A HODGE THEORETIC GENERALIZATION OF $\mathbb{Q}$-HOMOLOGY MANIFOLDS

BRADLEY DIRKS, SEBASTIÁN OLANO, AND DEBADITYA RAYCHAUDHURY

Abstract. We study a natural Hodge theoretic generalization of rational (or $\mathbb{Q}$-)homology manifolds through an invariant $\mathrm{HRH}(Z)$ where Z is a complex algebraic variety. The defining property of this notion encodes the difference between higher Du Bois and higher rational singularities for local complete intersections, which are two classes of singularities that have recently gained much attention.

We show that $\mathrm{HRH}(Z)$ can be characterized when the variety Z is embedded into a smooth variety using the local cohomology mixed Hodge modules. Near a point, this is also characterized by the local cohomology of Z at the point, and hence, by the cohomology of the link. We give an application to partial Poincaré duality.

In the case of local complete intersection subvarieties, we relate $\mathrm{HRH}(Z)$ to various invariants. In the hypersurface case it turns out that $\mathrm{HRH}(Z)$ can be completely characterized by these invariants. However for higher codimension subvarieties, the behavior is rather subtle, and in this case we relate $\mathrm{HRH}(Z)$ to these invariants through inequalities and give some conditions on when equality holds.

Contents

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A. Introduction

The cohomology of a compact, oriented real manifold X of dimension $2n$ satisfies the following incredible symmetry, called Poincaré duality:

$$H^{n-k}(X,\mathbb{Q}) \cong H^{n+k}(X,\mathbb{Q})^{\vee},$$

where the right hand side is the dual vector space.

In the non-compact setting, such a duality still holds, but it compares singular cohomology with compactly supported cohomology:

$$H^{n-k}(X,\mathbb{Q}) \cong H^{n+k}_{\mathrm{c}}(X,\mathbb{Q})^{\vee}.$$

The above applies to smooth complex algebraic varieties of dimension n. For singular varieties (in this paper, meaning reduced, finite type schemes over $\mathbb{C}$), the duality need not hold for singular cohomology of the associated analytic space. Goresky-MacPherson's theory of intersection cohomology provides a replacement of singular cohomology which still admits a Poincaré duality isomorphism. This theory associates to any purely d-dimensional complex variety Z a pair of graded vector spaces $\mathrm{IH}^*(Z,\mathbb{Q})$ and $\mathrm{IH}^*_{\mathrm{c}}(Z,\mathbb{Q})$ (which, in the smooth case agree with the singular cohomology and compactly supported cohomology, respectively), such that there are natural isomorphisms

$$\mathrm{IH}^{d-k}(Z,\mathbb{Q}) \cong \mathrm{IH}^{d+k}_{c}(Z,\mathbb{Q})^{\vee}.$$

As in [BBD82], these vector spaces can be realized as the hypercohomology of the intersection complex perverse sheaf, which is self-dual as a perverse sheaf.

One of the many achievements of Saito's theory of mixed Hodge modules [Sai88, Sai90] is that it endows these intersection cohomology spaces with natural mixed Hodge structures. The important aspects of this theory will be reviewed in Section B below.

Let Z be a purely d-dimensional complex algebraic variety. In §3, we explain the construction of a natural morphism in $D^b(\mathrm{MHM}(Z))$ (due to Saito [Sai90, (4.5.12)]):

$$\psi_Z \colon \mathbb{Q}^H_Z[d] \to \mathbf{D}_Z(\mathbb{Q}^H_Z[d])(-d),$$

which is a sheaf-theoretic incarnation of the Poincaré duality morphism.

By applying the associated graded de Rham functor $\mathrm{Gr}^F_{-p}\mathrm{DR}(-)$ to the morphism ψ_Z, we obtain morphisms in $D^b_{\mathrm{coh}}(\mathscr{O}_Z)$:

$$\phi^p \colon \underline{\Omega}^p_Z \to \mathbb{D}_Z(\underline{\Omega}^{d-p}_Z)[-d],$$

where $\underline{\Omega}^p_Z$ is the p-th Du Bois complex of Z and $\mathbb{D}_Z = R\mathcal{H}om_{\mathscr{O}_Z}(-,\omega^\bullet_Z)$ is the Grothendieck duality functor. We review the Du Bois complexes in Section B below. For now, these should be thought of as replacements of the sheaf of Kähler differentials which are better behaved from a Hodge theoretic point of view.

Definition 0.1. Let Z be a pure d-dimensional variety. We say Z is a *rational homology manifold to Hodge degree* k, or k*-Hodge rational homology variety* if ϕ^p is a quasi-isomorphism for $0 \leq p \leq k$.

Further, define the *HRH level of* Z to be

$$\mathrm{HRH}(Z) := \sup\{k \in \mathbb{Z}_{\geq 0} \mid \phi^p \text{ is a quasi-isomorphism for } 0 \leq p \leq k\}$$

where we follow the convention that $\mathrm{HRH}(Z) = -1$ if ϕ^0 is not a quasi-isomorphism.

We will see in the sequel that $\mathrm{HRH}(Z) = +\infty$ if and only if Z is a rational homology manifold (sometimes called *rationally smooth* in the literature). By duality, we have equality $\mathbb{D}(\phi^p) = \phi^{d-p}$, hence we have the upper bound

$$\mathrm{HRH}(Z) < +\infty \iff \mathrm{HRH}(Z) < \frac{d}{2}$$

(see Theorem G or Remark 4.8 (4)(c) for a better bound).

It turns out that if Z has rational singularities then $\mathrm{HRH}(Z) \geq 0$ (see Remark 4.8 (3) for a more general statement). However, we note that there are varieties with $\mathrm{HRH}(Z) \geq 0$ whose singularities are not even Du Bois, see Example 13.9 (2).

As the reader might have guessed from the above discussions, we have partial Poincaré duality for k-Hodge rational homology varieties (see Theorem 7.2 for a more elaborate formulation). Throughout the article, a variety Z is *embeddable* if there exists a closed embedding $i\colon Z \to X$ with X a smooth variety.

Theorem A. *Let Z be an embeddable complex algebraic variety with* $\mathrm{HRH}(Z) \geq k$*. Then for all $i \in \mathbb{Z}$, we have isomorphisms*

$$F^{d-k}H^{d-i}(Z) \cong F^{-k}H_c^{d+i}(Z)^\vee.$$

Remark 0.2. This notion is different from another weakening of Poincaré duality, due to Kato [Kat77]. Indeed, the notion we study is related to which Hodge filtered pieces are Poincaré dual to each other (in all cohomological degrees), whereas Kato's notion is asking for which cohomological degrees Poincaré duality holds (in all Hodge levels).

The notion of k-Hodge rational homology varieties is also studied in the recent article [PP24] through an equivalent defining property that is phrased as "$(*)_k$-condition" (see Remark 5.2 for the equivalence of $(*)_k$-condition with $\mathrm{HRH}(Z) \geq k$), and important consequences, such as symmetry of Hodge-Du Bois numbers and Lefschetz properties are discussed in detail. However, in this paper we will take a different point of view, and will mainly be concerned with characterizations of HRH level via various singularity invariants.

HRH level via local cohomology in the embedded case. Our main results focus on the case when $i\colon Z \hookrightarrow X$ is an embedding of a singular variety Z in a smooth variety X. In this setting, our first result relates these notions to the *local cohomology mixed Hodge modules* of $\mathscr{O}_X$ along Z.

Local cohomology modules have been recently studied [MP22, CDMO24, CDM22] from a Hodge theoretic perspective due to their relation to the classes of higher Du Bois and higher rational singularities (see Definition 3.2 below).

Let $(\mathcal{H}^j_Z(\mathscr{O}_X), F, W)$ be the bi-filtered local cohomology $\mathscr{D}_X$-module. These are potentially non-zero only for $j \geq \mathrm{codim}_X(Z)$. In what follows, we will be indexing the Hodge filtration following conventions for right $\mathscr{D}$-modules.

Theorem B. *Let $i\colon Z \hookrightarrow X$ be a closed embedding of a pure c-codimensional variety Z inside a smooth variety X of dimension n. Then* $\mathrm{HRH}(Z) \geq k$ *if and only if*

$$F_{k-n}\mathcal{H}^j_Z(\mathscr{O}_X) = 0 \text{ for all } j > c \quad \text{and} \quad F_{k-n}W_{n+c}\mathcal{H}^c_Z(\mathscr{O}_X) = F_{k-n}\mathcal{H}^c_Z(\mathscr{O}_X).$$

In the hypersurface case, the above result immediately shows that $\mathrm{HRH}(Z)$ can be detected using weighted Hodge ideals introduced in [Ola23] (see Corollary 5.4). Moreover, when $Z \subseteq X$ is a complete intersection variety of pure codimension c, this condition resembles the one in the main theorem of

[CDM22]. This similarity reflects the fact that the invariant $\mathrm{HRH}(Z)$ differentiates the notions of k-Du Bois and k-rational singularities (see Remark 4.8 below). For this reason, we expect this should be satisfied by a further iteration of the definitions of higher Du Bois and higher rational singularities. Furthermore, higher rational singularities yield a lower bound for the invariant $\mathrm{HRH}(Z)$. In fact, if a normal variety Z is pre-k-rational —a weakening of the k-rational condition— then $\mathrm{HRH}(Z) \geq k$ (see Remark 3.4 and Remark 4.8).

Applying Theorem B to a result of Mustaţă-Popa [MP22, Thm. C], we get the following. In the statement, the "Ext" filtration $E_\bullet \mathcal{H}^q_Z(\mathscr{O}_X)$ is defined using the Ext description of local cohomology:

$$\mathcal{H}^q_Z(\mathscr{O}_X) = \varinjlim_p \mathcal{E}xt^q(\mathscr{O}_X/\mathcal{J}_Z^{p+1}, \mathscr{O}_X),$$

where $\mathcal{J}_Z$ is the ideal sheaf of Z in X, and the filtration is defined by

$$E_\bullet \mathcal{H}^q_Z(\mathscr{O}_X) = \mathrm{Im}\left[\mathcal{E}xt^q(\mathscr{O}_X/\mathcal{J}_Z^{\bullet+1}, \mathscr{O}_X) \to \mathcal{H}^q_Z(\mathscr{O}_X)\right].$$

Corollary C. *If $Z \subseteq X$ is a closed embedding of a pure c-codimensional Cohen-Macaulay variety Z inside a smooth variety X of dimension n, then*

$$Z \text{ has rational singularities if and only if } F_{-n}W_{n+c}\mathcal{H}^c_Z(\mathscr{O}_X) = E_0\mathcal{H}^c_Z(\mathscr{O}_X).$$

HRH level via local cohomology at points and link invariants. We can also study the invariant $\mathrm{HRH}(Z)$ near $x \in Z$ in terms of the local cohomology $H^*_{\{x\}}(Z)$. It is well-known that Z is a rational homology manifold if and only if for every $x \in Z$, we have

$$H^i_{\{x\}}(Z) = \begin{cases} 0 & i < 2d, \\ \mathbb{Q} & i = 2d. \end{cases} \tag{0.3}$$

It turns out that the condition $\mathrm{HRH}(Z) \geq k$ is indeed the natural generalization of (0.3).

Theorem D. *Let Z be a purely d-dimensional variety. Then for any $k \geq 0$, we have* $\mathrm{HRH}(Z) \geq k$ *if and only if for every $x \in Z$, we have*

$$F_{k-d}H^i_{\{x\}}(Z) = \begin{cases} 0 & i < 2d, \\ \mathbb{Q} & i = 2d. \end{cases}$$

A slightly extended version of the above is proven in Theorem 6.5, which can also be related to the *link invariants* $\ell^{p,q}$ as defined in [FL24a]. We review the definition of the cohomology of the link L_x of Z at x in §6 (following [DS90]), and the definition of the link invariants in (6.7). Recall that the *local cohomological defect* of Z is defined by

$$\mathrm{lcdef}(Z) := \max\{j \mid \mathcal{H}^{c+j}_Z(\mathscr{O}_X) \neq 0, i\colon Z \to X \text{ codimension } c \text{ embedding, } X \text{ is smooth}\}$$

It turns out that the above description does not depend on choice of the embedding, and in fact

$$\mathrm{lcdef}(Z) = \max\{j \mid \mathcal{H}^{-j}(\mathbb{Q}^H_Z[d]) \neq 0\}.$$

This notion admits a local version $\mathrm{lcdef}_x(Z) = \min_{x \in U \subseteq Z} \mathrm{lcdef}(U)$ for any point $x \in Z$, where the minimum runs over all Zariski open neighborhoods of x in Z. We obtain the following generalization of [FL24a, Thm. 1.15(i)] via this invariant.

Theorem E. *Let Z be a purely d-dimensional variety and let $x \in Z$ be an isolated singular point. Let $a = \mathrm{lcdef}_x(Z)$. Then for any $k \geq 0$, we have $\mathrm{HRH}_x(Z) \geq k$ if and only if*

$$\begin{cases} \ell^{d-i,q-d+i} = 0 & i \leq k,\, q \in [d, d+a] \\ \ell^{d-i,d-1+i} = 0 & 1 \leq i \leq k \end{cases}. \tag{0.4}$$

Thus, Z has k-rational singularities near x if and only if it has k-Du Bois singularities near x and the vanishing (0.4) holds.

It is worth mentioning here that there is a result of Brion [Bri99, Prop. A1] which states that if Z is a rational homology manifold near x, then it is irreducible near x. In fact, using a bound on the local cohomological defect due to [PP24], we conclude the following (compare with the fact that if Z has rational singularities near x, then it is normal, hence irreducible, near x):

Theorem F. *Let Z be a purely d-dimensional variety and let $x \in Z$. If $\mathrm{HRH}_x(Z) \geq 0$, then*

$$\dim H^{2d-\ell-1}(L_x) = \dim H^{2d-\ell}_{\{x\}}(Z) = \begin{cases} 1 & \ell = 0 \\ 0 & 0 < \ell \leq 2\mathrm{HRH}_x(Z) + 1 \end{cases}.$$

In particular, if $\mathrm{HRH}_x(Z) \geq 0$, then Z is irreducible at x.

We also introduce a "generic variant" of the invariant $\mathrm{lcdef}(Z)$ in §5 that we call $\mathrm{lcdef}_{\mathrm{gen}}(Z)$ (see Definition 8.1). It is a non-negative integer satisfying the inequality

$$\mathrm{lcdef}_{\mathrm{gen}}(Z) \leq \mathrm{lcdef}(Z) \tag{0.5}$$

(equality holds in the case of isolated singularities, but strict inequality is also possible in the above, see §14 for explicit examples). In [PP24], it has been proven that the codimension of the locus Z_{nRS} where Z is not a rational homology manifold is bounded below by $2\,\mathrm{HRH}(Z) + 3$ (here nRS stands for non-rationally smooth). We obtain the following improvement of this bound via $\mathrm{lcdef}_{\mathrm{gen}}(Z)$.

Theorem G. *Let Z be a purely d-dimensional variety with $\mathrm{HRH}(Z) \geq 0$. Then we have the inequality*

$$\mathrm{lcdef}_{\mathrm{gen}}(Z) + 2\,\mathrm{HRH}(Z) + 3 \leq \mathrm{codim}_Z(Z_{\mathrm{nRS}}).$$

A more elaborate version of the above is proven in Proposition 8.5. There are instances when equality holds (see the examples in §14, Example 15.1, Example 15.2), however strict inequality also occurs (Example 15.3).

HRH level via integer invariants in the LCI case. For the remainder of the introduction, we fix an embedding $i \colon Z \hookrightarrow X$ as a complete intersection subvariety of pure codimension r with $\dim(X) = n$. Moreover, we fix $f_1, \ldots, f_r \in \mathscr{O}_X(X)$ such that $Z = V(f_1, \ldots, f_r)$. In this setting, there are various singularity invariants of Z defined through $\mathscr{D}$-module and mixed Hodge module theory, and our results compare these invariants. We will not mention the precise definitions in this introduction.

It is illustrative to first discuss hypersurface singularities. In this case, the complete picture can be understood with the well-known properties of the V-filtration (reviewed in §2 below). This is discussed in §9 and should motivate the definitions in the local complete intersection case, as we try to generalize the following initial result. Recall that for any object A with bounded below filtration $F_\bullet A$, we let

$$p(A, F) = \min\{p \mid F_p A \neq 0\}.$$

In the theorem statement, B_f is the Hodge module push-forward of the trivial Hodge module $\mathbb{Q}_X^H[\dim X]$ along the graph embedding $\Gamma\colon X \to X \times \mathbb{A}^1$.

Theorem H. *Let Z be a hypersurface in an n-dimensional smooth variety X defined by f. Then,*

$$\mathrm{HRH}(Z) \geq k-1 \iff F_{k-n}\,\mathrm{Gr}_V^0(B_f) = 0.$$

In particular, $\mathrm{HRH}(Z) = p(\mathrm{Gr}_V^0(B_f)) + n - 2$.

The central construction we use to study local complete intersections is the *unipotent Verdier specialization* of $\mathscr{O}_X$ along Z. This is a monodromic mixed Hodge module $\mathrm{Sp}_Z(\mathscr{O}_X)^{\mathbb{Z}}$ on $X \times \mathbb{A}_z^r$ which sits in a short exact sequence

$$0 \to i_*\mathbb{Q}^H_{Z\times\mathbb{A}^r_z}[n] \to \mathrm{Sp}_Z(\mathscr{O}_X)^{\mathbb{Z}} \to Q^{\mathbb{Z}} \to 0,$$

where the left-hand side is the trivial Hodge module on $Z \times \mathbb{A}_z^r$ and the module $Q^{\mathbb{Z}}$ is defined through this short exact sequence, though it was observed in [Dir23] that it is related to another important construction we will see below.

The first integer invariant we consider is $p(Q^{\mathbb{Z}}, F)$, where F is the Hodge filtration on the underlying $\mathscr{D}$-module of Q. In the hypersurface case, we have $p(Q^{\mathbb{Z}}, F) = p(\varphi_{f,1}(\mathscr{O}_X), F) - 1$. Up to a shift, $p(\varphi_{f,1}(\mathscr{O}_X), F)$ is the invariant $\widetilde{\alpha}^{\text{min.int}}(f)$ defined in [JKSY22].

The module $Q^{\mathbb{Z}}$ is related to the *integral spectrum of Z at $x \in Z$*, defined by [DMS11]. We let $\mathrm{Sp}_{\min,\mathbb{Z}}(Z, x)$ denote the smallest non-zero element of the integral spectrum of Z at x. In [JKSY22], it was noted that in the isolated hypersurface singularities case, $\widetilde{\alpha}^{\text{min.int}}(f) = \mathrm{Sp}_{\min,\mathbb{Z}}(Z, x)$. We give another proof of this below (see Remark 11.7).

The main result about this level is the following and is proved in Proposition 11.1 and Proposition 11.4.

Theorem I. *Let $x \in Z$ be a point in a local complete intersection subvariety of the smooth variety X. Then we have the following inequalities:*

(1) $p(Q^{\mathbb{Z}}, F) + n - 1 \leq \mathrm{HRH}(Z)$.
(2) $p(Q^{\mathbb{Z}}_x, F) + n + 1 \leq \mathrm{Sp}_{\min,\mathbb{Z}}(Z, x)$.

Note that in the hypersurface case, we always have $p(Q^{\mathbb{Z}}, F) + n - 1 = \mathrm{HRH}(Z)$. Furthermore, in the isolated hypersurface singularities case, we have $p(Q^{\mathbb{Z}}_x, F) + n + 1 = \mathrm{Sp}_{\min,\mathbb{Z}}(Z, x)$. For isolated local complete intersections, we partially recover the second equality (see Corollary 9.1).

Theorem J. *Let Z be a variety that is locally a complete intersection that has an isolated singularity at $x \in Z$. Then,*

$$\mathrm{HRH}(Z) \geq \mathrm{Sp}_{\min,\mathbb{Z}}(Z, x) - 2.$$

It is clear from Theorem I that $Q^{\mathbb{Z}} = 0$ implies Z is a rational homology manifold and that $\mathrm{Sp}_{\min,\mathbb{Z}}(Z, x) = +\infty$ for all $x \in Z$. However, this condition is too strong to be equivalent to rational smoothness of Z, as we see in Example 16.2 below. We can characterize the vanishing of $Q^{\mathbb{Z}}$, as we explain now.

In [Mus22, CDMO24], singularities of $f_1, \ldots, f_r \in \mathscr{O}_X(X)$ are related to those of the *general linear combination hypersurface* $g = \sum_{i=1}^r y_i f_i$ defined on $Y = X \times \mathbb{A}_y^r$. Let $U = Y \setminus (X \times \{0\})$. The hypersurface defined by $g|_U$ is used in [CDMO24, CDM22, Dir23] to study higher singularities of Z.

We have the following characterization of the vanishing of $Q^{\mathbb{Z}}$ in terms of the rational smoothness of $V(g|_U) \subseteq U$. We use the notation $\mathrm{HRH}(V(g|_U)) = \mathrm{HRH}(g|_U)$, and we let

$$\widetilde{\alpha}_{\mathbb{Z}}(g|_U) = \min\{j \in \mathbb{Z} \mid \widetilde{b}_{g|_U}(-\gamma) = 0\},$$

where $\widetilde{b}_{g|_U}(s) = b_{g|_U}(s)/(s+1)$ is the reduced Bernstein-Sato polynomial of $g|_U$.

Theorem K. *In the notation above, we have*

$$Q^{\mathbb{Z}} = 0 \iff V(g|_U) \text{ is a rational homology manifold} \iff \varphi_{g|_U,1}(\mathscr{O}_U) = 0.$$

Moreover, we have inequalities

(1) $\widetilde{\alpha}_{\mathbb{Z}}(g|_U) \leq p(Q^{\mathbb{Z}}, F) + n + r$,
(2) $\widetilde{\alpha}_{\mathbb{Z}}(g|_U) - r - 1 \leq \mathrm{HRH}(Z)$,
(3) $\mathrm{HRH}(g|_U) - r + 1 \leq \mathrm{HRH}(Z)$.

These results are a consequence of a more precise statement involving an invariant of the Hodge filtration of $Q^{\mathbb{Z}}$, or more precisely, of its monodromic pieces (see Remark 11.11).

The tuple $f_1, \dots, f_r$ has a *Bernstein-Sato polynomial* $b_f(s)$ which is divisible by $(s+r)$ in this case, and so we can consider the *reduced Bernstein-Sato polynomial* $\widetilde{b}_f(s) = b_f(s)/(s+r)$. Define

$$\widetilde{\alpha}_{\mathbb{Z}}(Z) = \min\{j \in \mathbb{Z} \mid \widetilde{b}_f(-j) = 0\},$$

and conventionally set $\widetilde{\alpha}_{\mathbb{Z}}(Z) = +\infty$ if there are no integer roots of $\widetilde{b}_f(s)$. For $x \in Z$, there are also local notions: $b_{f,x}(s) = \gcd_{x\in\mathcal{U}}(b_{f|_{\mathcal{U}}}(s))$ and $\widetilde{\alpha}_{\mathbb{Z},x}(f) = \min_{x\in\mathcal{U}} \widetilde{\alpha}_{\mathbb{Z}}(f|_{\mathcal{U}})$, where $\mathcal{U}$ varies over Zariski open neighborhoods of x. Regarding these invariants, we have the following:

Corollary L. *If Z is a hypersurface or has rational singularities, then*

(1) $\widetilde{\alpha}_{\mathbb{Z}}(Z) \leq p(Q^{\mathbb{Z}}, F) + n + r$,
(2) $\widetilde{\alpha}_{\mathbb{Z}}(Z) - r - 1 \leq \mathrm{HRH}(Z)$,
(3) $\widetilde{\alpha}_{\mathbb{Z},x}(Z) - r + 1 \leq \mathrm{Sp}_{\min,\mathbb{Z}}(Z, x)$.

If Z is a hypersurface, then $\widetilde{\alpha}_{\mathbb{Z}}(Z) = +\infty$ if and only if Z is a rational homology manifold. If Z has higher codimension, but has rational singularities, then $\widetilde{\alpha}_{\mathbb{Z}}(Z) = +\infty$ implies Z is a rational homology manifold.

In fact, we show a more precise statement, but it becomes rather technical and is discussed at the end of Section D below. We remark that even in the isolated hypersurface singularities case, it is possible to have strict inequality

$$\widetilde{\alpha}_{\mathbb{Z},x}(Z) < \mathrm{Sp}_{\min,\mathbb{Z}}(Z, x),$$

see [JKSY22, Rmk 3.4d]. Moreover, the converse to the last statement of the corollary is not true (Example 16.2).

Outline. Section B contains a review of the theory of mixed Hodge modules, the Specialization construction, the definition of the spectrum, and the definition of higher Du Bois and higher rational singularities.

Section C defines and studies the invariant $\mathrm{HRH}(Z)$. The proofs of Theorem B, Corollary C, and Theorem G (= Proposition 8.5) are given in §5. In the following §6, Theorem D (= Theorem 6.5), Theorem E, and Theorem F are proven. Moreover, an application to partial Poincaré duality on singular cohomology is given in §7 where we prove Theorem A (= Theorem 7.2). The following §8 contains a proof of Theorem G (= Proposition 8.5), the main observation being that $\mathrm{lcdef}_x(Z)$ is invariant under taking normal slices. The end of that section highlights some interesting behavior with references to the examples in §14.

Section D studies the integer invariants $\mathrm{HRH}(Z), \mathrm{Sp}_{\mathrm{min},\mathbb{Z}}(Z,x), \widetilde{\alpha}_{\mathbb{Z}}(Z)$ and $p(Q^{\mathbb{Z}}, F)$ when $Z = V(f_1, \dots, f_r) \subseteq X$ is defined by a regular sequence. The invariant $\mathrm{HRH}(Z)$ is characterized via the V-filtration and Theorem H is proven in §9. We prove Theorem I (= Proposition 11.1 and Proposition 11.4), Theorem K, and Corollary L in §11. The proof of Theorem J is contained in §12.

Section E provides examples of various features. Here we compute the Hodge rational homology levels of affine cones, determinantal varieties and other natural classes of examples. We also provide an example of a variety with $Q^{\mathbb{Z}} \neq 0$ but which is a rational homology manifold. Moreover, the case of varieties with k-liminal singularities (those which are k-Du Bois but not k-rational for some k) is studied in this section.

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B. Preliminaries

In this section, we give a brief overview of the background material needed in the rest of the paper. We will use without review the theory of perverse sheaves and $\mathscr{D}$-modules. For more information, see [BBD82] and [HTT08], respectively.

1. Mixed Hodge modules. The main objects used in this paper are mixed Hodge modules, defined by Saito [Sai88, Sai90]. We make the convention in this paper that all $\mathscr{D}$-modules are *left* modules, however, we will index the Hodge filtration following the conventions for right $\mathscr{D}$-modules. We will remind the reader about these conventions below, when necessary.

On a smooth complex algebraic variety X of dimension n, a *mixed Hodge module* consists of the data

$$M = (\mathcal{M}, F, W, (\mathcal{K}, W), \alpha)$$

where $\mathcal{M}$ is a regular holonomic $\mathscr{D}_X$-module, $F_\bullet \mathcal{M}$ is a good filtration on it, $W_\bullet \mathcal{M}$ is a finite filtration by $\mathscr{D}_X$-modules, $(\mathcal{K}, W)$ is an algebraically constructible $\mathbb{Q}$-perverse sheaf on X^{an} with a finite filtration $W_\bullet \mathcal{K}$, and α is a filtered isomorphism

$$\alpha \colon \mathbb{C} \otimes_{\mathbb{Q}} (\mathcal{K}, W) \to \mathrm{DR}_X^{\mathrm{an}}(\mathcal{M}, W)$$

of filtered $\mathbb{C}$-perverse sheaves. Recall that

$$\mathrm{DR}_X(\mathcal{M}) = \left[\mathcal{M} \xrightarrow{\nabla} \Omega_X^1 \otimes_{\mathscr{O}} \mathcal{M} \xrightarrow{\nabla} \dots \xrightarrow{\nabla} \omega_X \otimes_{\mathscr{O}} \mathcal{M}\right]$$

placed in degrees $-n, \dots, 0$, with filtration $W_i \mathrm{DR}_X(\mathcal{M}) = \mathrm{DR}_{\mathrm{X}}(W_i \mathcal{M})$. In a local choice of coordinates $x_1, \dots, x_n$ of X, the complex is the Koszul complex on the operators $\partial_{x_1}, \dots, \partial_{x_n}$.

The filtration $F_\bullet \mathcal{M}$ is the "Hodge filtration" and $W_\bullet \mathcal{M}$ is the "weight filtration". The $\mathbb{Q}$-perverse sheaf $\mathcal{K}$ is called the $\mathbb{Q}$-*structure*, and the functor $\mathrm{rat} \colon \mathrm{MHM}(X) \to \mathrm{Perv}(X)$ sending M to $\mathcal{K}$ is faithful.

These data are subject to various conditions, which we will not explain fully here. The essential idea is that mixed Hodge modules on a point should be exactly the graded-polarizable mixed Hodge structures, and then the definition for higher dimension varieties follows by induction on the dimension. The crucial constructions in the inductive step are the modules of *nearby and vanishing cycles*, explained in the §2.

The category $\mathrm{MHM}(X)$ is abelian. In fact, any morphism $\varphi\colon (\mathcal{M}, F, W) \to (\mathcal{N}, F, W)$ underlying a morphism of mixed Hodge modules is bi-strict with respect to F and W. We say a mixed Hodge module M is *pure of weight* w if $\mathrm{Gr}^W_i \mathcal{M} = 0$ for all $i \neq w$.

We let $D^b(\mathrm{MHM}(X))$ denote the bounded derived category of mixed Hodge modules on X.

The theory of mixed Hodge modules is endowed with a six functor formalism which, under the functor $\mathrm{rat}\colon D^b(\mathrm{MHM}(X)) \to D^b(\mathrm{Perv}(X))$ agrees with the six functor formalism on perverse sheaves and which agrees with the six functors on underlying $\mathscr{D}$-modules. In particular, given any morphism $f\colon X \to Y$ between smooth varieties, we have functors

$$f_*,\ f_!\colon D^b(\mathrm{MHM}(X)) \to D^b(\mathrm{MHM}(Y)),$$

$$f^*, f^!\colon D^b(\mathrm{MHM}(Y)) \to D^b(\mathrm{MHM}(X)),$$

with f^* left adjoint to f_*, $f_!$ left adjoint to $f^!$. Moreover, there is an exact functor

$$\mathbf{D}_X\colon \mathrm{MHM}(X)^{\mathrm{op}} \to \mathrm{MHM}(X)$$

so that $f^! = \mathbf{D}_X f^* \mathbf{D}_Y$, $f_! = \mathbf{D}_Y f_* \mathbf{D}_X$. The dual functor satisfies

$$\mathrm{Gr}^W_{-i} \mathbf{D}_X(M) \cong \mathbf{D}_X(\mathrm{Gr}^W_i M).$$

Using local embeddings into smooth varieties, the categories $\mathrm{MHM}(Z)$ and $D^b(\mathrm{MHM}(Z))$ make sense for an arbitrary complex variety Z, and admit six functor formalisms as described above. Similarly, the associated graded pieces $\mathrm{Gr}^F_p \mathrm{DR}_Z(M)$ give objects of $D^b_{\mathrm{coh}}(\mathscr{O}_Z)$ which are independent of the choice of local smooth embeddings.

