{|\alpha| \leq p} \frac{\alpha! h_\alpha}{f_1^{\alpha_1+1} \dots f_r^{\alpha_r+1}} \mid u = \sum_{|\alpha| \leq p} h_\alpha \partial_t^\alpha \delta_f \in {}^L V^{|L|} \mathcal{B}_f, L(t\partial_t)^{\ell+1} u \in {}^L V^{>|L|} \mathcal{B}_f \right\}.$$

An immediate corollary of this computation is the following:

Corollary C. *Let $Z = V(f_1, \dots, f_r) \subseteq X$ be a complete intersection of pure codimension r. Let L be a non-degenerate slope. Then*

$$F_{p+r} \mathcal{B}_f \subseteq {}^L V^{|L|} \mathcal{B}_f \iff Z \textit{ has } k\textit{-Du Bois singularities},$$

$$L(t\partial_t) F_{p+r} \mathcal{B}_f \subseteq {}^L V^{>|L|} \mathcal{B}_f \iff Z \textit{ has } k\textit{-rational singularities}.$$

Finally, the corollary allows us to prove the following, which is a generalization of the main result of [CDM24]. Note that, here, we do not give an exact formula for the minimal exponent, but what we show is enough for the singularity classification.

Let $f_1, \dots, f_r \in \mathbf{C}[x_1, \dots, x_n]$ be weighted homogeneous of (integer) degrees $d_1 \leq \dots \leq d_r$ with weights $(w_1, \dots, w_n) \in \mathbf{Z}^n_{\geq 1}$. Assume $Z = V(f_1, \dots, f_r) \subseteq \mathbf{A}^n_x$ is a complete intersection of pure codimension r with an isolated singularity at 0.

Corollary D. *The complete intersection Z has Du Bois (hence, log canonical) singularities if and only if $|w| = \sum_{i=1}^n w_i \geq d_1 + \dots + d_r$. In this case, let*

$$k = \left\lfloor \frac{|w| - \sum_{i=1}^r d_i}{d_r} \right\rfloor.$$

Then

$$r + k \leq \widetilde{\alpha}_0(Z) \leq r + \frac{|w| - \sum_{i=1}^r d_i}{d_r},$$

and in fact, $\widetilde{\alpha}_0(Z) = r + k$ *if and only if* $d_r \mid (|w| - \sum_{i=1}^r d_i)$.

In particular, for this value of k*, we see*

$$Z \text{ has } k\text{-liminal singularities near } 0 \iff d_r \mid \left(|w| - \sum_{i=1}^r d_i\right)$$

$$Z \text{ has } k\text{-rational singularities near } 0 \iff d_r \nmid \left(|w| - \sum_{i=1}^r d_i\right).$$

We expect that the equality $\widetilde{\alpha}_0(Z) = r + \frac{|w| - \sum_{i=1}^r d_i}{d_r}$ holds in the setting of Corollary D.

Outline. Section 2 contains the background needed for the proofs of the main results. In Subsection 2.1, we provide a review of the theory of mixed Hodge modules, including a review of hypersurface V-filtrations. Subsection 2.3 reviews the results for higher codimension V-filtrations. The definition and properties of the filtration ${}^LV^\bullet\mathcal{M}$ are given in Section 3. The Verdier specialization process is used in Subsection 3.1 to study ${}^LV$-filtrations on mixed Hodge modules, using the properties of hypersurface V-filtrations.

Section 4 contains the proof of Theorem A. It begins with an analysis of mixed Hodge modules under cyclic coverings $X \times \mathbf{A}^r_w \to X \times \mathbf{A}^r_t$ defined by

$$(x, w_1, \dots, w_r) \mapsto (x, w_1^{a_1}, \dots, w_r^{a_r}).$$

The main point is that the usual V-filtration on $\pi^!(\mathcal{M})$ can be related to the ${}^LV$-filtration on $\mathcal{M}$. Two difficulties are that $\pi^!(-)$ need not preserve pure modules (though we give a criterion for when a pure Hodge module pulls back to a pure module, in a special case) and the Hodge filtration is not easy to understand. We work around this by using the fact that, on restriction to the étale locus, things behave nicely.

The final Section 5 contains the proof of Theorem B, Corollary C and the example of weighted homogeneous complete intersections with isolated singularities. In particular, The proof of Corollary D is a combination of results in Section 5.1. Though we cannot give an exact computation of the minimal exponent in the latter example (except in special cases), we give an easy criterion to check whether such a subvariety has k-Du Bois or k-rational singularities.

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2. Preliminaries

We do not provide a review of the theory of $\mathcal{D}$-modules in this paper, though we will review the necessary notions as they arise. For reference, see [HTT08].

2.1. **Mixed Hodge modules and V-filtrations of hypersurfaces.** In this subsection, we discuss the relevant aspects of the theory of mixed Hodge modules. For details, one should consult Saito's papers [Sai88, Sai90] or Schnell's survey article [Sch14].

Let X be a smooth algebraic variety over $\mathbf{C}$ with $\dim(X) = n$. Saito associates to X an abelian category $\mathrm{MHM}(X)$, the category of *mixed Hodge modules* on X. A mixed Hodge module on X consists of the following data: a filtered regular holonomic $\mathcal{D}_X$-module $(\mathcal{M}, F)$, with a finite filtration $W_\bullet\mathcal{M}$ by sub-$\mathcal{D}_X$-modules, a finite filtered $\mathbf{Q}$-perverse sheaf $(\mathcal{K}, W)$, and a comparison morphism

$$\alpha\colon (\mathcal{K}, W) \otimes_{\mathbf{Q}} \mathbf{C} \to \mathrm{DR}_X(\mathcal{M}, W)$$

which is a filtered quasi-isomorphism. These data are subject to various conditions which we will not fully explain here. Some of the conditions are the following, which concern the interaction between the Hodge filtration $F_\bullet\mathcal{M}$ and the V-filtration $V^\bullet\mathcal{M}$ on $\mathcal{M}$ along any (locally defined) function $f \in \mathcal{O}_X$.

First, we recall the definition the V-filtration of $\mathcal{D}_{X\times\mathbf{A}^1_t}$-modules along t, which is the coordinate on $\mathbf{A}^1$. The $\mathcal{D}_{X\times\mathbf{A}^1_t}$-module $i_{f,+}(\mathcal{M})$ admits a V*-filtration along* t, which, following Saito, is a decreasing, discrete[1] and left continuous[2], $\mathbf{Q}$-indexed filtration $V^\bullet\mathcal{M}$ satisfying the following conditions:

(1) $V^\alpha\mathcal{M}$ is coherent over $V^0\mathcal{D}_{X\times\mathbf{A}^1} = \mathcal{D}_X[t, t\partial_t]$,
(2) $tV^\alpha\mathcal{M} \subseteq V^{\alpha+1}\mathcal{M}$ for all $\alpha \in \mathbf{Q}$, with equality for $\alpha \gg 0$,
(3) $\partial_t V^\alpha\mathcal{M} \subseteq V^{\alpha-1}\mathcal{M}$ for all $\alpha \in \mathbf{Q}$,
(4) the operator $t\partial_t - \alpha + 1$ is nilpotent on $\mathrm{Gr}_V^\alpha(\mathcal{M}) = V^\alpha\mathcal{M}/V^{>\alpha}\mathcal{M}$, where $V^{>\alpha}\mathcal{M} = \bigcup_{\beta>\alpha} V^\beta\mathcal{M}$.

Remark 2.1. Below, we will also use the operator $s = -\partial_t t$. This is more natural in the study of singularities and b-functions. We clearly have

$$s = -(t\partial_t + 1),$$

so condition 4 can be restated as requiring $s + \alpha$ to be nilpotent on $\mathrm{Gr}_V^\alpha(\mathcal{M})$.

Example 2.2. Let $\mathcal{M}$ be supported in $\{t = 0\}$. Then Kashiwara's equivalence (see [HTT08, Sect. 1.6]) tells us that $\mathcal{M} = i_+\mathcal{M}_0$ where $\mathcal{M}_0 = \ker(t) \subseteq \mathcal{M}$ is a $\mathcal{D}_{X\times\{0\}}$-module and $i\colon X \times \{0\} \to X \times \mathbf{A}^1_t$ is the closed embedding. Thus,

$$\mathcal{M} = \bigoplus_{k\geq 0} \mathcal{M}_0\partial_t^k,$$

and it is not hard to check that

$$V^\lambda\mathcal{M} = V^{\lceil\lambda\rceil}\mathcal{M} = \bigoplus_{k\leq -\lceil\lambda\rceil} \mathcal{M}_0\partial_t^k$$

is a V-filtration of $\mathcal{M}$ along t.

If a V-filtration on $\mathcal{M}$ along t exists, then it is unique. Hence, existence is an intrinsic property of the $\mathcal{D}$-module $\mathcal{M}$. The uniqueness implies the following:

[1] This means that there is an increasing sequence $\alpha_j \in \mathbf{Q}$ with $\lim_{j\to-\infty}\alpha_j = -\infty$ and $\lim_{j\to\infty}\alpha_j = \infty$, such that $V^\chi\mathcal{M}$ for $\chi \in (\alpha_j, \alpha_{j+1})$ only depends on j.

[2] Meaning $V^\chi\mathcal{M} = \bigcap_{\beta<\chi} V^\beta\mathcal{M}$. In other words, $V^\chi\mathcal{M}$ is constant for $\chi \in (\alpha_j, \alpha_{j+1}]$.

Lemma 2.3. *Let $0 \to \mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to 0$ be a short exact sequence of $\mathcal{D}_{X\times \mathbf{A}^1_t}$-modules such that $\mathcal{M}_i$ admits a V-filtration along t for $i = 1, 2, 3$. Then, for all $\lambda \in \mathbf{Q}$, the sequence*

$$0 \to V^\lambda \mathcal{M}_1 \to V^\lambda \mathcal{M}_2 \to V^\lambda \mathcal{M}_3 \to 0$$

is exact.

From this and Example 2.2, we see that positive pieces of the V-filtration only depend on the restriction to $\{t \neq 0\}$.

Lemma 2.4. *Let $\varphi\colon \mathcal{M} \to \mathcal{N}$ be a morphism of $\mathcal{D}_{X\times \mathbf{A}^1_t}$-modules such that $\varphi|_{\{t\neq 0\}}$ is an isomorphism.*

Then for all $\lambda > 0$, φ induces an isomorphism

$$\varphi\colon V^\lambda \mathcal{M} \cong V^\lambda \mathcal{N}.$$

For $f \in \mathcal{O}_X(X)$, let $i_f\colon X \to X \times \mathbf{A}^1_t$ be the graph embedding along f. We say that the V-filtration of a $\mathcal{D}_X$-module $\mathcal{M}$ along f exists if the V-filtration of $i_{f,+}(\mathcal{M})$ along t exists. For $\mathcal{D}_X$-modules underlying mixed Hodge modules, the V-filtration exists for any locally defined function $f \in \mathcal{O}_X$.

Returning to a $\mathcal{D}_{X\times\mathbf{A}^1}$-module $\mathcal{M}$, it is not hard to see using Condition 4 that the maps

$$t\colon \mathrm{Gr}_V^\alpha(\mathcal{M}) \to \mathrm{Gr}_V^{\alpha+1}(\mathcal{M}),$$
$$\partial_t\colon \mathrm{Gr}_V^{\alpha+1}(\mathcal{M}) \to \mathrm{Gr}_V^{\alpha}(\mathcal{M})$$

are isomorphisms for all $\alpha \neq 0$. In fact, Condition 2 shows that $t\colon V^\alpha\mathcal{M} \to V^{\alpha+1}\mathcal{M}$ is an isomorphism for all $\alpha > 0$.

If $(\mathcal{M}, F)$ is a filtered $\mathcal{D}_X$-module, then $i_{f,+}(\mathcal{M})$ is also a filtered $\mathcal{D}_{X\times\mathbf{A}^1_t}$-module. Indeed, we can write $i_{f,+}(\mathcal{M}) = \bigoplus_{k\geq 0} \mathcal{M}\partial_t^k \delta_f$, and we have

$$F_p i_{f,+}(\mathcal{M}) = \bigoplus_{k\geq 0} F_{p-k-1}\mathcal{M}\partial_t^k\delta_f,$$

where the shift by 1 is a normalizing convention due to the relative dimension of $i_f\colon X \to X \times \mathbf{A}^1_t$.

Saito imposes the following conditions on the filtration $F_\bullet\mathcal{M}$: for any $f \in \mathcal{O}_X(X)$, we have

(1) $\quad t\colon F_pV^\alpha i_{f,+}(\mathcal{M}) \to F_pV^{\alpha+1}i_{f,+}(\mathcal{M})$ is an isomorphism for all $\alpha > 0$,

(2) $\quad \partial_t\colon F_p\mathrm{Gr}_V^{\alpha+1}(i_{f,+}(\mathcal{M})) \to F_{p+1}\mathrm{Gr}_V^{\alpha}(i_{f,+}(\mathcal{M}))$ is an isomorphism for all $\alpha < 0$.

Remark 2.5. Another property imposed on filtered $\mathcal{D}$-modules underlying mixed Hodge modules is the following: for $(\mathcal{M}, F)$ underlying a mixed Hodge module on $X \times \mathbf{A}^1_t$, the filtration induced by $F_\bullet\mathcal{M}$ on $\mathrm{Gr}_V^\lambda(\mathcal{M})$ is a good filtration.

In fact, we have the following (see [Sai88, Cor. 3.4.7] and [SS, Prop. 10.7.3]): for any $\lambda \in \mathbf{Q}$, let $R_F(V^\lambda\mathcal{M}) = \bigoplus_{p\in\mathbf{Z}} F_pV^\lambda\mathcal{M}z^p$. This is a module over $R_F(V^0\mathcal{D}_{X\times\mathbf{A}^1_t})$, and in fact is coherent over that ring. It is even coherent over the subring $R_F(\mathcal{D}_{X\times\mathbf{A}^1_t/\mathbf{A}^1_t}) = R_F(\mathcal{D}_X[t])$.

For $i\colon H = V(f) \to X$ the inclusion of the hypersurface defined by f into X, we can restrict a mixed Hodge module M on X along i to get $i_*i^!M \in D^b(\mathrm{MHM}(X))$. In fact, for underlying filtered $\mathcal{D}$-modules, this restriction is given by the morphism

$$t\colon \mathrm{Gr}_V^0(i_{f,+}(\mathcal{M}, F)) \to \mathrm{Gr}_V^1(i_{f,+}(\mathcal{M}, F)),$$

which is the motivation for Theorem A.

In the remainder of this section, we will work with M a mixed Hodge module on $X \times \mathbf{A}^1_t$.

The following lemma comes immediately from conditions (1) and (2), respectively.

Lemma 2.6. *Let $(\mathcal{M}, F)$ underlie a mixed Hodge module on $X \times \mathbf{A}^1_t$. Let $j\colon \{t \neq 0\} \to X \times \mathbf{A}^1_t$. Then*

$$F_p V^\lambda \mathcal{M} = V^\lambda \mathcal{M} \cap j_*(F_p j^* \mathcal{M})$$

and

$$F_p \mathcal{M} = \sum_{k \geq 0} \partial_t^k (F_{p-k} V^0 \mathcal{M}).$$

A mixed Hodge module M is *pure of weight* d if $\mathrm{Gr}^W_i(M) = 0$ for all $i \neq d$. By definition, any pure Hodge module of weight d decomposes into its *strict support* decomposition. This means

$$M = \bigoplus_{Z \subseteq X \times \mathbf{A}^1_t} M_Z$$

where M_Z is a pure Hodge module of weight d with strict support $Z \subseteq X \times \mathbf{A}^1_t$. Here Z is an irreducible closed subset of $X \times \mathbf{A}^1_t$ and strict support means that the $\mathcal{D}$-module underlying M_Z admits no non-zero quotient or sub-object with support contained in a proper closed subset of Z. Equivalently, the underlying perverse sheaf is an intersection complex.

Lemma 2.7. *A $\mathcal{D}_{X \times \mathbf{A}^1_t}$-module $\mathcal{M}$ admits no non-zero sub-objects supported in $\{t = 0\}$ if and only if the map*

$$t\colon \mathrm{Gr}^0_V(\mathcal{M}) \to \mathrm{Gr}^1_V(\mathcal{M}) \text{ is injective.}$$

It admits no non-zero quotient object supported in $\{t = 0\}$ if and only if the map

$$\partial_t\colon \mathrm{Gr}^1_V(\mathcal{M}) \to \mathrm{Gr}^0_V(\mathcal{M}) \text{ is surjective,}$$

which holds if and only if $\mathcal{M} = \mathcal{D}_{X \times \mathbf{A}^1_t} \cdot V^{>0}\mathcal{M}$.

If $(\mathcal{M}, F)$ underlies a Hodge module, then the morphisms in the previous lemma statement are automatically strict with respect to the Hodge filtration. This gives the following:

Lemma 2.8. *If $(\mathcal{M}, F)$ underlies a Hodge module with no sub-object supported on $\{t = 0\}$, then*

$$F_p V^0 \mathcal{M} = V^0 \mathcal{M} \cap j_*(F_p j^*(\mathcal{M})).$$

If $(\mathcal{M}, F)$ underlies a Hodge module with no quotient supported on $\{t = 0\}$, then

$$F_p \mathcal{M} = \sum_{k \geq 0} \partial_t^k (F_{p-k} V^{>0} \mathcal{M}).$$

2.2. **Relative Monodromy Filtration.** An important construction in Hodge theory is the *monodromy filtration* of a nilpotent operator N on an object A in an abelian category. Given such an operator, there exists a unique increasing filtration $W(N)_\bullet A$ such that $N W(N)_\bullet A \subseteq W(N)_{\bullet-2}(A)$ and with the property that

$$N^i\colon \mathrm{Gr}_i^{W(N)}(A) \to \mathrm{Gr}_{-i}^{W(N)}(A)$$

is an isomorphism for all $i > 0$.

