# THE DU BOIS COMPLEX OF A HYPERSURFACE AND THE MINIMAL EXPONENT

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Abstract. We study the Du Bois complex of a hypersurface $Z$ in a smooth complex algebraic variety in terms of the minimal exponent $\widetilde{\alpha}(Z)$ and give various applications. We show that if $\widetilde{\alpha}(Z) \geq p+1$, then the canonical morphism $\Omega_Z^p \to \underline{\Omega}_Z^p$ is an isomorphism. On the other hand, if $Z$ is singular and $\widetilde{\alpha}(Z) > p \geq 2$, then $\mathcal{H}^{p-1}(\underline{\Omega}_Z^{n-p}) \neq 0$.

## 1. Introduction

One of the Hodge theoretic objects of great interest associated to a singular variety $Z$ is the Du Bois complex (or filtered de Rham complex) $\underline{\Omega}_Z^\bullet$, defined in [DB81], and later in a slightly different fashion in [GNAPGP88]. This is an object in the derived category of filtered complexes on $Z$; when $Z$ is smooth, it is given by the usual algebraic de Rham complex of $Z$, with its "stupid" filtration. In general, the (shifted) associated graded objects

$$\underline{\Omega}_Z^p := \mathrm{Gr}_F^p \underline{\Omega}_Z^\bullet[p]$$

are objects in the derived category of coherent sheaves which provide useful generalizations of the bundles of $p$-forms in the smooth case (for example, they feature in an extension of the Akizuki-Nakano vanishing theorem to singular varieties). The 0-th filtered piece $\underline{\Omega}_Z^0$ appears extensively in the literature, as it is related to what has become a quite important class of singularities; recall that $Z$ is said to have Du Bois singularities if the natural morphism $\mathcal{O}_Z \to \underline{\Omega}_Z^0$ is a quasi-isomorphism. See for instance [KS11] for a nice overview of Du Bois singularities and their role in birational geometry. Besides some formal statements and some special classes of singularities, little is known about $\underline{\Omega}_Z^p$ with $p \geq 1$.

The aim of this paper is to study the behavior of these higher filtered graded pieces of $\underline{\Omega}_Z^\bullet$ when $Z$ is a hypersurface in a smooth complex variety $X$, using methods from the theory of Hodge modules. We give both vanishing and non-vanishing statements about various cohomologies of these complexes, in terms of a singularity invariant derived from the Bernstein-Sato polynomial $b_Z(s)$, namely the *minimal exponent* $\widetilde{\alpha}(Z)$. This is defined as the negative of the greatest root of the reduced Bernstein-Sato polynomial $\widetilde{b}_Z(s) = b_Z(s)/(s+1)$; it has been studied extensively in [Sai94], [Sai16], [MP19], [MP20a]. M. Saito has shown that $Z$ has Du Bois singularities if and only if $\widetilde{\alpha}(Z) \geq 1$, which is equivalent to the pair $(X, Z)$ being log-canonical as $Z$ is a hypersurface in a smooth variety (he also showed that $Z$ has rational singularities if and only if $\widetilde{\alpha}(Z) > 1$). Our main result says that a part of the Du Bois complex becomes similarly well behaved as the minimal exponent gets larger.

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**Theorem 1.1.** *Let $Z$ be a reduced hypersurface in a smooth complex algebraic variety $X$. If $p$ is a nonnegative integer such that $\widetilde{\alpha}(Z) \geq p+1$, then the canonical morphism*

$$\Omega_Z^q \to \underline{\Omega}_Z^q$$

*is a quasi-isomorphism*[1] *for all $q$ with $0 \leq q \leq p$.*

For any non-negative integer $p$, the singularities for which $\widetilde{\alpha}(Z) \geq p+1$ are sometimes called *$p$-log canonical*, by analogy with the case $p = 0$. Note that the minimal exponent can be explicitly bounded, and can also be computed for certain singularity types. For example, we have $\widetilde{\alpha}(Z) = (\dim\, Z + 1)/m$ for an ordinary singularity of multiplicity $m \geq 2$, and $\widetilde{\alpha}(Z) = \sum w_i$ for a weighted homogeneous isolated singularity of weights $w_1, \ldots, w_n$; see Section 2.4 for details.

Theorem 1.1 is in fact a special case of a stronger statement, in which the vanishing of each individual $\mathcal{H}^i(\underline{\Omega}_Z^q)$ with $i > 0$ is derived from a suitable lower bound on the codimension of the locus in $Z$ where the minimal exponent is $< (p+1)$ (i.e. the co-support of the Hodge ideal $I_p(Z)$); see Theorem 3.4 for the precise statement. One consequence, see Corollary 3.5, is that if the singular locus of $Z$ has dimension $s$, then for all $p \geq 0$ we have

$$\mathcal{H}^i(\underline{\Omega}_Z^p) = 0 \quad \text{for} \quad 0 < i < n - s - p - 2.$$

In particular this applies to non-Du Bois singularities as well; see Remark 3.6.