For any two smooth varieties X, Y and for any $M \in \mathrm{MHM}(X)$, the functor

$$M \boxtimes -\colon \mathrm{MHM}(Y) \to \mathrm{MHM}(X \times Y)$$

is exact. On underlying filtered objects, it is given by convolution of filtrations: we have

$$F_k(\mathcal{M} \boxtimes \mathcal{N}) = \sum_{i+j=k} F_i\mathcal{M} \boxtimes F_j\mathcal{N},$$

$$W_k(\mathcal{M} \boxtimes \mathcal{N}) = \sum_{i+j=k} W_i\mathcal{M} \boxtimes W_j\mathcal{N}.$$

Example 1.1. For X a smooth variety of dimension n, the *trivial Hodge module* is

$$\mathbb{Q}^H_X[n] = (\mathscr{O}_X, F, W, \underline{\mathbb{Q}}_{X^{\mathrm{an}}}[n]),$$

where $\mathrm{Gr}^F_{-\bullet} \mathscr{O}_X = \mathrm{Gr}^W_\bullet \mathscr{O}_X = 0$ except for $\bullet = n$.

In general, given the trivial Hodge structure $\mathbb{Q}^H \in \mathrm{MHM}(\mathrm{pt})$, if $a_Z\colon Z \to \mathrm{pt}$ is the constant map, then the trivial Hodge module on Z is actually an object in $D^b(\mathrm{MHM}(Z))$ given by

$$\mathbb{Q}^H_Z = a_Z^* \mathbb{Q}^H,$$

which might have many non-zero cohomology modules and those modules may not be pure.

Example 1.2. ([Sai90, (4.4.2)]) Assume X and Y are smooth varieties. Let $p\colon X \times Y \to Y$ be the projection. Then the pullback functor p^* is given by $\mathbb{Q}^H_X \boxtimes -$.

Example 1.3. ([Sai90, (4.4.3)]) Consider a Cartesian diagram

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow{\scriptstyle g_Y} & & \downarrow{\scriptstyle g_X} \\ Y & \xrightarrow{f} & X \end{array} .$$

Then there are natural, canonical isomorphisms of functors

$$g_X^* f_! = f'_! g_Y^*, \quad g_X^! f_* = f'_* g_Y^!.$$

Example 1.4. For any $M \in \mathrm{MHM}(X)$ and $j \in \mathbb{Z}$, we can define another mixed Hodge module $M(j) \in \mathrm{MHM}(X)$, which is called the *Tate twist of M by j*. It has the same underlying $\mathscr{D}_X$-module $\mathcal{M}$, but the filtrations are shifted:

$$F_\bullet(\mathcal{M}(j)) = F_{\bullet-j}(\mathcal{M}),\, W_\bullet(\mathcal{M}(j)) = W_{\bullet+2j}\mathcal{M}.$$

If M is a pure Hodge module of weight w on X, then by definition it is *polarizable*, which implies that there exists an isomorphism of pure Hodge modules of weight $-w$:

$$\mathbf{D}_X(M) \cong M(w).$$

The Hodge filtration induces a filtration on $\mathrm{DR}_X(\mathcal{M})$ by

$$F_p\mathrm{DR}_X(\mathcal{M}) = \left[F_{p-n}\mathcal{M} \xrightarrow{\nabla} \Omega^1_X \otimes_{\mathscr{O}} F_{p-n+1}\mathcal{M} \xrightarrow{\nabla} \dots \xrightarrow{\nabla} \omega_X \otimes_{\mathscr{O}} F_p\mathcal{M}\right],$$

so that $\mathrm{Gr}^F_p\mathrm{DR}_X(\mathcal{M})$ is actually a bounded complex of coherent $\mathscr{O}$-modules with $\mathscr{O}$-linear differentials. The functor $\mathrm{Gr}^F_p\mathrm{DR}_X(-)$ extends to an exact functor

$$\mathrm{Gr}^F_p\mathrm{DR}_X(-)\colon D^b(\mathrm{MHM}(X)) \to D^b_{\mathrm{coh}}(\mathscr{O}_X),$$

where the right hand side is the bounded derived category of $\mathscr{O}_X$-modules with coherent cohomology.

Moreover, the functor is well-behaved under various operations, as given by the following proposition:

Proposition 1.5 ([Sai88, Lem. 2.3.6])**.** *Let $f\colon X \to Y$ be a proper morphism between smooth varieties and let $M^\bullet \in D^b(\mathrm{MHM}(X))$. Then, for any $p \in \mathbb{Z}$ there is a quasi-isomorphism*

$$\mathrm{Gr}^F_p\mathrm{DR}_Y(f_*(M^\bullet)) \cong Rf_*\mathrm{Gr}^F_p\mathrm{DR}_X(M^\bullet),$$

where Rf_* is the right derived functor of the usual $\mathscr{O}$-module push-forward.

Moreover, [Sai88, Sect. 2.4]

$$\mathbb{D}_X\mathrm{Gr}^F_p\mathrm{DR}_X(M^\bullet) \cong \mathrm{Gr}^F_{-p}\mathrm{DR}_X(\mathbf{D}_X(M^\bullet)),$$

where $\mathbb{D}_X(-) = R\mathcal{H}om_{\mathscr{O}_X}(-, \omega_X[\dim X])$ is the Grothendieck duality functor on X.

Given any object A with bounded below filtration $F_\bullet A$, we let $p(A, F) = \min\{p \mid F_pA \neq 0\}$. For M a mixed Hodge module, we let $p(M) = p(\mathcal{M}, F)$ where $F_\bullet\mathcal{M}$ is the Hodge filtration on the underlying $\mathscr{D}$-module. For $M^\bullet \in D^b(\mathrm{MHM}(X))$, we have $p(M^\bullet) = \min_{i\in\mathbb{Z}} p(\mathcal{H}^i(M^\bullet))$.

Lemma 1.6. *Let $M^\bullet \in D^b(\mathrm{MHM}(X))$. Then*

$$p(M^\bullet) = \min\{p \mid \mathrm{Gr}^F_p\mathrm{DR}_X(M^\bullet) \text{ is not acyclic.}\}$$

Proof. The argument is standard, but we include it for convenience of the reader.

We want to see that

$$F_k M^\bullet = 0 \text{ if and only if } \mathrm{Gr}^F_\ell \mathrm{DR}_X(M^\bullet) = 0 \text{ for all } \ell \leq k,$$

where the first expression is equivalent to saying $p(M^\bullet) > k$. Indeed, if we represent $M^\bullet$ by a bounded complex of morphisms of mixed Hodge modules, then

$$F_k \mathcal{H}^j(\mathcal{M}^\bullet) = \mathcal{H}^j(F_k\mathcal{M}^\bullet)$$

by strictness of morphisms.

We have the spectral sequence (shown using the standard truncation functors on $D^b(\mathrm{MHM}(X))$):

$${}^\ell E_2^{p,q} = \mathcal{H}^p \mathrm{Gr}^F_\ell \mathrm{DR}_X(\mathcal{H}^q M^\bullet) \implies \mathcal{H}^{p+q}\mathrm{Gr}^F_\ell \mathrm{DR}_X(M^\bullet).$$

Note that if $F_k M^\bullet = 0$ (meaning $F_k\mathcal{H}^j M^\bullet = 0$ for all $j \in \mathbb{Z}$), then ${}^\ell E_2^{p,q} = 0$ for all $\ell \leq k$ and all p, q. The spectral sequence then shows that $\mathrm{Gr}^F_\ell \mathrm{DR}_X(M^\bullet) = 0$ for all $\ell \leq k$, as desired.

Conversely, assume $\mathrm{Gr}^F_\ell \mathrm{DR}_X(M^\bullet) = 0$ for all $\ell \leq k$. Let $\sigma_0 = \min\{\sigma \mid F_\sigma M^\bullet \neq 0\}$, which is a finite value because $M^\bullet$ is a bounded complex (meaning there are only finitely many cohomology modules to consider). The claim is that $\sigma_0 > k$. If not, then $\sigma_0 \leq k$, and so by our assumed vanishing, we have

$${}^{\sigma_0}E_\infty^{p,q} = 0.$$

But for any fixed q, the only non-zero ${}^{\sigma_0}E_2^{p,q}$ is for $p = 0$. Indeed, the last few terms of the associated graded de Rham complex are

$$\cdots \to \Omega_X^{n-1} \otimes \mathrm{Gr}^F_{\sigma_0 - 1}\mathcal{H}^q(M^\bullet) \xrightarrow{d} \omega_X \otimes \mathrm{Gr}^F_{\sigma_0}\mathcal{H}^q(M^\bullet),$$

and by definition of σ_0, all the leftmost terms are 0. Thus, ${}^{\sigma_0}E_\infty^{p,q} = {}^{\sigma_0}E_2^{p,q} = 0$.

So we have reduced to checking the claim when M is a mixed Hodge module. But it is easy to see that $F_k M = 0$ if and only if $\mathrm{Gr}^F_\ell \mathrm{DR}_X M = 0$ for all $\ell \leq k$. ☐

Corollary 1.7. *Let $\psi \colon M^\bullet \to N^\bullet$ be a morphism in $D^b(\mathrm{MHM}(X))$. Then $F_k\psi$ is a quasi-isomorphism if and only if $\mathrm{Gr}^F_\ell \mathrm{DR}_X(\psi)$ is a quasi-isomorphism for all $\ell \leq k$.*

Proof. Apply Lemma 1.6 to the cone of the morphism ψ in $D^b(\mathrm{MHM}(X))$. ☐

The following two lemmas are proven in a way similar to [Sai90, Rmk. 4.6(1)]. The main idea is to establish the result for variations of mixed Hodge structures and to use induction on the dimension of the support.

Lemma 1.8. *Let $M^\bullet \in D^b(\mathrm{MHM}(Z))$. Then*

$$p(M^\bullet) \geq j \iff p(i_x^! M^\bullet) \geq j \text{ for all } x \in Z.$$

Proof. The implication

$$p(M^\bullet) \geq j \implies p(i_x^! M^\bullet) \geq j \text{ for all } x \in Z$$

is obvious, by definition of the functor $i_x^!$ for mixed Hodge modules (see, for example, Proposition 2.6 below).

To prove the converse, assume for all x that $p(i_x^! M^\bullet) \geq j$.

We use induction on $\dim(Z)$. For $\dim(Z) = 0$, the claim is obvious.

For $\dim(Z) > 0$, there exists a Zariski open cover $Z = \bigcup_{\alpha \in I} U_\alpha$ such that, for each $\alpha \in I$, there exists $g_\alpha \in \mathscr{O}(U_\alpha)$ so that the subset $U'_\alpha = \{g_\alpha \neq 0\} \subseteq U_\alpha$ is a smooth and dense open and with the property that the restriction $\mathcal{H}^i(M)|_{U'_\alpha}$ is a variation of mixed Hodge structures for all $i \in \mathbb{Z}$.

It suffices to prove the claim locally, so we can replace Z with U_α. We have reduced to the case that there exists $g \in \mathscr{O}_Z(Z)$ such that $U' = \{g \neq 0\} \subseteq Z$ is a smooth, Zariski open dense subset so that $\mathcal{H}^i(M)|_{U'}$ is a variation of mixed Hodge structures for all $i \in \mathbb{Z}$.

Let $j\colon U' \to Z$ and $i\colon \{g = 0\} \to Z$ be the natural embeddings, with exact triangle

$$i_* i^! M^\bullet \to M^\bullet \to j_*(M^\bullet|_{U'}) \xrightarrow{+1} .$$

If $p(M^\bullet) < j$, then either $p(i_* i^! M^\bullet) < j$ or $p(j_*(M^\bullet|_{U'})) < j$. Note that for all $x \in \{g = 0\}$, we have

$$i_x^! i_* i^! M^\bullet = i_x^! M^\bullet,$$

and so by induction on the dimension we conclude that $p(i_* i^! M^\bullet) \geq j$.

For all $x \in U'$, we have

$$i_x^! M^\bullet = i_x^! j_*(M^\bullet) = \iota_x^!(M^\bullet|_{U'}),$$

where $\iota_x\colon \{x\} \to U'$ is the inclusion. We see using the spectral sequence

$$E_2^{p,q} = \mathcal{H}^p \iota_x^! \mathcal{H}^q(M^\bullet|_{U'}) \implies \mathcal{H}^{p+q} \iota_x^!(M^\bullet|_{U'}),$$

and the fact that ι_x is non-characteristic for each cohomology module $\mathcal{H}^q(M^\bullet|_{U'})$ (implying that the spectral sequence degenerates at E_2) that $p(M^\bullet|_{U'}) \geq j$, and so $p(j_*(M^\bullet|_{U'})) \geq j$, too.

Thus, we have shown that $p(M^\bullet) \geq j$. □

Lemma 1.9. *Let $M \in \mathrm{MHM}(Z)$. Then for any $x \in Z$, we have $\mathcal{H}^j i_x^! M = 0$ for all $j > \dim Z$.*

Proof. Fix $x \in Z$. The claim is obvious if $\dim Z = 0$.

As above, we will use the definition of mixed Hodge modules in [Sai13]. We can replace Z by a Zariski open neighborhood of x in Z because the question is local near x. Take such a neighborhood U such that there exists a function $g \in \mathscr{O}(U)$ with the property that on $U' = \{g \neq 0\}$, the module M restricts to a variation of mixed Hodge structures.

We have the exact triangle

$$i_* i^! M \to M \to j_*(M|_{U'}) \xrightarrow{+1} .$$

If $x \in U'$, then we have

$$i_x^! M = i_x^! j_*(M|_{U'}) = i_{x,U'}^!(M|_{U'})$$

and the claim is obvious as U' is smooth of dimension $\dim(Z)$.

If $x \notin U'$, then

$$i_x^! i_* i^! M = i_x^! M$$

and so we can use induction on the dimension, using that each cohomology of $i_* i^! M$ is supported on $\{g = 0\}$ which has strictly smaller dimension than Z. Then one uses induction on the dimension of the support and the spectral sequence:

$$E_2^{i,j} = \mathcal{H}^i i_x^! \mathcal{H}^j(i_* i^! M) \implies \mathcal{H}^{i+j} i_x^! M.$$

Note that as i is the inclusion of a divisor, $E_2^{i,j} \neq 0$ implies $j = 0, 1$. Thus, by induction on the dimension of the support, $E_2^{i,j} \neq 0$ implies $i + j \leq 1 + \dim\{g = 0\} = \dim Z$. □

Lemma 1.10. *Let $M^\bullet \in D^b(\mathrm{MHM}(Z))$ be such that there exists $c \in \mathbb{Z}, \ell \in \mathbb{Z}_{\geq 0}$ such that*

$$\dim \mathrm{Supp}(\mathcal{H}^{i+c} M^\bullet) \leq \ell - i.$$

Then $\mathcal{H}^{j+c} i_x^! M^\bullet = 0$ for all $j > \ell$ and all $x \in Z$.

Proof. By shifting $M^\bullet$ we can assume $c = 0$. By the spectral sequence

$$E_2^{i,j} = \mathcal{H}^i i_x^! \mathcal{H}^j M^\bullet \implies \mathcal{H}^{i+j} i_x^! M^\bullet,$$

we want to show that $\mathcal{H}^i i_x^! \mathcal{H}^j M^\bullet = 0$ for all $i + j > \ell$. By the previous lemma and the assumption on $\dim \mathrm{Supp}(\mathcal{H}^j M^\bullet)$, we get the desired vanishing. □

2. V-filtration, specialization and spectrum. To define nearby and vanishing cycles for mixed Hodge modules, Saito uses the V-filtration of Kashiwara and Malgrange. As this is arguably the most important construction for what follows, we remind the reader of its definition.

For the smooth variety X, let $T = X \times \mathbb{A}^r_t$ be the trivial vector bundle over X with fiber coordinates $t_1, \dots, t_r$. We have $\mathscr{D}_T = \mathscr{D}_X\langle t_1, \dots, t_r, \partial_{t_1}, \dots, \partial_{t_r}\rangle$, where as usual $[\partial_{t_i}, t_j] = \delta_{ij}$, the Kronecker delta. This ring carries a $\mathbb{Z}$-indexed, decreasing filtration

$$V^\bullet \mathscr{D}_T = \left\{ \sum_{\beta,\gamma} P_{\beta,\gamma} t^\beta \partial_t^\gamma \mid P_{\beta,\gamma} \in \mathscr{D}_X, \, |\beta| \geq |\gamma| + \bullet \right\},$$

so that, for example, $t_i \in V^1\mathscr{D}_T$, $\partial_{t_j} \in V^{-1}\mathscr{D}_T$, and

$$V^j \mathscr{D}_T \cdot V^k \mathscr{D}_T \subseteq V^{j+k}\mathscr{D}_T.$$

If $\mathcal{M}$ is a regular holonomic $\mathscr{D}_T$-module underlying a mixed Hodge module, then it admits a $\mathbb{Q}$-indexed V-filtration along $(t_1, \dots, t_r)$. This is the unique exhaustive, decreasing, $\mathbb{Q}$-indexed filtration $(V^\alpha \mathcal{M})_{\alpha \in \mathbb{Q}}$ which is discrete[1] and left continuous[2], and which satisfies the following properties:

(1) For any $\alpha \in \mathbb{Q}, j \in \mathbb{Z}$, we have containment $V^j\mathscr{D}_T \cdot V^\alpha \mathcal{M} \subseteq V^{\alpha+j}\mathcal{M}$.
(2) For $\alpha \gg 0, j \in \mathbb{Z}_{\geq 0}$, we have equality $V^j\mathscr{D}_T \cdot V^\alpha \mathcal{M} = V^{\alpha+j}\mathcal{M}$.
(3) For all $\alpha \in \mathbb{Q}$, the $V^0\mathscr{D}_T$-module $V^\alpha\mathcal{M}$ is coherent.
(4) For $s = -\sum_{i=1}^r \partial_{t_i} t_i$ and for any $\alpha \in \mathbb{Q}$, there exists some $N \gg 0$ such that

$$(s+\alpha)^N V^\alpha \mathcal{M} \subseteq \bigcup_{\beta > \alpha} V^\beta \mathcal{M} = V^{>\alpha}\mathcal{M}.$$

In other words, $s + \alpha$ is nilpotent on $\mathrm{Gr}_V^\alpha(\mathcal{M}) = V^\alpha\mathcal{M}/V^{>\alpha}\mathcal{M}$.

Example 2.1. If $\sigma\colon X \to X \times T$ is the inclusion of the zero section, then for any mixed Hodge module N on X, the push forward $\sigma_* N$ underlies a mixed Hodge module on T. Its $\mathscr{D}_T$-module can be written

$$\sigma_+ \mathcal{N} = \bigoplus_{\alpha \in \mathbb{N}^r} \mathcal{N} \partial_t^\alpha \delta_0,$$

where δ_0 is a formal symbol which is annihilated by $t_1, \dots, t_r$. Then

$$V^\lambda \sigma_+ \mathcal{N} = \bigoplus_{|\alpha| = 0}^{\lfloor -\lambda \rfloor} \mathcal{N} \partial_t^\alpha \delta_0.$$

[1]Meaning there exists $\{\alpha_j\}_{j \in \mathbb{Z}}$ with $\lim_{j \to \pm\infty} \alpha_j = \pm\infty$ so that $V^\alpha\mathcal{M}$ is constant for $\alpha \in (\alpha_j, \alpha_{j+1})$.
[2]Meaning $V^\alpha \mathcal{M} = \bigcap_{\beta < \alpha} V^\beta \mathcal{M}$.

For $(\mathcal{M}, F)$ a filtered $\mathscr{D}_T$-module underlying a mixed Hodge module on T, we define Koszul-like complexes

$$F_pA^\chi(\mathcal{M}) = \left[F_pV^\chi\mathcal{M} \xrightarrow{t} \bigoplus_{i=1}^r F_pV^{\chi+1}\mathcal{M} \xrightarrow{t} \dots \xrightarrow{t} F_pV^{\chi+r}\mathcal{M}\right],$$

$$F_pB^\chi(\mathcal{M}) = F_pA^\chi(\mathcal{M})/F_pA^{>\chi}(\mathcal{M}) = \left[F_p\mathrm{Gr}_V^\chi(\mathcal{M}) \xrightarrow{t} \bigoplus_{i=1}^r F_p\mathrm{Gr}_V^{\chi+1}(\mathcal{M}) \xrightarrow{t} \dots \xrightarrow{t} F_p\mathrm{Gr}_V^{\chi+r}(\mathcal{M})\right],$$

$$F_pC^\chi(\mathcal{M}) = \left[F_{p-r}\mathrm{Gr}_V^{\chi+r}(\mathcal{M}) \xrightarrow{\partial_t} \bigoplus_{i=1}^r F_{p-r+1}\mathrm{Gr}_V^{\chi+r-1}(\mathcal{M}) \xrightarrow{\partial_t} \dots \xrightarrow{\partial_t} F_p\mathrm{Gr}_V^\chi(\mathcal{M})\right].$$

The last condition in the definition of the V-filtration leads to the following acyclicity results for these complexes.

Proposition 2.2 ([CD23, Thm. 3.1, 3.2])**.** *For all $\chi \neq 0$, the complexes $B^\chi(\mathcal{M}), C^\chi(\mathcal{M})$ are acyclic. For $\chi > 0$, the complex $A^\chi(\mathcal{M})$ is acyclic.*

When $r = 1$, this acyclicity means we have isomorphisms

$$t\colon V^\alpha\mathcal{M} \cong V^{\alpha+1}\mathcal{M} \text{ for } \alpha > 0$$
$$t\colon \mathrm{Gr}_V^\alpha(\mathcal{M}) \cong \mathrm{Gr}_V^{\alpha+1}(\mathcal{M}) \text{ for } \alpha \neq 0$$
$$\partial_t\colon \mathrm{Gr}_V^{\alpha+1}(\mathcal{M}) \cong \mathrm{Gr}_V^\alpha(\mathcal{M}) \text{ for } \alpha \neq 0.$$

For $r = 1$, one of the properties which the filtered module $(\mathcal{M}, F)$ must satisfy to underlie a mixed Hodge module on T is that these isomorphisms are *filtered isomorphisms* in certain ranges: specifically, Saito imposes that

$$t\colon F_pV^\alpha\mathcal{M} \cong F_pV^{\alpha+1}\mathcal{M} \text{ for } \alpha > 0$$
$$\partial_t\colon F_p\mathrm{Gr}_V^{\alpha+1}(\mathcal{M}) \cong F_{p+1}\mathrm{Gr}_V^\alpha(\mathcal{M}) \text{ for } \alpha < 0.$$

Some immediate consequences of these conditions are the following:

Proposition 2.3 ([Sai88, Sect. 3])**.** *Let $(\mathcal{M}, F)$ underlie a mixed Hodge module on $T = X \times \mathbb{A}^1_t$. Let $j\colon X \times \mathbb{G}_m \to X \times \mathbb{A}^1_t$ be the inclusion of the complement of the zero section. Then*

(1) For any $\lambda > 0$ and $p \in \mathbb{Z}$, we have

$$F_pV^\lambda\mathcal{M} = V^\lambda\mathcal{M} \cap j_*(j^*(F_p\mathcal{M})). \tag{2.4}$$

(2) For all $p \in \mathbb{Z}$, we have

$$F_p\mathcal{M} = \sum_{i \geq 0} \partial_t^i(F_{p-i}V^0\mathcal{M}).$$

If $\mathcal{M}$ has no sub-module supported on $\{t = 0\}$, then the equality (2.4) holds with $\lambda = 0$. In this case, we have

$$F_p\mathcal{M} = \sum_{i \geq 0} \partial_t^i(V^0\mathcal{M} \cap j_*j^*(F_{p-i}\mathcal{M})).$$

For $r > 1$, filtered acyclicity still holds in the corresponding ranges.

Proposition 2.5 ([CD23, CDS23])**.** *For all $\chi < 0$, the complex $C^\chi(\mathcal{M}, F)$ is filtered acyclic. For $\chi > 0$, the complexes $A^\chi(\mathcal{M}, F)$ and $B^\chi(\mathcal{M}, F)$ are filtered acyclic.*

The complexes $B^0(\mathcal{M}), C^0(\mathcal{M})$ are related to the restriction functors for mixed Hodge modules.

Proposition 2.6 ([CD23, CDS23])**.** *Let $M \in \mathrm{MHM}(T)$. Then $B^0(\mathcal{M}, F), C^0(\mathcal{M}, F)$ are strictly filtered complexes.*

Let $\sigma\colon X \to X \times \mathbb{A}^r_t$ be the zero section. We have filtered isomorphisms

$$F_p\mathcal{H}^j\sigma^!(\mathcal{M}) \cong F_p\mathcal{H}^j B^0(\mathcal{M}) \text{ for all } j \in [0, r],$$

$$F_p\mathcal{H}^j\sigma^*(\mathcal{M}) \cong F_p\mathcal{H}^j C^0(\mathcal{M}) \text{ for all } j \in [-r, \dots, 0].$$

The results of this paper make use of the Verdier specialization functor, which we review here. For details, see [Sai90, BMS06, CD23].

Let $Z \subseteq X$ be a (possibly singular) subvariety defined by $f_1, \dots, f_r \in \mathscr{O}_X(X)$. This defines a graph embedding $\Gamma\colon X \to T$ by $x \mapsto (x, f_1(x), \dots, f_r(x))$. Given M a mixed Hodge module on X, we obtain $\Gamma_* M$ a mixed Hodge module on T.

The *Verdier specialization* of M along Z (or, of $\Gamma_* M$ along $X \times \{0\}$) is a mixed Hodge module on $X \times \mathbb{A}^r_z$, where $z_1, \dots, z_r$ are the fiber coordinates. The module is denoted $\mathrm{Sp}_Z(M) = \mathrm{Sp}(\Gamma_*(M))$. Its underlying filtered $\mathscr{D}$-module is

$$F_p\mathrm{Sp}(\Gamma_*(\mathcal{M})) = \bigoplus_{\chi \in \mathbb{Q}} F_p\mathrm{Gr}_V^\chi(\Gamma_*(\mathcal{M})),$$

where $V^\bullet\Gamma_*(\mathcal{M})$ is the V-filtration along $t_1, \dots, t_r$ and $F_p\mathrm{Gr}_V^\chi(\Gamma_*(\mathcal{M})) = \frac{F_pV^\chi\Gamma_*(\mathcal{M})}{F_pV^{>\chi}\Gamma_*(\mathcal{M})}$. The $\mathscr{D}$-module action is given on $\overline{m} \in \mathrm{Gr}_V^\chi(\Gamma_*(\mathcal{M}))$ by

$$P\overline{m} = \overline{Pm}, \text{ for } P \in \mathscr{D}_X,$$

$$z_i\overline{m} = \overline{t_i m} \in \mathrm{Gr}_V^{\chi+1}(\Gamma_*(\mathcal{M})),$$

$$\partial_{z_i}\overline{m} = \overline{\partial_{t_i} m} \in \mathrm{Gr}_V^{\chi-1}(\Gamma_*(\mathcal{M})).$$

This gives an example of a *monodromic mixed Hodge module* on $X \times \mathbb{A}^r_z$. Recall that a mixed Hodge module is monodromic if its underlying $\mathscr{D}$-module is, which means that it decomposes into generalized eigenspaces for the Euler operator $\theta_z = \sum_{i=1}^r z_i\partial_{z_i}$. If $\mathcal{N}$ is monodromic, for any $\chi \in \mathbb{Q}$, we let

$$\mathcal{N}^\chi = \bigcup_{j \geq 1} \ker((\theta_z - \chi + r)^j), \text{ so that } \mathcal{N} = \bigoplus_{\chi \in \mathbb{Q}} \mathcal{N}^\chi.$$

The V-filtration along $z_1, \dots, z_r$ is particularly easy to understand for monodromic modules: indeed, it is given by

$$V^\lambda\mathcal{N} = \bigoplus_{\chi \geq \lambda} \mathcal{N}^\chi, \quad \mathrm{Gr}_V^\lambda(\mathcal{N}) \cong \mathcal{N}^\lambda.$$

We have particular interest in the case $M = \mathbb{Q}_X^H[n]$. We let $\Gamma_*(\mathbb{Q}_X^H[n]) = B_f$ in this case for ease of notation. For $Z \subseteq X$ a complete intersection (meaning $f_1, \dots, f_r \in \mathscr{O}_X(X)$ form a regular sequence), the module $\mathrm{Sp}(B_f)$ admits a morphism $L \to \mathrm{Sp}(B_f)$, where $L = i_*\mathbb{Q}^H_{Z \times \mathbb{A}^r_z}[n]$ is the trivial Hodge module on $i\colon Z \times \mathbb{A}^r_z \hookrightarrow X \times \mathbb{A}^r_z$.

Lemma 2.7. *Let*

$$(\mathcal{K}, F) = \ker((\mathrm{Gr}_V^r(\mathcal{B}_f), F[r]) \xrightarrow{\partial_{t_i}} \bigoplus_{i=1}^r (\mathrm{Gr}_V^{r-1}(\mathcal{B}_f), F[r-1])).$$

Then $(\mathcal{K}, F)$ *underlies* $i_*\mathbb{Q}_Z^H[n-r]$.

Moreover, the $\mathscr{D}$*-module* $\mathcal{L} = \mathcal{K} \boxtimes \mathscr{O}_{\mathbb{A}_z^r} = \mathcal{K}[z_1, \dots, z_r]$ *underlies* L*, with filtration given by*

$$F_p\mathcal{L} = (F_{p+r}\mathcal{K})[z_1, \dots, z_r].$$

Proof. The first claim follows by applying Proposition 2.6 to $B_f = \Gamma_*(\mathbb{Q}_X^H[n])$, using Base Change (Example 1.3) to see that

$$\sigma^*\Gamma_*(\mathbb{Q}_X^H[n]) = i_*i^*\mathbb{Q}_X^H[n] = i_*\mathbb{Q}_Z^H[n].$$

The second claim follows by definition, using Example 1.2. □

The map $L \to \mathrm{Sp}(B_f)$ is injective, and we let Q be the cokernel of the map. Then we have a short exact sequence of monodromic mixed Hodge modules on $X \times \mathbb{A}_z^r$:

$$0 \to L \to \mathrm{Sp}(B_f) \to Q \to 0.$$

The monodromic pieces of $\mathcal{L}$ satisfy $\mathcal{L}^\chi = 0$ for $\chi \notin \mathbb{Z}_{\geq r}$. Thus, we see that $\mathrm{Sp}(\mathcal{B}_f)^\chi = \mathcal{Q}^\chi$ for all $\chi \notin \mathbb{Z}_{\geq r}$. The interesting part of this short exact sequence is then

$$0 \to L \to \mathrm{Sp}(B_f)^{\mathbb{Z}} \to Q^{\mathbb{Z}} \to 0 \tag{2.8}$$

where for any monodromic mixed Hodge module, we use the superscript $\mathbb{Z}$ to denote the "unipotent part", which is the direct sum of the monodromic pieces with integer indices. We also use a superscript $\alpha + \mathbb{Z}$ to denote the filtered direct summand of a monodromic module which is obtained by collecting all summands with indices in $\alpha + \mathbb{Z}$.