If instead A is itself already filtered by sub-objects $L_\bullet A \subseteq A$ in the abelian category $\mathcal{A}$, there is the notion of *relative monodromy filtration* for any nilpotent operator N such that $NL_\bullet \subseteq L_\bullet$.

The relative monodromy filtration $W(N,L)_\bullet A$ is the unique, increasing filtration with the property that $NW(N,L)_\bullet \subseteq W(N,L)_{\bullet-2}$ and so that, for all $k \in \mathbf{Z}$ and $i \in \mathbf{Z}_{>0}$, the map

$$N^i \colon \mathrm{Gr}^{W(N,L)}_{k+i} \mathrm{Gr}^L_k(A) \to \mathrm{Gr}^{W(N,L)}_{k-i} \mathrm{Gr}^L_k(A)$$

is an isomorphism.

Such a filtration need not always exist, though an inductive criterion for existence is given in [SZ85] (see also [Sai90, Lem. 1.2] and [Kas86, Lem. 3.1.1]).

One of the most useful observations in the theory of relative monodromy filtrations is due to Kashiwara [Kas86, Thm. 3.2.9], which allows one to obtain a *canonical splitting* of the filtration induced by $L_\bullet$ on $\mathrm{Gr}^{W(N,L)}_k(A)$ for any $k \in \mathbf{Z}$. This splitting (with an alternative proof) is also given in [Sai90, Prop. 1.5]. In other words, there is a canonical isomorphism

$$L_i \mathrm{Gr}^{W(N,L)}_k(A) \cong \bigoplus_{j \leq i} \mathrm{Gr}^L_j \mathrm{Gr}^{W(N,L)}_k(A). \tag{3}$$

The above definitions and results hold in any exact category. This extension is important when considering (bi)-filtered $\mathcal{D}$-modules $(\mathcal{M}, F, W)$, though it makes the notation a bit cumbersome. For the details on this extension, consult [Sai90, Ch. 1].

In the theory of mixed Hodge modules, another property required of any $(\mathcal{M}, F, W)$ which underlies a mixed Hodge module is the following: for any $\lambda \in [0,1]$, the relative monodromy filtration on $\mathrm{Gr}^\lambda_V(\mathcal{M})$ with respect to the nilpotent operator $t\partial_t - \lambda + 1$ for the filtration

$$L_\bullet \mathrm{Gr}^\lambda_V(\mathcal{M}) = \begin{cases} \mathrm{Gr}^0_V(W_\bullet \mathcal{M}) & \lambda = 0 \\ \mathrm{Gr}^\lambda_V(W_{\bullet+1}\mathcal{M}) & \lambda \in (0,1] \end{cases}$$

should exist. The shift by 1 in the case $\lambda \in (0,1]$ is incredibly important to the theory. This relative monodromy filtration is then the weight filtration on $\mathrm{Gr}^\lambda_V(\mathcal{M})$ as a mixed Hodge module.

2.3. **Higher codimension V-filtrations.** In this subsection, we review the results of [CD23, CDS23] concerning Koszul-like complexes of higher codimension V-filtrations. For ease, we work always on $T = X \times \mathbf{A}^r_t$ with coordinates $t_1, \ldots, t_r$ on $\mathbf{A}^r_t$. By the graph embedding trick, this is always the local situation.

If $\mathcal{M}$ is a $\mathcal{D}_{X\times \mathbf{A}^r_t}$-module underlying a mixed Hodge module, then it admits a V-filtration along $t_1, \ldots, t_r$. For details on this V-filtration, see [BMS06, CD23].

The V-filtration is the unique decreasing, discretely and left continuously $\mathbf{Q}$-indexed filtration $V^\bullet \mathcal{M}$ such that

(1) $V^\chi \mathcal{M}$ is finitely generated over $V^0 \mathcal{D}_{X\times \mathbf{A}^r_t} = \mathcal{D}_X[t_1, \ldots, t_r]\langle t_i \partial_{t_j} \mid i,j \in \{1, \ldots, r\}\rangle$.
(2) $(t_1, \ldots, t_r) V^\chi \mathcal{M} \subseteq V^{\chi+1}\mathcal{M}$ for all $\chi \in \mathbf{Q}$, with equality for $\chi \gg 0$.
(3) $\partial_{t_i} V^\chi \mathcal{M} \subseteq V^{\chi-1}\mathcal{M}$ for all $i \in \{1, \ldots, r\}$ and $\chi \in \mathbf{Q}$.
(4) Let $\theta = \sum_{i=1}^r t_i \partial_{t_i}$. Then the operator $\theta - \chi + r$ is nilpotent on
$$\mathrm{Gr}^\chi_V(\mathcal{M}) = V^\chi \mathcal{M} / V^{>\chi}\mathcal{M},$$
where $V^{>\chi}\mathcal{M} = \bigcup_{\beta > \chi} V^\beta \mathcal{M}$.

Remark 2.9. As in the hypersurface case, we will also use the operator $s = \sum_{i=1}^r s_i$ where $s_i = -\partial_{t_i} t_i$. As $s = -(\theta + r)$, we can restate the last condition as requiring that $s + \chi$ is nilpotent on $\mathrm{Gr}_V^\chi(\mathcal{M})$.

As in the introduction, we define

$$A^\chi(\mathcal{M}) = \left[V^\chi \mathcal{M} \xrightarrow{t_i} \bigoplus_{i=1}^r V^{\chi+1}\mathcal{M} \xrightarrow{t_i} \dots \xrightarrow{t_i} V^{\chi+r}\mathcal{M} \right],$$

$$B^\chi(\mathcal{M}) = \left[\mathrm{Gr}_V^\chi(\mathcal{M}) \xrightarrow{t_i} \bigoplus_{i=1}^r \mathrm{Gr}_V^{\chi+1}(\mathcal{M}) \xrightarrow{t_i} \dots \xrightarrow{t_i} \mathrm{Gr}_V^{\chi+r}(\mathcal{M}) \right].$$

Proposition 2.10. *Let $\sigma \colon X \times \{0\} \to X \times \mathbf{A}_t^r$ be the inclusion of the zero section. We have quasi-isomorphisms $\sigma^!(\mathcal{M}) = \mathrm{Kosz}(\mathcal{M}, t) \cong A^0(\mathcal{M}) \cong B^0(\mathcal{M})$.*

In fact, for $\chi \neq 0$, the complex $B^\chi(\mathcal{M})$ is acyclic and for $\chi > 0$, the complex $A^\chi(\mathcal{M})$ is acyclic.

Proof. As $\theta - \chi - j + r$ is nilpotent on $\mathrm{Gr}_V^\chi(\mathcal{B}_f)$, we see that $\theta - j + r$ is an automorphism of $\mathrm{Gr}_V^{\chi+j}(\mathcal{M})$ for all $\chi \neq 0$. This allows us to define an automorphism of the complex $B^\chi(\mathcal{M})$. But using the ∂_{t_i} maps, it is easy to see that this automorphism is a null-homotopy, proving that $B^\chi(\mathcal{M})$ is acyclic for $\chi \neq 0$.

The claim for $A^\chi(\mathcal{M})$ being acyclic is easy to show for $\chi \gg 0$ using a strict surjection (on the $\mathbf{Z}$-indexed part)

$$\bigoplus_I (\mathcal{D}_{X\times \mathbf{A}^r}, V[\beta_i]) \to (\mathcal{M}, V),$$

and using the fact that such acyclicity is trivial to check for the ring $\mathcal{D}_{X\times\mathbf{A}^r}$.

Finally, the definition of $\sigma^!(\mathcal{M})$ is as the derived $\mathcal{O}$-module pull-back of $\mathcal{M}$ along $\sigma \colon X \times \{0\} \to X \times \mathbf{A}_t^r$. Using the Koszul resolution of $\mathcal{O}_{X\times\{0\}}$, we see that $\sigma^!(\mathcal{M}) = \mathrm{Kosz}(\mathcal{M}, t)$.

Using what we have already shown, it is obvious that $A^0(\mathcal{M}) \to B^0(\mathcal{M})$ is a quasi-isomorphism. By discreteness of the filtration $V^\bullet\mathcal{M}$, we can check that for all $\chi < 0$, the inclusion $A^0(\mathcal{M}) \to A^\chi(\mathcal{M})$ is a quasi-isomorphism. Taking the inductive limit as $\chi \to -\infty$ proves the claim. ☐

The main results of [CD23, CDS23] are to extend the results of the previous lemma to include the Hodge and weight filtrations.

For $(\mathcal{M}, F, W)$ a bi-filtered $\mathcal{D}_{X\times\mathbf{A}_t^r}$-module underlying a mixed Hodge module, we define filtered complexes

$$F_pA^\chi(\mathcal{M}) = \left[F_{p+r}V^\chi \mathcal{M} \xrightarrow{t_i} \bigoplus_{i=1}^r F_{p+r}V^{\chi+1}\mathcal{M} \xrightarrow{t_i} \dots \xrightarrow{t_i} F_{p+r}V^{\chi+r}\mathcal{M} \right],$$

$$F_pB^\chi(\mathcal{M}) = \left[F_{p+r}\mathrm{Gr}_V^\chi(\mathcal{M}) \xrightarrow{t_i} \bigoplus_{i=1}^r F_{p+r}\mathrm{Gr}_V^{\chi+1}(\mathcal{M}) \xrightarrow{t_i} \dots \xrightarrow{t_i} F_{p+r}\mathrm{Gr}_V^{\chi+r}(\mathcal{M}) \right].$$

Using Verdier specialization (as in Subsection 3.1 below), we can easily see that the relative monodromy filtration for $N = \theta - \chi + r$ on $\mathrm{Gr}_V^\chi(\mathcal{M})$ for the filtration $L_\bullet \mathrm{Gr}_V^\chi(\mathcal{M}) = \mathrm{Gr}_V^\chi(W_\bullet\mathcal{M})$ exists. We denote it by $W_\bullet\mathrm{Gr}_V^\chi(\mathcal{M})$.

We have the sub-complexes $L_\bullet B^0(\mathcal{M}) = B^0(W_\bullet\mathcal{M}) \subseteq B^0(\mathcal{M})$, and since the morphisms in $B^0(\mathcal{M})$ also commute with N (in the obvious way), we see that these morphisms preserve the relative monodromy filtration. Thus, we get a weight filtration

$$W_\bullet B^0(\mathcal{M}) = \left[W_\bullet \mathrm{Gr}_V^0(\mathcal{M}) \xrightarrow{t_i} \bigoplus_{i=1}^r W_\bullet \mathrm{Gr}_V^1(\mathcal{M}) \xrightarrow{t_i} \dots \xrightarrow{t_i} W_\bullet \mathrm{Gr}_V^r(\mathcal{M})\right].$$

With these filtrations on the complexes, we have the following results:

Theorem 2.11. *The complexes*

$$A^\chi(\mathcal{M},F) \text{ and } B^\chi(\mathcal{M},F) \text{ are filtered acyclic for all } \chi > 0.$$

Moreover, we have filtered quasi-isomorphisms

$$\sigma^!(\mathcal{M},F) \cong A^0(\mathcal{M},F) \cong B^0(\mathcal{M},F),$$

and the latter two complexes are strictly filtered.

Finally, the filtration $W_\bullet \mathcal{H}^i B^0(\mathcal{M},F)$ induced by $W_\bullet B^0(\mathcal{M},F)$ satisfies

$$\mathrm{Gr}_k^W \mathcal{H}^i B^0(\mathcal{M},F) \cong \mathrm{Gr}_{k+i}^W \mathcal{H}^i \sigma^!(\mathcal{M},F),$$

as filtered $\mathcal{D}_X$-modules underlying polarizable Hodge modules of weight $k+i$.

We remark that the filtered complex $W_\bullet B^0(\mathcal{M})$ need not be strict. However, the weight spectral sequence degenerates at E_2, see, for example [Sai00, Prop. 2.3].

3. Generalized V-filtrations

Throughout this section, we work on $T = X \times \mathbf{A}_t^r$, and let $t_1,\dots,t_r$ be the coordinates on $\mathbf{A}_t^r$.

We call a linear form $L(s) = \sum_{i=1}^r a_i s_i$ a *slope* if $a_i \in \mathbf{Z}_{\geq 0}$ for all i. It is non-degenerate if $a_i \neq 0$ for all i. Given a slope L, we obtain a $\mathbf{Z}$-indexed filtration on $\mathcal{D}_T$ by

$${}^LV^j\mathcal{D}_T = \left\{\sum_{\beta,\gamma} P_{\beta,\gamma} t^\beta \partial_t^\gamma \mid P_{\beta,\gamma} \in \mathcal{D}_X, L(\beta) \geq L(\gamma) + j.\right\}$$

If $\mathcal{M}$ is a module over $\mathcal{D}_T$, we say that a filtration $U^\bullet\mathcal{M}$ is *compatible* if

$${}^LV^j\mathcal{D}_T \cdot U^k\mathcal{M} \subseteq U^{k+j}\mathcal{M},$$

for example, the filtration ${}^LV^\bullet\mathcal{D}_T$ is compatible (in other words, it is a multiplicative filtration).

We define the *Rees ring* $R_L(\mathcal{D}_T) = \bigoplus_{k\in\mathbf{Z}} {}^LV^k\mathcal{D}_T u^{-k}$. It is a $\mathbf{Z}$-graded ring, by the multiplicative property of the filtration. Given a module $\mathcal{M}$ with a compatible filtration $U^\bullet\mathcal{M}$, we define the *Rees module* $R_U(\mathcal{M}) = \bigoplus_{k\in\mathbf{Z}} U^k\mathcal{M}u^{-k}$. The filtration $U^\bullet\mathcal{M}$ is *good* if $R_U(\mathcal{M})$ is coherent over $R_L(\mathcal{D}_T)$. We will also say that $(\mathcal{M},U)$ is a good filtered $(\mathcal{D}_T, {}^LV)$-module in this case.

For a filtered module $(\mathcal{M},U)$ and $k \in \mathbf{Z}$, we set $(\mathcal{M},U[k])$ for the filtration with $U[k]^\bullet\mathcal{M} = U^{\bullet-k}\mathcal{M}$. The following is immediate.

Lemma 3.1. *An exhaustive filtration $U^\bullet\mathcal{M}$ is good if and only if there exist $m_1, \dots, m_N$ and $k^{(1)}, \dots, k^{(N)} \in \mathbf{Z}$ such that we have*

$$U^\bullet\mathcal{M} = \sum_{i=1}^N {}^LV^{\bullet - k^{(i)}}\mathcal{D}_T \cdot m_i.$$

Equivalently, we have a strict surjection $\bigoplus_{i=1}^N (\mathcal{D}_T, {}^LV[k^{(i)}]) \to (\mathcal{M}, U)$.

From this, we get comparability of good filtrations:

Lemma 3.2. *Let $U_1^\bullet\mathcal{M}, U_2^\bullet\mathcal{M}$ be two good filtrations of the $\mathcal{D}_T$-module $\mathcal{M}$. Then there exists $k \in \mathbf{Z}$ such that*

$$U_1^{\bullet+k}\mathcal{M} \subseteq U_2^\bullet\mathcal{M} \subseteq U_1^{\bullet-k}\mathcal{M}.$$

The following can be thought of as the analogue of the Artin-Rees lemma:

Lemma 3.3. *The ring $R_L(\mathcal{D}_T)$ is Noetherian. Thus, if $(\mathcal{M}, U)$ is a good filtered $(\mathcal{D}_T, {}^LV)$-module and $\mathcal{N} \subseteq \mathcal{M}$ is a sub $\mathcal{D}_T$-module, the induced filtration*

$$U^\bullet\mathcal{N} = \mathcal{N} \cap U^\bullet\mathcal{M} \text{ is good.}$$

Proof. The second claim is immediate from the first. The first follows from Lemma 3.14 below, which we postpone until we discuss specialization constructions. □

Definition 3.4. Let $(\mathcal{M}, U)$ be a good filtered $(\mathcal{D}_T, {}^LV)$-module. The b-function for $U^\bullet\mathcal{M}$ is the monic polynomial $p(w) \in \mathbf{C}[w]$ of least degree such that

$$p(L(s) + k)U^k\mathcal{M} \subseteq U^{k+1}\mathcal{M}, \tag{4}$$

where $L(s) = \sum_{i=1}^r -a_i\partial_{t_i}t_i$.

We say $(\mathcal{M}, U)$ is *specializable* if it admits a b-function. For any subfield $A \subseteq \mathbf{C}$, we say $(\mathcal{M}, U)$ is A-specializable if the b-function splits into linear factors over A.

Lemma 3.2 can be used to show the following, which says that being A-specializable is a property of the module, not the filtration:

Proposition 3.5. *If $(\mathcal{M}, U)$ is a good filtered $(\mathcal{D}_T, {}^LV)$-module which is A-specializable, then any other good filtration is also A-specializable.*

Lemma 3.3 shows that if $\mathcal{N} \subseteq \mathcal{M}$ is a submodule and $\mathcal{M}$ is A-specializable, then $\mathcal{N}$ is, too. This applies in particular to $\mathcal{D}_T \cdot u \subseteq \mathcal{M}$ for any element $u \in \mathcal{M}$, which leads to the following definition.