For $Z \subseteq X$ a local complete intersection subvariety, we review the definition of the spectrum of Z at $x \in Z_{\mathrm{sing}}$ due to Dimca, Maisonobe and Saito. For details, consult [DMS11] and [Dir23].

Using the monodromy endomorphism and the mixed Hodge structure on the cohomology of the Milnor fiber, Steenbrink [Ste89] defined, in the isolated hypersurface singularities case, the *spectrum* of the hypersurface singularity. This is an invariant of the singularity given by a multiset of positive rational numbers, encoding the eigenspaces of the monodromy operator.

In [DMS11], Dimca, Maisonobe and Saito defined the spectrum for any variety Z at a point $x \in Z$ using the theory of mixed Hodge modules. The definition is rather technical, but when we assume Z is a local complete intersection variety it is slightly simpler.

Let $Z \subseteq X$ be a local complete intersection subvariety of pure codimension r. Let $x \in Z$, and locally around x we can write $Z = V(f_1, \dots, f_r)$ where $f_1, \dots, f_r \in \mathscr{O}_X(X)$ form a regular sequence. Let $\xi \in \{x\} \times \mathbb{A}_z^r$ be a sufficiently general element, and set $i_\xi \colon \{\xi\} \to X \times \mathbb{A}_z^r$ to be the inclusion. Then define the *non-reduced Spectrum of* Z *at* x by

$$\widehat{\mathrm{Sp}}(Z, x) = \sum_{\alpha \in \mathbb{Q}_{>0}} m_{\alpha,x} t^\alpha,$$

where

$$m_{\alpha,x} = \sum_{k \in \mathbb{Z}} (-1)^k \dim_{\mathbb{C}} \mathrm{Gr}^F_{\lceil \alpha \rceil - \dim Z - 1} \mathcal{H}^{k-r} i_\xi^*(\mathrm{Sp}(\mathcal{B}_f)^{\alpha+\mathbb{Z}}),$$

and we define the *reduced spectrum* by

$$\mathrm{Sp}(Z, x) = \widehat{\mathrm{Sp}}(Z, x) + (-t)^{\dim Z + 1}.$$

This definition clearly extends to an arbitrary monodromic module M, where we write $m_{\alpha,x}(M)$ for the alternating sum

$$m_{\alpha,x}(M) = \sum_{k\in\mathbb{Z}} (-1)^k \dim_{\mathbb{C}} \mathrm{Gr}^F_{\lceil\alpha\rceil-\dim Z-1} \mathcal{H}^{k-r} i_\xi^*(\mathcal{M}^{\alpha+\mathbb{Z}}).$$

Moreover, if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence of monodromic mixed Hodge modules, then we see that

$$\widehat{\mathrm{Sp}}(M_2, x) = \widehat{\mathrm{Sp}}(M_1, x) + \widehat{\mathrm{Sp}}(M_3, x).$$

For M a monodromic mixed Hodge module, we let its *integer spectrum* be denoted

$$\widehat{\mathrm{Sp}}_{\mathbb{Z}}(M, x) = \sum_{j\in\mathbb{Z}_{>0}} m_{j,x}(M) t^j.$$

Applying the additivity to the short exact sequence (2.8) we get

$$\widehat{\mathrm{Sp}}(Q, x) = \widehat{\mathrm{Sp}}(Z, x) - \widehat{\mathrm{Sp}}(L, x),$$

and in particular,

$$\widehat{\mathrm{Sp}}_{\mathbb{Z}}(Q, x) = \widehat{\mathrm{Sp}}_{\mathbb{Z}}(Z, x) - \widehat{\mathrm{Sp}}(L, x),$$

which by an easy computation of the non-reduced spectrum of L, gives

$$\widehat{\mathrm{Sp}}(Q, x) = \mathrm{Sp}(Z, x).$$

We end this subsection by stating a criterion for vanishing of integer spectral numbers, which is a special case of [Dir23, Lem. 2.7]:

Lemma 2.9. *Let M be a monodromic mixed Hodge module. Assume $F_{p-1-\dim X}\mathcal{M}^{\mathbb{Z}} = 0$. Then for all $j \in \mathbb{Z}_{<p}$, we have $m_{j,x}(M) = 0$.*

3. Poincaré duality and higher singularities.

Let Z be a pure d-dimensional complex variety. The trivial Hodge module $\mathbb{Q}_Z^H \in D^b(\mathrm{MHM}(Z))$ satisfies the property that if $a_Z \colon Z \to \mathrm{pt}$ is the constant map, then

$$\mathcal{H}^j(a_Z)_*\mathbb{Q}_Z^H \in \mathrm{MHM}(\mathrm{pt}) = \mathrm{MHS}$$

gives Deligne's mixed Hodge structure on the cohomology $H^j(Z,\mathbb{Q})$. Moreover,

$$\mathcal{H}^j(a_Z)_!\mathbb{Q}_Z^H \in \mathrm{MHS}$$

gives the canonical mixed Hodge structure on compactly supported cohomology $H^j_c(Z,\mathbb{Q})$.

Although $\mathbb{Q}_Z^H[d] \in D^b(\mathrm{MHM}(Z))$ is not necessarily a single mixed Hodge module, it is known (for example, by comparing to the underlying perverse sheaf) that $\mathcal{H}^j(\mathbb{Q}_Z^H[d]) \neq 0$ implies $j \le 0$. Moreover, $\mathcal{H}^0(\mathbb{Q}_Z^H[d]) \in \mathrm{MHM}(Z)$ satisfies the property that $\mathrm{Gr}^W_d\mathcal{H}^0(\mathbb{Q}_Z^H[d])$ has underlying perverse sheaf equal to IC_Z, the intersection complex perverse sheaf of [BBD82]. By the weight formalism for mixed Hodge modules, we know that $\mathrm{Gr}^W_\ell\mathcal{H}^0(\mathbb{Q}_Z^H[d]) \neq 0$ implies $\ell \le d$.

We write

$$\mathrm{IC}_Z^H = \mathrm{Gr}^W_d\mathcal{H}^0(\mathbb{Q}_Z^H[d]),$$

and so in particular, there is a natural epimorphism

$$\gamma_Z \colon \mathcal{H}^0(\mathbb{Q}_Z^H[d]) \to \mathrm{IC}_Z^H.$$

Being a pure, polarizable Hodge module of weight d means there exists an isomorphism

$$\mathbf{D}_Z(\mathrm{IC}_Z^H) \cong \mathrm{IC}_Z^H(d),$$

where $\mathbf{D}_Z$ is the duality of mixed Hodge modules on Z. On underlying perverse sheaves, it is Verdier duality.

By duality, we can also realize IC_Z^H as $\mathrm{Gr}_d^W(\mathcal{H}^0(\mathbf{D}_Z(\mathbb{Q}_Z^H[d])(-d)))$. Again by the weight formalism, we see that IC_Z^H can be written as $W_d(\mathcal{H}^0(\mathbf{D}_Z(\mathbb{Q}_Z^H[d])(-d)))$, and in particular, we have a monomorphism

$$\gamma_Z^\vee \colon \mathrm{IC}_Z^H \to \mathcal{H}^0(\mathbf{D}_Z(\mathbb{Q}_Z^H[d])(-d)),$$

which, up to Tate twist, is dual to γ_Z.

The composition $\gamma_Z^\vee \circ \gamma_Z$ gives a morphism

$$\mathcal{H}^0(\mathbb{Q}_Z^H[d]) \to \mathcal{H}^0(\mathbf{D}_Z(\mathbb{Q}_Z^H[d])(-d)),$$

which uniquely determines a morphism in $D^b(\mathrm{MHM}(Z))$:

$$\psi_Z \colon \mathbb{Q}_Z^H[d] \to \mathbf{D}_Z(\mathbb{Q}_Z^H[d])(-d).$$

Remark 3.1. Saito [Sai90, (4.5.14)] shows that the morphism ψ_Z is unique up to scalar multiplication on each irreducible component of Z.

If the morphism ψ_Z is a quasi-isomorphism, we say that the variety Z is a *rational* (or $\mathbb{Q}$-)*homology manifold* (this notion is also called *rational smoothness* in the literature). This can be checked on underlying perverse sheaves, and so does not require the theory of Hodge modules.

The morphism ψ_Z has recently found applications in the study of so-called "higher singularities". The reason for this comes from the connection with the Du Bois complex of the variety Z. We will not review the definition of the Du Bois complex here (aside from its connection to mixed Hodge modules), for such a review, see [MP22, SVV23, PS24].

For Z a pure d-dimensional variety, the p-th Du Bois complex is $\underline{\Omega}_Z^p \in D^b_{\mathrm{coh}}(\mathscr{O}_Z)$. An important consequence of the definition is that there is a natural comparison morphism in $D^b_{\mathrm{coh}}(\mathscr{O}_Z)$:

$$\alpha_p \colon \Omega_Z^p \to \underline{\Omega}_Z^p,$$

for all $0 \leq p \leq \dim(Z)$, where Ω_Z^p is the sheaf of Kähler differentials. Then α_p is a quasi-isomorphism for all p if Z is smooth.

Recall that $\mathrm{Gr}_\bullet^F \mathrm{DR}_Z(\mathbb{Q}_Z^H) \in D^b_{\mathrm{coh}}(\mathscr{O}_Z)$ can be defined using local embeddings into smooth varieties. Then there is a natural quasi-isomorphism ([Sai99]):

$$\underline{\Omega}_Z^p[d-p] \cong \mathrm{Gr}_{-p}^F \mathrm{DR}_Z(\mathbb{Q}_Z^H[d]),$$

and so we can use the theory of mixed Hodge modules to try to understand these Du Bois complexes.

By applying Grothendieck duality $\mathbb{D}_Z(-) = R\mathcal{H}om_{\mathscr{O}_Z}(-, \omega_Z^\bullet)$, where $\omega_Z^\bullet$ is the dualizing complex on Z, we can write (using Proposition 1.5)

$$\mathbb{D}_Z(\underline{\Omega}_Z^p[d-p]) = \mathrm{Gr}_p^F \mathrm{DR}_Z(\mathbf{D}_Z(\mathbb{Q}_Z^H[d])),$$

and using $d-p$ in place of p, we get

$$\mathbb{D}_Z(\underline{\Omega}_Z^{d-p}[p]) = \mathrm{Gr}_{d-p}^F \mathrm{DR}_Z(\mathbf{D}_Z(\mathbb{Q}_Z^H[d])) = \mathrm{Gr}_{-p}^F \mathrm{DR}_Z(\mathbf{D}_Z(\mathbb{Q}_Z^H[d])(-d)).$$

Thus, applying $\mathrm{Gr}^F_{-p}\mathrm{DR}_Z$ to the morphism ψ_Z, we obtain natural morphisms

$$\underline{\Omega}_Z^p[d-p] \cong \mathrm{Gr}^F_{-p}\mathrm{DR}_Z(\mathbb{Q}_Z^H[d])) \to \mathrm{Gr}^F_{-p}\mathrm{DR}_Z(\mathbf{D}_Z(\mathbb{Q}_Z^H[d])(-d)) \cong \mathbb{D}_Z(\underline{\Omega}_Z^{d-p}[p])$$

which are identified with $\phi^p[d-p]$ in the notation of [FL24b] and the introduction.

We now define the classes of higher singularities, though we focus on the case when Z has local complete intersection singularities.

Definition 3.2 ([JKSY22, FL24b])**.** A pure d-dimensional variety Z with isolated or local complete intersection singularities has k-Du Bois singularities if for all $p \leq k$, the maps $\alpha_p \colon \Omega_Z^p \to \underline{\Omega}_Z^p$ are quasi-isomorphisms.

Such a variety Z has k-rational singularities if it has k-Du Bois singularities and if, for all $p \leq k$, the maps $\phi^p \colon \underline{\Omega}_Z^p \to \mathbb{D}_Z(\underline{\Omega}_Z^{d-p})[-d]$ are quasi-isomorphisms. Equivalently, Z has k-rational singularities if for all $p \leq k$, the composition $\phi^p \circ \alpha_p$ is a quasi-isomorphism.

Example 3.3. If Z is smooth, then we mentioned above that α_p is a quasi-isomorphism for all p. But then the morphism

$$\underline{\Omega}_Z^p \to \mathbb{D}_Z(\underline{\Omega}_Z^{d-p})[-d]$$

is the well-known isomorphism of locally free sheaves

$$\Omega_Z^p \cong \mathcal{H}om_{\mathscr{O}_Z}(\Omega_Z^{d-p}, \omega_Z).$$

Thus, smooth varieties are k-rational for all k.

Remark 3.4 (Non-LCI Setting)**.** In the non-local complete intersection case, these notions have been studied for isolated singularities by [FL24a], and in general by [SVV23] (see also [ORS24, Tig24] for some interesting examples).

There are several relevant notions: in [SVV23], the definitions of k-Du Bois and k-rational that we gave above are called *strict* k-Du Bois and *strict* k-rational, respectively. However, comparing with the Kähler differentials is rather restrictive: there are not many examples of such singularities which are not local complete intersection.

Without comparing to the Kähler differentials, there is the notion of *pre-k-*Du Bois and *pre-k-*rational. Following [SVV23], pre-k-Du Bois is the condition that $\mathcal{H}^i(\underline{\Omega}_Z^p) = 0$ for all $i > 0$ and $p \leq k$, and pre-k-rational is the condition that $\mathcal{H}^i(\mathbb{D}_Z(\underline{\Omega}_Z^{d-p})[-d]) = 0$ for all $i > 0$ and $p \leq k$.

Then a normal variety Z is pre-k-rational if and only if it is pre-k-Du Bois and the maps $\phi^p \colon \underline{\Omega}_Z^p \to \mathbb{D}_Z(\underline{\Omega}_Z^{d-p})[-d]$ defined above are quasi-isomorphisms for all $p \leq k$.

Remark 3.5. If Z has hypersurface singularities, [MOPW23, JKSY22, FL24b] have related these classes of singularities to the *minimal exponent of* Z, which is a positive rational number $\widetilde{\alpha}(Z)$ refining the log canonical threshold. In the isolated singularities case, this minimal exponent agrees with the minimal non-zero spectral number of the hypersurface.

Similarly, when Z has local complete intersection singularities, [MP22, CDMO24, CDM22] have related these classes of singularities to the minimal exponent.

C. Hodge Rational Homology Manifold Level

4. Definition and basic properties. As mentioned above, a pure d-dimensional variety Z is a *rational* (or $\mathbb{Q}$-)*homology manifold*, or *rationally smooth*, if the morphism

$$\psi_Z \colon \mathbb{Q}_Z^H[d] \to (\mathbf{D}_Z\mathbb{Q}_Z^H[d])(-d)$$

is a quasi-isomorphism. This is equivalent to requiring the map on the underlying $\mathbb{Q}$-structure to be a quasi-isomorphism. This section is devoted to defining and studying a natural weakening of this notion.

We observed above that $\mathbb{Q}^H_Z[d] \in D^{\leq 0}(\mathrm{MHM}(Z))$, and by duality this implies $\mathbf{D}_Z(\mathbb{Q}^H_Z[d]) \in D^{\geq 0}(\mathrm{MHM}(Z))$. The following elementary lemma allows us to understand the map ψ_Z.

Lemma 4.1. *Let $\mathcal{A}$ be an abelian category and let $A \in D^{\leq 0}(\mathcal{A}), B \in D^{\geq 0}(\mathcal{A})$ be objects in the derived category. Let $\psi\colon A \to B$ be a morphism.*

Then ψ is a quasi-isomorphism if and only if $\mathcal{H}^0\psi\colon \mathcal{H}^0 A \to \mathcal{H}^0 B$ is an isomorphism in $\mathcal{A}$ and $\mathcal{H}^{-i}A = \mathcal{H}^i B = 0$ for all $i > 0$.

Recall that we have factored $\mathcal{H}^0\psi_Z = \gamma_Z^\vee \circ \gamma_Z$, where

$$\gamma_Z \colon \mathcal{H}^0(\mathbb{Q}^H_Z[d]) \to \mathrm{IC}^H_Z, \quad \gamma_Z^\vee = \mathbf{D}_Z(\gamma_Z)(-d).$$

In particular, by duality, we have the relation

$$\mathbf{D}_Z(\psi_Z) = \psi_Z(d).$$

Proposition 4.2. *We have the following:*

- *The map γ_Z is an isomorphism if and only if it is a monomorphism.*
- *The map $\gamma_Z^\vee$ is an isomorphism if and only if it is an epimorphism.*

Either condition is equivalent to $\mathcal{H}^0\psi_Z$ being an isomorphism.

Thus, the variety Z is a rational homology manifold if and only if $\mathbb{Q}_Z[d]$ is perverse and either γ_Z or $\gamma_Z^\vee$ is an isomorphism.

Proof. The first three claims are immediate.

The condition that $\mathbb{Q}_Z[d]$ is perverse is equivalent to saying that $\mathcal{H}^j(\mathbb{Q}^H_Z[d]) = 0$ for all $j < 0$, and so the last claim follows by the previous lemma. ☐

We will now define the weakening of rational smoothness that is slightly different but equivalent to Definition 0.1, and compare its behavior to that when Z is actually a rational homology manifold.

Definition 4.3. Let Z be a pure d-dimensional variety. We say Z is a *rational homology manifold to Hodge degree k*, or for short *k-Hodge rational homology* if the morphism

$$\mathrm{Gr}^F_{-p}\mathrm{DR}_Z(\psi_Z)\colon \mathrm{Gr}^F_{-p}\mathrm{DR}_Z(\mathbb{Q}^H_Z[d]) \to \mathrm{Gr}^F_{d-p}\mathrm{DR}_Z(\mathbf{D}_Z(\mathbb{Q}^H_Z[d]))$$

is a quasi-isomorphism for all $p \leq k$. We set

$$\mathrm{HRH}(Z) = \sup\{k \mid Z \text{ is a } k\text{-Hodge rational homology variety}\}$$

with $\mathrm{HRH}(Z) = -1$ if Z is not 0-Hodge rational homology.

Remark 4.4. By the discussion at the end of Section B, this condition is equivalent to having $\phi^p\colon \underline{\Omega}^p_Z \to \mathbb{D}_Z(\underline{\Omega}^{d-p}_Z)[-d]$ be a quasi-isomorphism for all $p \leq k$.

We will see in Remark 5.2 that this notion is the same as the one studied in [PP24, PSV24], where it is called condition $(*)_k$.

Remark 4.5. Let $f\colon \tilde{Z} \to Z$ be a strong log resolution with reduced exceptional divisor E. Then $U := Z \setminus Z_{\mathrm{sing}} \cong \tilde{Z} \setminus E$. Let $j\colon U \to Z$ and $j'\colon U \to \tilde{Z}$ be the inclusions. The map $j_!\mathbb{Q}_U^H[d] \to \mathbb{Q}_Z^H[d]$ by duality yields

$$\mathbf{D}_Z(\mathbb{Q}_Z^H[d])(-d) \to j_*\mathbb{Q}_U^H[d].$$

The above fits into the commutative diagram:

$$\begin{array}{ccc} \mathbb{Q}_Z^H[d] & \longrightarrow & f_*\mathbb{Q}_{\tilde{Z}}^H[d] \\ \Big\downarrow{\scriptstyle \psi_Z} & & \Big\downarrow \\ \mathbf{D}_Z(\mathbb{Q}_Z^H[d])(-d) & \longrightarrow & j_*\mathbb{Q}_U^H[d] \end{array} \tag{4.6}$$

Applying $\mathrm{Gr}^F_{-k}\mathrm{DR}_Z$ and appropriate shifts, we obtain the commutative diagram:

$$\begin{array}{ccc} \underline{\Omega}_Z^k & \longrightarrow & Rf_*\Omega_{\tilde{Z}}^k \\ \Big\downarrow{\scriptstyle \phi^k} & & \Big\downarrow \\ \mathbb{D}_Z(\underline{\Omega}_Z^{d-k})[-d] & \longrightarrow & Rf_*\Omega_{\tilde{Z}}^k(\log E) \end{array} \tag{4.7}$$

Note that the right vertical map is induced by the residue sequence on $\tilde{Z}$ (it is an isomorphism for $k = 0$).

To motivate the reader, we describe some useful properties of k-Hodge rational homology varieties that are proven in [PP24] (thanks to the fact that $\mathrm{HRH}(Z) \geq k$ is equivalent to Z satisfying $(*)_k$ in the sense of [PP24], see Remark 5.2):

Remark 4.8. Let Z be a variety of pure dimension d.

(1) Z is a rational homology manifold if and only if $\mathrm{HRH}(Z) \geq k$ for all k (i.e. $\mathrm{HRH}(Z) = +\infty$).
(2) If Z is quasi-projective with general hyperplane section Z' and $\mathrm{HRH}(Z) \geq k$, then we also have $\mathrm{HRH}(Z') \geq k$.
(3) If Z is normal and its singularities are pre-k-rational, then $\mathrm{HRH}(Z) \geq k$.
(4) Assume $\mathrm{HRH}(Z) \geq k$. Then:
   (a) Its singularities are pre-k-Du Bois (resp. strict-k-Du Bois) if and only if they are pre-k-rational (resp. strict-k-rational).
   (b) ϕ^p is an isomorphism for $d - k - 1 \leq p \leq d$.
   (c) $\mathrm{codim}_Z(Z_{\mathrm{nRS}}) \geq 2k+3$ where Z_{nRS} is the locus of Z where it is not a rational homology manifold. In particular
   $$\mathrm{HRH}(Z) < +\infty \text{ if and only if } \mathrm{HRH}(Z) \leq \frac{d-3}{2}.$$
   (d) $\mathrm{lcdef}(Z) \leq \max\{d - 2k - 3, 0\}$ where $\mathrm{lcdef}(Z)$ is the local cohomological defect of Z, given by
   $$\mathrm{lcdef}(Z) = \max\{a \mid \mathcal{H}^{-a}\mathbb{Q}_Z^H[d] \neq 0\}.$$
   (e) (Symmetry of Hodge-Du Bois numbers) The Hodge-Du Bois numbers $\underline{h}^{p,q}(Z) := \mathbb{H}^q(Z, \underline{\Omega}_Z^p)$ are partially equipped with the full symmetry:
   $$\underline{h}^{p,q}(Z) = \underline{h}^{q,p}(Z) = \underline{h}^{d-p,d-q}(Z) = \underline{h}^{d-q,d-p}(Z) \text{ for all } 0 \leq p \leq k, 0 \leq q \leq d.$$

We end this subsection with some general remarks on the behavior of the invariant $\mathrm{HRH}(Z)$.

Remark 4.9. By duality, we see that $\mathrm{HRH}(Z) \geq k$ if

$$\mathbb{D}_Z(\mathrm{Gr}^F_{-p}\mathrm{DR}_Z(\psi_Z)) = \mathrm{Gr}^F_p\mathrm{DR}_Z(\mathbf{D}(\psi_Z))$$

is a quasi-isomorphism for all $p \leq k$. As $\mathbf{D}_Z(\psi_Z) = \psi_Z(d)$, this is equivalent to having

$$\mathrm{Gr}^F_{p-d}\mathrm{DR}_Z(\psi_Z)$$

be a quasi-isomorphism for all $p \leq k$.

We proceed with some comments about the behavior of $\mathrm{HRH}(X)$ under certain geometric operations.

Recall the notation $p(A, F) = \min\{p \mid F_pA \neq 0\}$ for any (A, F) where $F_\bullet A$ is a bounded below filtration.

Lemma 4.10. *Let Z_1, Z_2 be two pure dimensional complex algebraic varieties. Assume either that* $\mathrm{HRH}(Z_1) \neq \mathrm{HRH}(Z_2)$ *or that* $\mathrm{lcdef}(Z_i) = 0$ *for* $i = 1, 2$. *Then*

$$\mathrm{HRH}(Z_1 \times Z_2) = \min\{\mathrm{HRH}(Z_1), \mathrm{HRH}(Z_2)\}.$$

Before beginning the proof, we recall the structure of the exterior product for (complexes of) mixed Hodge modules.

For any two mixed Hodge modules M_i on smooth varieties X_i, the underlying filtered $\mathscr{D}$-module of $M_1 \boxtimes M_2$ is $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ with convolution Hodge filtration

$$F_p(\mathcal{M}_1 \boxtimes \mathcal{M}_2) = \sum_{i+j=p} F_i\mathcal{M}_1 \boxtimes F_j\mathcal{M}_2,$$

and so

$$\mathrm{Gr}^F_p(\mathcal{M}_1 \boxtimes \mathcal{M}_2) = \bigoplus_{i+j=p} \mathrm{Gr}^F_i\mathcal{M}_1 \boxtimes \mathrm{Gr}^F_j\mathcal{M}_2.$$

Given two complexes $M_i^\bullet \in D^b(\mathrm{MHM}(X_i))$, we have

$$\mathcal{H}^k(M_1^\bullet \boxtimes M_2^\bullet) = \bigoplus_{i+j=k} \mathcal{H}^iM_1^\bullet \boxtimes \mathcal{H}^jM_2^\bullet.$$

Thus, we have

$$\mathrm{Gr}^F_p\mathcal{H}^k(\mathcal{M}_1^\bullet \boxtimes \mathcal{M}_2^\bullet) = \bigoplus_{a+b=p}\bigoplus_{i+j=k} \mathrm{Gr}^F_a\mathcal{H}^i(\mathcal{M}_1^\bullet) \boxtimes \mathrm{Gr}^F_b\mathcal{H}^j(\mathcal{M}_2^\bullet).$$

Consequently, we get

$$p(M_1^\bullet \boxtimes M_2^\bullet) = p(M_1^\bullet) + p(M_2^\bullet) \tag{4.11}$$

Now, for Z_1, Z_2 not necessarily smooth, consider the exact triangle in $D^b(\mathrm{MHM}(Z_i))$:

$$\mathbb{Q}^H_{Z_i}[d_i] \to \mathbf{D}_{Z_i}(\mathbb{Q}^H_{Z_i}[d_i])(-d_i) \to S^\bullet_{Z_i} \xrightarrow{+1}.$$

It is easy to see (for example, by restriction to the regular locus) that

$$p(\mathbb{Q}^H_{Z_i}[d_i]) = p(\mathbf{D}_{Z_i}(\mathbb{Q}^H_{Z_i}[d_i])(-d_i)) = -d_i.$$

Moreover, we have

$$\mathbb{Q}^H_{Z_1}[d_1] \boxtimes \mathbb{Q}^H_{Z_2}[d_2] = \mathbb{Q}^H_{Z_1\times Z_2}[d_1 + d_2]$$

$$\mathbf{D}_{Z_1}(\mathbb{Q}^H_{Z_1}[d_1])(-d_1) \boxtimes \mathbf{D}_{Z_i}(\mathbb{Q}^H_{Z_2}[d_2])(-d_2) = \mathbf{D}_{Z_1\times Z_2}(\mathbb{Q}^H_{Z_i}[d_1+d_2])(-d_1-d_2).$$

We make use of the following easy lemma.

Lemma 4.12. *For $i = 1,2$, let $A_i \xrightarrow{\alpha_i} B_i \xrightarrow{\beta_i} C_i \xrightarrow{+1}$ be an exact triangle. Let $S = \mathrm{cone}(\alpha_1 \boxtimes \alpha_2)$. Then there are exact triangles*

$$A_1 \boxtimes C_2 \to S \to C_1 \boxtimes B_2 \xrightarrow{+1}$$
$$C_1 \boxtimes A_2 \to S \to B_1 \boxtimes C_2 \xrightarrow{+1}.$$

Proof. This follows from the octahedral axiom and the exactness of the functor $-\boxtimes M$ for any object M. ☐

Proof of Lemma 4.10. As noted above, $\psi_{Z_1} \boxtimes \psi_{Z_2} = \psi_{Z_1\times Z_2}$ up to non-zero scalar multiples on the connected components, so that we can identify

$$\mathrm{cone}(\psi_{Z_1} \boxtimes \psi_{Z_2}) = S^\bullet_{Z_1\times Z_2}.$$

By Lemma 4.12, we have exact triangles

$$\mathbb{Q}^H_{Z_1}[d_1] \boxtimes S^\bullet_{Z_2} \to S^\bullet_{Z_1\times Z_2} \to S^\bullet_{Z_1} \boxtimes \mathbf{D}_{Z_2}(\mathbb{Q}^H_{Z_2}[d_2])(-d_2) \xrightarrow{+1}$$
$$S^\bullet_{Z_1} \boxtimes \mathbb{Q}^H_{Z_2}[d_2] \to S^\bullet_{Z_1\times Z_2} \to \mathbf{D}_{Z_1}(\mathbb{Q}^H_{Z_1}[d_1])(-d_1) \boxtimes S^\bullet_{Z_2} \xrightarrow{+1}.$$

We can use either exact triangle to conclude the proof: let $k = \min\{\mathrm{HRH}(Z_1), \mathrm{HRH}(Z_2)\}$, so, by definition, $p(S^\bullet_{Z_i}) \geq k - d_i$ with equality for (at least) one of $i = 1,2$. Thus, the outer two terms of either triangle satisfy $p(-) \geq k - d_1 - d_2$, and so we see by (4.11) that $p(S^\bullet_{Z_1\times Z_2}) \geq k - d_1 - d_2$. Moreover, equality obviously holds if $\mathrm{HRH}(Z_1) \neq \mathrm{HRH}(Z_2)$ or if $\mathrm{lcdef}(Z_i) = 0$ for $i = 1,2$, the latter implying that the exact triangles are actually short exact sequences. ☐

For example, if Z_2 is a rational homology manifold, then $\mathrm{HRH}(Z_1 \times Z_2) = \mathrm{HRH}(Z_1)$. For smooth morphisms which are not projections from a product, we will see in Lemma 6.10 that HRH is preserved.

5. Embedded case. The condition in Remark 4.9 resembles the condition in Corollary 1.7, though we cannot talk about $F_k\mathbb{Q}^H_Z[d]$ without using a fixed local embedding. Indeed, the terms in these complexes are not $\mathscr{O}_Z$-modules, and the morphisms are not $\mathscr{O}_Z$-linear. To continue this discussion, we will assume $i\colon Z \hookrightarrow X$ is a closed embedding of Z inside a smooth variety X.

We focus on $i_*\psi_Z\colon i_*\mathbb{Q}^H_Z[d] \to i_*\mathbf{D}_Z(\mathbb{Q}^H_Z[d])(-d)$, which is a morphism of objects in $D^b(\mathrm{MHM}(X))$. Then $\mathrm{HRH}(Z) \geq k$ if and only if $\mathrm{Gr}^F_{p-d}\mathrm{DR}_X(i_*\psi_Z)$ is a quasi-isomorphism for all $p \leq k$. By Corollary 1.7, this is equivalent to $F_{k-d}i_*\psi_Z$ being a quasi-isomorphism.

This condition is equivalent to $F_{k-d}\mathcal{H}^{-j}(i_*\mathbb{Q}^H_Z[d]) = F_{k-d}\mathcal{H}^j(i_*\mathbf{D}_Z(\mathbb{Q}^H_Z[d])(-d)) = 0$ for $j > 0$ and $F_{k-d}\mathcal{H}^0(i_*\psi_Z)$ being an isomorphism. From here, we can give the lower bound on $\mathrm{HRH}(Z)$ in terms of local cohomology in Theorem B. Recall that we are indexing our Hodge filtrations following the conventions for right $\mathscr{D}$-modules.

Remark 5.1. Before giving the proof, note that $W_{n+c}\mathcal{H}^c_Z(\mathscr{O}_X) = \mathrm{IC}^H_Z(-c)$. The theorem is essentially saying that $\mathrm{HRH}(Z)$ is controlled by the morphism $F_{k-d}i_*\gamma^\vee_Z$.

Although duality is used in the argument, it is important to note that it is not equivalent to study $F_{k-d}i_*\gamma_Z$. Indeed, in the non-rational homology manifold hypersurface case, we will see below that $F_{k-d}i_*\gamma_Z$ can be an isomorphism when $F_{k-d}i_*\gamma^\vee_Z$ is not.

Remark 5.2. At this point, we can see that $\mathrm{HRH}(Z) \geq k$ if and only if Z satisfies the condition $(*)_k$ of [PP24, PSV24]. Indeed, as both notions are local, we can assume $i\colon Z \to X$ is a closed embedding into a smooth variety X. Then we have

$$\mathrm{HRH}(Z) \geq k \text{ if and only if } F_{k-d} i_* \gamma_Z^\vee \text{ is a quasi-isomorphism,}$$

which is true if and only if

$$\mathrm{Gr}^F_{p-d}\mathrm{DR}_X(i_*\gamma_Z^\vee) \text{ is a quasi-isomorphism for all } p \leq k.$$

By duality, this is equivalent to

$$\mathrm{Gr}^F_{d-p}\mathrm{DR}_X(i_*\mathbf{D}(\gamma_Z^\vee)) \text{ being a quasi-isomorphism for all } p \leq k,$$

and finally, using that $\mathbf{D}(\gamma_Z^\vee) = \gamma_Z(d)$, we have that this is equivalent to the natural map

$$\mathrm{Gr}^F_{-p}\mathrm{DR}_X(i_*\mathbb{Q}_Z^H[d]) \to \mathrm{Gr}^F_{-p}\mathrm{DR}_X(i_*\mathrm{IC}_Z^H) \text{ being a quasi-isomorphism for all } p \leq k,$$

which is the condition $(*)_k$.

Remark 5.3. In [CDM22], it is shown that when Z is a complete intersection, then

$$F_{k-n}W_{n+c}\mathcal{H}^c_Z(\mathscr{O}_X) = F_{k-n}\mathcal{H}^c_Z(\mathscr{O}_X) = P_k\mathcal{H}^c_Z(\mathscr{O}_X)$$

is equivalent to Z having k-rational singularities. Here $P_k\mathcal{H}^c_Z(\mathscr{O}_X)$ is the *pole order filtration*, consisting of elements which are annihilated by $\mathcal{I}_Z^{k+1}$, where $\mathcal{I}_Z \subseteq \mathscr{O}_X$ is the ideal sheaf defining Z in X. The comparison with the pole order filtration is essentially the same as the map α_k comparing the Kähler differentials to the Du Bois complex.

Proof of Theorem B. We have $d = n - c$ by definition of the codimension of Z in X, and so we are interested in

$$i_*\mathbb{Q}_Z^H[n-c] \to i_*\mathbf{D}_Z(\mathbb{Q}_Z^H[n-c])(c-n).$$

By definition, $\mathbb{Q}_Z^H[n-c] = i^*(\mathbb{Q}_X^H[n])[-c]$. Applying duality and using the fact that X is smooth (so that $\mathbb{Q}_X^H[n]$ is pure, polarizable of weight n), we get

$$\mathbf{D}_Z(\mathbb{Q}_Z^H[n-c]) = i^!(\mathbf{D}_X(\mathbb{Q}_X^H[n]))[c] = i^!(\mathbb{Q}_X^H[n](n))[c]$$

so that $i_*\psi_Z$ can be identified with the morphism

$$i_*\psi_Z \colon i_*\mathbb{Q}_Z^H[n-c] \to i_*i^!(\mathbb{Q}_X^H[n])[c](c).$$

Using that $\mathcal{H}^j(i_*i^!(\mathbb{Q}_X^H[n])) = \mathcal{H}^j_Z(\mathscr{O}_X)$ as mixed Hodge modules, we see that

$$F_{k-d}\mathcal{H}^j(i_*i^!(\mathbb{Q}_X^H[n])(c)) = F_{k-d-c}\mathcal{H}^j_Z(\mathscr{O}_X) = F_{k-n}\mathcal{H}^j_Z(\mathscr{O}_X).$$

Thus, we see that $\mathrm{HRH}(Z) \geq k$ if and only if $F_{k-d}i_*\psi_Z$ is a quasi-isomorphism if and only if $F_{k-d}\mathcal{H}^{-j}(i_*\mathbb{Q}_Z^H[n-c]) = F_{k-d}\mathcal{H}^j(i_*i^!(\mathbb{Q}_X^H[n])(c)) = 0$ and $F_{k-d}\mathcal{H}^0(\psi_Z)$ is an isomorphism. As $F_{k-d}\mathcal{H}^0(\psi_Z) = F_{k-d}\gamma_Z^\vee \circ F_{k-d}\gamma_Z$, the last condition is equivalent to both $F_{k-d}\gamma_Z^\vee$ and $F_{k-d}\gamma_Z$ being isomorphisms.

Note that $F_{k-d}\gamma_Z^\vee$ is an isomorphism if and only if we have equality

$$F_{k-n}W_{n+c}\mathcal{H}^c_Z(\mathscr{O}_X) = F_{k-n}\mathcal{H}^c_Z(\mathscr{O}_X),$$

and so we see that $\mathrm{HRH}(Z) \geq k$ implies the conditions in the theorem statement.

For the converse, we assume

$$F_{k-n}\mathcal{H}^j_Z(\mathscr{O}_X) = 0 \text{ for all } j > c, \quad F_{k-n}W_{n+c}\mathcal{H}^c_Z(\mathscr{O}_X) = F_{k-n}\mathcal{H}^c_Z(\mathscr{O}_X),$$

and we want to show $\mathrm{HRH}(Z) \geq k$.

For ease of notation, let $A = \mathbb{Q}_Z^H[d]$ and let $B = \mathbf{D}(A)(-d)$. By the discussion at the beginning of the proof, to show that $\mathrm{HRH}(Z) \geq k$ it suffices under our assumption to show that $F_{k-d}\mathcal{H}^{-j}(A) = 0$ for all $j > 0$ and that $F_{k-d}\gamma_Z$ is an isomorphism. For this, we follow the argument of [CDM22, Lem. 3.5]. To prove $F_{k-d}\mathcal{H}^{-j}(A) = 0$ it suffices to prove for all $\ell \in \mathbb{Z}$ that $F_{k-d}\mathrm{Gr}_\ell^W \mathcal{H}^{-j}(A) = 0$. To prove that $F_{k-d}\gamma_Z$ is an isomorphism it suffices to prove that $F_{k-d}\mathrm{Gr}_\ell^W(\mathcal{H}^0(A)) = 0$ for all $\ell < d$.

Our assumption is equivalent to $F_{k-d}\mathcal{H}^j(B) = 0$ for all $j > 0$ and that $F_{k-d}\mathrm{Gr}_\ell^W \mathcal{H}^0(B) = 0$ for all $\ell > d$.

By polarizability of the pure Hodge module $\mathrm{Gr}_\ell^W \mathcal{H}^j(B)$, we have an isomorphism

$$\mathbf{D}(\mathrm{Gr}_\ell^W \mathcal{H}^j B) \cong (\mathrm{Gr}_\ell^W \mathcal{H}^j B)(\ell).$$

But we also have

$$\mathbf{D}(\mathrm{Gr}_\ell^W \mathcal{H}^j B) \cong \mathrm{Gr}_{-\ell}^W \mathcal{H}^{-j}(\mathbf{D}(B)).$$

We have that $\mathbf{D}(B) \cong A(d)$, and so if we apply F_p to this isomorphism (keeping in mind the Tate twists), we have

$$F_{p-\ell}\mathrm{Gr}_\ell^W \mathcal{H}^j B \cong F_{p-d}\mathrm{Gr}_{2d-\ell}^W \mathcal{H}^{-j} A.$$

By the weight formalism, we know that $\mathrm{Gr}_\ell^W \mathcal{H}^j(B) \neq 0$ implies $\ell \geq d+j$ (and the same inequality is implied when $\mathrm{Gr}_{2d-\ell}^W \mathcal{H}^{-j}(A) \neq 0$). For $j > 0$, we trivially have $\ell > d$, and for $j = 0$, we only need to consider $\ell > d$. Plugging in $p = k - d + \ell$, we get

$$0 = F_{k-d}\mathrm{Gr}_\ell^W \mathcal{H}^j(B) = F_{k-2d+\ell}(\mathrm{Gr}_{2d-\ell}^W \mathcal{H}^{-j}(A)),$$

and so because W is exhaustive (for the $j > 0$ case) and $\ell > d$, this gives the desired vanishing. □

Proof of Corollary C. Assume Z is a Cohen-Macaulay subvariety of X of pure codimension c.

Recall the notation of the corollary statement: the filtration $E_\bullet \mathcal{H}_Z^q(\mathscr{O}_X)$ is defined by

$$E_\bullet \mathcal{H}_Z^q(\mathscr{O}_X) = \mathrm{Im}\left[\mathcal{E}xt^q(\mathscr{O}_X/I_Z^{\bullet+1}, \mathscr{O}_X) \to \mathcal{H}_Z^q(\mathscr{O}_X)\right],$$

where $I_Z \subseteq \mathscr{O}_X$ is the ideal sheaf defining Z in X.

Under the Cohen-Macaulay assumption, we get $F_{-n}\mathcal{H}_Z^q(\mathscr{O}_X) = 0$ for $q > c$. Moreover, the result of [MP22, Thm. C] says that, under the Cohen-Macaulay assumption, Z is Du Bois if and only if it satisfies $F_{-n}\mathcal{H}_Z^c(\mathscr{O}_X) = E_0\mathcal{H}_Z^c(\mathscr{O}_X)$.

Thus, either assumption in the corollary statement implies that Z is Du Bois, so we can assume Z is Du Bois. Then Z has rational singularities if and only if $\mathrm{HRH}(Z) \geq 0$, and so the claim follows from the previous theorem. □

Let D be a hypersurface inside a smooth variety X. The notion of Hodge ideals $I_p(D)$ was introduced in [MP19], and subsequently their weighted variants $I_p^{W_l}(D)$ were introduced [Ola23]. It is interesting to note that $\mathrm{HRH}(D)$ can be detected through the weighted Hodge ideals:

Corollary 5.4. *Let D be a hypersurface inside a smooth variety X of dimension n. Then* $\mathrm{HRH}(D) \geq k$ *if and only if $I_p^{W_1}(D) = I_p(D)$ for all $0 \leq p \leq k$.*

Proof. Observe that $\mathrm{HRH}(D) \geq k$ if and only if $F_{p-n}\mathrm{Gr}_{n+l}^W \mathcal{H}_D^1(\mathscr{O}_X) = 0$ for $0 \leq p \leq k$, $l \geq 2$ by Theorem B. The assertion is an immediate consequence of the exact sequence

$$0 \to I_p^{W_{l-1}}(D) \otimes \mathscr{O}_X((p+1)D) \to I_p^{W_l}(D) \otimes \mathscr{O}_X((p+1)D) \to F_{p-n}\mathrm{Gr}_{n+l}^W \mathscr{O}_X(*D) \to 0$$

given by [Ola23, (6.1)], and the fact $\mathcal{H}^1_D(\mathscr{O}_X) = \mathscr{O}_X(*D)/\mathscr{O}_X$. □

In the case of isolated singularities, the connection between this value and the lack of Poincaré duality of D was noted in [Ola23, Remark 6.8].

6. Local cohomology at a point and link invariants. We can now relate $\mathrm{HRH}(Z)$ to the local cohomology at a point $x \in Z$. Using the results of [DS90], it can be related to the cohomology of the link.

Let $S^\bullet = \mathrm{cone}(\psi_Z)$, where $\psi_Z \colon \mathbb{Q}^H_Z[d] \to \mathbf{D}_Z(\mathbb{Q}^H_Z[d])(-d)$. If Z is embeddable into a smooth variety X, then we saw above that $\mathrm{HRH}(Z) \geq k$ if and only if $F_{k-d}\psi_Z$ is a quasi-isomorphism, which holds (by strictness of morphisms with respect to the Hodge filtration) if and only if $F_{k-d}S^\bullet = 0$.

Let $i_x \colon \{x\} \to Z$ be the inclusion of a point. The local cohomology of Z at x is given by $\mathcal{H}^k(i_x^!\mathbb{Q}^H_Z)$ and denoted $H^k_{\{x\}}(Z)$. For any neighborhood $x \in V \subseteq Z$, if $i_{x,V} \colon \{x\} \to V$ is the inclusion, then $i_x^!\mathbb{Q}^H_Z = i_{x,V}^!\mathbb{Q}^H_V$. Thus, as far as local cohomology is concerned, we can replace Z with any open neighborhood of x. In particular, we can choose an affine neighborhood, so that we can assume Z is embeddable into a smooth variety. We will need the following lemma:

Lemma 6.1. *Let $\phi \colon M^\bullet \to N^\bullet$ be a morphism in $D^b(\mathrm{MHM}(X))$ where X is a smooth algebraic variety. Let $i \colon Y \to X$ be a closed embedding. Assume $F_k\phi$ is a quasi-isomorphism. Then $F_k i_* i^!(\phi), F_k i_* i^*(\phi)$ are quasi-isomorphisms. If Y is a smooth subvariety, then $F_k i^!(\phi), F_k i^*(\phi)$ are quasi-isomorphisms.*

Proof. It suffices to prove the following: let $C^\bullet \in D^b(\mathrm{MHM}(X))$ be such that $F_k C^\bullet$ is acyclic. Then $F_k i_* i^*(C^\bullet)$ and $F_k i_* i^!(C^\bullet)$ are acyclic. Indeed, this implies the lemma by applying this claim to the cone of ϕ and using that $i_*i^!, i_*i^*$ are exact functors between triangulated categories.

Again, using that $i_*i^!$ and i_*i^* are exact functors, this reduces to the claim when C is a single mixed Hodge module. Indeed, assume $C^\bullet \in D^{[a,b]}(\mathrm{MHM}(X))$ and we use induction on $b-a$. We have the natural exact triangle

$$\tau_{\leq b-1}C^\bullet \to C^\bullet \to \mathcal{H}^b(C^\bullet)[-b] \xrightarrow{+1},$$

where, by assumption, F_k applied to any term in the triangle is acyclic. Induction handles the outer two terms. So we can assume $C^\bullet = C \in \mathrm{MHM}(X)$.

As this is a local statement, we can assume $Y = V(f_1, \ldots, f_r) \subseteq X$ for some $f_1, \ldots, f_r \in \mathscr{O}_X(X)$.

Let $\Gamma \colon X \to X \times \mathbb{A}^r_t$ be the graph embedding along $f_1, \ldots, f_r$. If $\sigma \colon X \times \{0\} \to X \times \mathbb{A}^r_t$ is the inclusion of the zero section, then by Base Change (Example 1.3) we have isomorphisms $\sigma^*\Gamma_* \cong i_*i^*, \sigma^!\Gamma_* \cong i_*i^!$. Moreover, we know that $F_k\Gamma_*(C) = 0$ by definition of the direct image for mixed Hodge modules (recall that we index like right $\mathscr{D}$-modules). Thus, we have reduced to the case that $Y \subseteq X$ is a smooth subvariety defined by $t_1, \ldots, t_r$. The claim then follows by Proposition 2.6. □

Now we can prove the connection with the local description of being a rational homology manifold using local cohomology at $x \in Z$ in (0.3). We note that this invariant is related to the question of whether $H^{2d}_{\{x\}}(Z)$ is one dimensional, which by a result of Brion [Bri99, Prop. A1] is equivalent to Z being irreducible near x.

First, we prove a lemma which strengthens [PP24, Prop. 6.4]. It is simply a corollary of [PP24, Cor. 7.5]. We have the following exact triangles considering the RHM defect object and its dual:

$$K_Z^\bullet \to \mathbb{Q}_Z^H[d] \xrightarrow{\gamma_Z} \mathrm{IC}_Z^H \xrightarrow{+1},$$

$$\mathrm{IC}_Z^H \xrightarrow{\gamma_Z^\vee} \mathbf{D}_Z(\mathbb{Q}_Z^H[d])(-d) \to \mathbf{D}_Z(K_Z^\bullet)(-d) \xrightarrow{+1},$$

$$\mathbb{Q}_Z^H[d] \xrightarrow{\psi_Z} \mathbf{D}_Z(\mathbb{Q}_Z^H[d])(-d) \to S^\bullet \xrightarrow{+1} .$$

The octahedral axiom gives an exact triangle

$$K_Z^\bullet[1] \to S^\bullet \to \mathbf{D}_Z(K_Z^\bullet)(-d) \xrightarrow{+1},$$

and so because the first object has non-zero cohomology only in strictly negative degrees and the third object only has non-zero cohomology in non-negative degrees, we have isomorphisms

$$\mathcal{H}^{-i}K_Z^\bullet \cong \mathcal{H}^{-i-1}S^\bullet \text{ for all } i > 0, \quad \mathcal{H}^i\mathbf{D}_Z(K_Z^\bullet)(-d) \cong \mathcal{H}^iS^\bullet \text{ for all } i \geq 0.$$

Lemma 6.2. *We have the following inequality:*

$$\dim \mathrm{Supp}(\mathcal{H}^{-i}K_Z^\bullet) = \dim \mathrm{Supp}(\mathcal{H}^i\mathbf{D}_Z(K_Z^\bullet)(-d)) = \dim \mathrm{Supp}(\mathcal{H}^{\pm i}S^\bullet) \leq d - 2\mathrm{HRH}(Z) - 3 - i.$$

Proof. It suffices by duality to prove the claim for $K_Z^\bullet$. For ease of notation let $k = \mathrm{HRH}(Z)$.

First of all, [PP24, Cor. 7.5(ii)] shows that

$$\mathcal{H}^{2k+2-d}K_Z^\bullet = 0.$$

The claim is local, so we can assume Z is quasi-projective. Note that $\mathrm{HRH}(Z)$ doesn't decrease under restriction to a general hyperplane L. Using that $\iota^*K_Z^\bullet = K_{Z\cap L}^\bullet[1]$ as shown in [PP24, Lem. 6.6], the argument of [PP24, Prop. 6.4] gives the desired claim. ☐

Recall that the local cohomological defect is given by

$$\mathrm{lcdef}(Z) = \max\{a \mid \mathcal{H}^{-a}(\mathbb{Q}_Z^H[d]) \neq 0\}.$$

It admits a local version $\mathrm{lcdef}_x(Z) = \min_{x\in U\subseteq Z} \mathrm{lcdef}(U)$, where the minimum runs over Zariski open neighborhoods of x in Z.

Remark 6.3. By the Hartshorne-Lichtenbaum Theorem [Har68, Thm. 3.1] [Ogu73, Cor. 2.10], which has also been recovered by Mustaţă-Popa in [MP22, Cor. 11.9], we have the following bound for the local cohomological defect :

$$\mathrm{lcdef}(Z) \leq d - 1 \text{ if and only if no irreducible component of } Z \text{ is a point.}$$

As we are working with Z purely d-dimensional with $d > 0$, this is automatic for us.

We begin with the following observation in the isolated singularities setting, which is a warm-up to the proof of Theorem F:

Lemma 6.4. *Let $x \in Z$ be such that $Z \setminus \{x\}$ is a rational homology manifold. Then Z is irreducible near x if and only if* $\mathrm{lcdef}_x(Z) \leq d - 2$.

Proof. If Z is a rational homology manifold, then [Bri99, Prop. A1] shows that Z is irreducible. Moreover, $\mathrm{lcdef}(Z) = 0$ in this case. So we assume Z is not a rational homology manifold at x.

Let $i_x\colon \{x\} \to Z$ be the inclusion. Apply $i_x^!$ to the triangle

$$\mathbb{Q}_Z^H[d] \to \mathbf{D}_Z(\mathbb{Q}_Z^H[d])(-d) \to S^\bullet \xrightarrow{+1}$$

to get

$$i_x^!\mathbb{Q}_Z^H[d] \to \mathbb{Q}^H(-d)[-d] \to i_x^! S^\bullet \xrightarrow{+1} .$$

This gives the exact sequence

$$0 \to \mathcal{H}^{d-1} i_x^! S^\bullet \to H_x^{2d}(Z) \to \mathbb{Q}(-d) \to \mathcal{H}^d i_x^! S^\bullet \to 0,$$

and so it suffices to prove that $\mathcal{H}^{d-1} i_x^! S^\bullet = \mathcal{H}^d i_x^! S^\bullet = 0$ if and only if $\mathrm{lcdef}(Z) \leq d-2$.

As $Z \setminus \{x\}$ is a rational homology manifold, we know $S^\bullet$ is supported at x. Thus, we can write $S^\bullet = i_{x*}S'$ for some $S' \in D^b(\mathrm{MHM}(\{x\}))$ and so $i_x^! S^\bullet = S'$. Thus, it suffices to prove that

$$\mathcal{H}^{d-1}S' = \mathcal{H}^d S' = 0 \text{ if and only if } \mathrm{lcdef}(Z) \leq d-2.$$

Again, as $i_{x*}S' = S^\bullet$, it suffices to consider $\mathcal{H}^\bullet S^\bullet$ itself. But then by definition (using that $S^\bullet \neq 0$), we have

$$\mathrm{lcdef}(Z) = \max\{a \mid \mathcal{H}^a(\mathbf{D}_Z(K_Z^\bullet)) \neq 0\} = \max\{a \mid \mathcal{H}^a S^\bullet \neq 0\},$$

so the claim follows. □

In the non-isolated setting, we see that $\mathrm{HRH}_x(Z)$ can also guarantee irreducibility, as well as vanishing of some higher local cohomology spaces of Z at x. Before stating the theorem, we introduce the *link invariants*, as they are closely related to the mixed Hodge structures on $H^\bullet_{\{x\}}(Z)$.

The cohomology of the link L_x of Z at x can be described, following [DS90], by the cohomology of the cone

$$\mathrm{cone}(i_x^!\mathbb{Q}_Z^H \to i_x^*\mathbb{Q}_Z^H) \in D^b(\mathrm{MHM}(\{x\})).$$

In other words, if $U = Z \setminus \{x\}$ with inclusion $j\colon U \to Z$, we have equality

$$H^k(L_x) = \mathcal{H}^k i_x^* j_*(\mathbb{Q}_U^H).$$

Immediately from the definition, we get a long exact sequence of mixed Hodge structures [DS90, Prop. 3.5]:

$$\dots \to H^k_{\{x\}}(Z) \to H^k(\{x\}) \to H^k(L_x) \to \dots .$$

As $\{x\}$ is simply a point, we get isomorphisms of mixed Hodge structures

$$\mathbb{Q} \cong H^0(L_x), \quad H^1_{\{x\}}(Z) = 0, \quad H^k(L_x) \cong H^{k+1}_{\{x\}}(Z) \text{ for all } k \geq 1.$$

Proof of Theorem F. Apply $i_x^!$ to the triangle

$$\mathbb{Q}_Z^H[d] \to \mathbf{D}_Z(\mathbb{Q}_Z^H[d])(-d) \to S^\bullet \xrightarrow{+1},$$

which gives

$$i_x^!\mathbb{Q}_Z^H[d] \to \mathbb{Q}^H(-d)[-d] \to i_x^! S^\bullet \xrightarrow{+1},$$

and thus, by the long exact sequence in cohomology, we get an exact sequence

$$0 \to \mathcal{H}^{d-1} i_x^! S^\bullet \to H_x^{2d}(Z) \to \mathbb{Q}(-d) \to \mathcal{H}^d i_x^! S^\bullet \to 0$$

and for $i \geq 2$, we get isomorphisms

$$\mathcal{H}^{d-i} i_x^! S^\bullet \cong \mathcal{H}_x^{2d-i+1}(Z).$$

By Lemma 1.10 and Lemma 6.2 we get (using that $\mathrm{HRH}_x(Z) \geq 0$):

$$\mathcal{H}^j i_x^! S^\bullet = 0 \text{ for all } j \geq d - 2\mathrm{HRH}_x(Z) - 2,$$

and so we get the desired vanishing. □

Theorem 6.5. *Let Z be a complex variety of pure dimension d and let $x \in Z$ be a point. Then the following are equivalent for $k \in \mathbb{Z}_{\geq 0}$:*

(1) $\mathrm{HRH}(Z) \geq k$

(2) $F_{k-d} H^i_{\{x\}}(Z) = \begin{cases} 0 & i < 2d \\ \mathbb{Q} & i = 2d. \end{cases}$

(3) $F_{k-d} H^{i-1}(L_x) = \begin{cases} 0 & 0 < i < 2d \\ \mathbb{Q} & i = 2d. \end{cases}$

Proof. As the claim is local, we can assume Z is embeddable into a smooth variety. Let $i_x \colon \{x\} \to Z$ be the inclusion of a point x.

We have $\mathrm{HRH}_x(Z) \geq 0$ iff in a neighborhood of x, the map

$$F_{k-d}\psi_Z \colon F_{k-d}\mathbb{Q}_Z^H[d] \to F_{k-d}(\mathbf{D}_Z(\mathbb{Q}_Z^H[d])(-d))$$

is a quasi-isomorphism, which is equivalent to $F_{k-d}S^\bullet = 0$.

By Lemma 1.10, we have $F_{k-d}S^\bullet = 0$ if and only if $F_{k-d} i_x^! S^\bullet = 0$ for all $x \in Z$, and the latter claim is true if and only if $F_{k-d} i_x^! \mathbb{Q}_Z^H[d] \to F_k(\mathbb{Q}^H[-d])$ is a quasi-isomorphism, giving the result. □

The above proposition gives a descent result for $\mathrm{HRH}(Z)$, and hence for pre-k-rational singularities which should be compared to [SVV23, Prop. 4.2 (2)].

Corollary 6.6. *let $\pi \colon Z \to W$ be the quotient of a variety Z by the action of a finite group G. Then,*

$$\mathrm{HRH}(W) \geq \mathrm{HRH}(Z).$$

In particular, if Z and W are in addition normal, and Z has pre-k-rational singularities, then the singularities of W are also pre-k-rational.

Proof. Let $x \in Z$ such that $\pi(x)$ is singular. Using [Bri99, Proof of Prop. A1], we have the isomorphisms

$$H^i_{\{x\}}(Z)^{G_x} \cong H^i_{Gx}(Z)^G \cong H^i_{\{\pi(x)\}}(W).$$

The assertion follows by taking Hodge pieces and applying Theorem 6.5.

The last assertion follows by combining this with [SVV23, Prop. 4.2 (1)]. □

In [FL24a], the *link invariants* of Z at x are defined by

$$\ell^{p,q} = \dim_{\mathbb{C}} \mathrm{Gr}_F^p H^{p+q}(L_x). \tag{6.7}$$

We can restate the condition $\mathrm{HRH}_x(Z) \geq k$ in terms of these invariants, and give a generalization of [FL24a, Thm. 1.15(i)].

Proof of Theorem E. The first statement trivially implies the second.

For the first statement, note that $\ell^{d-i,q-d+i} = 0$ for all $i \leq k$ if and only if $F^{d-i}H^q(L_x) = 0$ for all $i \leq k$ if and only if $F_{k-d}H^q(L_x) = 0$. Similarly, $\ell^{d-i,d-1+i} = 0$ for all $1 \leq i \leq k$ is equivalent to $F_{k-d}H^q(L_x) = \mathbb{Q}$. So this shows that $\mathrm{HRH}_x(Z) \geq k$ implies those vanishings.

For the converse, we need to also check that $F_{k-d}H^p(L_x) = 0$ for $p \in [d-a, d-1]$. We apply [FL24a, Prop. 2.8] and Serre duality [FL24a]. The duality says that $\ell^{p,q} = \ell^{d-p,d-q-1}$.

Thus, our assumption $\ell^{d-i,q-d+i} = 0$ gives $\ell^{i,2d-q-i-1} = 0$ for all $i \leq k$ and $q \in [d, d+a]$. In other words, for $p \in [d-a-1, d-1]$, we have $\ell^{i,p-i} = 0$ for all $i \leq k$. Then [FL24a, Prop. 2.8] gives

$$\sum_{i=0}^{k} \ell^{p-i,i} \leq \sum_{i=0}^{k} \ell^{i,p-i} = 0$$

and so $\ell^{p-i,i} = 0$ for all $i \leq k$, too. Thus,

$$F_{k-d}H^p(L_x) \subseteq F_{k-p}H^p(L_x) = 0,$$

where the first containment uses that $p \leq d-1$. This completes the proof. □

Remark 6.8. If $x \in Z$ is an isolated singular point and Z has local complete intersection singularities near x, [FL24a, Thm. 1.15(i)] says that Z is k-rational near x if and only if Z is k-Du Bois near x and $\ell^{k,d-k-1} = 0$.

In the local complete intersection setting, $\mathrm{lcdef}_x(Z) = 0$ and $H^{2d-1}(L) = \mathbb{Q}$. Thus, the condition that $\ell^{d-i,d-1+i} = 0$ for all $1 \leq i \leq k$ is true. Moreover, by the Serre duality relation $\ell^{p,q} = \ell^{d-p,d-q-1}$, the condition is equivalent to $\ell^{d-i,i} = 0$.

In fact, this argument shows that for any Z with an isolated singular point at x and satisfying $\mathrm{lcdef}_x(Z) = 0$, the result of Friedman-Laza holds.

Using the criteria for $\mathrm{HRH}(Z) \geq k$ in terms of $H^{\bullet}_{\{x\}}(Z)$, we see easily that HRH is preserved under étale morphisms.

Lemma 6.9. *Let $\varphi\colon Z_1 \to Z_2$ be a surjective étale morphism. Then we have* $\mathrm{HRH}(Z_1) = \mathrm{HRH}(Z_2)$.