Definition 3.6. Let $\mathcal{M}$ be A-specializable. For any $u \in \mathcal{M}$, the b-function of u is the monic polynomial of least degree $b(w) \in \mathbf{C}[w]$ such that

$$b(s)u \in {}^LV^1\mathcal{D}_T \cdot u,$$

which we denote by $b_u(w)$. Such a polynomial exists (and splits over A) for any section $u \in \mathcal{M}$ of an A-specializable module.

For the remainder, we assume $\mathcal{M}$ is $\mathbf{Q}$-specializable, though the same constructions can be made with $A = \mathbf{R}$. Given $u \in \mathcal{M}$, let $b_u(w)$ be the b-function of u, which we factor as

$$b_u(w) = (w + \gamma_1) \dots (w + \gamma_N),$$

with $\gamma_1 \leq \cdots \leq \gamma_N$. Then define the *$L$-order* of u to be $\mathrm{ord}_L(u) = \gamma_1$.

This leads to a $\mathbf{Q}$-indexed filtration ${}^L V^\bullet \mathcal{M}$ defined by

$${}^L V^\lambda \mathcal{M} = \{u \in \mathcal{M} \mid \mathrm{ord}_L(u) \geq \lambda\},$$

whose $\mathbf{Z}$-indexed part is characterized by the following proposition.

Proposition 3.7. *Let $\mathcal{M}$ be $\mathbf{Q}$-specializable. Then there exists a unique good filtration $U^\bullet \mathcal{M}$ whose b-function satisfies the following: $b_U(-\gamma) = 0 \implies \gamma \in [0,1)$.*

Moreover, for all integers $j \in \mathbf{Z}$, this filtration satisfies

$${}^L V^j \mathcal{M} = U^j \mathcal{M}.$$

Remark 3.8. The $\mathbf{Z}$-indexed filtration in the proposition statement can be refined in the following way to a $\mathbf{Q}$-indexed filtration, which agrees with the $\mathbf{Q}$-indexed filtration ${}^L V^\bullet \mathcal{M}$. The main idea is to lift generalized eigenspaces of the operator $L(s)$ on the associated graded pieces $\mathrm{Gr}_U^j(\mathcal{M})$.

To be precise, let $U^\bullet \mathcal{M}$ be a good $\mathbf{Z}$-indexed filtration whose b-function $b_U(w)$ satisfies $b_U(-\gamma) = 0$ implies $\gamma \in [0,1)$. Write $b_U(w) = (w+\gamma_1)^{m_1} \dots (w+\gamma_N)^{m_N}$ for $0 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_N < 1$.

For any $j \in \mathbf{Z}$, define $U^{\gamma_N + j}\mathcal{M} = \{u \in \mathcal{M} \mid (L(s)+\gamma_N+j)^{m_N} u \in U^{j+1}\mathcal{M}\}$. Inductively, we then define

$$U^{\gamma_i + j}\mathcal{M} = \{u \in \mathcal{M} \mid (L(s)+\gamma_i+j)^{m_i} u \in U^{j+\gamma_{i+1}}\mathcal{M}\}.$$

For any $\chi \in \mathbf{Q}$, set $j = \lfloor \chi \rfloor$ and $\varepsilon = \chi - j \in [0,1)$. First, if $\varepsilon \leq \gamma_N$, let i be minimal such that $\varepsilon \leq \gamma_i$ and set $U^\chi \mathcal{M} = U^{j+\gamma_i}\mathcal{M}$. Otherwise, if $\varepsilon > \gamma_N$, set $U^\chi \mathcal{M} = U^{j+1+\gamma_1}\mathcal{M}$.

It is an easy exercise to see that, in this case, $U^\bullet \mathcal{M}$ is a decreasing, discrete and left-continuous $\mathbf{Q}$-indexed filtration.

We call ${}^L V^\bullet \mathcal{M}$ "the (canonical) ${}^L V$-filtration" of $\mathcal{M}$. For example, for any $I \subseteq \{1, \dots, r\}$, if $L = \sum_{i \in I} s_i$, this filtration is the Kashiwara-Malgrange V-filtration of $\mathcal{M}$ along the subvariety $V(t_i \mid i \in I)$.

Define Koszul-like complexes

$$A_L^\gamma(\mathcal{M}) = \left[{}^L V^\gamma \mathcal{M} \xrightarrow{t_i} \bigoplus_{i=1}^r {}^L V^{\gamma + a_i}\mathcal{M} \xrightarrow{t_i} \dots \xrightarrow{t_i} {}^L V^{\gamma + |L|}\mathcal{M} \right]$$

$$B_L^\gamma(\mathcal{M}) = \left[\mathrm{Gr}_L^\gamma(\mathcal{M}) \xrightarrow{t_i} \bigoplus_{i=1}^r \mathrm{Gr}_L^{\gamma + a_i}(\mathcal{M}) \xrightarrow{t_i} \dots \xrightarrow{t_i} \mathrm{Gr}_L^{\gamma + |L|}(\mathcal{M}) \right]$$

placed in degree $0, 1, \dots, r$.

The following can be shown in the same way as Proposition 2.10 above, using the fact that $L(s) + \lambda$ is nilpotent on $\mathrm{Gr}_L^\lambda(\mathcal{M})$.

Lemma 3.9. *Let $\mathcal{M}$ admit an ${}^L V$-filtration. Then $B_L^\chi(\mathcal{M})$ is acyclic for all $\chi \neq 0$. Moreover, $A_L^\chi(\mathcal{M})$ is acyclic for all $\chi > 0$.*

Let $\mathcal{M}$ admit an ${}^L V$-filtration. Then

$$\sigma^!(\mathcal{M}) \cong A_L^0(\mathcal{M}) \cong B_L^0(\mathcal{M})$$

for any slope L.

The following proposition gives the characterizing properties of the canonical ${}^LV$-filtration, as well as a useful test for containment. As above, instead of $L(s)+\chi$ being nilpotent on $\mathrm{Gr}_L^\chi(\mathcal{M})$, we could ask for $\theta_L - \chi + |L| = \sum_{i=1}^r a_i t_i \partial_{t_i} - \chi + |L|$ to be nilpotent.

Proposition 3.10. *Let $\mathcal{M}$ be a $\mathcal{D}_T$-module. Assume $U^\bullet\mathcal{M}$ is a discrete, left-continuous $\mathbf{Q}$-indexed filtration such that the following conditions hold:*

(1) ${}^LV^k\mathcal{D}_T \cdot U^\chi\mathcal{M} \subseteq U^{k+\chi}\mathcal{M}$,
(2) for $\chi \gg 0$ we have equality $U^\chi\mathcal{M} = \sum_{i=1}^r t_i U^{\chi - a_i}\mathcal{M}$,
(3) for any $\chi \in \mathbf{Q}$, the operator $L(s)+\chi$ is nilpotent on $\mathrm{Gr}_U^\chi(\mathcal{M})$.

Then ${}^LV^\chi\mathcal{M} \subseteq U^\chi\mathcal{M}$ for all $\chi \in \mathbf{Q}$. If, moreover, we assume that $U^\chi\mathcal{M}$ is coherent over ${}^LV^0\mathcal{D}_T$ for all $\chi \in \mathbf{Q}$, then equality holds.

Proof. The claim follows from the observation that if $U_1^\bullet\mathcal{M}$ is a filtration satisfying all conditions in the proposition statement and $U_2^\bullet\mathcal{M}$ is another filtration satisfying just the first three conditions, then $U_1^\bullet\mathcal{M} \subseteq U_2^\bullet\mathcal{M}$. This can be shown similarly to [CDM24, Prop. 3.14] and we leave the checking of details to the reader.

To see why this observation implies the desired result, it suffices to note that ${}^LV^\bullet\mathcal{M}$ satisfies all the conditions in the proposition statement. Indeed, by construction, $L(s)+\chi$ is nilpotent on $\mathrm{Gr}_L^\chi(\mathcal{M})$. By definition of ${}^LV^\chi\mathcal{M}$ in Remark 3.8, it is easy to see that the $\mathbf{Q}$-indexed filtration ${}^LV^\chi\mathcal{M}$ is compatible, using the fact that $Pt^\beta\partial_t^\gamma L(s) = L(s+\beta-\gamma)Pt^\beta\partial_t^\gamma$ for $P \in \mathcal{D}_X$. By the goodness of the $\mathbf{Z}$-indexed filtration ${}^LV^\bullet\mathcal{M}$ and Noetherianity of the ring ${}^LV^0\mathcal{D}_T$ (which holds because it is the 0th graded piece of a Noetherian $\mathbf{Z}$-graded ring), we see that each ${}^LV^\chi\mathcal{M}$ is ${}^LV^0\mathcal{D}_T$-coherent.

From this, using the acyclicity of Lemma 3.9, we see that the Koszul-like complex

$$A_L^\gamma(\mathcal{M}) = \left[{}^LV^\gamma\mathcal{M} \xrightarrow{t_i} \bigoplus_{i=1}^r {}^LV^{\gamma+a_i}\mathcal{M} \xrightarrow{t_i} \dots \xrightarrow{t_i} {}^LV^{\gamma+|L|}\mathcal{M}\right]$$

is acyclic for all $\gamma > 0$, where $|L| = \sum_{i=1}^r a_i$. In particular, by the vanishing of the rightmost cohomology, we see that for all $\chi > |L|$, we have the equality

$${}^LV^\chi\mathcal{M} = \sum_{i=1}^r t_i\, {}^LV^{\chi - a_i}\mathcal{M}.$$

□

Example 3.11. Let $\mathcal{M}$ be supported on $V(t_1,\dots,t_r) \subseteq T$. Then we can write $\mathcal{M} = \bigoplus_{\alpha\in\mathbf{N}^r}\mathcal{M}_0\partial_t^\alpha\delta_0$ where $t_i(\eta\delta_0) = 0$ for all $\eta \in \mathcal{M}_0$. In particular,

$$L(s)\eta\delta_0 = \sum_{i=1}^r -a_i\partial_{t_i}t_i(\eta\delta_0) = 0.$$

It is not hard to check that $\mathcal{M}$ is $\mathbf{Q}$-specializable. Using that $L(s)\partial_t^\alpha\eta\delta_0 = \partial_t^\alpha L(s+\alpha)\eta\delta_0 = L(\alpha)\partial_t^\alpha\eta\delta_0$, we see in fact that

$${}^LV^\lambda\mathcal{M} = {}^LV^{\lceil\lambda\rceil}\mathcal{M} = \bigoplus_{L(\alpha)\le -\lceil\lambda\rceil}\mathcal{M}_0\partial_t^\alpha\delta_0,$$

which has b-function equal to $b(w) = w$.

Example 3.12. Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_T$-module supported on $V(t_i) \subseteq T$. Then we have $\mathcal{M} = \bigoplus_{j\in \mathbf{N}} \mathcal{N}\partial_{t_i}^j \delta_0$ where $t_i(\eta\delta_0) = 0$ for all $\eta \in \mathcal{N}$. In particular, we have $L(s)(\eta\delta_0) = (\ell(s)\eta)\delta_0$, where if $L = \sum_{i=1}^r a_i s_i$ then $\ell = \sum_{j\neq i} a_j s_j$. More generally,

$$L(s)(\eta\partial_{t_i}^j\delta_0) = (\ell(s) + a_i j)(\eta)\partial_{t_i}^j \delta_0.$$

From this it is easy to see

$${}^L V^\lambda \mathcal{M} = \bigoplus_{j\geq 0} {}^\ell V^{\lambda + a_i j}\mathcal{N}\partial_{t_i}^j\delta_0.$$

The following lemma applies in particular to the case when $\mathcal{M}$ is $\mathcal{O}$-coherent, or when $\mathcal{M} = \mathcal{N} \boxtimes \mathcal{O}_{\mathbf{A}_t^r}$ for $\mathcal{N}$ a $\mathcal{D}_X$-module.

Lemma 3.13. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_T$-module. Then the ${}^L V$-filtration is "t-adic", in the sense that*

$${}^L V^\lambda \mathcal{M} = \begin{cases} \mathcal{M} & \lambda \leq |L| \\ \sum_{i=1}^r t_i {}^L V^{\lambda - a_i}\mathcal{M} & \lambda > |L|, \end{cases}$$

if and only if $\mathcal{M}$ is coherent over ${}^L V^0 \mathcal{D}_T$.

Proof. If the filtration is t-adic, then ${}^L V^{|L|}\mathcal{M} = \mathcal{M}$ is, by definition, coherent over ${}^L V^0\mathcal{D}_T$. Note that another way to write the t-adic filtration is

$${}^L V^\lambda \mathcal{M} = \left(t^\beta \mid L(\beta + \underline{1}) \geq \lambda\right) \cdot \mathcal{M}.$$

For the converse, note that if $\mathcal{M}$ is coherent over ${}^L V^0 \mathcal{D}_T$ and we define

$$U^\lambda \mathcal{M} = \left(t^\beta \mid L(\beta + \underline{1}) \geq \lambda\right) \cdot \mathcal{M},$$

then each $U^\lambda\mathcal{M}$ is coherent over ${}^L V^0\mathcal{D}_T$. Then we need to check the remaining properties of the ${}^L V$-filtration to conclude.

Assume $L(\beta + \underline{1}) \geq \lambda$. Then $L(\beta + e_i + \underline{1}) \geq \lambda + a_i$, so we see that $t_i U^\lambda \mathcal{M} \subseteq U^{\lambda + a_i}\mathcal{M}$.

Let $L(\beta+\underline{1}) \geq \lambda$. Then for $t^\beta m \in U^\lambda\mathcal{M}$, when we apply ∂_{t_i}, there are two options. Either $\beta_i = 0$, in which case

$$\partial_{t_i}(t^\beta m) = t^\beta(\partial_{t_i} m) \in U^\lambda \mathcal{M} \subseteq U^{\lambda - a_i}\mathcal{M},$$

or $\beta_i > 0$, in which case

$$\partial_{t_i}(t^\beta m) = t^\beta(\partial_{t_i} m) + \beta_i t^{\beta - e_i} m,$$

and so since $L(\beta - e_i + \underline{1}) \geq \lambda - a_i$, this shows that $\partial_{t_i} U^\lambda\mathcal{M} \subseteq U^{\lambda - a_i}\mathcal{M}$.

Each $U^\lambda \mathcal{M}$ is clearly stable by $\mathcal{D}_X$, so we only need to prove the nilpotency of $L(s) + \lambda$ on $\mathrm{Gr}_U^\lambda \mathcal{M}$. Note that $\mathrm{Gr}_U^\lambda\mathcal{M} \neq 0$ if and only if there exists $\beta \in \mathbf{N}^r$ with $L(\beta + \underline{1}) = \lambda$. Take such a β and consider an element $t^\beta m$. Then

$$(L(s) + \lambda)(t^\beta m) = L(s + \beta + \underline{1})(t^\beta m) = t^\beta L(s + \underline{1})m,$$

and $s + \underline{1} = (s_1 + 1, \ldots, s_r + 1) = (-t_1\partial_{t_1}, \ldots, -t_r\partial_{t_r})$. Hence,

$$L(s + \underline{1})m = -\sum_{i=1}^r a_i t_i \partial_{t_i}(m),$$

and we have that $a_i t^{\beta + e_i}(\partial_{t_i} m) \in U^{>\lambda}\mathcal{M}$, proving the claim. ∎

3.1. **Specialization Constructions.** In this section, we use deformation to the normal bundle to allow us to use established tools to study ${}^LV$-filtrations. The idea goes back at least to Verdier. The papers [BMS06] and [CD23] use these ideas to study the V-filtration when $L = \sum_{i=1}^r s_i$, and Wu's article [Wu22] discusses the case of arbitrary slopes L.

As above, consider $T = X \times \mathbf{A}^r_t$. Let $L = \sum_{i=1}^r a_i s_i$ be a non-degenerate slope. For $T = X \times \mathbf{A}^r_t$, define

$$\widetilde{T}^L = X \times \mathbf{A}^r_{t/u^a} \times \mathbf{A}^1_u,$$

where the coordinates on the $\mathbf{A}^r$ term are $t_1/u^{a_1}, \dots, t_r/u^{a_r}$. This is naturally a deformation to the normal bundle of $X \times \{0\} \subseteq T$.

This carries a map $\widetilde{T}^L \to T$ sending $(x, t/u^a, u)$ to (x,t). Over $\{u \neq 0\}$, this is isomorphic to the projection $T \times \mathbf{G}_m \to T$. Let $j_L \colon \{u \neq 0\} \to \widetilde{T}^L$ be the open embedding.

The projection $\widetilde{T}^L \to \mathbf{A}^1_u$ is clearly smooth, so we can consider the ring of relative differential operators $\mathcal{D}_{\widetilde{T}^L/\mathbf{A}^1_u}$.

We have an identification of the relative differential operators with the Rees ring of $\mathcal{D}_T$ for the ${}^LV$-filtration. As relative differential operator rings are Noetherian, this proves the first claim of Lemma 3.3.

Lemma 3.14. *We have a filtered isomorphism of rings*

$$(\mathcal{D}_{\widetilde{T}_L/\mathbf{A}^1_u}, F) \cong (R_L(\mathcal{D}_T), F).$$

Hence, we also have an isomorphism

$$R_F(\mathcal{D}_{\widetilde{T}_L/\mathbf{A}^1_u}) \cong R_{F,L}(\mathcal{D}_T) = \bigoplus_{k,j} F_k {}^LV^j \mathcal{D}_T z^k u^{-j}.$$

Proof. As the coefficients in $\mathcal{D}_X$ are unimportant in this proof, we assume X is a point.