Proof. We have an isomorphism of mixed Hodge structures for any $x \in Z_1$

$$H^i_x(Z_1) \cong H^i_{f(x)}(Z_2),$$

and so the claim follows from Theorem 6.5. □

Lemma 6.10. *Let $\varphi\colon Z_1 \to Z_2$ be a smooth morphism. Then* $\mathrm{HRH}(Z_1) = \mathrm{HRH}(Z_2)$.

Proof. As the question is local, we can reduce to the previous lemma and Lemma 4.10. □

In a similar vein, we have the following about pre-k-Du Bois singularities under smooth morphisms.

Lemma 6.11. *Let $\varphi\colon Z_1 \to Z_2$ be a smooth morphism. Then Z_1 is pre-k-Du Bois if and only if Z_2 is.*

Proof. The question is local, so we can prove the claim in two steps. First of all, if φ is an étale morphism, then $\underline{\Omega}^p_{Z_1} = \varphi^*(\underline{\Omega}^p_{Z_2})$, and because φ is faithfully flat, we see that the claim is true in this setting.

On the other hand, if $\varphi\colon Z_1 = Z_2 \times Y \to Z_2$ is a smooth projection, where Y is a smooth variety, we want to understand $\underline{\Omega}^p_{Z_1}$ in terms of $\underline{\Omega}^\bullet_{Z_2}$ and $\underline{\Omega}^\bullet_Y = \Omega^\bullet_Y$. To do this, locally embed $i\colon Z_2 \subseteq W$ with W a smooth variety. Let $\pi\colon W \times Y \to W$ be the smooth projection and let $i\colon Z_2 \times Y \to W \times Y$ be the closed embedding.

One can check the following: given $M^\bullet \in D^b(\mathrm{MHM}(W))$ and $N^\bullet \in D^b(\mathrm{MHM}(Y))$, there is a natural quasi-isomorphism

$$\mathrm{Gr}^F_k \mathrm{DR}(M^\bullet \boxtimes N^\bullet) \cong \bigoplus_{i+j=k} \mathrm{Gr}^F_i \mathrm{DR}(M^\bullet) \boxtimes \mathrm{Gr}^F_j \mathrm{DR}(N^\bullet).$$

Applying this with $M^\bullet = i_*\mathbb{Q}^H_{Z_2}$ and $N^\bullet = \mathbb{Q}^H_W$, we get

$$\underline{\Omega}^k_{Z_2\times Y} = \bigoplus_{i+j=k} \underline{\Omega}^i_{Z_2} \boxtimes \underline{\Omega}^j_Y = \bigoplus_{i+j=k} \underline{\Omega}^i_{Z_2} \boxtimes \Omega^j_Y,$$

where the latter equality is due to the smoothness of Y. This follows from the fact that

$$i_*\mathbb{Q}^H_{Z_2\times Y} = \pi^*(i_*\mathbb{Q}^H_{Z_2}) = i_*\mathbb{Q}^H_{Z_2} \boxtimes \mathbb{Q}^H_W$$

and the equalities

$$\begin{aligned} \underline{\Omega}^k_{Z_2\times Y} &= \mathrm{Gr}^F_{-k}\mathrm{DR}(\mathbb{Q}^H_{Z_2\times Y})[k], \\ \underline{\Omega}^i_{Z_2} &= \mathrm{Gr}^F_{-i}\mathrm{DR}(\mathbb{Q}^H_{Z_2})[i], \\ \underline{\Omega}^j_Y &= \mathrm{Gr}^F_{-j}\mathrm{DR}(\mathbb{Q}^H_Y)[j]. \end{aligned}$$

As Ω^j_Y is a sheaf, we see that $\underline{\Omega}^\ell_{Z_2\times Y}$ is a sheaf for all $\ell \le k$ if and only if $\underline{\Omega}^\ell_{Z_2}$ is a sheaf for all $\ell \le k$, proving the claim by definition of pre-k-Du Bois. □

We immediately obtain the following

Corollary 6.12. *Let $\varphi\colon Z_1 \to Z_2$ be a smooth morphism between normal varieties. Then Z_2 is pre-k-rational if and only if Z_1 is pre-k-rational.*

7. Partial Poincaré duality. This short subsection proves a partial Poincaré duality result under the assumption $\mathrm{HRH}(Z) \ge k$. Our goal is to understand how the condition that $F_{k-d}\psi_Z$ is a quasi-isomorphism behaves under the direct image $(a_Z)_*$.

Proposition 7.1. *Let $f\colon X \to Y$ be a morphism of embeddable algebraic varieties. Let $\varphi\colon M^\bullet \to N^\bullet$ be such that $F_p\varphi$ is a quasi-isomorphism. Then $F_p f_*\varphi$ is a quasi-isomorphism.*

Proof. By [Par23, Lem. 3.4], if $C^\bullet$ is the cone of φ, then our assumption implies

$$\mathrm{Gr}^F_\ell \mathrm{DR}_Y(f_*C^\bullet) = 0 \text{ for all } \ell \le p,$$

and so, by Lemma 1.6, we get

$$F_p f_* C^\bullet = 0.$$

Finally, as f_* is an exact functor between triangulated categories, we have the exact triangle

$$f_*M^\bullet \to f_*N^\bullet \to f_*C^\bullet \xrightarrow{+1},$$

and the result follows by looking at the long exact sequence in cohomology, using strictness of morphisms between Hodge modules. □

We can now prove the main result concerning Poincaré duality. We state the result using decreasing Hodge filtrations, as is the convention for the mixed Hodge structure on singular cohomology.

Theorem 7.2. *Let Z be an embeddable complex algebraic variety and assume* $\mathrm{HRH}(Z) \geq k$. *Then for any $i \in \mathbb{Z}$, the natural map*

$$H^{d-i}(Z) \to \mathrm{IH}^{d-i}(Z) \to (H_c^{d+i}(Z)^\vee)(-d)$$

induces isomorphisms

$$F^{d-k}H^{d-i}(Z) \to F^{d-k}\mathrm{IH}^{d-i}(Z) \to F^{-k}H_c^{d+i}(Z)^\vee.$$

Proof. This follows by applying $\mathcal{H}^{-i}(a_Z)_*$ to the map ψ_Z, where $a_Z \colon Z \to \mathrm{pt}$ is the constant map.

After taking some embedding $i\colon Z \to X$ into a smooth variety, the condition $\mathrm{HRH}(Z) \geq k$ implies that the maps

$$F_{k-d}\mathbb{Q}_Z^H[d] \to F_{k-d}\mathrm{IC}_Z^H \to F_{k-d}\mathbf{D}_Z(\mathbb{Q}_Z^H[d])(-d),$$

are quasi-isomorphisms. Applying $(a_Z)_*$ and using Proposition 7.1, we get that

$$F_{k-d}(a_Z)_*\mathbb{Q}_Z^H[d] \to F_{k-d}(a_Z)_*\mathrm{IC}_Z^H \to F_{k-d}(a_Z)_*\mathbf{D}_Z(\mathbb{Q}_Z^H[d])(-d)$$

are quasi-isomorphisms, too. By taking $\mathcal{H}^{-i}$, we get the desired claim. □

8. Generic local cohomological defect. We provide in this section a generalization of Remark 4.8(4)(c) in the embedded setting. The key idea is that one can improve the important codimension bound on the non-rational homology manifold locus in [PP24] by incorporating the "generic" local cohomological defect, which we introduce next.

Let $Z \subseteq X$ be an embedding of the purely d-dimensional variety Z into a smooth connected variety X.

Recall that $K_Z^\bullet$ is the RHM defect object, which lies in the exact triangle

$$K_Z^\bullet \to \mathbb{Q}_Z^H[d] \to \mathrm{IC}_Z^H \xrightarrow{+1} .$$

Let $\mathcal{S} = \{S_\alpha\}_{\alpha \in I}$ be a Whitney stratification of X such that Z is a union of strata and so that we have containment

$$\mathrm{Ch}(\mathcal{K}_Z^\bullet) = \bigcup_{\ell \in \mathbb{Z}} \mathrm{Ch}(\mathcal{H}^\ell \mathcal{K}_Z^\bullet) \subseteq \bigcup_{\alpha \in I} T^*_{S_\alpha}(X),$$

where $\mathcal{K}_Z^\bullet$ is the underlying complex of $\mathscr{D}_X$-modules for $K_Z^\bullet$.

This implies for all $j, \ell \in \mathbb{Z}$ and $\alpha \in I$, we have

$$\mathcal{H}^j \mathrm{DR}_{S_\alpha}(i_\alpha^* \mathcal{H}^\ell K_Z^\bullet)$$

is locally constant, where $i_\alpha \colon S_\alpha \to X$ is the locally closed embedding. Then the function

$$x \mapsto \mathrm{lcdef}_x(Z)$$

is constructible with respect to this stratification.

Definition 8.1. Let $\mathcal{S} = \{S_\alpha\}_{\alpha \in I}$ be a Whitney stratification as above and $\mathrm{lcdef}_{S_\alpha}(Z)$ the value of lcdef on S_α. We denote

$$\mathrm{lcdef}_{\mathrm{gen}}(Z) = \max_{\dim S_\alpha = \dim Z_{\mathrm{nRS}}} \mathrm{lcdef}_{S_\alpha}(Z).$$

This invariant can be detected through local cohomology in our embedded setting.

Lemma 8.2. *Let $Z \subseteq X$ be an embedding into a smooth variety X with* $\operatorname{codim}_X(Z) = c$. *Let $\{S_\alpha\}$ be a Whitney stratification of X as above. Then*

$$\operatorname{lcdef}_{S_\alpha}(Z) = \max\{i \mid S_\alpha \subseteq \operatorname{Supp} \mathcal{H}^{c+i}_Z(\mathscr{O}_X)\}.$$

Proof. This is immediate by choice of stratification. ☐

This invariant can also be detected without mention of a Whitney stratification.

Lemma 8.3. *Let $Z \subseteq X$ be an embedding into a smooth variety X, with* $\operatorname{codim}_X(Z) = c$. *For $i \geq 0$, define*

$$d(i) = \begin{cases} \dim \operatorname{Supp} \mathcal{H}^{c+i}_Z(\mathscr{O}_X) & i > 0 \\ \dim \operatorname{Supp} \mathcal{H}^{c}_Z(\mathscr{O}_X)/\mathrm{IC}^H_Z & i = 0 \end{cases}.$$

Then

$$\operatorname{lcdef}_{\mathrm{gen}}(Z) = \max\{i \mid d(i) \geq d(j) \text{ for all } j \geq 0\}.$$

Proof. By definition, we have

$$Z_{\mathrm{nRS}} = \operatorname{Supp}(K^\bullet_Z) = \operatorname{Supp}(\mathbf{D}_Z(K^\bullet_Z)) = \operatorname{Supp}(\mathcal{H}^c_Z(\mathscr{O}_X)/\mathrm{IC}^H_Z) \cup \bigcup_{i>0} \operatorname{Supp}\mathcal{H}^{c+i}_Z(\mathscr{O}_X).$$

Note that, for any i which satisfies $d(i) \geq d(j)$ for all $j \geq 0$, we have $d(i) = \dim Z_{\mathrm{nRS}}$. Thus, the claim is immediate by the previous lemma and the fact that the support of Z_{nRS} is a union of strata. ☐

Now, we show that the value $\operatorname{lcdef}_x(Z)$ for $x \in S_\alpha$ is unchanged upon taking a normal slice.

Proposition 8.4. *Let $\{S_\alpha\}$ be a Whitney stratification of X as above and fix $x \in S_\alpha$. Let $T_\alpha \subseteq X$ be a normal slice through x, meaning a smooth subvariety of dimension* $\dim X - \dim S_\alpha$ *such that $T_\alpha \cap S_\alpha = \{x\}$ is a transverse intersection. Then*

$$\operatorname{lcdef}_x(Z) = \operatorname{lcdef}_x(Z \cap T_\alpha).$$

Proof. By replacing X with a neighborhood of x, we can assume the stratification $\{S_\alpha\}$ is finite. By further replacing X by $X \setminus \bigcup_{\beta \in B} \overline{S_\beta}$, where $B = \{\gamma \mid S_\alpha \not\subseteq \overline{S_\beta}\}$, we can assume S_α is a minimal stratum in the sense that $S_\alpha \subseteq \overline{S_\gamma}$ for all γ. As the support of each local cohomology is a union of strata and closed, we see then that after this restriction, we have

$$\operatorname{lcdef}_x(Z) = \operatorname{lcdef}_{S_\alpha}(Z) = \operatorname{lcdef}(Z).$$

In this case, Whitney's condition (a) implies that the normal slice T_α has transverse intersection with all strata. If $\iota\colon T_\alpha \to X$ is the closed embedding, this implies that ι is non-characteristic with respect to $\mathcal{H}^j(K^\bullet_Z)$ for all $j \in \mathbb{Z}$.

Thus, the spectral sequence

$$E_2^{i,j} = \mathcal{H}^i \iota^* \mathcal{H}^j(K^\bullet_Z) \implies \mathcal{H}^{i+j}\iota^*(K^\bullet_Z)$$

degenerates at E_2, because the only non-zero terms must have $i = -\dim S_\alpha$, the relative dimension of the embedding ι. This gives equality

$$\mathcal{H}^{j-\dim S_\alpha}\iota^*(K^\bullet_Z) = \mathcal{H}^j K^\bullet_Z \otimes_{\mathscr{O}_X} \mathscr{O}_{T_\alpha}.$$

Again using that ι is non-characteristic, we have

$$\iota^*(K_Z^\bullet) = K_{Z\cap T_\alpha}^\bullet[\dim S_\alpha].$$

Finally, using that $\operatorname{lcdef}(Z) = \max\{i \mid \mathcal{H}^{-i}K_Z^\bullet \neq 0\}$ and the same formula for $\operatorname{lcdef}(Z\cap T_\alpha)$, we get the desired equality. □

Putting these together, we can show that the bound $\operatorname{codim}_Z(Z_{\mathrm{nRS}}) \geq 2\,\mathrm{HRH}(Z) + 3$ can be improved if one has knowledge of the "generic" local cohomological defect. This method of proof is inspired by [Sai94, Rmk. 2.11].

Proposition 8.5. *Let Z be a purely d-dimensional complex algebraic variety and let $\{S_\alpha\}$ be a Whitney stratification as above. Then for all $x \in S_\alpha$, we have*

$$\operatorname{lcdef}_{S_\alpha}(Z) \leq \max\{\operatorname{codim}_Z(S_\alpha) - 2\,\mathrm{HRH}_x(Z) - 3, 0\}.$$

In particular, we have

$$\operatorname{lcdef}_{\mathrm{gen}}(Z) \leq \max\{\operatorname{codim}_Z(Z_{\mathrm{nRS}}) - 2\,\mathrm{HRH}(Z) - 3, 0\}.$$

Proof. The first claim implies the second by first taking α such that $\operatorname{lcdef}_{S_\alpha}(Z) = \operatorname{lcdef}_{\mathrm{gen}}(Z)$ and $\dim S_\alpha = \dim Z_{\mathrm{nRS}}$, and then noting that $\mathrm{HRH}(Z) \leq \mathrm{HRH}_x(Z)$. So it suffices to prove the first claim.

For $x \in S_\alpha$, take a normal slice T_α through x. The first claim is then immediate from the fact that $\mathrm{HRH}_x(Z) \leq \mathrm{HRH}_x(Z\cap T_\alpha)$ and the inequality

$$\operatorname{lcdef}_{S_\alpha}(Z) = \operatorname{lcef}_x(Z) = \operatorname{lcef}_x(Z\cap T_\alpha) \leq \max\{\dim(T_\alpha) - 2\,\mathrm{HRH}_x(Z\cap T_\alpha) - 3, 0\}.$$

This completes the proof.

As an alternative proof, one can use Lemma 6.2. Indeed, assuming $\mathrm{HRH}(Z) \geq 0$ this lemma tells us that

$$\dim \operatorname{Supp}\mathcal{H}^i\mathbf{D}_Z(K_Z^\bullet) \leq \dim(Z) - 2\,\mathrm{HRH}(Z) - 3 - i,$$

and so choosing i maximal so that $S_\alpha \subseteq \operatorname{Supp}(\mathcal{H}^i\mathbf{D}_Z(K_Z^\bullet))$, we get

$$\dim S_\alpha \leq \dim \mathcal{H}^i\mathbf{D}_Z(K_Z^\bullet) \leq \dim(Z) - 2\,\mathrm{HRH}(Z) - 3 - i,$$

but by definition, such an i is $\operatorname{lcdef}_{S_\alpha}(Z)$. □

Remark 8.6. When Z_{nRS} locus is isolated, we have $\operatorname{lcdef}_{\mathrm{gen}}(Z) = \operatorname{lcdef}(Z)$ (in particular, if $\dim Z \leq 3$ and $\mathrm{HRH}(Z) \geq 0$, we always have equality). However, equality can also hold even when Z_{nRS} is non-isolated, see the examples in §14.

D. Local Complete Intersection Case

In this section, we discuss the invariant $\mathrm{HRH}(Z)$ of a variety Z that is locally a complete intersection. We introduce invariants of the singularities that partially capture the degree to which Z is rational homology. Before that, we discuss the case of a hypersurface to illustrate how the vanishing cycles capture the whole picture in this case. The results beyond this case are different ways of generalizing these results.

9. Hypersurfaces. Let Z be a hypersurface of an n-dimensional smooth variety X defined by f. Let

$$\mathcal{B}_f = \bigoplus_{j\geq 0} \mathscr{O}_X \partial_t^j \delta$$

as defined in §2, and

$$F_{k-n}\mathcal{B}_f = \bigoplus_{j\leq k} \mathscr{O}_X \partial_t^j \delta.$$

Recall that by Theorem B, Z is k-Hodge rational homology if

$$F_{k-n}W_{n+1}\mathcal{H}^1_Z(\mathscr{O}_X) = F_{k-n}\mathcal{H}^1_Z(\mathscr{O}_X).$$

The local cohomology is captured by the unipotent nearby and vanishing cycles and their corresponding filtrations. More precisely, we have a short exact sequence

$$0 \to \mathrm{Gr}^0_V(\mathcal{B}_f) \xrightarrow{t\cdot} \mathrm{Gr}^1_V(\mathcal{B}_f) \to \mathcal{H}^1_Z(\mathscr{O}_X) \to 0,$$

which is bi-strict with respect to the Hodge filtration and the weight filtration, where the weight filtration of the first two terms is induced by their monodromy operators, and underlies the sequence of mixed Hodge modules (see, for example, [Ola23, Thm. A])

$$0 \to \varphi_{f,1}\mathbb{Q}^H_X[n] \xrightarrow{Var} \psi_{f,1}\mathbb{Q}^H_X[n](-1) \to \mathcal{H}^1_Z(\mathbb{Q}^H_X[n]) \to 0.$$

Recall the convention for the Hodge filtration on nearby cycles, when indexing the Hodge filtration as for right $\mathscr{D}$-modules, is as follows:

$$F_p\psi_{f,1}(\mathscr{O}_X) = F_{p-1}\mathrm{Gr}^1_V(\mathcal{B}_f),$$

and that the weight filtration is defined as the monodromy filtration for the nilpotent operator (N or $t\partial_t$) centered at $n-1$. In general, the monodromy filtration satisfies (see [SZ85, Rmk. 2.3])

$$W_{n-1+i}\,\mathrm{Gr}^1_V(\mathcal{B}_f) = \sum_{\ell\geq \max\{0,-i\}} (t\partial_t)^\ell \ker((t\partial_t)^{1+i+2\ell}).$$

However, in our situation, $(t\partial_t)^\ell \colon (\mathrm{Gr}^1_V(\mathcal{B}_f), F) \to (\mathrm{Gr}^1_V(\mathcal{B}_f), F[-\ell])$ is strict for all $\ell \geq 1$. It is an easy exercise, then, to see

$$F_pW_{n-1+i}\,\mathrm{Gr}^1_V(\mathcal{B}_f) = \sum_{\ell\geq \max\{0,-i\}} (t\partial_t)^\ell F_{p-\ell} \ker((t\partial_t)^{1+i+2\ell}).$$

This discussion, by taking $i=0$, gives the containments

$$F_p \ker(t\partial_t) \subseteq F_pW_{n-1}\,\mathrm{Gr}^1_V(\mathcal{B}_f) \subseteq F_p \ker(t\partial_t) + t(F_p\,\mathrm{Gr}^0_V(\mathcal{B}_f)),$$

and hence, the formula

$$F_pW_{n+1}\mathcal{H}^1_Z(\mathscr{O}_X) = \frac{F_pW_{n-1}\mathrm{Gr}^1_V(\mathcal{B}_f) + t(F_p\,\mathrm{Gr}^0_V(\mathcal{B}_f))}{t(F_p\,\mathrm{Gr}^0_V(\mathcal{B}_f))} = \frac{F_p\ker(t\partial_t) + t(F_p\mathrm{Gr}^0_V(\mathcal{B}_f))}{t(F_p\mathrm{Gr}^0_V(\mathcal{B}_f))}.$$

Proof of Theorem H. Assume first that $F_{k-n}\mathrm{Gr}^0_V(\mathcal{B}_f) = 0$. This means that every element in $F_{k-1-n}\,\mathrm{Gr}^1_V(\mathcal{B}_f) = F_{k-1-n}(\psi_{f,1}(\mathscr{O}_X)(-1))$ lies in the subset $\ker\partial_t = \ker t\partial_t \subseteq W_{n-1}\mathrm{Gr}^1_V(\mathscr{O}_X) = W_{n+1}(\psi_{f,1}(\mathbb{Q}^H_X[n])(-1))$, where the containment follows by definition of the monodromy filtration centered at $n-1$. This implies that

$$F_{k-1-n}\mathcal{H}^1_Z(\mathscr{O}_X) \subseteq W_{n+1}\mathcal{H}^1_Z(\mathscr{O}_X),$$

and therefore, $\mathrm{HRH}(Z) \geq k-1$.

Suppose now that $F_{k-1-n}W_{n+1}\mathcal{H}^1_Z(\mathscr{O}_X) = F_{k-1-n}\mathcal{H}^1_Z(\mathscr{O}_X)$. By induction on k, we have that $F_{k-1-n}\mathrm{Gr}^0_V(\mathcal{B}_f) = 0$. Thus, we have an isomorphism

$$F_{k-1-n}\mathrm{Gr}^1_V(\mathcal{B}_f) \cong F_{k-1-n}\mathcal{H}^1_Z(\mathscr{O}_X).$$

To prove the claim, it suffices by strict surjectivity of $\partial_t\colon (\mathrm{Gr}^1_V(\mathcal{B}_f), F) \to (\mathrm{Gr}^0_V(B_f), F[-1])$ to prove that $F_{k-1-n}\mathrm{Gr}^1_V(\mathcal{B}_f) \subseteq \ker(\partial_t) = \ker(t\partial_t)$. As $F_{k-1-n}\mathrm{Gr}^0_V(\mathcal{B}_f) = 0$, we get

$$F_{k-1-n}\mathcal{H}^1_Z(\mathscr{O}_X) = F_{k-1-n}\ker(N),$$

which finishes the proof. □

Suppose now that Z has an isolated singularity. Let F be the Milnor fiber of Z. It is well-known that

$$\mathrm{DR}(\mathrm{Gr}^0_V(B_f)) \cong H^{n-1}(F)_1$$

supported on the singular point. Moreover, the dimension of the Hodge filtration of this cohomology is controlled by the spectral numbers. More precisely, if $\alpha_{f,i}$ are the spectral numbers, then

$$\#\{i \mid \alpha_{f,i} = k\} = \dim \mathrm{Gr}^p_F H^{n-1}(F)_1$$

for $p = n-k$ (and similarly for non-integer ones, see e.g., [Sai07, §3]). By using induction and the definition of the Hodge filtration and the de-Rham functor, the following result follows from Theorem H.

Corollary 9.1. *Let Z be a hypersurface of a smooth variety that has an isolated singularity at $x \in Z$. Then,*

$$\mathrm{HRH}(Z) = \mathrm{Sp}_{\min,\mathbb{Z}}(Z,x) - 2.$$

Remark 9.2. The results in this section can be proved with the techniques of the upcoming sections. The hypersurface case is a good illustration of objects introduced in the case of local complete intersections. See Remark 11.7.

10. Reinterpretation using specialization. We now focus on the local complete intersection case. Let $Z = V(f_1,\dots,f_r) \subseteq X$ be a complete intersection subvariety of pure codimension r, where X is a smooth irreducible variety of dimension n. We introduce in this section several integer invariants associated to Z and show how they relate to each other. In the hypersurface case, we will see that these numbers are all essentially the same, except for the invariant defined using the Bernstein-Sato polynomial.

We have the short exact sequence of monodromic mixed Hodge modules on $X \times \mathbb{A}^r_z$,

$$0 \to L \to \mathrm{Sp}(B_f)^{\mathbb{Z}} \to Q^{\mathbb{Z}} \to 0,$$

which in this case encodes the morphism $i_*\psi_Z$. To see this, let $\sigma\colon X \to X \times \mathbb{A}^r_z$ be the inclusion of the zero section. By Proposition 2.6 above, we see that $\sigma^!(M) = \sigma^!(M^{\mathbb{Z}})$ for any monodromic module M (for example, $\mathrm{Sp}(B_f)$ and Q).

We have by [Sai90, Pg. 269] that

$$\sigma^!(\mathrm{Sp}(B_f)) = \sigma^!(B_f) \cong i_*i^!\mathbb{Q}^H_X[n],$$

where we also use σ to denote the zero section in $X \times \mathbb{A}^r_t$ for the middle term, and the last equality follows by Base Change (Example 1.3).

Applying $\sigma^!$ to the short exact sequence, we get an exact triangle

$$\sigma^! L \to i_* i^! \mathbb{Q}_X^H[n] \to \sigma^! Q \xrightarrow{+1} .$$

Recalling that $L = i_* \mathbb{Q}_{Z \times \mathbb{A}_z^r}[n]$, it is a simple computation to check that

$$\sigma^! L = i_* \mathbb{Q}_Z^H[n-2r](-r) = (i_* \mathbb{Q}_Z^H[d])[-r](-r),$$

and so the morphism $\chi\colon \sigma^! L \to i_* i^! \mathbb{Q}_X^H[n]$ is (up to non-zero scalar multiplication on the irreducible components of Z) the map $\psi_Z[-r](-r)$. By looking at the long exact sequence in cohomology, the only non-zero terms are the rightmost four:

$$0 \to \mathcal{H}^{r-1}(\sigma^! Q) \to (i_* \mathbb{Q}_Z^H[d])(-r) \xrightarrow{\chi} \mathcal{H}_Z^r(\mathbb{Q}_X^H[n]) \to \mathcal{H}^r(\sigma^! Q) \to 0.$$

Proposition 10.1. *In the notation above, we have the following:*

(1) $F_{k-d}\gamma_Z$ is an isomorphism if and only if $F_{k-n}\mathcal{H}^{r-1}(\sigma^! Q) = 0$.
(2) $F_{k-d}\gamma_Z^\vee$ is an isomorphism if and only if $\mathrm{HRH}(Z) \geq k$ if and only if $F_{k-n}\mathcal{H}^r(\sigma^! Q) = 0$

In particular, $F_{k-n}\mathcal{H}^r(\sigma^! Q) = 0$ implies $F_{k-n}\mathcal{H}^{r-1}(\sigma^! Q) = 0$.

Proof. Recall that $F_{k-d}\gamma_Z$ is an isomorphism if and only if it is injective. This is true if and only if $F_{k-d}\psi_Z$ is injective. As $\psi_Z(-r)$ and χ agree up to non-zero scalar multiples, this is equivalent to $F_{k-n}\chi$ being injective, which by the exact sequence is equivalent to the vanishing $F_{k-n}\mathcal{H}^{r-1}(\sigma^! \mathcal{Q})$.

The other claim is shown similarly. □

We can rephrase the results of the proposition purely in terms of the V-filtration.

Proposition 10.2. *In the notation above, we have the following:*

(1) The map $F_{k-d}\gamma_Z$ is an isomorphism if and only if

$$\left(\sum_{i=1}^r t_i F_{k-n}\mathrm{Gr}_V^{r-1}(\mathcal{B}_f)\right) \cap \left(\bigcap_{i=1}^r \ker(\partial_{t_i}\colon F_{k-n}\mathrm{Gr}_V^r(\mathcal{B}_f) \to F_{k-n+1}\mathrm{Gr}_V^{r-1}(\mathcal{B}_f))\right) = 0.$$

(2) The map $F_{k-d}\gamma_Z^\vee$ is an isomorphism if and only if $\mathrm{HRH}(Z) \geq k$ if and only if

$$\left(\sum_{i=1}^r t_i F_{k-n}\mathrm{Gr}_V^{r-1}(\mathcal{B}_f)\right) + \left(\bigcap_{i=1}^r \ker(\partial_{t_i}\colon F_{k-n}\mathrm{Gr}_V^r(\mathcal{B}_f) \to F_{k-n+1}\mathrm{Gr}_V^{r-1}(\mathcal{B}_f))\right) = F_{k-n}\mathrm{Gr}_V^r(\mathcal{B}_f).$$

In particular $\mathrm{HRH}(Z) \geq k$ if and only if

$$F_{k-n}\mathrm{Gr}_V^r(\mathcal{B}_f) = \left(\sum_{i=1}^r t_i F_{k-n}\mathrm{Gr}_V^{r-1}(\mathcal{B}_f)\right) \oplus \left(\bigcap_{i=1}^r \ker(\partial_{t_i}\colon F_{k-n}\mathrm{Gr}_V^r(\mathcal{B}_f) \to F_{k-n+1}\mathrm{Gr}_V^{r-1}(\mathcal{B}_f))\right)$$

Proof. These are immediate restatements of the conditions in the previous proposition, using the definition of the underlying filtered $\mathscr{D}$-module of Q. □

Thus, we get a V-filtration characterization of rational smoothness in the local complete intersection case.

Corollary 10.3. *In the above notation, Z is a rational homology manifold if and only if*

$$\mathrm{Gr}_V^r(\mathcal{B}_f) = \left(\sum_{i=1}^r t_i \mathrm{Gr}_V^{r-1}(\mathcal{B}_f)\right) \oplus \left(\bigcap_{i=1}^r \ker(\partial_{t_i}\colon \mathrm{Gr}_V^r(\mathcal{B}_f) \to \mathrm{Gr}_V^{r-1}(\mathcal{B}_f))\right).$$

In fact, we can say the following:

Theorem 10.4. *Let $Z \subseteq X$ be a complete intersection defined by $f_1, \ldots, f_r \in \mathscr{O}_X(X)$. The following are equivalent:*

(1) $\mathrm{Sp}(B_f)^{\mathbb{Z}}$ *is a pure Hodge module of weight* n*.*
(2) $N = 0$ *on* $\mathrm{Sp}(B_f)^{\mathbb{Z}}$*.*
(3) Q *is a pure Hodge module of weight* n*.*

Moreover, any of those equivalent conditions implies the following (all of which are equivalent to each other):

(1) Z *is a rational homology manifold.*
(2) $Z \times \mathbb{A}^r$ *is a rational homology manifold.*
(3) $L = \mathbb{Q}^H_{Z\times\mathbb{A}^r}[n]$ *is a pure Hodge module of weight* n*.*

If $r = 1$*, the converse holds.*

Proof. The equivalence of the second collection of three conditions is obvious.

The equivalence of the first and second conditions follows from the fact that the weight filtration on $\mathrm{Sp}(B_f)^{\mathbb{Z}}$ is the monodromy filtration for N centered at n. Either condition implies the third, because Q is a quotient of $\mathrm{Sp}(B_f)$ by definition.

To see that the third implies the first, recall that $W_n L = L$. Thus, for all $i \in Z$, we have a short exact sequence,

$$0 \to \mathrm{Gr}^W_{n+i} L \to \mathrm{Gr}^W_{n+i}\mathrm{Sp}(B_f)^{\mathbb{Z}} \to \mathrm{Gr}^W_{n+i} Q^{\mathbb{Z}} \to 0,$$

and for $i > 0$, we have an isomorphism $\mathrm{Gr}^W_{n+i}\mathrm{Sp}(B_f)^{\mathbb{Z}} = \mathrm{Gr}^W_{n+i} Q^{\mathbb{Z}}$. Thus, $W_n Q = Q$ implies $\mathrm{Sp}(B_f)$ is pure.

We see that the condition that $\mathrm{Sp}(B_f)^{\mathbb{Z}}$ is pure of weight n implies L is pure of weight n.

To prove the converse, assume $r = 1$. Then $Z = \{f = 0\}$ is a rational homology manifold if and only if $\varphi_{f,1}(\mathscr{O}_X) = 0$ if and only if $N = 0$ on $\psi_{f,1}(\mathscr{O}_X)$. But then

$$\mathrm{Sp}(B_f)^{\mathbb{Z}} = \bigoplus_{\ell \geq 1} \mathrm{Gr}^1_V(\mathcal{B}_f)$$

clearly has $N = 0$. ∎

Remark 10.5. The difficulty in proving the converse of the above theorem in $r > 1$ is the fact that we do not know if the inequality

$$\min\{i \mid (s+r)^i \mathrm{Gr}^r_V(\mathcal{B}_f) = 0\} \geq \min\{\ell \mid (s+r-1)^\ell \mathrm{Gr}^{r-1}_V(\mathcal{B}_f) = 0\}$$

is strict. This is true when $r = 1$.

The inequality cannot be strict in general: indeed, if Z is a rational homology manifold, then by Corollary 10.3, we have

$$\mathrm{Gr}^r_V(\mathcal{B}_f) = \ker(s+r) + \sum_{i=1}^{r} t_i \mathrm{Gr}^{r-1}_V(\mathcal{B}_f),$$

and so there are two possibilities: $\mathrm{Gr}^{r-1}_V(\mathcal{B}_f) = 0$, in which case $Q^{\mathbb{Z}} = 0$, or the inequality is an equality.

11. Integer invariants and relations among them. We can now introduce the second integer invariant of our interest, which is $p(Q^{\mathbb{Z}}, F)$. Immediately we obtain the following:

Proposition 11.1. *We have an inequality*

$$p(Q^{\mathbb{Z}}, F) + n - 1 \leq \mathrm{HRH}(Z).$$

Proof. By definition, $F_{p(Q^{\mathbb{Z}},F)-1} Q^{\mathbb{Z}} = 0$, so $F_{p(Q^{\mathbb{Z}},F)-1} \mathcal{H}^r \sigma^!(\mathcal{Q}) = 0$ by Proposition 2.6. □

Before introducing the next integer invariant, we study how the invariant $p(Q^{\mathbb{Z}}, F)$ relates to the invariants $p(\mathcal{Q}^k, F) = \min\{p \mid F_p \mathcal{Q}^k \neq 0\}$.

Lemma 11.2. *The following hold:*

1. $p(Q^{\mathbb{Z}}, F) = p(\mathcal{Q}^r, F) = p(\mathcal{Q}^\ell, F)$ *for all* $\ell \geq r$.
2. $p(\mathcal{Q}^{1-\ell}, F) = p(\mathcal{Q}^1, F) + \ell$ *for all* $\ell \geq 0$.
3. *For all* $k \in [1, r) \cap \mathbb{Z}$, *we have*

$$p(\mathcal{Q}^{k+1}) \leq p(\mathcal{Q}^k) \leq p(\mathcal{Q}^{k+1}) + 1.$$

Proof. The filtered acyclicity of $B^j(\mathcal{Q}, F)$ for $j > 0$, given by Proposition 2.5 above, tells us that there are surjections

$$\bigoplus_{i=1}^r F_p \mathcal{Q}^{r+j-1} \xrightarrow{z_i} F_p \mathcal{Q}^{r+j},$$

which inductively proves $p(\mathcal{Q}^\ell, F) \geq p(\mathcal{Q}^r, F)$ for all $\ell \geq r$. The same filtered acyclicity tells us that the map

$$F_p \mathcal{Q}^j \xrightarrow{z_i} \bigoplus_{i=1}^r F_p \mathcal{Q}^{j+1}$$

is injective for all $j > 0$, so that

$$p(\mathcal{Q}^j, F) \geq p(\mathcal{Q}^{j+1}, F) \text{ for all } j > 0.$$

These together show that $p(\mathcal{Q}^r, F) = p(\mathcal{Q}^\ell, F)$ for all $\ell \geq r$.

The filtered acyclicity of $C^j(\mathcal{Q}, F)$ for all $j < 0$ gives surjections

$$\bigoplus_{i=1}^r F_p \mathcal{Q}^{j+1} \xrightarrow{\partial_{z_i}} F_{p+1} \mathcal{Q}^j,$$

for all $j < 0$, which gives $p(\mathcal{Q}^j, F) \geq p(\mathcal{Q}^{j+1}, F) + 1$ for all $j < 0$. Moreover, it gives injections

$$F_p \mathcal{Q}^{j+r} \xrightarrow{\partial_{z_i}} F_{p+1} \mathcal{Q}^{j+r-1}$$

so that $p(\mathcal{Q}^j, F) \leq p(\mathcal{Q}^{j+1}, F) + 1$ for all $j < r$.

This proves almost all claims, except we need to show that $p(\mathcal{Q}^0, F) = p(\mathcal{Q}^1, F) + 1$.

The long exact sequence in cohomology for σ^* applied to the short exact sequence (2.8) gives isomorphisms

$$\mathcal{H}^0 \sigma^*(Q) \cong \mathcal{H}^0 \sigma^*(\mathrm{Sp}(\mathcal{B}_f)) \cong \mathcal{H}^0 \sigma^*(\mathcal{B}_f).$$

and all three modules vanish, because $\mathcal{B}_f$ has no quotient object supported on $X \times \{0\}$.

Thus, $\mathcal{H}^0(C^0(\mathcal{Q},F))=0$, giving a surjection

$$\bigoplus_{i=1}^r F_p\mathcal{Q}^1 \xrightarrow{\partial_{z_i}} F_{p+1}\mathcal{Q}^0,$$

and proving the last remaining claim. ☐

This leads to a natural condition for equality in Proposition 11.1.

Proposition 11.3. *Assume $p(\mathcal{Q}^{r-1},F)=p(Q^{\mathbb{Z}},F)+1$. Then*

$$\mathrm{HRH}(Z)=p(Q^{\mathbb{Z}},F)+n-1.$$

Proof. The assumption tells us that $F_{p(Q^{\mathbb{Z}},F)}\mathcal{Q}^{r-1}=F_{p(Q^{\mathbb{Z}},F)}\mathrm{Gr}_V^{r-1}(\mathcal{B}_f)=0$. Then

$$F_{p(Q^{\mathbb{Z}},F)}\mathcal{H}^r(\sigma^!(\mathcal{Q}))=F_{p(Q^{\mathbb{Z}},F)}\mathcal{Q}^r/\sum_{i=1}^r z_iF_{p(Q^{\mathbb{Z}},F)}\mathcal{Q}^{r-1}=F_{p(Q^{\mathbb{Z}},F)}\mathcal{Q}^r\neq 0,$$

proving that $\mathrm{HRH}(Z)<p(Q^{\mathbb{Z}},F)+n$. ☐

The next integer invariant is the *minimal integer spectral number* of Z at a point $x\in Z$, which we denote by $\mathrm{Sp}_{\min,\mathbb{Z}}(Z,x)$.

An immediate application of Lemma 2.9 above gives the following lower bound. Here

$$p(Q^{\mathbb{Z}}_x,F)=\min\{p\in\mathbb{Z}\mid (F_p\mathcal{Q}^{\mathbb{Z}})_x\neq 0\}$$

is the lowest non-vanishing index of the stalk of the Hodge filtration at x.

Proposition 11.4. *Let $x\in Z$ be a point in the local complete intersection variety Z. Then*

$$\mathrm{Sp}_{\min,\mathbb{Z}}(Z,x)\geq p(Q^{\mathbb{Z}}_x,F)+n+1.$$

The same condition in Proposition 11.3 above gives a condition which allows us to ensure equality in the proposition above in the isolated singularities case.

Proposition 11.5. *Let $x\in Z$ be an isolated singular point. Assume $p(\mathcal{Q}^{r-1}_x,F)=p(Q^{\mathbb{Z}}_x,F)+1$. Then*

$$\mathrm{Sp}_{\min,\mathbb{Z}}(Z,x)=p(Q^{\mathbb{Z}}_x,F)+n+1.$$

Proof. The assumption allows us to use [Dir23, Lem. 2.6], which tells us that

$$\mathrm{Supp}_{\mathbb{A}^r_z}(F_{p(\mathcal{Q},F)}\mathcal{Q}^{\mathbb{Z}})=\mathbb{A}^r_z.$$

Then the proof goes through in exactly the same way as the last step of the proof of [Dir23, Thm. 1.1]. ☐

Remark 11.6. We have the following: $Q^{\mathbb{Z}}=0$ if and only if $p(Q^{\mathbb{Z}},F)=+\infty$, which implies that $\mathrm{Sp}_{\min,\mathbb{Z}}(Z,x)=+\infty$ and that Z is a rational homology manifold.

It is easy to check that $Q^{\mathbb{Z}}=0$ if and only if $\mathrm{Gr}_V^{r-1}(\mathcal{B}_f)=0$. Example 16.2 below shows, then, that $Q^{\mathbb{Z}}=0$ is too strong: there is a complete intersection variety Z which is a rational homology manifold but with $Q^{\mathbb{Z}}\neq 0$.

Remark 11.7. In the hypersurface case, the situation simplifies immensely. Let $Z = V(f) \subseteq X$. It is well known that Z is a rational homology manifold if and only if $\varphi_{f,1}(\mathscr{O}_X) = 0$, see, for example, [JKSY22, Thm. 3.1]. By the previous remark, this is true if and only if $Q^{\mathbb{Z}} = 0$. In this remark, we give another proof of that fact.

We use the following notation as we have fixed a defining function for Z:

$$\mathrm{Sp}_{\min,\mathbb{Z}}(Z,x) = \mathrm{Sp}_{\min,\mathbb{Z}}(f,x),$$

$$\mathrm{HRH}(Z) = \mathrm{HRH}(f).$$

If $Q^{\mathbb{Z}} \neq 0$, then in this case we have $p(\mathrm{Gr}_V^0(\mathcal{B}_f), F) = p(Q^{\mathbb{Z}}, F)+1$. Indeed, this follows immediately from the fact that

$$\mathrm{can} = \partial_t \colon \mathrm{Gr}_V^1(\mathcal{B}_f, F) \to \mathrm{Gr}_V^0(\mathcal{B}_f, F[-1])$$

is strictly surjective.

By Proposition 11.3, we see that $\mathrm{HRH}(f) = p(Q^{\mathbb{Z}}, F) + n - 1 = p(\varphi_{f,1}(\mathscr{O}_X), F) + n - 2$, which shows that $\mathrm{HRH}(f) = +\infty$ if and only if $p(\varphi_{f,1}(\mathscr{O}_X), F) = +\infty$, proving the equivalence mentioned at the beginning of the remark. Moreover, if Z has an isolated singularity at $x \in Z$, then we get

$$\mathrm{Sp}_{\min,\mathbb{Z}}(Z,x) = p(Q^{\mathbb{Z}}, F) + n + 1 = p(\varphi_{f,1}(\mathscr{O}_X), F) + n.$$

This gives another proof of the equality noted in [JKSY22, Formula (13)].

In [CDMO24], when defining the minimal exponent for the complete intersection subvariety $Z = V(f_1, \dots, f_r) \subseteq X$, an auxiliary construction is used. This is the *general linear combination hypersurface*, defined by $g = \sum_{i=1}^r y_i f_i$ on $Y = X \times \mathbb{A}^r_y$. Let $U = Y \setminus (X \times \{0\})$ with open embedding $j \colon U \to Y$ and let $\sigma \colon X \times \{0\} \to Y$ be the inclusion of the zero section. Then we have the exact triangle

$$\sigma_*\sigma^! \varphi_{g,1}(\mathbb{Q}_Y^H[n+r]) \to \varphi_{g,1}(\mathbb{Q}_Y^H[n+r]) \to j_*\varphi_{g|_U,1}(\mathbb{Q}_U^H[n+r]) \xrightarrow{+1}$$

In [Dir23], the $\mathscr{D}$-module $\varphi_{g,1}(\mathscr{O}_Y)$ is compared to the Fourier-Laplace transform of $\mathrm{Sp}(\mathcal{B}_f)$. This comparison implies that $\sigma_*\sigma^!\varphi_{g,1}(\mathbb{Q}_Y^H[n+r])$ has a unique non-vanishing cohomology module in the local complete intersection case, namely, the 0-th one. This is due to the fact that $\sigma_*\sigma^*\mathrm{Sp}(\mathcal{B}_f)$ has a unique non-vanishing cohomology module, and the Fourier-Laplace transform interchanges the two types of restriction to the zero section.

Thus, this exact triangle is actually a short exact sequence

$$0 \to \sigma_*\sigma^! \varphi_{g,1}(\mathbb{Q}_Y^H[n+r]) \to \varphi_{g,1}(\mathbb{Q}_Y^H[n+r]) \to j_*\varphi_{g|_U,1}(\mathbb{Q}_U^H[n+r]) \to 0.$$

Each term in the short exact sequence is monodromic along the variables $y_1, \dots, y_r$. In fact, this is the Fourier-Laplace transform of the short exact sequence (2.8). As a corollary of this fact, we get the following.

Corollary 11.8. *In the above notation, we see*

$Q^{\mathbb{Z}} = 0$ *if and only if* $\varphi_{g|_U,1}(\mathscr{O}_U) = 0$ *if and only if* $V(g|_U) \subseteq U$ *is a rational homology manifold.*

In particular, $V(g|_U)$ *being a rational homology manifold implies* Z *is one, too.*

As explained in the proof of [Dir23, Prop. 3.4], there is an isomorphism

$$F_p\varphi_{g,1}(\mathscr{O}_Y)^{r-k} \cong F_{p-k+r}\mathrm{Gr}_V^k(\mathcal{B}_f),$$

where the left hand side is the corresponding monodromic piece. By the short exact sequence and strictness of morphisms with respect to the Hodge filtration, there is also an isomorphism

$$F_p j_*\varphi_{g|_U,1}(\mathscr{O}_U)^{r-k} \cong F_{p-k+r}\mathcal{Q}^k, \tag{11.9}$$

which we can use to prove the following.

Proposition 11.10. *Let* $p(\varphi_{g|_U}(\mathscr{O}_U)) = \min\{p \mid F_p\varphi_{g|_U}(\mathscr{O}_U) \neq 0\} = \min\{p \mid F_p j_*(\varphi_{g|_U}(\mathscr{O}_U)) \neq 0\}$. *Then*

$$p(\varphi_{g|_U}(\mathscr{O}_U)) = \min_{\ell\in[0,r-1]}\{p(\mathcal{Q}^{r-\ell},F)-\ell\} = p(\mathcal{Q}^1,F)-r+1.$$

Proof. The last equality follows from Lemma 11.2, which implies

$$p(\mathcal{Q}^1,F) \leq p(\mathcal{Q}^\ell,F)+\ell-1 \text{ for all } \ell\in[1,r]\cap\mathbb{Z},$$

and so $p(\mathcal{Q}^1,F)-r+1 \leq p(\mathcal{Q}^{r-\ell},F)-\ell$ for all $\ell\in[0,r-1]$.

Note that $p(j_*\varphi_{g|_U,1}(\mathscr{O}_U),F) = p(\varphi_{g|_U,1}(\mathscr{O}_U),F)$.

We have the formula (11.9) from above

$$F_p j_*\varphi_{g,1}(\mathscr{O}_U)^{r-k} \cong F_{p-k+r}\mathcal{Q}^k,$$

and so by definition, we get

$$p(\varphi_{g|_U}(\mathscr{O}_U)) = \min_{k\in\mathbb{Z}}\{p(\mathcal{Q}^{r-k},F)-k\}.$$

Thus, we need to check that this minimum is achieved for $k\in[0,r-1]\cap\mathbb{Z}$. For $k\geq r$, we have

$$p(\mathcal{Q}^{r-k},F) = p(\mathcal{Q}^1,F)+1+k-r,$$

and so

$$p(\mathcal{Q}^{r-k},F)-k = p(\mathcal{Q}^{r-(r-1)},F)-(r-1),$$

and we get that $p(\varphi_{g|_U}(\mathscr{O}_U)) = \min_{k\in\mathbb{Z}_{<r}} p(\mathcal{Q}^{r-k},F)-k$.

For $k\leq 0$, we have

$$p(\mathcal{Q}^{r-k},F) = p(\mathcal{Q}^r,F),$$

and so

$$p(\mathcal{Q}^{r-k},F)-(r-k) > p(\mathcal{Q}^r,F)-r,$$

proving the desired equality. ◻

Remark 11.11. Another way to phrase the result is the following: let

$$j(Z) = \#\{\ell\in[1,r-1]\cap\mathbb{Z} \mid p(\mathcal{Q}^{r-\ell},F) = p(\mathcal{Q}^{r-\ell+1},F)\}$$

be the number of times when the lowest Hodge piece doesn't jump in the interval $[1,r-1]$. Then

$$p(Q^{\mathbb{Z}},F) = p(\varphi_{g|_U}(\mathscr{O}_U))+j.$$

Indeed, this follows from the observation

$$p(\mathcal{Q}^{r-\ell},F) = p(Q^{\mathbb{Z}},F)+\ell-|\{j\leq\ell \mid p(\mathcal{Q}^{r-j},F) = p(\mathcal{Q}^{r-j+1},F)\}|,$$

so that $p(\mathcal{Q}^{r-\ell},F)-\ell = p(Q^{\mathbb{Z}},F)-$(the number of missed jumps up to ℓ). The minimum of this is clearly when ℓ is maximal, proving the claim.

Corollary 11.12. *Let $j(Z)$ be the number of missed jumps in the interval $[1, r-1]$ as defined above. Then*

(1) $\mathrm{HRH}(Z) \geq \mathrm{HRH}(g|_U) - r + j(Z) + 1$.
(2) *If $p(\mathcal{Q}^{r-1}, F) = p(Q^{\mathbb{Z}}, F) + 1$ (which is true if $j(Z) = 0$), then $0 \leq j(Z) < r-1$ and we have equality*
$$\mathrm{HRH}(Z) = \mathrm{HRH}(g|_U) - r + j(Z) + 1.$$

Proof. We have equality
$$p(Q, F) = p(\varphi_{g|_U,1}(\mathscr{O}_U), F) + j(Z),$$
and by Remark 11.7 the right hand side is equal to $\mathrm{HRH}(g|_U) - (n+r) + 2 + j(Z)$. Thus, we have
$$\mathrm{HRH}(Z) \geq p(Q, F) + n - 1 = \mathrm{HRH}(g|_U) - r + j(Z) + 1,$$
as claimed.

For the last claim, equality holds by Proposition 11.3. ☐

We now describe the relation of the Bernstein-Sato polynomial to the G-filtration when $r = 1$. Let $f \in \mathscr{O}_X(X)$ define a non-empty hypersurface. The module $\mathcal{B}_f = \bigoplus_{k\in\mathbb{N}} \mathscr{O}_X \partial_t^k \delta_f$ admits an exhaustive filtration $G^\bullet \mathcal{B}_f$ indexed by $\mathbb{Z}$ (defined below also for $r \geq 1$). This has the property that the Bernstein-Sato polynomial $b_f(s)$ of f is the minimal polynomial of the action of $-\partial_t t$ on $\mathrm{Gr}_G^0(\mathcal{B}_f)$. As f defines a non-empty hypersurface, it is easy to see that $(s+1) \mid b_f(s)$ using the usual description in terms of functional equations:
$$b_f(s) f^s = P(s) f^{s+1} \text{ for some } P(s) \in \mathscr{D}_X[s].$$

The reduced Bernstein-Sato polynomial $\widetilde{b}_f(s) = b_f(s)/(s+1)$ is an invariant of the singularities of f. The minimal exponent (as mentioned in Remark 3.5 above) is
$$\widetilde{\alpha}(f) = \min\{\lambda \in \mathbb{Q} \mid \widetilde{b}_f(-\lambda) = 0\},$$
and so we can consider a similar formula, only considering integer roots:
$$\widetilde{\alpha}_{\mathbb{Z}}(f) = \min\{j \in \mathbb{Z} \mid \widetilde{b}_f(-j) = 0\}.$$

We clearly have an inequality $\widetilde{\alpha}(f) \leq \widetilde{\alpha}_{\mathbb{Z}}(f)$.

The G-filtration has the property that
$$\mathrm{Gr}_V^\lambda \mathrm{Gr}_G^0(\mathcal{B}_f) \neq 0 \text{ if and only if } (s+\lambda) \mid b_f(s).$$

To study the reduced polynomial, Saito [Sai94] introduced the *microlocalization* $\widetilde{\mathcal{B}}_f = \mathcal{B}_f[\partial_t^{-1}]$. This object carries a *microlocal V-filtration* and a G-filtration. Importantly, these filtrations satisfy: the minimal polynomial of $-\partial_t t$ on $\mathrm{Gr}_G^0(\widetilde{\mathcal{B}}_f)$ is $\widetilde{b}_f(s)$ by [Sai94, Prop. 0.3], and
$$\mathrm{Gr}_V^\lambda \mathrm{Gr}_G^0(\widetilde{\mathcal{B}}_f) \neq 0 \text{ if and only if } (s+\lambda) \mid \widetilde{b}_f(s).$$

Using the isomorphisms $\partial_t^j \colon \mathrm{Gr}_V^\lambda \mathrm{Gr}_G^k(\widetilde{\mathcal{B}}_f) \cong \mathrm{Gr}_V^{\lambda-j} \mathrm{Gr}_G^{k-j}(\widetilde{\mathcal{B}}_f)$, we see then that
$$\widetilde{\alpha}_{\mathbb{Z}}(f) \geq j \text{ if and only if } G^{-j+1}\mathrm{Gr}_V^0(\widetilde{\mathcal{B}}_f) = 0.$$

Another important aspect of the microlocalization functor is that the natural map
$$\mathrm{Gr}_V^0(\mathcal{B}_f, F) \to \mathrm{Gr}_V^0(\widetilde{\mathcal{B}}_f, F)$$

is an isomorphism, where $F_\bullet\widetilde{\mathcal{B}}_f = \bigoplus_{k\leq \bullet+n} \mathscr{O}_X\partial_t^k\delta_f$.

Using this property and the fact that $F_{k-n}\widetilde{\mathcal{B}}_f \subseteq G^{-k}\widetilde{\mathcal{B}}_f$, it follows that $p(\mathrm{Gr}_V^0(\mathcal{B}_f), F) \geq \widetilde{\alpha}_{\mathbb{Z}}(f)-n$. By Proposition 11.4, we get

$$\widetilde{\alpha}_{\mathbb{Z}}(f) \leq \mathrm{Sp}_{\min,\mathbb{Z}}(f,x).$$

Even in the isolated hypersurface singularities case, strict inequality $\widetilde{\alpha}_{\mathbb{Z}}(f) < \mathrm{Sp}_{\min,\mathbb{Z}}(f,x)$ is possible [JKSY22, Rmk. 3.4d].

Remark 11.13. In Remark 3.5 above, we mentioned that there exists a notion of minimal exponent for local complete intersection subvarieties. In fact, when $Z = V(f_1,\dots,f_r) \subseteq X$ is defined by a regular sequence, the definition is

$$\widetilde{\alpha}(Z) = \widetilde{\alpha}(g|_U),$$

where the minimal exponent for hypersurfaces was just defined. Associated to the tuple $f_1,\dots,f_r$, there is a Bernstein-Sato polynomial [BMS06]. It is related to the reduced Bernstein-Sato polynomial of g, by [Mus22]:

$$\widetilde{b}_g(s) = b_f(s).$$

Moreover, $(s+r) \mid b_f(s)$, so we can consider the *reduced Bernstein-Sato polynomial* $\widetilde{b}_f(s) = b_f(s)/(s+r)$. Then the minimal exponent satisfies $\widetilde{\alpha}(Z) = \min\{\lambda \in \mathbb{Q} \mid \widetilde{b}_f(-\lambda) = 0\}$ ([Dir23]).

One might ask whether $\widetilde{\alpha}_{\mathbb{Z}}(g|_U)$ satisfies a similar formula, that is, whether it is equal to

$$\widetilde{\alpha}_{\mathbb{Z}}(Z) = \min\{j \in \mathbb{Z} \mid \widetilde{b}_f(-j) = 0\}.$$

This would be beneficial to rewrite the results of Corollary 11.12 in terms of invariants of Z rather than invariants of $g|_U$.

It is not true, however, that $\widetilde{b}_f(s) = \widetilde{b}_{g|_U}(s)$, and so the question is slightly subtle. In general, we have [Dir23, Lem. 5.2]:

$$b_f(s) = \widetilde{b}_g(s) = \widetilde{b}_{g|_U}(s)\prod_{i\in I}(s+r+i),$$

where $I \subseteq \mathbb{Z}_{\geq 0}$ is some finite subset. So we can see that we have equality $\widetilde{\alpha}_{\mathbb{Z}}(Z) = \widetilde{\alpha}_{\mathbb{Z}}(g|_U)$ if either one is strictly less than r.

There are two cases: either $0 \in I$ or $0 \notin I$. If $0 \in I$, then we get

$$\widetilde{b}_f(s) = \widetilde{b}_{g|_U}(s)\prod_{i\in I\setminus\{0\}}(s+r+i),$$

and so we have inequality $\widetilde{\alpha}_{\mathbb{Z}}(Z) \leq \widetilde{\alpha}_{\mathbb{Z}}(g|_U)$.

In the other case, we get

$$\widetilde{b}_f(s) = \frac{\widetilde{b}_{g|_U}(s)}{(s+r)}\prod_{i\in I}(s+r+i),$$

and so we only get

$$\widetilde{\alpha}_{\mathbb{Z}}(g|_U) = \min\{r, \widetilde{\alpha}_{\mathbb{Z}}(Z)\}.$$

This case is possible, as Example 11.21 shows below.

Corollary 11.14. *In the above notation, let $j(Z)$ be the number of missed jumps. Then*

(1) $\mathrm{HRH}(Z) \geq \widetilde{\alpha}_{\mathbb{Z}}(g|_U) - r + j(Z) - 1$.

(2) If $\widetilde{\alpha}_{\mathbb{Z}}(g|_U) > r$, then we have inequality

$$\mathrm{HRH}(Z) \geq \widetilde{\alpha}_{\mathbb{Z}}(Z) - r + j(Z) - 1.$$

Proof. We have

$$\widetilde{\alpha}_{\mathbb{Z}}(g|_U) \leq p(\varphi_{g|_U,1}(\mathscr{O}_U), F) + (n+r),$$

and so we get that

$$p(Q^{\mathbb{Z}}, F) - j(Z) + (n+r) \geq \widetilde{\alpha}_{\mathbb{Z}}(g|_U),$$

giving $p(Q^{\mathbb{Z}}, F) + n - 1 \geq \widetilde{\alpha}_{\mathbb{Z}}(g|_U) + j(Z) - r - 1$. Then the claim follows from Proposition 11.1.

If we assume $\widetilde{\alpha}_{\mathbb{Z}}(g|_U) > r$, this implies that $(s+r) \nmid \widetilde{b}_{g|_U}(s)$, and so we are in the case $0 \in I$. Thus, we have inequality $\widetilde{\alpha}_{\mathbb{Z}}(Z) \leq \widetilde{\alpha}_{\mathbb{Z}}(g|_U)$. ☐

As a consequence of these results, we obtain Theorem K and Corollary L.

Proof of Theorem K. The first statement is Corollary 11.8. The first inequality follows from the proof of Corollary 11.14, and the second inequality from the statement of the same corollary. Finally, the last inequality follows from Corollary 11.12. ☐

Proof of Corollary L. The inequalities follow from Theorem K and the last line of Corollary 11.14.

The implication $\widetilde{\alpha}_{\mathbb{Z}}(Z) = +\infty$ implies rational homology manifold follows from the inequality $\widetilde{\alpha}_{\mathbb{Z}}(Z) - r - 1 \leq \mathrm{HRH}(Z)$. In the hypersurface case, the converse follows from the observation that $\widetilde{\alpha}_{\mathbb{Z}}(Z) = +\infty$ if and only if for all $j \in \mathbb{Z}$, we have $\mathrm{Gr}^j_G \mathrm{Gr}^0_V(\mathcal{B}_f) = 0$, but then by exhaustiveness of the filtration G, this is true if and only if $\mathrm{Gr}^0_V(\mathcal{B}_f) = 0$, which is equivalent to the rational homology manifold condition for hypersurfaces. ☐

Remark 11.15. As noted in Remark 4.8 above, we have

$$\mathrm{HRH}(Z) < +\infty \text{ if and only if } \mathrm{HRH}(Z) \leq \frac{d-3}{2}.$$

Combined with the inequality above, we see that if

$$\widetilde{\alpha}_{\mathbb{Z}}(g|_U) > \frac{d-3}{2} + r - j(Z) + 1 = \frac{n+r-1}{2} - j(Z),$$

then Z is a rational homology manifold.