For ease of notation, write $z_i = \frac{t_i}{u^{a_i}}$.

In this case, we have $\mathcal{D}_{\widetilde{T}^L/\mathbf{A}^1_u} = \mathbf{C}[u, z_1, \dots, z_r]\langle \partial_{z_1}, \dots, \partial_{z_r}\rangle$. Define a $\mathbf{C}[u]$-linear morphism

$$\mathcal{D}_{\widetilde{T}^L/\mathbf{A}^1_u} \to R_L(\mathcal{D}_T),$$

$$z_i \mapsto \frac{t_i}{u^{a_i}}, \quad \partial_{z_i} \mapsto \partial_{t_i} u^{a_i}.$$

For the inverse, we want it to satisfy

$$t^\beta \partial_t^\gamma u^{L(\gamma-\beta)} \mapsto z^\beta \partial_z^\gamma,$$

and so for arbitrary elements

$$t^\beta \partial_t^\gamma u^{-j} = u^{-j-L(\gamma-\beta)}(t^\beta \partial_t^\gamma u^{L(\gamma-\beta)}) \in R_L(\mathcal{D}_T),$$

we simply have to note that, by definition, $t^\beta \partial_t^\gamma \in {}^LV^j \mathcal{D}_T$, so that $L(\beta-\gamma) \geq j$. Thus, $u^{-j-L(\gamma-\beta)} \in \mathbf{C}[u]$, and so we can define the map by sending $t^\beta \partial_t^\gamma u^{L(\gamma-\beta)} \to z^\beta \partial_z^\gamma$ and extending $\mathbf{C}[u]$-linearly.

It is clear that this isomorphism preserves the order filtration on both sides. $\square$

Let $\mathcal{M}$ be a $\mathcal{D}_T$-module with $\mathbf{Z}$-indexed filtrations $F_\bullet\mathcal{M}, U^\bullet\mathcal{M}$ such that $F_\bullet\mathcal{M}$ is compatible with $F_\bullet\mathcal{D}_T$ and $U^\bullet\mathcal{M}$ is compatible with ${}^LV^\bullet\mathcal{D}_T$. We can define

$$F_pA^\chi(\mathcal{M},U) = \left[F_{p+r}U^\chi\mathcal{M} \xrightarrow{t_i} \bigoplus_{i=1}^r F_{p+r}U^{\chi+a_i}\mathcal{M} \xrightarrow{t_i} \dots \xrightarrow{t_i} F_{p+r}U^{\chi+|L|}\mathcal{M}\right].$$

The Noetherianity of the ring $R_{F,L}(\mathcal{D}_T)$ allows us to prove acyclity of $A^\chi(\mathcal{M},U,F)$ for $\chi \gg 0$.

Lemma 3.15. *Let $(\mathcal{M},U,F)$ be a bi-filtered $\mathcal{D}_T$-module as above such that $R_{F,U}(\mathcal{M}) = \bigoplus_{k,j\in\mathbf{Z}} F_kU^j\mathcal{M}z^ku^{-j}$ is coherent over $R_{F,L}(\mathcal{D}_T)$. Then there exists $k_0 \in \mathbf{Z}$ such that $k \geq k_0$ implies $A^k(\mathcal{M},U,F)$ is F-filtered acyclic.*

Proof. It is a simple computation to see that $A^k(\mathcal{D}_T, {}^LV, F)$ is filtered acyclic for all $k \geq 0$. Indeed, by taking $\mathrm{Gr}^F_\bullet$, this boils down to the claim that variables in a polynomial ring form a regular sequence, hence the corresponding Koszul complex is acyclic (except at the right). For the vanishing of the right-most cohomology, use that $k \geq 0$.

Now, by coherence of $R_{F,U}(\mathcal{M})$ over $R_{F,L}(\mathcal{D}_T)$, we have a finite indexing set I and a strict surjection

$$\bigoplus_{i\in I}(\mathcal{D}_T, {}^LV[b_i], F[c_i]) \to (\mathcal{M},U,F) \to 0.$$

By Noetherianity of $R_{F,L}(\mathcal{D}_T)$, the kernel $\mathcal{K}$ with its induced filtrations $F_\bullet\mathcal{K}$ and $U^\bullet\mathcal{K}$ also satisfies $R_{F,U}(\mathcal{K})$ is coherent over $R_{F,L}(\mathcal{D}_T)$. We have the F-strict short exact sequence of complexes

$$0 \to A^k(\mathcal{K},U,F) \to \bigoplus_{i\in I} A^{k-b_i}(\mathcal{D}_T, {}^LV, F[c_i]) \to A^k(\mathcal{M},U,F) \to 0,$$

and so for $k \geq k_0 = \max\{b_i\}$, the middle term is filtered acyclic. Hence, we get the vanishing of $F_p\mathcal{H}^rA^k(\mathcal{M},U)$ for all p and filtered isomorphisms

$$\mathcal{H}^j(A^k(\mathcal{M},U),F) \cong \mathcal{H}^{j+1}(A^k(\mathcal{K},U),F).$$

Repeating the argument with $(\mathcal{K},F,U)$ in place of $(\mathcal{M},F,U)$, and possibly increasing k_0, we get $\mathcal{H}^{r+1}(A^k(\mathcal{M},U),F) \cong \mathcal{H}^r(A^k(\mathcal{K},U),F) = 0$ for all $k \geq k_0$. Repeating again $r-1$ more times, this completes the proof. □

Given a system of coordinates $x_1,\dots,x_n$ on X, the variety $\widetilde{T}^L$ has local coordinates $x_1,\dots,x_n, z_1 = \frac{t_1}{u^{a_1}},\dots,z_r = \frac{t_r}{u^{a_r}}, u$. The open subset $T \times \mathbf{G}_m$ has the simpler system of coordinates $x_1,\dots,x_n,t_1,\dots,t_r,u$, and the change of variables formula yields (using $\widetilde{(-)}$ to denote the functions $x_1,\dots,x_n,u$ viewed in the second system of coordinates):

$$\partial_{\widetilde{x_i}} = \partial_{x_i},$$

$$\partial_{t_i} = \frac{1}{u^{a_i}}\partial_{z_i},$$

$$\partial_{\widetilde{u}} = \partial_u + \sum_{j=1}^r \partial_{\widetilde{u}}(z_j)\partial_{z_j} = \partial_u - \sum_{j=1}^r a_j\frac{t_j}{u^{a_j+1}}\partial_{z_j} = \partial_{v_i} - \frac{1}{u}L(t\partial_t).$$

For M a mixed Hodge module on T, consider $\widetilde{M}_L = j_{L*}(M \boxtimes \mathbf{Q}^H_{\mathbf{G}_m}[1])$ a mixed Hodge module on $\widetilde{T}_L$. The underlying $\mathcal{D}$-module $\widetilde{\mathcal{M}}_L$ is the $\mathcal{O}$-module

$$\bigoplus_{k\in\mathbf{Z}} \mathcal{M}u^k,$$

on which, thanks to the computation of the coordinate change above, the action is given by

$$z_i(mu^k) = (t_i m)u^{k-a_i},$$
$$\partial_{z_i}(mu^k) = \partial_{t_i}(m)u^{k+a_i},$$
$$\partial_u(mu^k) = (k + L(t\partial_t))(m)u^{k-1}.$$

We have the following:

Proposition 3.16. *Let $V^\bullet\widetilde{\mathcal{M}}_L$ be the V-filtration along u. Then*

$$V^\lambda\widetilde{\mathcal{M}}_L = \bigoplus_{k\in\mathbf{Z}} {}^LV^{\lambda+|L|-k-1}\mathcal{M}u^k.$$

For any $\lambda \geq 0$, we have

$$F_pV^\lambda\widetilde{\mathcal{M}}_L = \bigoplus_{k\in\mathbf{Z}} F_p{}^LV^{\lambda+|L|-k-1}\mathcal{M}u^k$$

Proof. The second claim follows from the first using the fact that for all $p \in \mathbf{Z}$ and $\lambda \geq 0$, we have by Lemma 2.8 equality

$$F_pV^\lambda\widetilde{\mathcal{M}}_L = V^\lambda\widetilde{\mathcal{M}}_L \cap j_{L*}(F_p(\mathcal{M} \boxtimes \mathcal{O}_{\mathbf{G}_m})).$$

For the first claim, define

$$U^\lambda\widetilde{\mathcal{M}}_L = \bigoplus_{k\in\mathbf{Z}} {}^LV^{\lambda+|L|-k-1}\mathcal{M}u^k.$$

We show that $U^\lambda\widetilde{\mathcal{M}}_L$ satisfies the properties of the V-filtration along u. A simple computation shows

$$uU^\lambda = U^{\lambda+1}, \quad \partial_uU^\lambda \subseteq U^{\lambda-1}.$$

As

$$u\partial_u(mu^k) = (k + L(t\partial_t))(m)u^k$$

it is easy to see that $u\partial_u - \lambda + 1$ is nilpotent on $\mathrm{Gr}^\lambda_U(\widetilde{\mathcal{M}}_L)$. By Proposition 3.10, this proves the containment $V^\lambda\widetilde{\mathcal{M}}_L \subseteq U^\lambda\widetilde{\mathcal{M}}_L$

For the other containment, for any fixed k, define a filtration

$$\mathcal{U}^\lambda\mathcal{M} = \{m \in \mathcal{M} \mid mu^k \in V^{\lambda+k+1-|L|}\widetilde{\mathcal{M}}_L\}.$$

Let $mu^k \in V^{\lambda+k+1-|L|}\widetilde{\mathcal{M}}_L$. Then by applying $u^{a_i}z_i$, we see that

$$(t_im)u^k \in V^{\lambda+a_i+k+1-|L|}\widetilde{\mathcal{M}}_L,$$

so that $t_i\mathcal{U}^\lambda \subseteq \mathcal{U}^{\lambda+a_i}$. Applying ∂_{z_i}, we get that $\partial_{t_i}(m)u^{k+a_i} \in V^{\lambda+a_i+k+1-|L|}\widetilde{\mathcal{M}}_L$.

As u acts invertibly on $\widetilde{\mathcal{M}}_L$, we know that $u\colon V^\chi\widetilde{\mathcal{M}}_L \to V^{\chi+1}\widetilde{\mathcal{M}}_L$ is an isomorphism. Thus, having $\partial_{t_i}(m)u^{k+a_i} \in V^{\lambda+k+1-|L|}\widetilde{\mathcal{M}}_L$ implies that $\partial_{t_i}(m)u^k \in V^{\lambda-a_i+k+1-|L|}\widetilde{\mathcal{M}}_L$.

By definition of the V-filtration, we know that $V^{\lambda+k+1-|L|}\widetilde{\mathcal{M}}_L$ is coherent over $V^0\mathcal{D}_{\widetilde{T}^L} = \mathcal{D}_X[z,u]\langle\partial_z, u\partial_u\rangle$. For some fixed λ, choose generators $m_1u^{\ell_1},\dots,m_au^{\ell_a}$ for $V^{\lambda+k+1-|L|}\widetilde{\mathcal{M}}_L$ over V^0. As $u\colon V^\chi\widetilde{\mathcal{M}}_L \to V^{\chi+1}\widetilde{\mathcal{M}}_L$ is an isomorphism for all $\chi\in\mathbf{Q}$, we see then that

$$m_1u^{\ell_1+j},\dots,m_au^{\ell_a+j}$$

are generators of $V^{(\lambda+j)+k+1-|L|}\widetilde{\mathcal{M}}_L$ for all $j\in\mathbf{Z}$.

Let $\ell=\min\{\ell_1,\dots,\ell_a\}$. Note that the only operators in V^0 which decrease the power of u are $z_1,\dots,z_r$. Thus, we see that for any $b<\ell+j$, we have

$$mu^b\in V^{(\lambda+j)+k+1-|L|}\widetilde{\mathcal{M}}_L \implies mu^b\in(z_1,\dots,z_r)V^{(\lambda+j)+k+1-|L|}\widetilde{\mathcal{M}}_L,$$

and so for all j with $j>k-\ell$, we have

$$m\in\mathcal{U}^{\lambda+j}\mathcal{M} \implies m\in\sum_{i=1}^r t_i\mathcal{U}^{\lambda+j-a_i}\mathcal{M}.$$

Finally, it is clear that $L(t\partial_t)-\lambda+|L|$ is nilpotent on $\mathrm{Gr}_{\mathcal{U}}^\lambda(\mathcal{M})$. By Proposition 3.10, this shows

$${}^LV^\bullet\mathcal{M}\subseteq\mathcal{U}^\bullet\mathcal{M},$$

which finishes the proof of the claim. □

Lemma 3.17. *Consider the $\mathbf{Z}$-indexed ${}^LV$-filtration ${}^LV^\bullet\mathcal{M}$ on $\mathcal{M}$. Then $R_{F,L}(\mathcal{M})$ is coherent over $R_{F,L}(\mathcal{D}_T)$.*

In particular, for integer $k\gg0$, the complex $A_L^k(\mathcal{M},F)$ is filtered acyclic.

Proof. The second claim follows immediately from the first using Lemma 3.15.

The first claim follows from the observation that, up to a shift of grading, we have $R_{F,L}(\mathcal{M})=R_F(V^0\widetilde{\mathcal{M}}_L)$ and the isomorphism $R_F(\mathcal{D}_{\widetilde{T}^L/\mathbf{A}^1_u})\cong R_{F,L}(\mathcal{D}_T)$ from Lemma 3.14. We know by Remark 2.5 that $R_F(V^0\widetilde{\mathcal{M}}_L)$ is coherent over $R_F(\mathcal{D}_{\widetilde{T}^L/\mathbf{A}^1_u})$, proving the claim. □

Define $\mathrm{Sp}_L(M)=\psi_u(\widetilde{M}_L)$, which is a mixed Hodge module on $X\times\mathbf{A}^r_z$, where we use $z_i=\frac{t_i}{u^{a_i}}$ as above. Its underlying filtered $\mathcal{D}$-module is given by

$$F_p\mathrm{Sp}_L(\mathcal{M})=\bigoplus_{\chi\in\mathbf{Q}}F_p\mathrm{Gr}_L^\chi(\mathcal{M}).$$

This is an example of an *L-monodromic* mixed Hodge module, i.e., one whose underlying $\mathcal{D}$-module is L-monodromic. Recall that this means that every local section m is annihilated by some polynomial in $L(z\partial_z)=\sum_{i=1}^r a_iz_i\partial_{z_i}$.

Such modules decompose into generalized eigenspaces for the operator $L(z\partial_z)$: we write

$$\mathcal{N}=\bigoplus_{\chi\in\mathbf{Q}}\mathcal{N}^\chi,$$

where $\mathcal{N}^\chi=\bigcup_{j\geq1}\ker((L(z\partial_z)-\chi+|L|)^j)$. Any L-monodromic module carries a nilpotent $\mathcal{D}$-linear endomorphism N which acts on $\mathcal{N}^\chi$ by $L(z\partial_z)-\chi+|L|$.

We record the following useful fact about pure L-monodromic mixed Hodge modules.

Lemma 3.18. *Assume M is pure and L-monodromic on $X \times \mathbf{A}^r_z$. Then $N = 0$ on M, i.e., $L(z\partial_z)$ acts semi-simply on $\mathcal{M}$.*

Proof. As the $\mathcal{D}$-module underlying M is semi-simple (see [CD23, Pf. of Prop. 5.7]) it suffices to assume that the $\mathcal{D}$-module underlying M is simple, as any sub-module of an L-monodromic module is also L-monodromic. But then because N is nilpotent, the claim is obvious. □

We use the fact that the complex $B^0_L(\mathcal{M})$ computes the $\mathcal{D}$-module theoretic restriction. This shows that $\mathrm{Sp}_L(\mathcal{M})$ can be used to compute the restriction. The proposition below is shown following the proof for the usual V-filtration [Sai90, Pg. 269]:

Proposition 3.19. *Let M be a mixed Hodge module on T. There is a canonical quasi-isomorphism*

$$\sigma^!(M) \cong \sigma^! \mathrm{Sp}_L(M).$$

Proof. The claim follows for underlying $\mathcal{D}$-modules by Lemma 3.9. Indeed,

$$\sigma^!(\mathcal{M}) \cong B^0_L(\mathcal{M}) = B^0_L(\mathrm{Sp}_L(\mathcal{M})) \cong \sigma^! \mathrm{Sp}_L(\mathcal{M}).$$

Kashiwara's equivalence shows that $\mathrm{Sp}_L \circ \sigma_* = \sigma_*$, i.e., that specialization is the identity on modules supported on $X \times \{0\}$.

Let $j\colon T \setminus (X \times \{0\}) \to T$ be the inclusion of the complement of the zero section. We have morphisms

$$j_* j^* \mathrm{Sp}_L(M) \to j_* j^* \mathrm{Sp}(j_* j^*(M))$$
$$\mathrm{Sp}_L(j_* j^*(M)) \to j_* j^* \mathrm{Sp}(j_* j^*(M)).$$

The cones of these morphisms vanish because their underlying complexes of $\mathcal{D}$-modules do. Thus, these morphisms are quasi-isomorphisms.

Thus, starting with

$$\sigma_* \sigma^!(M) \to M \to j_* j^*(M) \xrightarrow{+1},$$

when we apply $\mathrm{Sp}_L(-)$, we get

$$\sigma_* \sigma^!(M) \to \mathrm{Sp}_L(M) \to j_* j^* \mathrm{Sp}_L(M) \xrightarrow{+1},$$

which gives a canonical isomorphism

$$\sigma_* \sigma^!(M) \cong \sigma_* \sigma^! \mathrm{Sp}_L(M),$$

by, for example, [Dir24, Lem. 4.4]. □

We end this section by mentioning the existence of the relative monodromy filtration on $\mathrm{Gr}^\lambda_L(\mathcal{M})$ for the nilpotent operator $L(t\partial_t) - \lambda + |L|$ and for the induced filtration

$$M_\bullet \mathrm{Gr}^\lambda_L(\mathcal{M}) = \mathrm{Gr}^\lambda_L(W_\bullet \mathcal{M}).$$

We denote this filtration $W_\bullet \mathrm{Gr}^\lambda_L(\mathcal{M})$.

As above, this allows us to define $W_\bullet B^0_L(\mathcal{M})$, which will be the weight filtration on the complex $B^0_L(\mathcal{M})$.

The following can be shown exactly as in the proof of [CD23, Lem. 6.2]. It says that there exists a splitting of $M_\bullet$ on $\mathrm{Gr}^W_k \mathrm{Gr}^\lambda_L(\mathcal{M})$ which is functorial in a certain sense.

Lemma 3.20. *There exists a splitting of $M_\bullet \mathrm{Gr}_k^W \mathrm{Gr}_L^\lambda(\mathcal{M})$ which is functorial with respect to the morphisms*

$$t_i \colon \mathrm{Gr}_k^W \mathrm{Gr}_L^\lambda(\mathcal{M}) \to \mathrm{Gr}_k^W \mathrm{Gr}_L^{\lambda+a_i}(\mathcal{M}).$$

In particular, the complex $\mathrm{Gr}_k^W B_L^0(\mathcal{M})$ splits into its associated graded pieces.

This lemma will be the key to allow us to reduce to the pure case. Indeed, we have that $\mathrm{Gr}_i^M \mathrm{Gr}_k^W B_L^0(\mathcal{M}) = \mathrm{Gr}_k^W B_L^0(\mathrm{Gr}_i^W \mathcal{M})$.

4. Cyclic Coverings

Let $(a_1, \ldots, a_r) \in \mathbf{Z}_{\geq 1}^r$. We consider the map $\pi \colon X \times \mathbf{A}_w^r \to X \times \mathbf{A}_t^r$ which sends (x, w) to (x, w^a). For any $(b_1, \ldots, b_r) \in \mathbf{Z}_{\geq 0}^r$, we define the slope $\ell = \sum_{i=1}^r b_i s_i$ and consider $L = \ell * a = \sum_{i=1}^r (a_i b_i) s_i$.

Given $\mathcal{M}$ a regular holonomic $\mathcal{D}_{X \times \mathbf{A}_t^r}$-module, the pull-back $\pi^! \mathcal{M}$ agrees with the $\mathcal{O}$-module pull-back, and is given by

$$\pi^! \mathcal{M} = \bigoplus_{0 \leq \beta \leq \underline{a-1}} \mathcal{M} w^\beta,$$

where $\beta \leq \underline{a-1}$ means $\beta_i \leq a_i - 1$ for all $1 \leq i \leq r$.

The $\mathcal{D}$-module action is given by [HTT08, Sect. 1.3]

$$P(mw^\beta) = P(m) w^\beta \text{ for all } P \in \mathcal{D}_X,$$

$$w_i(mw^\beta) = \begin{cases} mw^{\beta+e_i} & \beta_i < a_i - 1 \\ t_i m w^{\beta-(a_i-1)e_i} & \beta_i = a_i - 1 \end{cases},$$

$$\partial_{w_i}(mw^\beta) = \begin{cases} (\beta_i + a_i t_i \partial_{t_i})(m) w^{\beta-e_i} & \beta_i > 0 \\ a_i \partial_{t_i}(m) w^{\beta+(a_i-1)e_i} & \beta_i = 0 \end{cases}.$$

Thus, $w_i \partial_{w_i}(mw^\beta) = (\beta_i + a_i t_i \partial_{t_i})(m) w^\beta$ and so

$$\ell(w\partial_w)(mw^\beta) = (\ell(\beta) + L(t\partial_t))(m) w^\beta.$$

The first main result of this section is the following:

Theorem 4.1. *Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_{X \times \mathbf{A}_t^r}$-module. For any slope $\ell = \sum_{i=1}^r b_i s_i$, write $L = \ell * a$. Then, for all $\lambda \in \mathbf{Q}$, we have*

$${}^\ell V^\lambda \pi^! \mathcal{M} = \bigoplus_{0 \leq \beta \leq \underline{a-1}} {}^L V^{\lambda + |L| - \ell(\beta) - |\ell|} \mathcal{M} w^\beta.$$

Remark 4.2. We will only apply this in the case where $\mathcal{M}$ is L-monodromic, where it becomes essentially trivial.

Before proving this, we study cyclic coverings of the corresponding deformations to the normal bundle. Let $T = X \times \mathbf{A}_t^r$ and $W = X \times \mathbf{A}_w^r$. We consider

$$\widetilde{T}^L = X \times \mathbf{A}_{t/u^{ab}}^r \times \mathbf{A}_u^1$$

$$\widetilde{W}^\ell = X \times \mathbf{A}_{w/u^b}^r \times \mathbf{A}_u^1.$$

By Lemma 3.14 above, we have isomorphisms

$$\mathcal{D}_{\widetilde{T}^L/\mathbf{A}^1_u} \cong R_{{}^LV}(\mathcal{D}_T),$$
$$\mathcal{D}_{\widetilde{W}^\ell/\mathbf{A}^1_u} \cong R_{{}^\ell V}(\mathcal{D}_W).$$

We have a finite morphism $\Pi\colon \widetilde{W}^\ell \to \widetilde{T}^L$ over $X \times \mathbf{A}^1_u$ defined by the map $\mathbf{A}^r_{w/u^b} \to \mathbf{A}^r_{t/u^{ab}}$ which sends

$$t_i/u^{b_i a_i} \mapsto (w_i/u^{b_i})^{a_i}.$$

The functor $\Pi^!$ for relative $\mathcal{D}$-modules agrees with the (derived) $\mathcal{O}$-module pull-back. By flatness, it is the usual pull-back of $\mathcal{O}$-modules.

The functor $\Pi^!$ need not preserve coherence of relative $\mathcal{D}$-modules. Indeed, even imposing relative holonomicity, coherence is not preserved, see [MFS17, Example 2.4].

However, [MFS17, Thm. 2] shows that the pull-back of a *regular* relative holonomic $\mathcal{D}$-module is regular relative holonomic, in particular, coherent.

Lemma 4.3. *Let $\mathcal{M}$ be a regular holonomic $\mathcal{D}_T$-module. Then $R_{{}^LV}(\mathcal{M})$ is a relative regular holonomic $\mathcal{D}_{\widetilde{T}^L/\mathbf{A}^1_u}$-module.*

Proof. For $\lambda \in \mathbf{C}$, let $i_\lambda\colon \{\lambda\} \to \mathbf{A}^1_u$ be the inclusion of the point λ. We have

$$i^!_\lambda(R_{{}^LV}(\mathcal{M})) = \begin{cases} \mathcal{M} & \lambda \neq 0 \\ \mathrm{Sp}_L(\mathcal{M}) & \lambda = 0 \end{cases},$$

where we use that ${}^LV^k\mathcal{M}/{}^LV^{k+1}\mathcal{M} = \bigoplus_{\chi \in [k,k+1)} \mathrm{Gr}^\chi_L(\mathcal{M})$ for all $k \in \mathbf{Z}$.

As $\mathcal{M}$ and $\mathrm{Sp}_L(\mathcal{M})$ are regular holonomic, this proves the claim. □

Proof of Theorem 4.1. We will show that

$$U^\bullet \pi^! \mathcal{M} = \bigoplus_{0 \leq \beta \leq \underline{a-1}} {}^LV^{\bullet + |L| - \ell(\beta) - |\ell|} \mathcal{M} w^\beta$$

satisfies the defining properties of the ${}^\ell V$-filtration.

It is trivial to see that $w_i U^\bullet \subseteq U^{\bullet+1}$, $\partial_{w_i} U^\bullet \subseteq U^{\bullet-1}$, and that $(\ell(w\partial_w) - \lambda + |\ell|)$ is nilpotent on $\mathrm{Gr}^\lambda_U(\pi^!\mathcal{M})$. The last claim follows from

$$(\ell(w\partial_w) - \lambda + |\ell|)(mw^\beta) = (L(t\partial_t) - \lambda + \ell(\beta) + |\ell|)(mw^\beta).$$

The last remaining claim is coherence of $U^\bullet$ over $V^0\mathcal{D}_{X \times \mathbf{A}^r_w}$. We prove this for the underlying $\mathbf{Z}$-indexed filtration.

By Lemma 4.3, the module $R_{{}^LV}(\mathcal{M})$ is regular relative holonomic on $\widetilde{T}^L$. Hence, $\Pi^!$ applied to it remains regular relative holonomic, in particular, coherent. It is a simple computation to see that this pull-back is isomorphic to $R_U(\pi^!\mathcal{M})$, proving the claim. □

Remark 4.4. Let $\ell = e_i$, so that ${}^\ell V^\bullet \pi^! \mathcal{M}$ is the V-filtration along w_i.

Then $L = a_i e_i$, and it is not hard to check that

$${}^LV^\lambda \mathcal{M} = V_i^{\frac{\lambda}{a_i}} \mathcal{M},$$

where the right hand side is the canonical V-filtration along t_i. Thus, Theorem 4.1 gives the formula

$$V_i^{\lambda}\pi^!(\mathcal{M}) = \bigoplus_{0\leq \beta \leq \underline{a-1}} V_i^{1+\frac{\lambda-\beta_i-1}{a_i}}\mathcal{M}w^{\beta}.$$

For now, we let $r=1$. We will apply this case later with $X\times \mathbf{A}_t^{r-1}$ in place of X. Let $j_w\colon \{w\neq 0\}\to W = X\times \mathbf{A}^1_w$ and $j_t\colon \{t\neq 0\}\to T = X\times\mathbf{A}^1_t$ be the open embeddings. Define the restriction $\pi|_{\{w\neq 0\}}$ to be $\rho\colon \{w\neq 0\}\to\{t\neq 0\}$, which is finite étale.

Let M be a pure Hodge module of weight d on $X\times\mathbf{A}^1_t$ with strict support not contained in $\{t=0\}$. In general, $\pi^! M$ need not remain pure, but we know that $W_{d-1}\pi^! M = 0$.

Lemma 4.5. *In the situation above, we have that $Q = \pi^!(M)/W_d\pi^!(M)$ is supported on $\{w=0\}$.*

Moreover, $W_d\pi^!(\mathcal{M})$ has no non-zero sub-module or quotient module supported in $\{w=0\}$.

Proof. The first claim is obvious by restricting $\pi^!(M)$ to $\{w\neq 0\}$, which gives $\rho^!(M|_{\{t\neq 0\}})$. As ρ is étale, this module is pure of weight d.

Note that $W_d\pi^!(\mathcal{M})\subseteq \pi^!(\mathcal{M})$, and $\pi^!(\mathcal{M})$ has no sub-modules supported in $\{w=0\}$ by adjunction, using the fact that $\mathcal{M}$ has no sub-modules supported on $\{t=0\}$.

Thus, $W_d\pi^!(\mathcal{M})$ has no sub-modules supported in $\{w=0\}$. By polarizability, it also admits no quotient objects supported in $\{w=0\}$, proving the claim. □

Thus, we have $W_d\pi^!(\mathcal{M}) = \mathcal{D}_W\cdot V^{>0}W_d\pi^!(\mathcal{M})$ by Lemma 2.7, where $V^\bullet$ is the V-filtration along w. Note that

$$V^{>0}W_d\pi^!(\mathcal{M}) = V^{>0}\pi^!(\mathcal{M}) = \bigoplus_{0\leq b\leq a-1} V^{>1-\frac{b+1}{a}}\mathcal{M}w^b,$$

where the second equality comes from Remark 4.4. The first one comes from Lemma 2.4.

We see from the $\mathcal{D}_W$-module action that $W_d\pi^!(\mathcal{M})$ is a graded sub-module of $\pi^!(\mathcal{M})$. Throughout, we let

$$W_d\pi^!(\mathcal{M})^b \subseteq \pi^!(\mathcal{M})^b$$

denote the bth graded piece, i.e., the coefficient of w^b.

Lemma 4.6. *We have*

$$V^0W_d\pi^!(\mathcal{M}) = \bigoplus_{0\leq b< a-1}\left(N^{(b)}\cdot V^{1-\frac{b+1}{a}}\mathcal{M} + V^{>1-\frac{b+1}{a}}\mathcal{M}\right)w^b \oplus V^0\mathcal{M}w^{a-1},$$

where $N^{(b)} = (t\partial_t - (1-\frac{b+1}{a})+1)$ is the nilpotent endomorphism of $\mathrm{Gr}_V^{1-\frac{b+1}{a}}(\mathcal{M})$.

Proof. This follows from $V^0W_d\pi^!(\mathcal{M}) = \partial_w V^1\pi^!(\mathcal{M}) + V^{>0}\pi^!(\mathcal{M})$, which holds by the last claim of Lemma 4.5 and the surjectivity

$$\partial_w\colon \mathrm{Gr}_V^1(W_d\pi^!(\mathcal{M}))\to \mathrm{Gr}_V^0(W_d\pi^!(\mathcal{M}))$$

which holds by Lemma 2.7. To identify the coefficient of w^{a-1}, we use the similar formula $V^0\mathcal{M} = \partial_t V^1\mathcal{M} + V^{>0}\mathcal{M}$ which also follows by Lemma 2.7. □

Lemma 4.7. *The inclusion*

$$W_d\pi^!(\mathcal{M}) \to \pi^!(\mathcal{M})$$

induces an equality for all $p \in \mathbf{Z}$ *and* $\lambda > 0$,

$$F_pV^\lambda W_d\pi^!(\mathcal{M}) = F_pV^\lambda\pi^!(\mathcal{M}) = \bigoplus_{0\leq b\leq a-1} F_pV^{1+\frac{\lambda-b-1}{a}}(\mathcal{M})w^b,$$

and for $\lambda = 0$*, we get equality for all* $p \in \mathbf{Z}$*:*

$$F_pV^0W_d\pi^!(\mathcal{M})^{a-1} = F_pV^0\pi^!(\mathcal{M})^{a-1} = F_pV^0(\mathcal{M})w^{a-1}.$$

Proof. We already saw the equality

$$F_pV^\lambda W_d\pi^!(\mathcal{M}) = F_pV^\lambda\pi^!(\mathcal{M}) \text{ for all } \lambda > 0$$

above, using the fact that both restrict to $\rho^!(\mathcal{M}|_{\{t\neq 0\}})$ on $\{w \neq 0\}$.

For $\lambda = 0$, we have by Lemmas 4.5 and 2.8 the equality

$$F_pV^0W_d\pi^!(\mathcal{M}) = \partial_{w_r}(F_{p-1}V^1W_d\pi^!(\mathcal{M})) + F_pV^{>0}W_d\pi^!(\mathcal{M})$$

and by what we mentioned at the beginning of this proof, the right hand side is equal to

$$\partial_{w_r}(F_{p-1}V^1\pi^!(\mathcal{M})) + F_pV^{>0}\pi^!(\mathcal{M}).$$

Using the fact that $\pi^!(\mathcal{M})$ has no w_r-torsion, we know by Lemma 2.8 that for all $\lambda \geq 0$ that we have

$$F_pV^\lambda\pi^!(\mathcal{M}) = V^\lambda\pi^!(\mathcal{M}) \cap j_*(F_p\rho^!(\mathcal{M}|_{t_r\neq 0\}})) = \bigoplus_{0\leq b\leq a-1} F_pV^{1+\frac{\lambda-b-1}{a}}(\mathcal{M})w^b.$$

Finally, using that $\mathcal{M}$ has strict support not contained in $\{t_r = 0\}$, we get by Lemma 2.8 $F_pV^0\mathcal{M} = \partial_t(F_{p-1}V^1\mathcal{M}) + F_pV^{>0}\mathcal{M}$, which is the coefficient of w^{a-1} in $F_pV^0W_d\pi^!(\mathcal{M})$. ☐

By Kashiwara's equivalence and Lemma 4.5, we have $Q = i_*Q_0$ where $i\colon \{w = 0\} \to W$ is the closed embedding and the underlying $\mathcal{D}$-module of Q_0 is $\mathcal{Q}_0 = \ker(w) \subseteq \mathcal{Q}$. Note that by Example 2.2, we have $V^0\mathcal{Q} = \mathcal{Q}_0$.

Corollary 4.8. *Let* $Q = \pi^!(\mathcal{M})/W_d\pi^!(\mathcal{M})$ *with underlying* $\mathcal{D}$*-module* $\mathcal{Q}$*. Then*

$$\mathcal{Q}_0 = V^0\mathcal{Q} = \bigoplus_{0\leq b< a-1} \text{coker}\left(N^{(b)}\colon \text{Gr}_V^{1-\frac{b+1}{a}}(\mathcal{M}) \to \text{Gr}_V^{1-\frac{b+1}{a}}(\mathcal{M})\right).$$

In particular, $\pi^!(\mathcal{M})$ *is pure if and only if the* V*-filtration of* $\mathcal{M}$ *along* t *has no jumping numbers in* $\frac{1}{a}\mathbf{Z} \setminus \mathbf{Z}$*, i.e.,* $\text{Gr}_V^\lambda(\mathcal{M}) \neq 0$ *implies* $\lambda \notin \frac{1}{a}\mathbf{Z} \setminus \mathbf{Z}$*.*

Proof. We have $V^0\mathcal{Q} = V^0\pi^!(\mathcal{M})/V^0W_d\pi^!(\mathcal{M})$, so the formula is obvious by the previous lemma and Lemma 2.3.