Example 11.16. The previous remark is most clear when $r = 2$. Indeed, in that case, we have $j \in \{0, 1\}$. If $j = 0$, then by Proposition 11.3, we see that $\mathrm{HRH}(Z) = p(Q^{\mathbb{Z}}, F) + n - 1$. Otherwise, if $j = 1$, then we see that if Z is not a rational homology manifold, we have the inequality

$$\widetilde{\alpha}_{\mathbb{Z}}(g|_U) \leq \frac{n-1}{2}.$$

Now, we give some partial results on the invariant $\widetilde{\alpha}_{\mathbb{Z}}(Z)$. As in the hypersurface case above, the polynomial $b_f(s)$ is controlled by the G-filtration on $\mathcal{B}_f$. This is defined as follows: the ring $\mathscr{D}_T$ has a $\mathbb{Z}$-indexed filtration

$$V^k \mathscr{D}_T = \left\{ \sum_{\beta,\gamma} P_{\beta,\gamma} t^\beta \partial_t^\gamma \mid P_{\beta,\gamma} \in \mathscr{D}_X, |\beta| \geq |\gamma| + k \right\},$$

and so we can define an exhaustive filtration

$$G^\bullet \mathcal{B}_f = (V^\bullet \mathscr{D}_T) \cdot \delta_f.$$

Then $b_f(s)$ is the minimal polynomial of the action of $s = -\sum_{i=1}^r \partial_{t_i} t_i$ on $\mathrm{Gr}_G^0(\mathcal{B}_f)$. Moreover, we have

$$\mathrm{Gr}_V^{\lambda}\mathrm{Gr}_G^0(\mathcal{B}_f) \neq 0 \text{ if and only if } (s+\lambda) \mid b_f(s). \tag{11.17}$$

Thus, it is worthwhile to study the induced G-filtration on $\mathrm{Sp}(\mathcal{B}_f)^{\mathbb{Z}}$. We define it as

$$G^{\bullet}\mathrm{Sp}(\mathcal{B}_f)^{\mathbb{Z}} = \bigoplus G^{\bullet+k}\mathrm{Gr}_V^{r+k}(\mathcal{B}_f),$$

where the shift by k is to make it so that $G^{\bullet}\mathrm{Sp}(\mathcal{B}_f)^{\mathbb{Z}}$ is a filtration by sub-$\mathscr{D}_T$-modules. This induces G-filtrations on $\mathcal{L}$ and $\mathcal{Q}$ by the short exact sequence 2.8. The G-filtration on $\mathcal{L}$ is rather simple by [Dir23, Lem. 5.1]: it satisfies $G^0\mathcal{L} = \mathcal{L}$. In particular, for all $j > 0$, we have an isomorphism

$$\mathrm{Gr}_G^{-j}(\mathrm{Sp}(\mathcal{B}_f)) \cong \mathrm{Gr}_G^{-j}(\mathcal{Q}).$$

This immediately leads to the following:

Proposition 11.18. *Assume $(s+r+j) \mid b_f(s)$ for some $j > 0$. Then $Q^{\mathbb{Z}} \neq 0$.*

Proof. As $(s+r+j) \mid b_f(s)$, this means that $\mathrm{Gr}_G^0\mathrm{Gr}_V^{r+j}(\mathcal{B}_f) \neq 0$. This is the $r+j$th monodromic piece of $\mathrm{Gr}_G^{-j}(\mathrm{Sp}(\mathcal{B}_f))$, proving that

$$\mathrm{Gr}_G^{-j}(\mathrm{Sp}(\mathcal{B}_f)^{\mathbb{Z}}) = \mathrm{Gr}_G^{-j}(\mathcal{Q}^{\mathbb{Z}}) \neq 0,$$

and so $\mathcal{Q}^{\mathbb{Z}} \neq 0$. □

Remark 11.19. The result is a bit surprising, in view of the equality

$$b_f(s) = \widetilde{b}_{g|_U}(s) \prod_{i\in I}(s+r+i).$$

Indeed, the claim says that if there is any factor $(s+r+j) \mid b_f(s)$ with $j > 0$, then there is also an integer root in $\widetilde{b}_{g|_U}(s)$, because $V(g|_U)$ is not a rational homology manifold in this case. So any non-zero element of I implies the existence of an integer root in $\widetilde{b}_{g|_U}(s)$, and hence the existence of another integer root in $b_f(s)$.

This remark naturally leads to the following conjecture:

Conjecture 11.20. *For any $f_1, \ldots, f_r \in \mathscr{O}_X(X)$ a regular sequence, we have*

$$b_g(s) = \begin{cases} b_{g|_U}(s)(s+r) \\ b_{g|_U}(s) \end{cases}.$$

In other words, the set I is either empty or equal to the singleton $\{0\}$.

If $r = 1$, then we always have $b_g(s) = b_{g|_U}(s)(s+1)$ by [Lee24]. For $r > 1$, the case $b_g(s) = b_{g|_U}(s)$ is possible, as we see in the following example.

Example 11.21. There is a reduced, irreducible complete intersection variety $Z = V(f_1, f_2)$ with $b_g(s) = b_{g|_U}(s)$. Indeed, if we let $f_1 = x^2 + y^3$ and $f_2 = xy + zw$, then Macaulay2 shows that

$$b_g(s) = b_{g|_U}(s) = (s+1)(s+2)^2\left(s+\frac{5}{2}\right)\left(s+\frac{7}{3}\right)\left(s+\frac{8}{3}\right)\left(s+\frac{11}{6}\right)^2\left(s+\frac{13}{6}\right)^2.$$

Note that, in this example, Z is not normal.

Remark 11.22. Note that if Z has rational singularities, then $0 \in I$. Indeed, [CDMO24] tells us that Z having rational singularities is equivalent to $\widetilde{\alpha}(Z) = \widetilde{\alpha}(g|_U) > r$. But if

$$b_g(s) = b_{g|_U}(s) \prod_{i \in I}(s + r + i) \text{ with } I \subseteq \mathbb{Z}_{>0},$$

then we have

$$(s + r) \mid b_f(s) = \widetilde{b}_g(s) = \widetilde{b}_{g|_U}(s) \prod_{i \in I}(s + r + i),$$

which forces $(s + r) \mid \widetilde{b}_{g|_U}(s)$ and so $\widetilde{\alpha}(g|_U) \leq r$, a contradiction.

Corollary 11.23. *Assume Z has rational singularities. Then*

$$\mathrm{HRH}(Z) \geq \widetilde{\alpha}_{\mathbb{Z}}(Z) - r + j(Z) - 1,$$

where $j(Z)$ is the number of missed jumps.

Proof. As Z has rational singularities, we have $\widetilde{\alpha}(Z) = \widetilde{\alpha}(g|_U) > r$. Moreover, by the discussion above, $\widetilde{\alpha}_{\mathbb{Z}}(Z) = \widetilde{\alpha}_{\mathbb{Z}}(g|_U) \geq \widetilde{\alpha}(g|_U) > r$. So the result follows by Corollary 11.14. □

In any case, we can collect our findings in the following corollary.

Corollary 11.24. *The following hold:*

(1) If $(s + \ell) \mid b_f(s)$ for some integer $\ell \neq r$, then $Q^{\mathbb{Z}} \neq 0$.
(2) If $b_g(s) = b_{g|_U}(s)(s + r)$, then $\widetilde{\alpha}_{\mathbb{Z}}(Z) < +\infty$ if and only if $Q^{\mathbb{Z}} \neq 0$.
(3) If $b_g(s) = b_{g|_U}(s)$, then $Q^{\mathbb{Z}} \neq 0$.

Proof. For the first claim, if $\ell > r$, then this is the result of Proposition 11.18.

For $\ell < r$, use the equality

$$b_f(s) = \widetilde{b}_{g|_U}(s) \prod_{i \in I}(s + r + i),$$

where $I \subseteq \mathbb{Z}_{\geq 0}$ implies that $(s + \ell) \mid \widetilde{b}_{g|_U}(s)$. Then we use Corollary 11.8 and Remark 11.7 to conclude.

For the second claim, we have $\widetilde{b}_f(s) = \widetilde{b}_g(s)/(s + r) = \widetilde{b}_{g|_U}(s)$, and so $\widetilde{\alpha}_{\mathbb{Z}}(Z) = \widetilde{\alpha}_{\mathbb{Z}}(g|_U)$. So the claim follows by Corollary 11.8 and Remark 11.7.

For the last claim, we have $(s + r) \mid b_f(s) = \widetilde{b}_{g|_U}(s)$, so that $\widetilde{\alpha}_{\mathbb{Z}}(g|_U) < +\infty$ and again we use Corollary 11.8 and Remark 11.7. □

12. The case of isolated singularities. In this section, we discuss a deeper relation between the spectrum of an isolated local complete intersection singularity and the Hodge Rational Homology degree.

Recall that, in this case, we have the Milnor fiber, which can be defined in the following way. Let (Z, x) be the germ of the isolated singularity, and $\rho\colon (\mathcal{X}, x) \to \Delta$ a smoothing with central fiber Z. We then let the Milnor fiber F be the topological space $\mathcal{X}_t$, for $t \neq 0$, and note that its cohomology can be endowed with a mixed Hodge structure. The cohomology is nonzero only in degrees 0 and $d = \dim Z$, and is independent of the smoothing along with its Hodge filtration. For more details, see [FL24a, §2.2 and §4].

Following Steenbrink and the notation in [FL24a], let

$$s_p = \dim \mathrm{Gr}_F^p H^d(F).$$

These invariants are deeply connected to the link invariants and can be used to describe $\mathrm{HRH}(Z)$.

Proposition 12.1. *Let (Z, x) be a normal isolated local complete intersection singularity. Then $\mathrm{HRH}(Z) \geq k$ if and only if $s_{d-p} - s_p = 0$ for all $0 \leq p \leq k$.*

Proof. We will use that

$$s_{d-p} - s_p = \ell^{p,d-p-1} - \ell^{p,d-p} \tag{12.2}$$

[FL24a, Prop. 2.11 ii)], and

$$\sum_{p=0}^{k} s_{d-p} \geq \sum_{p=0}^{k} s_p \tag{12.3}$$

and if equality holds, then $\ell^{d-k-1,k} = \ell^{k+1,d-k-1} = 0$ [FL24a, Prop. 2.12].

Suppose $\mathrm{HRH}(Z) \geq k$. Then by Theorem E, $0 = \ell^{d-p,p} = \ell^{p,d-p-1}$ for $p \leq k$, and by combining (12.2), (12.3), and using induction on k implies $s_{d-p} - s_p = 0$ for all $0 \leq p \leq k$.

Conversely, suppose $s_{d-p} - s_p = 0$ for all $0 \leq p \leq k$. We proceed by induction. For $k = 0$,

$$0 = s_d - s_0 = \ell^{0,d-1} - \ell^{0,d} = \ell^{d,0},$$

since $\ell^{0,d} = 0$. By Theorem E, we obtain the conclusion. Suppose now that we know the result up until degree $(k-1)$. Then

$$s_{d-k} - s_k = \ell^{k,d-k-1} - \ell^{k,d-k} = \ell^{d-k,k} - \ell^{(k-1)+1,d-(k-1)-1}.$$

By the second part of (12.3), we obtain that $\ell^{d-k,k} = 0$, and by Theorem E we conclude. ☐

We note that in this setting, the spectral numbers are identified with the cohomology of the Milnor fiber. Indeed,

$$m_{\alpha,x} = \dim \mathrm{Gr}_F^p H^d(F)_\lambda,$$

where $p = \lfloor d + 1 - \alpha \rfloor$, and $\lambda = \exp(-2\pi\alpha)$, since the pullback to a point $\xi \in \{x\} \times \mathbb{A}^r$ corresponds to picking a nearby fiber of a 1-parameter smoothing of Z (see [DMS11, Rmk. 1.3 (i)] for more details). Furthermore, spectral numbers partially recover the duality classically known for hypersurface singularities.

Proposition 12.4. *Let (Z, x) be an isolated local complete intersection singularity. Then, for $\alpha \notin \mathbb{Z}$,*

$$m_{\alpha,x} = m_{d+1-\alpha,x}.$$

Proof. This is a consequence of duality applied to the module $\mathcal{Q}$. We also show in the proof what the construction yields for $\alpha \in \mathbb{Z}$.

Since we are working locally around isolated singularity $x \in Z$, we have that $\mathcal{Q}$ is supported on $\{x\} \times \mathbb{A}^r$, hence $\mathcal{Q} = i_{x*}\mathcal{N}$ and so if $j_\xi \colon \{\xi\} \to \{x\} \times \mathbb{A}^r$, we have the identification

$$i_\xi^* \mathcal{Q} = j_\xi^* \mathcal{N},$$

and

$$i_\xi^* \mathbf{D}\mathcal{Q} = j_\xi^*(\mathbf{D}\mathcal{N}) = \mathbf{D} j_\xi^! \mathcal{N} = \mathbf{D}(j_\xi^* \mathcal{N})(r)[2r],$$

the last equality following from the fact that j_ξ is non-characteristic.

We get

$$m_{\alpha,x}(\mathcal{Q}) = \sum_{k \in \mathbb{Z}} (-1)^k \dim \mathrm{Gr}^F_{\lceil \alpha \rceil - d - 1} \mathcal{H}^{k-r} i_\xi^* (\mathcal{Q}^{\alpha + \mathbb{Z}})$$

$$= \sum_{k\in\mathbb{Z}} (-1)^k \dim \mathrm{Gr}^F_{\lceil\alpha\rceil-d-1} \mathcal{H}^{k-r} j_\xi^*(\mathcal{N}^{\alpha+\mathbb{Z}}).$$

Note that for any mixed Hodge structure H, have the relation

$$\mathrm{Gr}^F_\bullet \mathbf{D}(H) = \mathbf{D}\,\mathrm{Gr}^F_{-\bullet}(H),$$

and in particular,

$$\dim \mathrm{Gr}^F_\bullet \mathbf{D}(H) = \dim \mathrm{Gr}^F_{-\bullet}(H).$$

Hence, we have

$$m_{\alpha,x}(\mathbf{D}\mathcal{Q}) = \sum_{k\in\mathbb{Z}} (-1)^k \dim \mathrm{Gr}^F_{\lceil\alpha\rceil-d-1} \mathcal{H}^{k-r} i_\xi^*((\mathbf{D}\mathcal{Q})^{\alpha+\mathbb{Z}})$$

$$= \sum_{k\in\mathbb{Z}} (-1)^k \dim \mathrm{Gr}^F_{\lceil\alpha\rceil-d-1} \mathcal{H}^{k-r} (\mathbf{D}(j_\xi^*(\mathcal{N}^{-\alpha+\mathbb{Z}}))(r)[2r])$$

$$= \sum_{k\in\mathbb{Z}} (-1)^k \dim \mathrm{Gr}^F_{\lceil\alpha\rceil-d-r-1} \mathcal{H}^{k+r} (\mathbf{D}(j_\xi^*(\mathcal{N}^{-\alpha+\mathbb{Z}})))$$

$$= \sum_{k\in\mathbb{Z}} (-1)^k \dim \mathbf{D}\,\mathrm{Gr}^F_{d+r+1-\lceil\alpha\rceil} \mathcal{H}^{-k-r} (j_\xi^*(\mathcal{N}^{-\alpha+\mathbb{Z}})).$$

$$= \sum_{k\in\mathbb{Z}} (-1)^k \dim \mathrm{Gr}^F_{d+r+1-\lceil\alpha\rceil} \mathcal{H}^{k-r} (j_\xi^*(\mathcal{N}^{-\alpha+\mathbb{Z}})).$$

Write $d+r+1-\lceil\alpha\rceil = \lceil\mu\rceil - d - 1$ where $\mu+\alpha\in\mathbb{Z}$. Thus,

$$2d+r+2-\lceil\alpha\rceil = \lceil\mu\rceil.$$

If $\alpha\in\mathbb{Z}$, then $\mu\in\mathbb{Z}$ and we get $\mu = d+n+2-\alpha$.

Otherwise, we write $\alpha = p-\varepsilon$ with $\varepsilon\in(0,1)$, then $\lceil\alpha\rceil = p$ and so we must have

$$\mu = \lfloor\mu\rfloor + \varepsilon = 2d+r+2-p-1+\varepsilon = 2d+r+1-p+\varepsilon = d+n+1-\alpha.$$

Hence, we get

$$m_{\alpha,x}(\mathbf{D}\mathcal{Q}) = \begin{cases} m_{d+n+2-\alpha}(\mathcal{Q}) & \alpha\in\mathbb{Z} \\ m_{d+n+1-\alpha}(\mathcal{Q}) & \alpha\notin\mathbb{Z} \end{cases}.$$

Using the isomorphism $\mathcal{Q}^{\neq\mathbb{Z}} \cong \mathrm{Sp}(\mathcal{B}_f)^{\neq\mathbb{Z}}$ and $\mathbf{D}(\mathcal{Q}^{\neq\mathbb{Z}}) \cong \mathrm{Sp}(\mathcal{B}_f)^{\neq\mathbb{Z}}(n)$, we have

$$\widehat{\mathrm{Sp}}(\mathcal{Q}^{\neq\mathbb{Z}}, x) = \widehat{\mathrm{Sp}}(\mathrm{Sp}(\mathcal{B}_f)^{\neq\mathbb{Z}}, x) = t^{-n}\widehat{\mathrm{Sp}}(\mathrm{Sp}(\mathcal{B}_f)^{\neq\mathbb{Z}}(n), x) = t^{-n}\widehat{\mathrm{Sp}}(\mathbf{D}\mathcal{Q}^{\neq\mathbb{Z}}, x).$$

The result follows. □

Remark 12.5. The same duality might not hold for $m_{k,x}$, $k\in\mathbb{Z}$ [DMS11, Rmk. 1.3 (iv)], and depends on the Milnor fiber of a generic 1-parameter smoothing of the 1-parameter smoothing of Z. Furthermore, to compare the integer spectrum numbers using the proof above, we would need to consider the spectral numbers of $\mathcal{L}$ and its dual.

Proof of Theorem J. We show that if $\mathrm{Sp}_{\min,\mathbb{Z}}(Z,x) \geq k+2$, then $\mathrm{HRH}(Z) \geq k$ around x. By Proposition 12.1, it is enough to verify $s_{d-p} - s_p = 0$ for all $p \leq k$. It is immediate to see that

$$s_p = \sum_{\alpha \in (d-p, d-p+1]} m_{\alpha,x}.$$

Therefore,

$$s_{d-p} - s_p = \sum_{\beta \in (p,p+1]} m_{\beta,x} - \sum_{\alpha \in (d-p,d-p+1]} m_{\alpha,x} = m_{p+1,x} - m_{d-p+1,x},$$

where the last equality follows from Proposition 12.4.

Suppose $\mathrm{Sp}_{\min,\mathbb{Z}}(Z,x) \geq k+2$, that is, $m_{1,x} = \cdots = m_{k+1,x} = 0$. Then by (12.3) we have

$$0 \leq \sum_{p=0}^{k} s_{d-p} - \sum_{p=0}^{k} s_p = -(m_{d+1,x} + \cdots + m_{d-k,x}),$$

and thus, $m_{d+1,x} = \cdots = m_{d-k,x} = 0$. Therefore, $s_{d-p} = s_p = 0$ for all $p \leq k$. □

E. Examples

This section is devoted to providing various examples with different features.

13. Affine cones, toric and secant varieties. We first calculate the HRH level of affine cones over smooth projective varieties following the treatment of [SVV23].

Proposition 13.1. *Let X be a smooth projective variety of dimension n, and L be an ample line bundle on X. Let*

$$Z = C(X,L) := \mathrm{Spec}\left(\bigoplus_{m\geq 0} H^0(X, L^m)\right)$$

be the affine cone with conormal bundle L. Then $\mathrm{HRH}(Z) \geq k$ for some $k \leq \frac{n+1}{2}$ if and only if the following two conditions are satisfied:

(1) $H^i(\Omega_X^p) = 0$ for all $i \neq p, 0 \leq p \leq k$,
(2) $H^0(\mathcal{O}_X) \xrightarrow{\cup c_1(L)} H^1(\Omega_X^1) \xrightarrow{\cup c_1(L)} \cdots \xrightarrow{\cup c_1(L)} H^k(\Omega_X^k)$ are all isomorphisms.

Proof. The blow up $f\colon \tilde{Z} \to Z$ at the cone point v is a strong log resolution of Z with $E \cong X$. Note that by our assumption, $k < \mathrm{codim}_Z(Z_{\mathrm{sing}}) = n+1$, whence the bottom map in (4.7) is an isomorphism by [SVV23, Lem. 2.4]. In what follows, we use (4.7) without any further reference.

Let us first prove the assertion when $k=0$. We have the distinguished triangle

$$\underline{\Omega}_Z^0 \to Rf_*\mathcal{O}_{\tilde{Z}} \oplus \mathcal{O}_v \to Rf_*\mathcal{O}_X \xrightarrow{+1}.$$

Since Z is affine, using the arguments of [SVV23, Appendix A.1], we see that the above induces

$$0 \to \Gamma(\mathcal{H}^0(\underline{\Omega}_Z^0)) \to H^0(\mathcal{O}_{\tilde{Z}}) \oplus \mathbb{C} \to H^0(\mathcal{O}_X) \to 0, \tag{13.2}$$

$$0 \to \Gamma(\mathcal{H}^i(\underline{\Omega}_Z^0)) \to H^i(\mathcal{O}_{\tilde{Z}}) \to H^i(\mathcal{O}_X) \to 0 \quad \forall i \geq 1. \tag{13.3}$$

By *loc. cit.*, $H^0(\mathcal{O}_{\tilde{Z}}) = \bigoplus_{m\geq 0} H^0(L^m)$ and the map $H^0(\mathcal{O}_{\tilde{Z}}) \oplus \mathbb{C} \to H^0(\mathcal{O}_X)$ in (13.2) sends $(x, \alpha) \mapsto \varphi(x) - \alpha$ where $\varphi\colon \bigoplus_{m\geq 0} H^0(L^m) \to H^0(\mathcal{O}_X)$ is the projection. Thus the composed map

$$\Gamma(\mathcal{H}^0(\underline{\Omega}_Z^0)) \to H^0(\mathcal{O}_{\tilde{Z}}) \oplus \mathbb{C} \to H^0(\mathcal{O}_{\tilde{Z}})$$

is an isomorphism. Finally, by (13.3), for $i \geq 1$, $\Gamma(\mathcal{H}^i(\underline{\Omega}_Z^0)) \to H^i(\mathcal{O}_{\tilde{Z}})$ is an isomorphism if and only if $H^i(\mathcal{O}_X) = 0$, which concludes the proof for $k = 0$.

We use induction on k. Assume $k \geq 1$ and note the distinguished triangle induced by the residue sequence on $\tilde{Z}$:

$$Rf_*\Omega^k_{\tilde{Z}} \to Rf_*\Omega^k_{\tilde{Z}}(\log X) \to Rf_*\Omega^{k-1}_X \xrightarrow{+1} . \tag{13.4}$$

It is shown in [SVV23, Appendix A.2] that

$$H^i(\Omega^k_{\tilde{Z}}) = \bigoplus_{m\geq 0} H^i(\Omega^k_X \otimes L^m) \oplus \bigoplus_{m\geq 1} H^i(\Omega^{k-1}_X \otimes L^m) \tag{13.5}$$

and the map

$$\Gamma(\mathcal{H}^i(\underline{\Omega}^k_Z)) \to H^i(\Omega^k_{\tilde{Z}}) \tag{13.6}$$

induced from

$$\underline{\Omega}^k_Z \to Rf_*\Omega^k_{\tilde{Z}} \to Rf_*\Omega^k_X \xrightarrow{+1}$$

realizes $\Gamma(\mathcal{H}^i(\underline{\Omega}^k_Z))$ as the following direct summand of $H^i(\Omega^k_{\tilde{Z}})$ through (13.5):

$$\Gamma(\mathcal{H}^i(\underline{\Omega}^k_Z)) = \bigoplus_{m\geq 1} H^i(\Omega^k_X \otimes L^m) \oplus \bigoplus_{m\geq 1} H^i(\Omega^{k-1}_X \otimes L^m).$$

Thus, we are reduced to showing that the composition

$$\Gamma(\mathcal{H}^i(\underline{\Omega}^k_Z)) \hookrightarrow H^i(\Omega^k_{\tilde{Z}}) \to H^i(\Omega^k_{\tilde{Z}}(\log X)) \tag{13.7}$$

(the first map is (13.6) and the second one is induced by (13.4)) is an isomorphism for all i if and only if the two conditions in the statement are satisfied. If $i \neq k-1, k$, then we see from our induction hypothesis and (13.4) that the second map above is an isomorphism, whence the composition is an isomorphism if and only if $H^i(\Omega^k_X) = 0$. Note that the connecting map

$$H^i(\Omega^{k-1}_X) \to H^{i+1}(\Omega^k_{\tilde{Z}})$$

arising from (13.4) is the cup product $\cup c_1(L)$ map and lands in the direct summand $H^i(\Omega^k_X)$. This is injective for $i = k-1$ by Hard Lefschetz, by our assumption on k. Thus the second map in (13.7) is an isomorphism for $i = k-1$, whence the composition is an isomorphism if and only if $H^{k-1}(\Omega^k_X) = 0$. Finally, by the above argument, (13.4) induces the exact sequence

$$0 \to H^{k-1}(\Omega^{k-1}_X) \to H^k(\Omega^k_{\tilde{Z}}) \to H^k(\Omega^k_{\tilde{Z}}(\log X)) \to 0.$$

It follows immediately from the description of $\Gamma(\mathcal{H}^i(\underline{\Omega}^k_Z))$ as a direct summand of $H^i(\Omega^k_{\tilde{Z}})$ that the composed map (13.7) is an isomorphism for $i = k$ if and only if

$$H^{k-1}(\Omega^{k-1}_X) \xrightarrow{\cup c_1(L)} H^k(\Omega^k_X)$$

is an isomorphism. That completes the proof. $\square$

Remark 13.8. As before, let X be a smooth projective variety of dimension n, L be an ample line bundle on X, and $Z = C(X, L)$. Since Z is singular only at the cone point, we have

$$\mathrm{lcdef}_{\mathrm{gen}}(Z) = \mathrm{lcdef}(Z)$$

$$= \min \left\{ c \in \mathbb{N} \;\middle|\; \begin{array}{c} H^i(X, \mathbb{C}) \xrightarrow{\cup c_1(L)} H^{i+2}(X, \mathbb{C}) \text{ are isomorphisms for all } -1 \le i \le n-3-c, \\ \text{and injective for } i = n-2-c \text{ with the convention that } H^{-1}(X, \mathbb{C}) = 0 \end{array} \right\}$$

where the last equality comes from [PS24, Thm. 6.1]. In fact, when $X \subseteq \mathbb{P}^N$, setting $Z = C(X) \subseteq \mathbb{A}^{N+1}$ to be the affine cone, we have the following by [PS24, Thm. A]

$$\mathrm{lcdef}_{\mathrm{gen}}(Z) = \mathrm{lcdef}(Z) = \min_{c \in \mathbb{N}} \left\{ H^i(\mathbb{P}^N, \mathbb{C}) \xrightarrow{\sim} H^i(X, \mathbb{C}) \, \forall \, i \le n-1-c \right\}.$$

Example 13.9. Here are two explicit examples of affine cones with interesting properties:

(1) There are varieties with k-Du Bois singularities that satisfy $\mathrm{HRH}(Z) \ge k$ but their singularities are not k-rational (in the sense of [SVV23]). For example, consider $Z = C(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$. Then Z is a rational homology manifold by the above proposition, but according to [SVV23, Prop. F], its singularities are 1-Du Bois but not 1-rational.

(2) Let X be a Godeaux surface. These are surfaces with $p_g = q = 0$, $h^{1,1}(X) = 9$ and ample canonical bundle ω_X. Thus $Z = C(X, \omega_X)$ satisfies $\mathrm{HRH}(Z) = 0$ by the above proposition. But their singularities are not pre-0-Du Bois by [SVV23, Prop. F] as $h^2(\omega_X) \neq 0$. This is an example of a variety with $\mathrm{HRH}(Z)$ strictly higher than its Du Bois level.

Example 13.10 (Normal affine toric varieties)**.** Let Z be a normal affine toric variety. By [SVV23, Prop. E] and Remark 4.8, we obtain

$$\mathrm{HRH}(Z) = \begin{cases} +\infty & \text{if } Z \text{ is simplicial,} \\ 0 & \text{otherwise.} \end{cases}$$

Example 13.11 (Secant varieties)**.** Let $X \subset \mathbb{P}^N$ be a smooth projective variety embedded by the complete linear series of a sufficiently positive line bundle (i.e. one that satisfies (Q_1)-property as per [ORS24, Def. 3.1]). Let Σ be its secant variety. Then by [ORS24] and Remark 4.8,

$$\mathrm{HRH}(\Sigma) = \begin{cases} +\infty & \text{if } X \cong \mathbb{P}^1, \\ 0 & \text{if } H^i(\mathcal{O}_X) = 0 \text{ for all } i \ge 1 \text{ and } X \ncong \mathbb{P}^1, \\ -1 & \text{otherwise.} \end{cases}$$

14. Ideals of generic, symmetric and skew-symmetric minors. In this subsection, we show how the main theorem of [RW16] can be used to compute $\mathrm{lcdef}_{\mathrm{gen}}(-)$ and to give a bound on $\mathrm{HRH}(-)$ for determinantal varieties.

We will be interested in subspaces defined by matrices of appropriate ranks of the following spaces:

(1) (Generic) $X = \mathrm{Mat}_{m,n}(\mathbb{C})$ with $m \ge n$,
(2) (Odd skew) $X = \mathrm{Mat}_n(\mathbb{C})^{\mathrm{skew}}$, n odd,
(3) (Even skew) $X = \mathrm{Mat}_n(\mathbb{C})^{\mathrm{skew}}$, n even,
(4) (Symmetric) $X = \mathrm{Mat}_n(\mathbb{C})^{\mathrm{sym}}$.

In cases (1) and (4), we let Z_p denote the subvariety of matrices of rank $\le p$ and in cases (2) and (3), we let Z_p denote the subvariety of matrices of rank $\le 2p$. In every case, Z_p is known to have rational singularities by [Bou87]. Thus, $\mathrm{HRH}(Z_p) \ge 0$.

Following [RW16], we let D_p be the intersection homology $\mathscr{D}_X$-module associated to the trivial local system on $Z_{p,\mathrm{reg}}$. We let $\Gamma(X)$ denote the Grothendieck group of holonomic $\mathscr{D}_X$-modules. For p fixed, we write

$$H_p(q) = \sum_{j\geq 0} \left[\mathcal{H}^j_{Z_p}(\mathscr{O}_X)\right] \cdot q^j \in \Gamma(X)[q].$$

Finally, for $a \geq b \geq 0$, let $\binom{a}{b}_q$ be the q-binomial coefficient, defined by

$$\binom{a}{b}_q = \frac{(1-q^a)\dots(1-q^{a-b+1})}{(1-q^b)\dots(1-q)}.$$

The following computation will be important below:

Lemma 14.1. *Let $a \geq b \geq 0$. Then*

$$\binom{a}{b}_{q^{-4}} = q^{-4b(a-b)} + \text{ higher order terms.}$$

Proof. We can write

$$\binom{a}{b}_{q^{-4}} = \frac{(1-q^{-4a})\dots(1-q^{-4(a-b+1)})}{(1-q^{-4b})\dots(1-q^{-4})}$$
$$= \frac{q^{-4(\sum_{i=1}^b (a-b+i))}}{q^{-4(\sum_{i=1}^b i)}} \cdot \frac{(q^{4a}-1)\dots(q^{4(a-b+1)}-1)}{(q^{4b}-1)\dots(q^4-1)}$$
$$= q^{-4b(a-b)}(1 + \text{ higher order terms})$$

which completes the proof. ∎

Now, we can state the main result of [RW16], which gives a formula for $H_p(q) \in \Gamma(X)[q]$.