The last claim follows because $\pi^!(\mathcal{M})$ is pure if and only if $\mathcal{Q} = 0$ if and only if $N^{(b)}$ is surjective on $\text{Gr}_V^{1-\frac{b+1}{a}}(\mathcal{M})$ for all $0 \leq b < a-1$, but since $N^{(b)}$ is nilpotent, this is equivalent to the vanishing of $\text{Gr}_V^{1-\frac{b+1}{a}}(\mathcal{M})$. ☐

4.1. **Proof of Theorem A.** By induction on r, we have the claims of Theorem A for any mixed Hodge modules on $X \times \mathbf{A}_t^{r-1}$ with coordinates $t_1, \dots, t_{r-1}$ on $\mathbf{A}_t^{r-1}$.

For any M a mixed Hodge module on $X \times \mathbf{A}_t^r$, by Proposition 3.19 we can replace M by $\mathrm{Sp}_L(M)$ to assume that M is L-monodromic. Indeed, we have $B_L^\chi(\mathcal{M}, F) = B_L^\chi(\mathrm{Sp}_L(\mathcal{M}), F)$ for all $\chi \geq 0$.

Moreover, by the functorial splitting of the relative monodromy filtration, we can assume M is L-monodromic and pure of weight d. By the strict support decomposition, we can assume M has strict support not contained in $\{t_r = 0\}$. Indeed, if M is supported in $\{t_r = 0\}$, the result follows by induction on r using Example 3.12.

Lemma 4.9. *Let $V_r^\bullet \mathcal{M}$ be the V-filtration along $t_r = 0$. The complex $V_r^0 B_L^\alpha(\mathcal{M}, F)$ is filtered quasi-isomorphic to $B_L^\alpha(\mathcal{M}, F)$ for $\alpha \geq 0$.*

Proof. The idea is essentially the same as the one in [CDS23]. We include a detailed proof for the reader's convenience. We argue inductively on r for Theorem A. Let $L' = L - a_r t_r \partial_r$. Denote by $\theta_r = t_r \partial_r$, $\theta_L = \sum_{i=1}^r a_i t_i \partial_i$ and $\theta_{L'} = \theta_L - a_r \theta_r$. As θ_L and θ_r commute, we have a decomposition

$$V_r^\beta \mathcal{M} = \bigoplus_{\alpha \in \mathbf{Q}} V_r^\beta \mathcal{M}^\alpha, \text{ where } V_r^\beta \mathcal{M}^\alpha = V_r^\beta \mathcal{M} \cap \mathcal{M}^\alpha.$$

Recall that $\mathcal{M}^\alpha$ is the component annihilated by some large power of $\theta_L - \alpha + |L|$. Hence, there is a decomposition

$$\mathrm{Gr}_{V_r}^\beta(\mathcal{M}) = \bigoplus_{\alpha \in \mathbf{Q}} \mathrm{Gr}_{V_r}^\beta(\mathcal{M}^\alpha).$$

This implies that $\mathrm{Gr}_{V_r}^\beta(\mathcal{M})$ is an L'-monodromic mixed Hodge module on $X \times \mathbf{A}_t^{r-1}$. Indeed, a local section m of $\mathrm{Gr}_{V_r}^\beta(\mathcal{M}^\alpha)$ is annihilated by a sufficiently large power of

$$(\theta_{L'} - (\alpha - a_r\beta) + |L'|) = \big((\theta_L - \alpha + |L|) - a_r(\theta_r - \beta + 1)\big).$$

In particular, the L'-graded piece $\big(\mathrm{Gr}_{V_r}^\beta(\mathcal{M})\big)^{\alpha - a_r\beta}$ is exactly $\mathrm{Gr}_{V_r}^\beta(\mathcal{M}^\alpha)$. It follows that we have an isomorphism between complexes $\mathrm{Gr}_{V_r}^\beta B_L^\alpha(\mathcal{M}, F)$ and the shifted mapping cone

$$\mathrm{Cone}\big[B_{L'}^{\alpha - a_r\beta}(\mathrm{Gr}_{V_r}^\beta(\mathcal{M}, F)) \xrightarrow{t_r} B_{L'}^{\alpha - a_r\beta}\mathrm{Gr}_{V_r}^{\beta+1}(\mathcal{M}, F)\big][-1].$$

Therefore, if $\beta > 0$, $\mathrm{Gr}_{V_r}^\beta B_L^\alpha(\mathcal{M}, F)$ is acyclic because $t_r \colon (V_r^\beta \mathcal{M}, F) \to (V_r^{\beta+1}\mathcal{M}, F)$ is an isomorphism; and if $\alpha - a_r\beta > 0$, the inductive hypothesis on r implies that again $\mathrm{Gr}_{V_r}^\beta B_L^\alpha(\mathcal{M}, F)$ is acyclic. By the distinguished triangle

$$V_r^{>\beta} B_L^\alpha(\mathcal{M}, F) \to V_r^\beta B_L^\alpha(\mathcal{M}, F) \to \mathrm{Gr}_{V_r}^\beta B_L^\alpha(\mathcal{M}, F) \xrightarrow{+1}$$

if $\alpha \geq 0$ and $\beta < 0$, the natural inclusion $V_r^{>\beta} B_L^\alpha(\mathcal{M}, F) \to V_r^\beta B_L^\alpha(\mathcal{M}, F)$ is a filtered quasi-isomorphism. Hence, taking the direct limit gives the filtered quasi-isomorphism

$$V_r^0 B_L^\alpha(\mathcal{M}, F) \xrightarrow{\cong} B_L^\alpha(\mathcal{M}, F)$$

for $\alpha \geq 0$. □

Note that by Lemma 3.17, in order to prove filtered acyclity of $A_L^\chi(\mathcal{M}, F)$ for all $\chi > 0$, it suffices to prove it for $B_L^\chi(\mathcal{M}, F)$ for all $\chi > 0$.

We will prove the claims by induction on $|I_L|$ where $I_L = \{i \mid a_i > 1\}$. The case $|I_L| = 0$ is [CD23, Thm. 1, 2]. So assume by induction that we can compute restriction functors and we have filtered acyclicity for any $\ell = \sum_{i=1}^r b_i s_i$ with $|I_\ell| < |I_L|$. Without loss of generality, we can assume $a_r = a > 1$ in L.

Let $\pi\colon W = X \times \mathbf{A}^r_w \to X \times \mathbf{A}^r_t = T$ be the map $(x, w) \mapsto (x, w_1, w_2, \dots, w_r^a)$. We can use the results of the above sub-section for this cyclic cover, as only one variable is being raised to a power.

Let $\sigma_w\colon X \to W$ and $\sigma_t\colon X \to T$ be the inclusion of the zero section. Then $\pi \circ \sigma_w = \sigma_t$. Thus, we have an isomorphism

$$\sigma_w^!(\pi^! M) \cong \sigma_t^!(M).$$

Note that $\pi^!(M)$ is ℓ-monodromic, where $\ell = s_r + \sum_{i=1}^{r-1} a_i s_i$, as we can easily see from the definition of the $\mathcal{D}$-module action.

Let $V_r^\bullet \mathcal{M}$ and $V_r^\bullet \pi^!(\mathcal{M})$ be the V-filtration along t_r (resp. w_r). By Lemma 4.9, for any $\chi \geq 0$, we have filtered quasi-isomorphisms

$$V_r^0 A_L^\chi(\mathcal{M}, F) \cong A_L^\chi(\mathcal{M}, F), \quad \text{and} \quad V_r^0 A_\ell^\chi(\pi^!(\mathcal{M}), F) \cong A_\ell^\chi(\pi^!(\mathcal{M}), F).$$

The latter two complexes are $\mathbf{Z}/a\mathbf{Z}$-graded because the morphisms $w_1, \dots, w_r$ preserve the grading up to a shift. We normalize the grading so that the bth graded piece of the complex is the sub-complex which starts on the left at the bth graded piece of the corresponding term.

By the right-most equalities in Lemma 4.7 and the definition of the $\mathcal{D}$-module action, we have equality

$$V_r^0 A_\ell^\chi(\pi^!(\mathcal{M}), F)^{a-1} = V_r^0 A_L^\chi(\mathcal{M}, F).$$

Proof of Theorem A, Filtered Acyclicity. Let $\chi > 0$, then we have the chain of filtered quasi-isomorphisms

$$A_L^\chi(\mathcal{M}, F) \cong V_r^0 A_L^\chi(\mathcal{M}, F) \cong V_r^0 A_\ell^\chi(\pi^!(\mathcal{M}), F)^{a-1} \cong A_\ell^\chi(\pi^!(\mathcal{M}), F)^{a-1}.$$

By the inductive hypothesis, the last complex is filtered acyclic, proving the claim. □

Proof of Theorem A, Restriction Functor. We begin by proving the claim for the Hodge filtration.

By the inductive hypothesis, we know $A_\ell^0(\pi^!(\mathcal{M}), F) \cong \sigma_w^!(\pi^!(\mathcal{M}, F)) = \sigma^!(\mathcal{M}, F)$. Moreover, the graded pieces which are not the $(a-1)$th are filtered acyclic. Indeed, for $b < a-1$, the map w_r from the w^b term to the w^{b+1} term is an isomorphism. Thus, we have

$$A_\ell^0(\pi^!(\mathcal{M}), F)^{a-1} \cong \sigma_w^!(\pi^!(\mathcal{M}, F)) = \sigma^!(\mathcal{M}, F).$$

To prove the claim for the Hodge filtration, note that we have the chain of filtered quasi-isomorphisms

$$\begin{aligned} A_L^0(\mathcal{M}, F) \cong V_r^0 A_L^0(\mathcal{M}, F) &\cong V_r^0 A_\ell^0(\pi^!(\mathcal{M}), F)^{a-1} \\ &\cong A_\ell^0(\pi^!(\mathcal{M}), F)^{a-1} \cong \sigma^!(\mathcal{M}, F). \end{aligned}$$

For the weight filtration, as M is pure L-monodromic, we know that the nilpotent endomorphism $N\colon M \to M$ is 0 by Lemma 3.18. Thus, the same is true for $\pi^!(M)$, though $\pi^!(M)$ need not be pure.

Thus, for any $\chi \in \mathbf{Z}$, the relative monodromy filtration on $\mathrm{Gr}_\ell^\chi(\pi^!(\mathcal{M}))$ is simply the induced filtration from $W_\bullet \pi^!(\mathcal{M})$, i.e.,

$$W_\bullet \mathrm{Gr}_\ell^\chi(\pi^!(\mathcal{M})) = \mathrm{Gr}_\ell^\chi(W_\bullet \pi^!(\mathcal{M})),$$

and so we have

$$W_\bullet B_\ell^0(\pi^!(\mathcal{M}), F) = B_\ell^0(W_\bullet \pi^!(\mathcal{M}), F).$$

By Lemma 4.10 below, the inclusion

$$W_d B_\ell^0(\pi^!(\mathcal{M}), F)^{a-1} \to B_\ell^0(\pi^!(\mathcal{M}), F)^{a-1}$$

is a filtered quasi-isomorphism. Thus, the cohomology of $B_\ell^0(\pi^!(\mathcal{M}), F)^{a-1}$ is pure, which proves the claim. □

Lemma 4.10. *Let $\mathcal{M}$ be a pure Hodge module of weight d on T with strict support not contained in $\{t_r = 0\}$. Using the notation above, the natural map*

$$B_\ell^0(W_d \pi^!(\mathcal{M}), F)^{a-1} \to B_\ell^0(\pi^!(\mathcal{M}), F)^{a-1}$$

is a quasi-isomorphism.

Proof. It suffices to check that $B_\ell^0(\mathcal{Q}, F)^{a-1} = 0$. By Kashiwara's equivalence, we have $(\mathcal{Q}, F) = \bigoplus_{k \geq 0} (\mathcal{Q}_0, F)\partial_w^k$.

We have by Example 3.12 the equality

$${}^{\ell}V^\lambda \mathcal{Q} = \bigoplus_{k \geq 0} {}^{\ell'}V^{\lambda+k} \mathcal{Q}_0 \partial_w^k,$$

where $\ell' = \sum_{i=1}^{r-1} a_i s_i$. For ease of notation, write ${}^{\ell'}V^{\lambda+k}\mathcal{Q}_0 = \mathcal{Q}_0^{\lambda+k}$ and write its w^b graded piece as $\mathcal{Q}_0^{b,\lambda+k}$. As ${}^{\ell}V^\lambda\mathcal{Q}$ splits into graded pieces, so does ${}^{\ell'}V^{\lambda+k}\mathcal{Q}_0$.

Then we have

$$\mathrm{Gr}_\ell^\lambda \mathcal{Q} = \bigoplus_{\beta=0}^{a-1} \left(\bigoplus_{b-j \equiv \beta \mod a} \partial_{w_r}^j (\mathcal{Q}_0^{b,\lambda+j}) \right).$$

For all $\lambda \in \mathbf{Q}$, the map

$$w_r \colon (\mathrm{Gr}_\ell^\lambda \mathcal{Q})^{a-1} \to (\mathrm{Gr}_\ell^{\lambda+1}\mathcal{Q})^0$$

is an isomorphism.

Indeed, note that we can write

$$(\mathrm{Gr}_\ell^\lambda \mathcal{Q})^{a-1} = \bigoplus_{b \equiv j-1 \mod a} \partial_{w_r}^j (\mathcal{Q}_0^{b,\lambda+b_r j})$$

and

$$(\mathrm{Gr}_\ell^{\lambda+1} \mathcal{Q})^{0} = \bigoplus_{b \equiv j \mod a} \partial_{w_r}^j (\mathcal{Q}_0^{b,\lambda+1+b_r j}).$$

Importantly, for the $(a-1)$th piece, there is no $j = 0$ term. Given any $j > 0$ and $m \in \mathcal{Q}_0^{b,\lambda+j} = \mathcal{Q}_0^{b,(\lambda+1)+(j-1)}$, we have

$$w_r \partial_{w_r}^j m = -j \partial_{w_r}^{j-1} m,$$

so the map is injective and surjective. As this is one of the morphisms appearing in the Koszul-like complex $B_\ell^0(\mathcal{Q}, F)^{a-1}$, this proves the claim. □

5. Singularities of Subvarieties

Let $Z = V(f_1, \dots, f_r) \subseteq X$ be a closed subvariety of the smooth variety X. Consider

$$\mathcal{B}_f = i_{f,+}(\mathcal{O}_X) = \bigoplus_{\alpha \in \mathbf{N}^r} \mathcal{O}_X \partial_t^\alpha \delta_f,$$

a regular holonomic $\mathcal{D}_T$-module. The $\mathcal{D}$-module action is given by

$$h(g\partial_t^\alpha \delta_f) = hg\partial_t^\alpha \delta_f \text{ for all } h \in \mathcal{O}_X,$$

$$D(g\partial_t^\alpha \delta_f) = D(g)\partial_t^\alpha \delta_f - \sum_{i=1}^r D(f_i) g \partial_t^{\alpha+e_i} \delta_f \text{ for all } D \in \mathrm{Der}_{\mathbf{C}}(\mathcal{O}_X),$$

$$t_i(g\partial_t^\alpha \delta_f) = f_i g \partial_t^\alpha \delta_f - \alpha_i g \partial_t^{\alpha - e_i} \delta_f,$$

$$\partial_{t_i}(g\partial_t^\alpha \delta_f) = g\partial_t^{\alpha+e_i}\delta_f.$$

The Hodge filtration is given by

$$F_{p+r}\mathcal{B}_f = \bigoplus_{|\alpha| \leq p} \mathcal{O}_X \partial_t^\alpha \delta_f,$$

where the shift by r is due to the relative dimension of the graph embedding and the fact that we use left $\mathcal{D}$-modules.

For any slope $L = \sum_{i=1}^r a_i s_i$, define a $\mathbf{Z}$-indexed filtration on $\mathcal{B}_f$ by

$${}^L G^\bullet(\mathcal{B}_f) = {}^L V^\bullet \mathcal{D}_T \cdot \delta_f.$$

Define the *b_L-function of* $f_1, \dots, f_r$ to be the monic minimal polynomial of the action of $L(s) = \sum_{i=1}^r -a_i \partial_{t_i} t_i$ on $\mathrm{Gr}^0_{{}^L G}(\mathcal{B}_f)$.

Lemma 5.1. *For any $j \in \mathbf{Z}$, we have*

$$b_L(w+j)\mathrm{Gr}^j_{{}^L G}(\mathcal{B}_f) = 0.$$

Proof. As

$${}^L V^j \mathcal{D}_T = \sum_{L(\beta) \geq L(\gamma)+j} {}^L V^0 \mathcal{D}_T \cdot t^\beta \partial_t^\gamma,$$

we have a surjection

$$\bigoplus_{L(\beta) \geq L(\gamma)+j} {}^L G^0 \mathcal{B}_f \xrightarrow{(t^\beta \partial_t^\gamma)} {}^L G^j(\mathcal{B}_f),$$

and if we compose with the projection to $\mathrm{Gr}^j_{{}^L G}(\mathcal{B}_f)$, we get a surjection

$$\bigoplus_{L(\beta) = L(\gamma)+j} {}^L G^0 \mathcal{B}_f \xrightarrow{(t^\beta \partial_t^\gamma)} \mathrm{Gr}^j_{{}^L G}(\mathcal{B}_f),$$

where we can take $=$ in the index set, as any terms with strict inequality necessarily map to 0 by definition. Finally, for any fixed β, γ, note that ${}^L G^1 \mathcal{B}_f$ maps to 0 in the associated graded piece, so we have a surjection

$$\bigoplus_{L(\beta) = L(\gamma)+j} \mathrm{Gr}^0_{{}^L G}(\mathcal{B}_f) \xrightarrow{\Phi} \mathrm{Gr}^j_{{}^L G}(\mathcal{B}_f).$$

As $t^\beta \partial_t^\gamma L(s) = (L(s+\beta-\gamma))t^\beta \partial_t^\gamma$, we see that $\Phi \circ L(s) = (L(s)+j)\Phi$, which proves the claim. □

Example 5.2. Let $L = as_i$ for some $a \in \mathbf{Z}_{>0}$. Let $b_{f_i}(s) = \prod(s+\gamma)^{m_\gamma}$ be the usual Bernstein-Sato polynomial of the hypersurface defined by f_i. Then it is easy to check that

$$b_{L,f}(s) = \prod(s+a\gamma)^{m_\gamma} = a^{\deg b_{f_i}} b_{f_i}\left(\frac{s}{a}\right).$$

This satisfies the following Thom-Sebastiani type property, similar to [BMS06, Thm. 5]. The proof is essentially the same, but we repeat it for convenience.

Proposition 5.3. *Let $f_1,\dots,f_r \in \mathcal{O}_X(X)$, $g_1,\dots,g_c \in \mathcal{O}_Y(Y)$ for X, Y two smooth complex algebraic varieties. Write $L_1 = \sum_{i=1}^r a_i s_i$ and $L_2 = \sum_{i=1}^c b_i s_i$ and let $L = \sum_{i=1}^r a_i s_i + \sum_{j=1}^c b_j s_{r+j}$.*

Let $b_{L_1,f}(w)$ be the b_{L_1}-function for $f_1,\dots,f_r$ and define similarly $b_{L_2,g}(w)$ and $b_{L,(f,g)}(w)$. Write

$$b_{L_1,f}(w) = \prod_\alpha (w+\alpha)^{m_\alpha^{(f)}}, \quad b_{L_2,g}(w) = \prod_\beta (w+\beta)^{m_\beta^{(g)}}.$$

Then

$$b_{L,(f,g)}(w) = \prod(w+\gamma)^{m_\gamma},$$

where $m_\gamma = \max\{m_\alpha^{(f)} + m_\beta^{(g)} - 1 \mid m_\alpha^{(f)}, m_\beta^{(g)} > 0, \alpha+\beta = \gamma\}$.

Proof. Let $i_{f,+}(\mathcal{O}_X) = \mathcal{B}_f$, $i_{g,+}(\mathcal{O}_Y) = \mathcal{B}_g$ and $i_{(f,g),+}(\mathcal{O}_{X\times Y}) = \mathcal{B}_{(f,g)}$. Then, as in the proof of [BMS06, Thm. 5] we have an isomorphism

$$\mathcal{B}_{(f,g)} \cong \mathcal{B}_f \boxtimes \mathcal{B}_g.$$

Moreover, the ${}^LG^\bullet$-filtration on the left is given by the convolution of the filtrations ${}^{L_1}G^\bullet\mathcal{B}_f$ and ${}^{L_2}G^\bullet\mathcal{B}_g$. In other words,

$${}^LG^k\mathcal{B}_{(f,g)} = \sum_{i+j=k} {}^{L_1}G^i\mathcal{B}_f \boxtimes {}^{L_2}G^j\mathcal{B}_g.$$

As in the proof of [BMS06, Thm. 5], we have

$$\mathrm{Gr}^k_{{}^LG}\mathcal{B}_{(f,g)} \cong \bigoplus_{i+j=k} \mathrm{Gr}^i_{{}^{L_1}G}\mathcal{B}_f \boxtimes \mathrm{Gr}^j_{{}^{L_2}G}\mathcal{B}_g.$$

Let $b'(w) = \prod(w+\gamma)^{m_\gamma}$ as defined in the proposition statement.

By Lemma 5.1, we see that $b'(L(s)+i+j)$ annihilates $\mathrm{Gr}^i_{{}^{L_1}G}(\mathcal{B}_f) \boxtimes \mathrm{Gr}^j_{{}^{L_2}G}(\mathcal{B}_g)$ for any i, j. We see then that $b_{L,(f,g)}(w) \mid b'(w)$.

On the other hand, by the binomial theorem we see that $b'(w)$ is the minimal polynomial of the action of $L(s)$ on $\mathrm{Gr}^0_{{}^{L_1}G}(\mathcal{B}_f) \boxtimes \mathrm{Gr}^0_{{}^{L_2}G}(\mathcal{B}_g)$. Thus, as $b_{L,(f,g)}(L(s))$ annihilates this term, we get the other divisibility. □

Next, we review the definitions of higher Du Bois and higher rational singularities.

Given any complex algebraic variety Z of pure dimension d_Z and any $0 \leq p \leq d_Z$, we have the pth Du Bois complex $\underline{\Omega}_Z^p \in D^b_{\mathrm{coh}}(\mathcal{O}_Z)$, with comparison morphisms $\alpha_p \colon \Omega_Z^p \to \underline{\Omega}_Z^p$, where Ω_Z^p is the sheaf of Kähler differentials on Z. These morphisms are quasi-isomorphisms when Z is smooth.

If Z has local complete intersection singularities, then following [JKSY22, MOPW23, FL22], we say Z has *k-Du Bois singularities* if α_p is a quasi-isomorphism for all $p \leq k$.

Using a resolution of singularities, one can define a morphism $\Omega_Z^p \xrightarrow{\gamma_p} \mathbf{D}_Z(\underline{\Omega}_Z^{d_Z-p})$ in $D^b_{\mathrm{coh}}(\mathcal{O}_Z)$. Here $\mathbf{D}_Z(-)$ is the shifted Grothendieck duality functor $R\mathcal{H}om(-,\omega_Z^\bullet)[-d_Z]$, where $\omega_Z^\bullet$ is the dualizing complex of Z. If Z is smooth, this map is the natural isomorphism $\Omega_Z^p \cong \mathcal{H}om(\Omega_Z^{d_Z-p},\omega_Z)$.

Assuming still that Z has local complete intersection singularities, we say that Z has k-rational singularities if the composition $\Omega_Z^p \to \underline{\Omega}_Z^p \to \mathbf{D}_Z(\underline{\Omega}_Z^{d_Z-p})$ is a quasi-isomorphism for all $p \leq k$. When Z is smooth, this is the usual isomorphism $\Omega_Z^p \cong \mathcal{H}om_{\mathcal{O}_Z}(\Omega_Z^{d_Z-p},\omega_Z)$ of locally free sheaves.

Now, let $Z = V(f_1,\dots,f_r) \subseteq X$ be a local complete intersection subvariety of a smooth variety X. Associated to this, we have the *local cohomology mixed Hodge module*

$$\mathcal{H}^r_Z(\mathbf{Q}^H_X[\dim X]),$$

with underlying bi-filtered $\mathcal{D}_X$-module denoted $(\mathcal{H}^r_Z(\mathcal{O}_X),F,W)$.

The standard description of the local cohomology module is as follows: let $\mathcal{O}_X[\frac{f_i}{f_1\dots f_r}]$ be the localization of $\mathcal{O}_X$ at all f_j with $j \neq i$ and let $\mathcal{O}_X[\frac{1}{f_1\dots f_r}]$ be the localization at all f_i. These modules naturally underlie mixed Hodge modules on X, and we have

$$\mathcal{H}^r_Z(\mathcal{O}_X) = \operatorname{coker}\left(\bigoplus_{i=1}^r \mathcal{O}_X\left[\frac{f_i}{f_1\dots f_r}\right] \to \mathcal{O}_X\left[\frac{1}{f_1\dots f_r}\right]\right).$$

This carries the *pole-order filtration*, defined by

$$P_k\mathcal{H}^r_Z(\mathcal{O}_X) = \{m \in \mathcal{H}^r_Z(\mathcal{O}_X) \mid (f_1,\dots,f_r)^{k+1}m = 0\}.$$

It is not hard to see that $F_k\mathcal{H}^r_Z(\mathcal{O}_X) \subseteq P_k\mathcal{H}^r_Z(\mathcal{O}_X)$, see [MP22, Prop. 7.1].

Our starting point is the following:

Theorem 5.4 ([CDMO24, CDM22, MP22])**.** *Let $(\mathcal{H}^r_Z(\mathcal{O}_X),F,W)$ be the local cohomology bi-filtered $\mathcal{D}$-module. Then*

$$\widetilde{\alpha}(Z) \geq r+k \iff F_k\mathcal{H}^r_Z(\mathcal{O}_X) = P_k\mathcal{H}^r_Z(\mathcal{O}_X) \iff Z \text{ has } k\text{-Du Bois singularities},$$

$$\widetilde{\alpha}(Z) > r+k \iff F_kW_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X) = P_k\mathcal{H}^r_Z(\mathcal{O}_X) \iff Z \text{ has } k\text{-rational singularities.}$$

In other words, the structure of the local cohomology mixed Hodge module allows us to give lower bounds on the minimal exponent of a local complete intersection, which controls these classes of singularities.

By Theorem A, we have the following:

Theorem 5.5. *Let $(\mathcal{H}^r_Z(\mathcal{O}_X),F,W)$ be the local cohomology bi-filtered $\mathcal{D}$-module. Then for any non-degenerate slope L, we have*

$$F_p\mathcal{H}^r_Z(\mathcal{O}_X) = \left\{\sum_{|\alpha|\leq p} \frac{\alpha! h_\alpha}{f_1^{\alpha_1+1}\dots f_r^{\alpha_r+1}} \mid \sum_{|\alpha|\leq p} h_\alpha \partial_t^\alpha \delta_f \in {}^LV^{|L|}\mathcal{B}_f\right\}$$

and for any $\ell \in \mathbf{Z}_{\geq 0}$, we have

$$W_{n+r+\ell}\mathcal{H}^r_Z(\mathcal{O}_X) = \left\{\sum_\alpha \frac{\alpha! h_\alpha}{f_1^{\alpha_1+1}\dots f_r^{\alpha_r+1}} \mid L(t\partial_t)^{\ell+1}\left(\sum_\alpha h_\alpha\partial_t^\alpha\delta_f\right) \in {}^LV^{>|L|}\mathcal{B}_f\right\}.$$

To see this, we must make explicit the isomorphism

$$ {}^LV^{|L|}\mathcal{B}_f \Big/ \sum_{i=1}^r t_i {}^LV^{|L|-a_i}\mathcal{B}_f \cong \mathcal{H}^r_Z(\mathcal{O}_X). $$

Note that, by [CDMO24, Lem. 5.1], if we give any isomorphism between these $\mathcal{D}$-modules, then any other one differs from this by a non-zero scalar multiple. So in terms of the Hodge and weight filtrations, any $\mathcal{D}$-module isomorphism we define will (up to scalar multiple) agree with the formal bi-filtered isomorphism given by Theorem A. We denote that isomorphism by $\rho\colon {}^LV^{|L|}\mathcal{B}_f/\sum_{i=1}^r t_i {}^LV^{|L|-a_i}\mathcal{B}_f \to \mathcal{H}^r_Z(\mathcal{O}_X)$.

In *loc. cit.*, a $\mathcal{D}_X$-linear morphism $\tau\colon \mathcal{B}_f \to \mathcal{H}^r_Z(\mathcal{O}_X)$ is defined as

$$ \sum_\beta h_\beta \partial_t^\beta \delta_f \mapsto \sum_\beta \frac{\beta! h_\beta}{f_1^{\beta_1+1}\dots f_r^{\beta_r+1}}. $$

In *loc. cit.* it is observed that $\sum_{i=1}^r t_i\mathcal{B}_f \subseteq \ker(\tau)$. In particular, $\sum_{i=1}^r (s_i+1)\mathcal{B}_f \subseteq \ker(\tau)$.

We define $\tau\colon {}^LV^{|L|}\mathcal{B}_f \to \mathcal{H}^r_Z(\mathcal{O}_X)$ by applying τ to ${}^LV^{|L|}\mathcal{B}_f \subseteq \mathcal{B}_f$, which vanishes on $\sum_{i=1}^r t_i {}^LV^{|L|-a_i}\mathcal{B}_f \subseteq \sum_{i=1}^r t_i\mathcal{B}_f$. Thus, we get an induced morphism

$$ \overline{\tau}\colon {}^LV^{|L|}\mathcal{B}_f \Big/ \sum_{i=1}^r t_i {}^LV^{|L|-a_i}\mathcal{B}_f \to \mathcal{H}^r_Z(\mathcal{O}_X). $$

Using the same argument as that in *loc. cit.*, we have the following:

Lemma 5.6. *The map $\overline{\tau}$ is surjective.*

Proof. Let $u = \frac{g}{(f_1\dots f_r)^m} \in \mathcal{O}_X[\frac{1}{f_1\dots f_r}]$ with $m \geq 1$, by definition, we have $u = \tau(v)$ where $v = \frac{g}{(m-1)!^r}(\partial_{t_1}\dots\partial_{t_r})^{m-1}\delta_f \in \mathcal{B}_f$. However, v needs not lie in ${}^LV^{|L|}\mathcal{B}_f$. If it does, we are done.

Otherwise, by discreteness of the ${}^LV$-filtration and nilpotency of $L(s)+\lambda$ on $\mathrm{Gr}^\lambda_L(\mathcal{B}_f)$, we can find $\alpha_1 \leq \dots \leq \alpha_N < |L|$ such that

$$ (L(s)+\alpha_1)\dots(L(s)+\alpha_N)v \in {}^LV^{|L|}\mathcal{B}_f. $$

By Bézout's relation, we have some $p(w), q(w)$ such that

$$ (w+|L|)p(w) + q(w)\prod_{i=1}^N (w+\alpha_i) = 1 \in \mathbf{C}[w]. $$

Plugging in $L(s)$ and applying to v, we get

$$ (L(s)+|L|)p(L(s))v + q(L(s))\prod_{i=1}^N (L(s)+\alpha_i)v = v. $$

Note that we have $(L(s)+|L|) = \sum_{i=1}^r a_i(s_i+1)$.

As $(L(s)+|L|)p(L(s))v = \sum_{i=1}^r (s_i+1)a_i p(L(s))v \in \sum_{i=1}^r (s_i+1)\mathcal{B}_f \subseteq \ker(\tau)$, we conclude. $\square$

Proof of Theorem B. We have shown above that $\overline{\tau}\circ\rho^{-1}\in \mathrm{End}(\mathcal{H}^r_Z(\mathcal{O}_X))$ is surjective, hence non-zero. It must then be multiplication by a non-zero scalar, and so we get

$$\overline{\tau} = \lambda\rho \text{ for some } \lambda\in \mathbf{C}^*,$$

which shows that $\overline{\tau}$ is a bi-filtered isomorphism, as desired.

The description of the Hodge filtration is an easy computation (keeping in mind the shift by r in the Hodge filtration in Theorem A). For the weight filtration, as $\mathcal{B}_f$ is pure of weight n, we are interested in the monodromy filtration (shifted by $n=\dim X$) on $\mathrm{Gr}_L^{|L|}(\mathcal{B}_f)$ with respect to $L(t\partial_t)$.