Theorem 14.2 ([RW16, Main Thm.])**.** *In the notation above, we have the following formula for $H_p(q)$ in the cases (1)-(4).*

(1) *(Generic) For all $0 \leq p < n$, we have*
$$H_p(q) = \sum_{s=0}^p [D_s] \cdot q^{(n-p)^2+(n-s)(m-n)} \binom{n-s-1}{p-s}_{q^2}.$$
(2) *(Odd skew) Write $n = 2m+1$, then for all $0 \leq p < m$, we have*
$$H_p(q) = \sum_{s=0}^p [D_s] \cdot q^{2(m-p)^2+(m-p)+2(p-s)} \binom{m-1-s}{p-s}_{q^4}.$$
(3) *(Even skew) Write $n = 2m$, then for all $0 \leq p < m$, we have*
$$H_p(q) = \sum_{s=0}^p [D_s] \cdot q^{2(m-p)^2-(m-p)} \binom{m-1-s}{p-s}_{q^4}.$$
(4) *(Symmetric) For all $0 \leq p < n$, we have*
$$H_p(q) = \sum_{\ell=0}^{\lfloor \frac{p}{2} \rfloor} [D_{p-2\ell}] \cdot q^{1+\binom{n-p+2\ell+1}{2}-\binom{2\ell+2}{2}} \binom{\lfloor \frac{n-p+2\ell-1}{2} \rfloor}{\ell}_{q^{-4}}.$$

As $\min\{j \mid \mathcal{H}^j_{Z_p}(\mathscr{O}_X) \neq 0\} = \operatorname{codim}_X(Z_p)$, these expressions immediately imply the following well-known formulas for the (co)dimension of Z_p in X (see [BV88] and [Wey03]).

Corollary 14.3. *In the notation above, we have the following formulas in cases (1)-(4).*

(1) (Generic) In this case, we have

$$\operatorname{codim}_X(Z_p) = (m-p)(n-p), \text{ and } \dim(Z_p) = p(m+n-p).$$

(2) (Odd skew) In this case, we have

$$\operatorname{codim}_X(Z_p) = (m-p)(2(m-p)+1), \text{ and } \dim(Z_p) = p(2(n-p)-1).$$

(3) (Even skew) In this case, we have

$$\operatorname{codim}_X(Z_p) = (m-p)(2(m-p)-1), \text{ and } \dim(Z_p) = p(2(n-p)-1).$$

(4) (Symmetric) In this case, we have

$$\operatorname{codim}_X(Z_p) = \binom{n-p+1}{2}, \text{ and } \dim(Z_p) = \frac{1}{2}p(2n-p+1).$$

We ignore $p = 0$ as in this case Z_p is smooth, and everywhere below we assume $p \geq 1$. In fact, in case (4), when $p = 1$, Theorem 14.2 implies Z_1 is a rational homology manifold (it is known that $\operatorname{lcdef}(Z_1) = 0$, and by the argument of Proposition 14.4, $\mathrm{IC}_{Z_1} = \mathcal{H}^{\operatorname{codim}_X(Z_1)}_{Z_1}(\mathscr{O}_X)$ which by Theorem B implies Z_1 is $\mathbb{Q}$-homology) so we will assume in case (4) that $p \geq 2$. More precisely, in what follows, our assumptions are as follows:

- In case (1), $1 \leq p < n$.
- In cases (2), (3), $1 \leq p < m$.
- In case (4), $2 \leq p < n$.

By definition, $D_p = \mathrm{IC}_{Z_p}$, and so we can study for which Z_p we have equality $\mathrm{IC}_{Z_p} = \mathcal{H}^{\operatorname{codim}_X(Z_p)}_{Z_p}(\mathscr{O}_X)$.

Proposition 14.4. *In the notation above, let $c_p = \operatorname{codim}_X(Z_p)$.*

(1) (Generic) We have equality $\mathrm{IC}_{Z_p} = \mathcal{H}^{c_p}_{Z_p}(\mathscr{O}_X)$ if and only if $m > n$.
(2) (Odd skew) We have equality $\mathrm{IC}_{Z_p} = \mathcal{H}^{c_p}_{Z_p}(\mathscr{O}_X)$ for every $1 \leq p < m$.
(3) (Even skew) We have inequality $\mathrm{IC}_{Z_p} \neq \mathcal{H}^{c_p}_{Z_p}(\mathscr{O}_X)$ for every $1 \leq p < m$.
(4) (Symmetric) We have equality $\mathrm{IC}_{Z_p} = \mathcal{H}^{c_p}_{Z_p}(\mathscr{O}_X)$ if and only if $n \equiv p \mod 2$ (when $p \geq 2$).

Proof. By definition, one needs only check for which $s < p$ the summand $[D_s]$ appears as a coefficient of q^{c_p} using Theorem 14.2. The first three claims are immediate, using that the lowest degree term of the q-binomial coefficient in each expression is the constant term. Note that in case (1), when $m = n$, every $[D_s]$ appears as a coefficient of $q^{c_p} = q^{(m-p)(n-p)}$.

For the case of symmetric matrices, we look at the lowest degree term in

$$q^{1+\binom{n-p+2\ell+1}{2}-\binom{2\ell+2}{2}} \binom{\lfloor \frac{n-p+2\ell-1}{2} \rfloor}{\ell}_{q^{-4}},$$

which by Lemma 14.1, is at degree

$$1 + \binom{n-p+2\ell+1}{2} - \binom{2\ell+2}{2} - 4\ell\left(\left\lfloor \frac{n-p+2\ell-1}{2} \right\rfloor - \ell\right). \tag{14.5}$$

We break into two cases depending on the class of $n-p$ mod 2. We write the difference as

$$n-p = \begin{cases} 2k \\ 2k+1 \end{cases},$$

so that the expression in (14.5) can simplify into

$$\begin{cases} 1+\binom{2k+2\ell+1}{2} - \binom{2\ell+2}{2} - 4\ell(k-1) & \text{when } n-p=2k \\ 1+\binom{2k+2\ell+2}{2} - \binom{2\ell+2}{2} - 4\ell k & \text{when } n-p=2k+1 \end{cases},$$

and these are easily seen to simplify to

$$\begin{cases} 2\ell + k(2k+1) & \text{when } n-p=2k \\ (k+1)(2k+1) & \text{when } n-p=2k+1 \end{cases}.$$

This proves the claim, as the expression in the second case doesn't depend on ℓ, meaning all $[D_{p-2\ell}]$ appear in $\mathcal{H}^{c_p}_{Z_p}(\mathscr{O}_X)$, and the first case is minimized for $\ell = 0$. □

Now we can compute $\mathrm{lcdef}_{\mathrm{gen}}(Z_p)$. To do this, in cases (1)-(3), we look for the highest index such that $[D_{p-1}]$ appears with non-zero coefficient. In case (4), we look for the highest index with $[D_{p-2}]$ having non-zero coefficient. To get the defect, we subtract the codimension.

Proposition 14.6. *In the notation above, we have the following formulas for* $\mathrm{lcdef}_{\mathrm{gen}}(Z_p)$*.*

(1) (Generic) $\mathrm{lcdef}_{\mathrm{gen}}(Z_p) = m+n-2p-2$*.*
(2) (Odd skew) $\mathrm{lcdef}_{\mathrm{gen}}(Z_p) = 4(m-p-1)+2$*.*
(3) (Even skew) $\mathrm{lcdef}_{\mathrm{gen}}(Z_p) = 4(m-p-1)$*.*
(4) (Symmetric) $\mathrm{lcdef}_{\mathrm{gen}}(Z_p) = 2(n-p-1)$ *(we assume* $p \geq 2$*).*

Proof. In the first three cases, we take $s = p-1$ and use that

$$(\text{Generic}): \binom{n-s-1}{1}_{q^2} = \frac{(1-q^{2(n-p)})}{(1-q)} = 1+q^2+\cdots+q^{2(n-p-1)}.$$

$$(\text{Skew}): \binom{m-s-1}{1}_{q^4} = \frac{(1-q^{4(m-p)})}{(1-q)} = 1+q^4+\cdots+q^{4(m-p-1)}.$$

In the symmetric case, we take $\ell = 1$ and use that the maximal degree term in the q-binomial coefficient (evaluated at q^{-4}) is the constant term, so the q-binomial coefficient can be ignored in this case.

In summary, the highest degree with $[D_{p-1}]$ (in cases (1)-(3)) or $[D_{p-2}]$ (in case (4)) is

- (Generic) $(n-p)^2 + (n-p+1)(m-n) + 2(n-p-1)$,
- (Odd skew) $2(m-p)^2 + (m-p) + 2 + 4(m-p-1)$,
- (Even skew) $2(m-p)^2 - (m-p) + 4(m-p-1)$,
- (Symmetric) $1 + \binom{n-p+3}{2} - \binom{4}{2} = \binom{n-p+3}{2} - 5$.

The claim then follows by subtracting $\mathrm{codim}_X(Z_p)$ from each of these expressions. □

In [RW16], a formula for the the local cohomological dimension

$$\mathrm{lcd}(X, Z_p) = \max\{j \mid \mathcal{H}^j_{Z_p}(\mathscr{O}_X) \neq 0\}$$

is given. Using this and the computation at the end of the previous proof, we can compute the difference $\mathrm{lcdef}(Z_p) - \mathrm{lcdef}_{\mathrm{gen}}(Z_p)$.

Proposition 14.7. *In the notation above, we have the following formulas.*

1. *(Generic)* $\mathrm{lcdef}(Z_p) - \mathrm{lcdef}_{\mathrm{gen}}(Z_p) = (p-1)(m+n-2p-2)$.
2. *(Odd skew)* $\mathrm{lcdef}(Z_p) - \mathrm{lcdef}_{\mathrm{gen}}(Z_p) = 2(p-1)(2(m-p-1)+1)$.
3. *(Even skew)* $\mathrm{lcdef}(Z_p) - \mathrm{lcdef}_{\mathrm{gen}}(Z_p) = 4(p-1)(m-p-1)$.
4. *(Symmetric)* $\mathrm{lcdef}(Z_p) - \mathrm{lcdef}_{\mathrm{gen}}(Z_p) = \begin{cases} (n-p-1)(p-2) & p \text{ even} \\ (n-p-1)(p-3) & p \text{ odd and } p \geq 3 \end{cases}$.

Proof. This follows immediately from the computations of $\mathrm{lcd}(X, Z_p)$ in [RW16], which is as follows:

- (Generic) $\mathrm{lcd}(X, Z_p) = mn - (p+1)^2 + 1$,
- (Odd skew) $\mathrm{lcd}(X, Z_p) = \binom{2m+1}{2} - \binom{2p+2}{2} + 1$
- (Even skew) $\mathrm{lcd}(X, Z_p) = \binom{2m}{2} - \binom{2p+2}{2} + 1$
- (Symmetric) $\mathrm{lcd}(X, Z_p) = \begin{cases} 1 + \binom{n+1}{2} - \binom{p+2}{2} & p \text{ even} \\ 1 + \binom{n}{2} - \binom{p+1}{2} & p \text{ odd} \end{cases}$.

The assertion follows by combining the above with Proposition 14.6. □

Using this, we can characterize which Z_p are rational homology manifolds. Indeed, by Theorem B this is equivalent to $\mathrm{lcdef}(Z_p) = 0$ and $\mathrm{IC}_{Z_p} = \mathcal{H}^{c_p}_{Z_p}(\mathscr{O}_X)$. The inequality $\mathrm{lcdef}_{\mathrm{gen}}(Z_p) \leq \mathrm{lcdef}(Z_p)$ shows that if $\mathrm{lcdef}(Z_p) = 0$, then the difference $\mathrm{lcdef}(Z_p) - \mathrm{lcdef}_{\mathrm{gen}}(Z_p)$ is also 0, and so we can use the previous proposition to simplify our computations.

Recall that in the next proposition we assume $p \geq 1$ in cases (1)-(3) and we assume $p \geq 2$ in case (4).

Proposition 14.8. *In all cases (1)-(4), the variety Z_p is not a rational homology manifold.*

Proof. For case (1), the difference $\mathrm{lcdef}(Z_p) - \mathrm{lcdef}_{\mathrm{gen}}(Z_p)$ and $\mathrm{lcdef}_{\mathrm{gen}}(Z_p)$ vanish if and only if $m + n = 2p + 2$. As $p \leq n-1$, this gives $2p + 2 \leq 2(n-1) + 2 = 2n$, so this equality is only possible if $m = n$ and $p = n-1$. But for $m = n$, we have observed that $\mathcal{H}^{c_p}_{Z_p}(\mathscr{O}_X) \neq \mathrm{IC}_{Z_p}$.

For case (2), we have that $\mathrm{lcdef}_{\mathrm{gen}}(Z_p) \geq 2 > 0$, so Z_p can never be a rational homology manifold.

In case (3), $\mathrm{IC}_{Z_p} \neq \mathcal{H}^{c_p}_{Z_p}(\mathscr{O}_X)$, so Z_p cannot be a rational homology manifold.

For case (4), $\mathrm{lcdef}_{\mathrm{gen}}(Z_p) = 0$ if and only if $p = n-1$. But then $\mathrm{IC}_{Z_p} \neq \mathcal{H}^{c_p}_{Z_p}(\mathscr{O}_X)$ as in this case, $p \not\equiv n \mod 2$. □

We conclude this subsection with an application of Theorem G to give bounds on $\mathrm{HRH}(Z_p)$. To do this, we must compute $\mathrm{codim}_{Z_p}(Z_{p,\mathrm{nRS}})$. Note that $Z_{p,\mathrm{nRS}}$ is equal to Z_{p-1} in cases (1)-(3) and Z_{p-2} in case (4).

Lemma 14.9. *In the notation above, we have the following formula for* $\mathrm{codim}_{Z_p}(Z_{p,\mathrm{nRS}})$.

1. *(Generic)* $\mathrm{codim}_{Z_p}(Z_{p,\mathrm{nRS}}) = m + n - 2p + 1$.
2. *(Odd skew)* $\mathrm{codim}_{Z_p}(Z_{p,\mathrm{nRS}}) = 4(m-p) + 3$.
3. *(Even skew)* $\mathrm{codim}_{Z_p}(Z_{p,\mathrm{nRS}}) = 4(m-p) + 1$.
4. *(Symmetric)* $\mathrm{codim}_{Z_p}(Z_{p,\mathrm{nRS}}) = 2(n-p) + 3$ *(when $p \geq 2$).*

Proof. This is immediate by computing the differences $\operatorname{codim}_X(Z_{p-1}) - \operatorname{codim}_X(Z_p)$ in cases (1)-(3), and $\operatorname{codim}_X(Z_{p-2}) - \operatorname{codim}_X(Z_p)$ in case (4) using Corollary 14.3. □

Proposition 14.10. *In the notation above, we have*

1. *(Generic)* $\operatorname{HRH}(Z_p) = 0$.
2. *(Odd skew)* $\operatorname{HRH}(Z_p) \in \{0, 1\}$.
3. *(Even skew)* $\operatorname{HRH}(Z_p) \in \{0, 1\}$.
4. *(Symmetric)* $\operatorname{HRH}(Z_p) \in \{0, 1\}$ *(when* $p \geq 2$*).*

Proof. As Z_p has rational singularities, we always have $\operatorname{HRH}(Z_p) \geq 0$, and we know Z_p is not a rational homology manifold for all p, so $\operatorname{HRH}(Z_p) < \infty$.

By Theorem G, we have inequality

$$\operatorname{lcdef}_{\mathrm{gen}}(Z_p) \leq \operatorname{codim}_{Z_p}(Z_{p,\mathrm{nRS}}) - 2\operatorname{HRH}(Z_p) - 3.$$

The assertion follows from Proposition 14.6 and Lemma 14.9. □

Remark 14.11. In the generic determinantal case, $\operatorname{HRH}(Z_p) = 0$ also follows directly using Theorem B and [Per24, Cor. 1.4]. For this reason, we believe it will be interesting to carry out a study of the Hodge structures on the local cohomology modules analogous to [Per24] in the cases (2)-(4).

Remark 14.12. Some remarks are in order:

1. Note that equality holds in Theorem G for generic determinantal varieties.
2. We observed earlier that the equality $\operatorname{lcdef}_{\mathrm{gen}}(Z) = \operatorname{lcdef}(Z)$ holds when Z_{nRS} is isolated. However, we see above that equality can also hold when Z_{nRS} is non-isolated. Indeed, take for example $m = 3, p = 2$ in case (3) and apply Proposition 14.7.
3. We also note that Proposition 14.7 gives many examples when $\operatorname{lcdef}_{\mathrm{gen}}(Z) < \operatorname{lcdef}(Z)$. The smallest dimension of such Z that we have in these classes of examples is 10 (generic determinantal with $m = 4, n = 3, p = 2$).

15. Examples concerning equality in Theorem G. Previously, we saw that equality holds in Theorem G for generic determinantal varieties. We thank Mihnea Popa for the following classes of examples where we also have equality:

- Any threefold with rational singularities which is not a rational homology manifold (equivalently not locally analytically $\mathbb{Q}$-factorial by [PP24, Thm. F], see also [GWS18, Thm. 5.8]).
- More generally, any local complete intersection $(2k+3)$-fold Z with k-rational singularities that is not a $\mathbb{Q}$-homology manifold. In this case, $\operatorname{HRH}(Z) = k$, as if $\operatorname{HRH}(Z) \geq k+1$, this would imply Z is a rational homology manifold. Thus, the non-rational homology manifold locus is isolated, and we have by the LCI assumption that $\operatorname{lcdef}_{\mathrm{gen}}(Z) = \operatorname{lcdef}(Z) = 0$.

Here are two more examples of this phenomenon:

Example 15.1. Let $f = x_1^2 + \cdots + x_{2m}^2$ for $m \in \mathbb{Z}_{\geq 2}$. Then $b_f(s) = (s+1)(s+m)$, and we have

$$\widetilde{\alpha}_{\mathbb{Z}}(f) = m = \operatorname{Sp}_{\min,\mathbb{Z}}(f) = \operatorname{HRH}(Z) + 2.$$

Let $Z = \{f = 0\} \subseteq \mathbb{A}^{2m}$. As Z is a hypersurface, we have

$$\operatorname{lcdef}(Z) = \operatorname{lcdef}_{\mathrm{gen}}(Z) = 0.$$

Thus, Theorem G gives

$$0 + 2(m-2) + 3 \leq \operatorname{codim}_Z(Z_{\mathrm{nRS}}).$$

But the right hand side is $\dim Z = 2m-1$, as Z has isolated singularities and is not a rational homology manifold. So again, we see that equality holds in Theorem G.

Example 15.2. Equality for Theorem G also holds in the example of [JKSY22, Rmk. 3.4d]. Indeed, there it is shown that for $h = x^6 + y^5 + x^3y^3 + z^5 + w^3$, we have $\mathrm{Sp}_{\min,\mathbb{Z}}(h) = 2$, and so

$$\mathrm{HRH}(\{h=0\}) = 0.$$

Equality holds because $\{h=0\}$ has dimension 3.

However, strict inequality is also possible in Theorem G:

Example 15.3 (An example of strict inequality)**.** Let $f = \sum_{i=1}^{2m} x_i^m$ for $m > 2$. This defines a hypersurface Z of dimension $2m-1$ with an isolated singularity. Moreover, its minimal exponent is

$$\widetilde{\alpha}(f) = \sum_{i=1}^{2m} \frac{1}{m} = 2,$$

so that Z is not a rational homology manifold. As f is weighted homogeneous, we have $\widetilde{\alpha}(f) = \mathrm{Sp}_{\min}(f) = \mathrm{Sp}_{\min,\mathbb{Z}}(f)$, which gives

$$\mathrm{HRH}(Z) = \mathrm{Sp}_{\min,\mathbb{Z}}(Z) - 2 = 0.$$

Thus, Z_{nRS} has codimension $2m-1$ in Z, which is strictly larger than $\mathrm{lcdef}_{\mathrm{gen}}(Z) + 2\,\mathrm{HRH}(Z) + 3 = 3$.

16. Thom-Sebastiani examples. Many of the singularity invariants are easier to control when using defining equations in separate collections of variables (of "Thom-Sebastiani" type), as we see now.

We will make use of the product formula for Verdier specializations, see [DMS11, Section 3] for details. Given $Z_i \subseteq X_i$, we consider the subvariety $Z_1 \times Z_2 \subseteq X_1 \times X_2$. Let $Z_1 = V(f_1, \dots, f_r)$ and let $Z_2 = V(g_1, \dots, g_\rho)$, so that $Z_1 \times Z_2 = V(f_1, \dots, f_r, g_1, \dots, g_\rho)$.

We consider modules $\mathcal{B}_f, \mathcal{B}_g$ and $\mathcal{B}_{(f,g)}$. Then [DMS11, Prop. 3.2] gives an isomorphism

$$\mathrm{Sp}(\mathcal{B}_{(f,g)}, F) = \mathrm{Sp}(\mathcal{B}_f, F) \boxtimes \mathrm{Sp}(\mathcal{B}_g, F).$$

In particular, there are isomorphisms

$$\mathrm{Gr}_V^{\alpha}(\mathcal{B}_{(f,g)}, F) = \bigoplus_{\alpha_1+\alpha_2=\alpha} \mathrm{Gr}_V^{\alpha_1}(\mathcal{B}_f, F) \boxtimes \mathrm{Gr}_V^{\alpha_2}(\mathcal{B}_g, F). \tag{16.1}$$

Example 16.2 (An example of Torelli)**.** In [Tor09], it is noted that there exists a complete intersection variety Z which is a rational homology manifold but such that $\widetilde{\alpha}_{\mathbb{Z}}(Z) < +\infty$. The example follows from the observation that if $f = x^2 + y^2 + z^2$ and $g = u^2 + v^2 + w^2$, then we have

$$b_f(s) = b_g(s) = (s+1)\left(s + \frac{3}{2}\right).$$

Hence, $V(f)$ and $V(g)$ are both rational homology manifolds, and their product $Z = V(f) \times V(g) = V(f,g) \subseteq \mathbb{A}^6$ is also a rational homology manifold.

The Thom-Sebastiani rule for the roots of the Bernstein-Sato polynomial [BMS06, Thm. 5] gives

$$b_{(f,g)}(s) = (s+2)\left(s + \frac{5}{2}\right)(s+3),$$

and so $\widetilde{b}_{(f,g)}(s)$ has an integer root. In this example, $\widetilde{\alpha}_{\mathbb{Z}}(Z) = 3$. As $3 \neq 2$, we see by Remark 11.19 above that this implies $\widetilde{\alpha}_{\mathbb{Z}}(g|_U) < +\infty$ (it is not hard to check that it is also equal to 3 in this case). Thus, $Q^{\mathbb{Z}} \neq 0$. Moreover, note that since $\widetilde{\alpha}_{\mathbb{Z}}(g|_U) = 3$, then Corollary 11.14 recovers the fact that Z is a rational homology manifold.

In fact, we can be rather explicit using the Product Formula (16.1) above. First of all, as f, g are homogeneous with an isolated singular point, their V-filtrations are easy to compute (see [Sai09, (4.2.1)]). As $V(f), V(g)$ are rational homology manifolds, we have that $\mathrm{Gr}_V^j(\mathcal{B}_f) = \mathrm{Gr}_V^j(\mathcal{B}_g) = 0$ for all $j \leq 0$. Thus, we have

$$\mathrm{Gr}_V^\lambda(\mathcal{B}_f) \neq 0 \implies \lambda \in (\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}) \cup \mathbb{Z}_{\geq 1},$$

and similarly for $\mathrm{Gr}_V^\lambda(\mathcal{B}_g)$.

Thus, we have

$$\mathrm{Gr}_V^1(\mathcal{B}_{(f,g)}) = \bigoplus_{\alpha \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}} \mathrm{Gr}_V^\alpha(\mathcal{B}_f) \boxtimes \mathrm{Gr}_V^{1-\alpha}(\mathcal{B}_g),$$

where we know we cannot include any $\alpha \in \mathbb{Z}$ in the direct sum because such an α would have to satisfy $\alpha > 0$ and $1 - \alpha > 0$ in order to be non-zero.

By similar reasoning, we have

$$\mathrm{Gr}_V^2(\mathcal{B}_{(f,g)}) = \bigoplus_{\alpha \in \frac{1}{2}\mathbb{Z}} \mathrm{Gr}_V^\alpha(\mathcal{B}_f) \boxtimes \mathrm{Gr}_V^{2-\alpha}(\mathcal{B}_g)$$

$$= (\mathrm{Gr}_V^1(\mathcal{B}_f) \boxtimes \mathrm{Gr}_V^1(\mathcal{B}_g)) \oplus \bigoplus_{\alpha \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}} \mathrm{Gr}_V^\alpha(\mathcal{B}_f) \boxtimes \mathrm{Gr}_V^{2-\alpha}(\mathcal{B}_g).$$

The claim is that this is the decomposition in Corollary 10.3 above. It is not hard to see that

$$\mathrm{Gr}_V^1(\mathcal{B}_f) \boxtimes \mathrm{Gr}_V^1(\mathcal{B}_g) = \bigcap_{i=1}^2 \ker\left(\partial_{t_i} \colon \mathrm{Gr}_V^2(\mathcal{B}_{(f,g)}) \to \mathrm{Gr}_V^1(\mathcal{B}_{(f,g)})\right),$$

using that Gr_V^0 vanishes for both $\mathcal{B}_f$ and $\mathcal{B}_g$.

Finally, for $\alpha \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, we have

$$t_1 \mathrm{Gr}_V^{\alpha-1}(\mathcal{B}_f) = \mathrm{Gr}_V^\alpha(\mathcal{B}_f),$$

and similarly for $\mathcal{B}_g$, showing that

$$\bigoplus_{\alpha \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}} \mathrm{Gr}_V^\alpha(\mathcal{B}_f) \boxtimes \mathrm{Gr}_V^{2-\alpha}(\mathcal{B}_g) = t_1 \mathrm{Gr}_V^1(\mathcal{B}_{(f,g)}) + t_2 \mathrm{Gr}_V^1(\mathcal{B}_{(f,g)}).$$

Recall from Remark 11.11, we say that the ℓ-th jump is missed for $\ell \in [1, r-1] \cap \mathbb{Z}$ if $p(\mathcal{Q}^{r-\ell}, F) = p(\mathcal{Q}^{r-\ell+1}, F)$. Note that in the above example, the first jump is missed: $p(Q^{\mathbb{Z}}, F) = p(\mathcal{Q}^{r-1}, F)$.

Example 16.3 (Failure of Thom-Sebastiani type rules)**.** An example related to the above shows that the property of being a (partial) rational homology manifold is not well-behaved under Thom-Sebastiani sums of hypersurfaces.

Indeed, if $f = x_1^2 + \cdots + x_n^2$ and $g = y_1^2 + \cdots + y_m^2$ with n, m both odd, then $V(f), V(g)$ are rational homology manifolds. However, their sum $f + g$ has Bernstein-Sato polynomial

$$b_{f+g}(s) = (s+1)\left(s + \frac{n+m}{2}\right),$$

and hence it has an extra integer root. In this way, the Thom-Sebastiani sum of two rational homology manifold hypersurfaces need not remain a rational homology manifold.

17. Examples with liminal singularities. Recall that a local complete intersection Z is said to have l-liminal singularities for some non-negative integer l if its singularities are l-Du Bois but not l-rational (this is equivalent to $\widetilde{\alpha}(Z) = r + \ell$). Below we study the jumps when Z has l-liminal singularities, or more generally when $\widetilde{\alpha}(Z)$ is an arbitrary integer.

Example 17.1. The jumps are rather explicit when $\widetilde{\alpha}(Z) = r + \ell \in \mathbb{Z}$.

For $\ell < 0$, this condition is equivalent to $\delta_f \in V^{r+\ell}\mathcal{B}_f \setminus V^{>r+\ell}\mathcal{B}_f$. In this case,

$$\widetilde{\alpha}(Z) = \mathrm{LCT}(X, Z) = r + \ell$$

(where LCT stands for *log-canonical threshold*) and we have

$$p(\mathcal{Q}^{r+\ell}, F) = p(\mathcal{Q}^{r+\ell+1}, F) = \cdots = p(Q^{\mathbb{Z}}, F) = -n.$$

The claim is that $p(\mathcal{Q}^{r+\ell-k}, F) = k - n$ for all $k \geq 0$. Indeed, as $\delta_f \in V^{r+\ell}\mathcal{B}_f$, we see that $F_{k-n}\mathcal{B}_f \subseteq V^{r+\ell-k}\mathcal{B}_f$ using that $F_{k-n}\mathcal{B}_f = \bigoplus_{|\alpha| \leq k} \mathscr{O}_X \partial_t^{\alpha} \delta_f$. As a result, $F_{(k-1)-n}\mathcal{B}_f \subseteq V^{>r+\ell-k}\mathcal{B}_f$, and so we get the desired claim. Thus, in this case, the first $-\ell$ jumps are missed, and after that, every jump is hit.

For $\ell \geq 0$, this condition is equivalent to $F_{\ell-n}\mathcal{B}_f \subseteq V^r\mathcal{B}_f$ and $F_{\ell+1-n}\mathcal{B}_f \not\subseteq V^{>r-1}\mathcal{B}_f$. The last condition implies that $F_{\ell-n}\mathcal{Q}^r \neq 0$, and in fact, it is easy to see that in this case, $\ell - n = p(Q^{\mathbb{Z}}, F)$. Moreover, by the same reasoning as above, we see that

$$p(\mathcal{Q}^{r-k}, F) = p(Q^{\mathbb{Z}}, F) + k \text{ for all } k \geq 0,$$

proving that, in this example, no jumps are missed.

In the case $\widetilde{\alpha}(Z) = r + \ell \geq r$, the variety Z is ℓ-Du Bois but not ℓ-rational. As the difference between these two is measured by the property that $\mathrm{HRH}(Z) \geq \ell$, we know that $\mathrm{HRH}(Z) \leq \ell - 1$. By Proposition 11.1, using that $p(Q^{\mathbb{Z}}, F) = \ell - n$, we see that we actually have equality $\mathrm{HRH}(Z) = \ell - 1$, which is guaranteed in this case also by Proposition 11.3, once we know that the first jump is not missed.

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School of Mathematics, 1 Einstein Drive, Princeton, New Jersey 08540, USA

Email address: bdirks@ias.edu

Department of Mathematics, University of Toronto, 40 St. George St., Toronto, Ontario Canada, M5S 2E4

Email address: seolano@math.toronto.edu

Department of Mathematics, University of Arizona, 617 N. Santa Rita Ave., Tucson, AZ 85721, USA

Email address: draychaudhury@math.arizona.edu