By [SZ85, Rmk. 2.3], we can write this filtration as

$$W_{n+\ell}\mathrm{Gr}_L^{|L|}(\mathcal{B}_f) = \sum_{j\geq \max\{0,-\ell\}} L(t\partial_t)^j \ker(L(t\partial_t)^{\ell+1+2j}).$$

Note that if $j>0$, then

$$L(t\partial_t)^j\ker(L(t\partial_t)^{\ell+1+2i}) \subseteq \mathrm{Im}(L(t\partial_t)) \subseteq \sum_{i=1}^r t_i\partial_{t_i}\mathrm{Gr}_V^{|L|}(\mathcal{B}_f)\subseteq \sum_{i=1}^r t_i \mathrm{Gr}_L^{|L|-a_i}(\mathcal{B}_f).$$

Thus,

$$W_{n+\ell+r}\mathcal{H}^r_Z(\mathcal{O}_X) = \frac{\ker(L(t\partial_t)^{\ell+1}) + \sum_{i=1}^r t_i\mathrm{Gr}_L^{|L|-a_i}(\mathcal{B}_f)}{\sum_{i=1}^r t_i\mathrm{Gr}_L^{|L|-a_i}(\mathcal{B}_f)},$$

which finishes the proof. Note that the shift by r comes from the fact that we are studying the rth cohomology in Theorem A. □

This immediately gives the following:

Corollary 5.7. *For $Z=V(f_1,\dots,f_r)\subseteq X$ a complete intersection of pure codimension r and any non-degenerate slope L, we have*

$$\widetilde{\alpha}(Z)\geq r+k \iff \partial_t^\beta\delta_f\in {}^LV^{|L|}\mathcal{B}_f \quad \forall |\beta|\leq k \iff Z \text{ has } k\text{-Du Bois singularities},$$

$$\widetilde{\alpha}(Z)> r+k \iff L(t\partial_t)\partial_t^\beta\delta_f\in {}^LV^{>|L|}\mathcal{B}_f \quad \forall |\beta|\leq k \iff Z \text{ has } k\text{-rational singularities}.$$

5.1. **Weighted homogeneous complete intersections.** Next, we prove Corollary D. We assume that $f_1,\dots,f_r\in\mathbf{C}[x_1,\dots,x_n]$ are weighted homogeneous of degrees $d_1\leq\dots\leq d_r$ which define $Z\subseteq \mathbf{A}^n_x$, a complete intersection such that $0\in Z$ is an isolated singular point. Here weighted homogeneous means there exist $w_1,\dots,w_n\in\mathbf{Z}_{>0}$ such that if $\theta_w = \sum_{i=1}^n w_i x_i\partial_{x_i}$, then

$$\theta_w f_j = d_j f_j \text{ for all } j.$$

We assume throughout this section that $d_1+\dots+d_r\leq |w|$. It was shown in [CDM24, Prop. 2.1] that this implies

$$\widetilde{\alpha}(Z)\leq r+\frac{|w|-\sum_{i=1}^r d_i}{d_r}.$$

Our goal is the following theorem, which is a strengthening of the result of [CDM24] in the case of Du Bois singularities:

Theorem 5.8. *Let $Z = V(f_1,\dots,f_r) \subseteq X = \mathbf{A}^n_x$ be defined by $f_1,\dots,f_r$ which are weighted homogeneous of degrees $2 \le d_1 \le \cdots \le d_r$ satisfying $|w| \ge d_1+\cdots+d_r$. Assume Z is a complete intersection with an isolated singularity at 0. Then*

$$r + \left\lfloor \frac{|w| - \sum_{i=1}^r d_i}{d_r} \right\rfloor \le \widetilde{\alpha}_0(Z) \le r + \frac{|w| - \sum_{i=1}^r d_i}{d_r}.$$

Thus, $\lfloor \widetilde{\alpha}_0(Z) \rfloor = r + \lfloor \frac{|w|-\sum_{i=1}^r d_i}{d_r} \rfloor$. Moreover, $\widetilde{\alpha}_0(Z) = \lfloor \widetilde{\alpha}_0(Z)\rfloor$ if and only if

$$d_r \,\Big|\, |w| - \sum_{i=1}^r d_i.$$

To prove the lower bound, we study the ${}^LV$-filtration on $\mathcal{B}_f = \bigoplus_{\alpha\in\mathbf{N}^r} \mathcal{O}_X \partial_t^\alpha \delta_f$. Here $L = \sum_{i=1}^r d_i s_i$. This is the natural slope to consider in this example, by observing

$$\theta_w(\delta_f) = L(s)\delta_f,$$

and so

$$\theta_w(x^\alpha \partial_t^\beta \delta_f) = (L(s-\beta) + w\cdot\alpha)(x^\alpha\partial_t^\beta\delta_f) \tag{5}$$

We have the following general observation:

Lemma 5.9. *Let $Z = V(f_1,\dots,f_r) \subseteq X$ be a reduced complete intersection of codimension r in a smooth variety X. Then for any non-degenerate slope $L = \sum_{i=1}^r a_i s_i$, we have*

$$\mathrm{Gr}_L^\lambda(\mathcal{B}_f) \text{ is supported on } Z_{\mathrm{sing}} \text{ for } \lambda \notin \mathbf{Z}_{\ge |L|},$$

where $|L| = \sum_{i=1}^r a_i$.

Proof. Let $U \subseteq X$ be an open subset such that $U \cap Z = Z_{\mathrm{reg}}$. Then $f_1,\dots,f_r$ are part of a system of coordinates on U. In this case, we have that $\mathcal{B}_f$ is ${}^LV^0\mathcal{B}_f$ coherent, which by Lemma 3.13 proves the claim. Indeed, it suffices to show that, for every $\beta \in \mathbf{N}^r$, the element $\partial_t^\beta \delta_f \in {}^LV^0\mathcal{D}\cdot\delta_f$. But we have

$$\partial_t^\beta \delta_f = \partial_f^\beta \delta_f \in \mathcal{D}_X \cdot \delta_f$$

over U, proving the claim. ☐

Return now to the case that $Z = V(f_1,\dots,f_r) \subseteq \mathbf{A}^n_x$ is a complete intersection with an isolated singular point at 0, such that each f_i is weighted homogeneous of degree d_i, i.e.,

$$\theta_w f_i = d_i f_i.$$

By reordering, we can assume $d_1 \le \cdots \le d_r$. The following observation is elementary:

Lemma 5.10. *In the situation above, for any $\alpha \in \mathbf{N}^n$ and $\beta \in \mathbf{N}^r$, we have*

$$x^\alpha \partial_t^\beta \delta_f \in {}^LV^{\min\{|L|, |w| + w\cdot\alpha - L(\beta)\}}\mathcal{B}_f,$$

where $|w| = \sum_{i=1}^n w_i$ and $w\cdot\alpha = \sum_{i=1}^n w_i\alpha_i$.

Proof. Assume $x^\alpha\partial_t^\beta\delta_f$ defines a non-zero element of $\mathrm{Gr}_L^\chi(\mathcal{B}_f)$ with $\chi < |L|$. Our goal is to establish the inequality $\chi \ge w\cdot\alpha + |w| - L(\beta)$.

As $\chi < |L|$, we have by Lemma 5.9 above that $\mathrm{Gr}_L^{\chi}(\mathcal{B}_f)$ is supported on $Z_{\mathrm{sing}} = \{0\}$. By Kashiwara's equivalence (which in this setting of possibly non-coherent $\mathcal{D}$-modules, just means studying the eigenspaces of the Euler operator), we can write

$$\mathrm{Gr}_L^{\chi}(\mathcal{B}_f) = \bigoplus_{\gamma \in \mathbf{N}^n} \mathcal{N} \partial_x^{\gamma} \delta_0,$$

where $\mathcal{N}$ is a complex vector space (possibly of infinite dimension). For any $\eta \in \mathcal{N}$, we have $x_i(\eta\delta_0) = 0$. Moreover, the Leibniz rule gives

$$\theta_w \partial_x^{\gamma} = \partial_x^{\gamma}(\theta_w - w \cdot \gamma).$$

Finally, $\theta_w = \sum_{i=1}^n w_i x_i \partial_{x_i} = \sum_{i=1}^n w_i \partial_{x_i} x_i - |w|$. Putting this together, we see that

$$\theta_w(\partial_x^{\gamma}(\eta\delta_0)) = \partial_x^{\gamma}(\theta_w \eta\delta_0) = (-|w| - w \cdot \gamma)\partial_x^{\gamma}\eta\delta_0.$$

Hence, in this situation, any generalized eigenvector of θ_w is actually an eigenvector and its eigenvalue is $\leq -|w|$.

Assume $x^{\alpha}\partial_t^{\beta}\delta_f$ defines a non-zero element of $\mathrm{Gr}_L^{\chi}(\mathcal{B}_f)$. Then

$$\theta_w(x^{\alpha}\partial_t^{\beta}\delta_f) = (L(s-\beta) + w \cdot \alpha)(x^{\alpha}\partial_t^{\beta}\delta_f),$$

and so

$$(L(s) + \chi)(x^{\alpha}\partial_t^{\beta}\delta_f) = (\theta_w + L(\beta) - w \cdot \alpha + \chi)(x^{\alpha}\partial_t^{\beta}\delta_f).$$

As $L(s) + \chi$ is nilpotent on $\mathrm{Gr}_L^{\chi}(\mathcal{B}_f)$, we see that $x^{\alpha}\partial_t^{\beta}\delta_f$ is a generalized eigenvector for θ_w with eigenvalue $w \cdot \alpha - L(\beta) - \chi$. This gives

$$w \cdot \alpha - L(\beta) - \chi \leq -|w|,$$

and so $\chi \geq w \cdot \alpha - L(\beta) + |w|$, proving the claim. ☐

Corollary 5.11. *For $k = \lfloor \frac{|w|-|L|}{d_r} \rfloor$, we have*

$$F_{k+r}\mathcal{B}_f \subseteq {}^L V^{|L|}\mathcal{B}_f, \quad F_{k+r+1}\mathcal{B}_f \not\subseteq {}^L V^{|L|}\mathcal{B}_f.$$

Proof. To prove this, we show that for any β with $|\beta| = k$ we have

$$\partial_t^{\beta}\delta_f \in {}^L V^{|L|}\mathcal{B}_f.$$

By the previous lemma, we want to understand when $|w| - L(\beta) \geq |L|$. By varying over all β with $|\beta| = k$, the maximal value of $L(\beta)$ is kd_r. Hence, for any $p \in \mathbf{Z}$ such that $|w| - |L| \geq pd_r$, we get $F_{p+r}\mathcal{B}_f \subseteq {}^L V^{|L|}\mathcal{B}_f$.

The second claim follows from the general upper bound on $\widetilde{\alpha}(Z)$. ☐

In summary, with the discussion above, we have the following result.

Corollary 5.12. *If $Z = V(f_1, \dots, f_r) \subseteq \mathbf{A}_x^n$ is a complete intersection with isolated singularity at 0, such that f_i is weighted homogeneous of degree d_i and $d_1 \leq \cdots \leq d_r$ and $|w| \geq d_1 + \cdots + d_r$, then*

$$\lfloor \widetilde{\alpha}_0(Z) \rfloor = r + \left\lfloor \frac{|w| - |L|}{d_r} \right\rfloor.$$

Proof. This follows immediately from $F_{k+r}\mathcal{B}_f \subseteq {}^L V^{|L|}\mathcal{B}_f$ when $k = \lfloor \frac{|w|-|L|}{d_r} \rfloor$ and the previous proposition. ☐

Using the general upper bound, we have the following:

Corollary 5.13. *If $d_r \big| |w| - |L|$, we have*

$$\widetilde{\alpha}_0(Z) = r + \frac{|w| - |L|}{d_r}.$$

As we can compute the weight filtration on $\mathcal{H}^r_Z(\mathcal{O}_X)$ using the nilpotent operator $L(t\partial_t)$ on $\mathrm{Gr}^{|L|}_L(\mathcal{B}_f)$, we also have the following:

Lemma 5.14. *If $d_r \nmid |w| - |L|$, then*

$$\widetilde{\alpha}_0(Z) > r + \frac{|w| - |L|}{d_r}.$$

Proof. Let $k = \lfloor \frac{|w|-|L|}{d_r} \rfloor$. We will show that, under the assumption $d_r \nmid |w| - |L|$, we have

$$\partial_t^{\beta+e_i}\delta_f \in {}^L V^{>|L|-d_i}\mathcal{B}_f \text{ for all } |\beta| = k, 1 \leq i \leq r.$$

Indeed, this implies that $L(t\partial_t)F_{k+r}\mathrm{Gr}^{|L|}_L(\mathcal{B}_f) = 0$, proving

$$F_k W_{n+r}\mathcal{H}^r_Z(\mathcal{O}_X) = F_k\mathcal{H}^r_Z(\mathcal{O}_X) = P_k\mathcal{H}^r_Z(\mathcal{O}_X).$$

Assume there exists β with $|\beta| = k$ and $1 \leq i \leq r$ such that

$$\partial_t^{\beta+e_i}\delta_f \in {}^L V^{|L|-d_i}\mathcal{B}_f \setminus {}^L V^{>|L|-d_i}\mathcal{B}_f.$$

By Lemma 5.10, this means $|L| - d_i \geq |w| - L(\beta + e_i)$, so that $L(\beta) \geq |w| - |L|$. But $L(\beta) \leq kd_r$, so we get

$$|w| - |L| \leq kd_r,$$

and so $\frac{|w|-|L|}{d_r} \leq \lfloor \frac{|w|-|L|}{d_r} \rfloor$, contradicting our assumption. ☐

We end with an explicit computation of the ${}^L V$-filtration in this setting.

Theorem 5.15. *Define a filtration*

$$U^\lambda \mathcal{B}_f = \begin{cases} \sum_{|w|+w\cdot\alpha-L(\beta)\geq\lambda} \mathcal{D}_X \cdot (x^\alpha \partial_t^\beta \delta_f) & \lambda \leq |L| \\ \sum_{i=1}^r t_i U^{\lambda-d_i}\mathcal{B}_f & \lambda > |L| \end{cases}.$$

In the setting above, we have ${}^L V^\lambda \mathcal{B}_f = U^\lambda \mathcal{B}_f$.

Proof. Lemma 5.10 shows $U^\bullet\mathcal{B}_f \subseteq V^\bullet\mathcal{B}_f$, so we need only prove the opposite inclusion. It is trivial to check that it satisfies the properties of Proposition 3.10 except possibly the coherence condition, but this gives the desired containment. ☐

References

[BMS06] Nero Budur, Mircea Mustaţă, and Morihiko Saito, *Bernstein-Sato polynomials of arbitrary varieties*, Compos. Math. **142** (2006), no. 3, 779–797. MR2231202 ↑1, 9, 16, 29

[CD23] Qianyu Chen and Bradley Dirks, *On V-filtration, Hodge filtration and Fourier transform*, Selecta Math. (N.S.) **29** (2023), no. 4, Paper No. 50, 76. MR4610959 ↑1, 2, 3, 9, 10, 16, 20, 26

[CDM22] Qianyu Chen, Bradley Dirks, and Mircea Mustaţă, *The minimal exponent and k-rationality for local complete intersections*, arXiv e-prints (December 2022), arXiv:2212.01898, available at 2212.01898. to appear in Journal de l'École Polytechnique. ↑1, 4, 30

[CDM24] ———, *The minimal exponent of cones over smooth complete intersection projective varieties*, arXiv e-prints (April 2024), arXiv:2404.18289, available at 2404.18289. ↑4, 14, 32

[CDMO24] Qianyu Chen, Bradley Dirks, Mircea Mustaţă, and Sebastián Olano, *V-filtrations and minimal exponents for local complete intersections*, J. Reine Angew. Math. **811** (2024), 219–256. MR4751121 ↑1, 4, 30, 31

[CDS23] Qianyu Chen, Bradley Dirks, and Morihiko Saito, *Verdier specialization and restrictions of Hodge modules*, arXiv e-prints (April 2023), arXiv:2304.13644, available at 2304.13644. ↑1, 2, 3, 9, 10, 25

[Dir24] Bradley Dirks, *Fourier transform and Radon transform for mixed Hodge modules*, arXiv e-prints (May 2024), arXiv:2405.19127. ↑20

[FL22] Robert Friedman and Radu Laza, *Higher Du Bois and higher rational singularities*, to appear in Duke Mathematical Journal (2022). with an appendix by Morihiko, Saito. ↑29

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki, *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics, vol. 236, Birkhäuser Boston, Inc., Boston, MA, 2008. Translated from the 1995 Japanese edition by Takeuchi. MR2357361 ↑5, 6, 21

[JKSY22] Seung-Jo Jung, In-Kyun Kim, Morihiko Saito, and Youngho Yoon, *Higher Du Bois singularities of hypersurfaces*, Proceedings of the London Mathematical Society **125** (2022), no. 3, 543—567. ↑29

[Kas86] Masaki Kashiwara, *A study of variation of mixed Hodge structure*, Publ. Res. Inst. Math. Sci. **22** (1986), no. 5, 991–1024. MR866665 ↑9

[MFS17] Teresa Monteiro Fernandes and Claude Sabbah, *Relative Riemann-Hilbert correspondence in dimension one*, Portugal. Math. (N.S.) **74** (2017), no. 2, 149–159. ↑22

[MOPW23] Mircea Mustaţă, Sebastián Olano, Mihnea Popa, and Jakub Witaszek, *The Du Bois complex of a hypersurface and the minimal exponent*, Duke Mathematical Journal **173** (2023), no. 2, 1411—1436. ↑29

[MP22] Mircea Mustaţă and Mihnea Popa, *Hodge filtration on local cohomology, Du Bois complex and local cohomological dimension*, Forum Math. Pi **10** (2022), Paper No. e22, 58. MR4491455 ↑1, 4, 30

[Sab87] Claude Sabbah, *Proximité évanescente. I. La structure polaire d'un $\mathcal{D}$-module*, Compositio Math. **62** (1987), no. 3, 283–328. MR901394 ↑2

[Sai00] Morihiko Saito, *Mixed Hodge complexes on algebraic varieties*, Math. Ann. **316** (2000), no. 2, 283–331. ↑11

[Sai88] ______, *Modules de Hodge polarisables*, Publ. Res. Inst. Math. Sci. **24** (1988), no. 6, 849–995 (1989). MR1000123 ↑6, 7

[Sai90] ______, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. **26** (1990), no. 2, 221–333. MR1047415 ↑6, 9, 20

[Sch14] Christian Schnell, *An overview of Morihiko Saito's theory of mixed Hodge modules*, arXiv e-prints (May 2014), arXiv:1405.3096, available at 1405.3096. ↑6

[SS] Claude Sabbah and Christian Schnell, *Mixed Hodge Modules Project.* Accessed: 2024-08-19. ↑7

[SZ85] Joseph Steenbrink and Steven Zucker, *Variation of mixed Hodge structure. I*, Invent. Math. **80** (1985), no. 3, 489–542. MR791673 ↑9, 32

[Wu22] Lei Wu, *Characteristic cycles associated to holonomic $\mathcal{D}$-modules*, Math. Z. **301** (2022), 2059—2098. ↑16

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