In view of the connection between the Du Bois complex and sheaves of forms with log poles on a resolution, established by Steenbrink [Ste85], Theorem 1.1 implies (in fact is almost equivalent to) a local vanishing result for direct images of such sheaves. Continuing to assume that $Z$ is a reduced hypersurface in the smooth variety $X$, let $f\colon Y \to X$ be a proper morphism that is an isomorphism over $X \smallsetminus Z$, such that $Y$ is smooth and $E = (f^*Z)_{\rm red}$ is a simple normal crossing divisor.

**Corollary 1.2.** *With the above notation, if $p$ is a nonnegative integer such that $\widetilde{\alpha}(Z) \geq p+1$, then for all $q$, with $0 \leq q \leq p$, we have*

$$R^i f_* \Omega_Y^q(\log E)(-E) = 0 \quad \textit{for} \quad i > 0.$$

This is an extension of the well-known fact that if $Z$ is Du Bois, then the canonical morphism $\mathcal{O}_Z \to \mathbf{R} f_* \mathcal{O}_E$ is a quasi-isomorphism,[2] which translates into the vanishing of $R^i f_* \mathcal{O}_Y(-E)$ for $i > 0$.

We also deduce from Theorem 1.1 the following version of global Akizuki-Nakano vanishing for hypersurfaces with high minimal exponent.

**Corollary 1.3.** *If $Z$ is a reduced hypersurface in a smooth, irreducible, projective complex algebraic variety $X$ of dimension $n$, and $p$ is a nonnegative integer such that $\widetilde{\alpha}(Z) \geq p+1$, then for every ample line bundle $L$ on $Z$, we have*

$$H^q(Z, \Omega_Z^p \otimes L) = 0 \quad \textit{for} \quad q > n - 1 - p.$$

Using the same approach as in the proof of Theorem 1.1, we also obtain a vanishing result under a slightly weaker assumption on the minimal exponent:

[1] As part of the proof we show that $\Omega_Z^q$ is reflexive when $\widetilde{\alpha}(Z) \geq p+1$.

[2] These two conditions are in fact equivalent when $Z$ is not necessarily a hypersurface in a smooth variety; see [Ste85] and [Sch07].

**Theorem 1.4.** *With the above notation, for every nonnegative integer $q \leq \widetilde{\alpha}(Z)$ we have*

$$\mathcal{H}^{n-q-1}(\underline{\Omega}_Z^q) = 0$$

*unless $q = n-1$ (this can only hold if either $Z$ is smooth or $q = 1$ and $Z$ is a nodal curve on a surface). In particular, we have*

$$R^{n-q} f_* \Omega_Y^q(\log E)(-E) = 0. \tag{1}$$

Note that $\underline{\Omega}_Z^q$ for $q = \lfloor \widetilde{\alpha}(Z) \rfloor$ is the first graded piece of the Du Bois complex which is not covered by Theorem 1.1. Theorem 1.4 shows that the top cohomology group thereof which could possibly be non-trivial in fact vanishes; indeed, it is known in general that $\mathcal{H}^p(\underline{\Omega}_Z^q) = 0$ for $p \geq n-q$ and every $q$. This general vanishing is related to a theorem of Steenbrink, see [Ste85, Theorem 2], stating that

$$R^i f_* \Omega_Y^j(\log E)(-E) = 0 \quad \text{for} \quad i + j > n$$

for $Z$ of arbitrary codimension in $X$ (in our special setting, this is easy to prove, see Section 2.6).

When $q = 0$, the vanishing in (1) is trivial, while for $q = 1$ it is a special case of a result of Greb, Kovács, Kebekus and Peternell, see [GKKP11, Theorem 14.1], which applies to general log canonical pairs. It is also interesting to note that a related result, namely

$$R^{n-q} f_* \Omega_Y^q(\log E) = 0 \quad \text{for} \quad q \leq \lceil \widetilde{\alpha}(Z) \rceil$$

appears in [MP20b, Corollary C]. Despite the similarity, its proof is of a very different flavor.

Changing gears, we also give a non-vanishing result for the cohomology of certain graded pieces of the Du Bois complex when the minimal exponent is large.

**Theorem 1.5.** *Let $Z$ be a reduced hypersurface in a smooth $n$-dimensional complex algebraic variety $X$, defined by $f \in \mathcal{O}_X(X)$. If $p \geq 2$ is an integer such that $\widetilde{\alpha}(Z) > p$, then for every singular point $x \in Z$, the following hold:*

- i) *We have an isomorphism $\mathcal{H}^{p-1}(\underline{\Omega}_Z^{n-p})_x \simeq \mathcal{O}_{X,x}/\big(J_f + (f)\big)$, where $J_f$ is the Jacobian ideal of $f$.*[3] *In particular, $\mathcal{H}^{p-1}(\underline{\Omega}_Z^{n-p})_x \neq 0$.*
- ii) *If $x$ is an isolated singularity of $Z$ and $p \geq 3$, then $\mathcal{H}^{p-2}(\underline{\Omega}_Z^{n-p})_x \simeq (J_f : f)/J_f$. In particular, $\mathcal{H}^{p-2}(\underline{\Omega}_Z^{n-p})_x \neq 0$ (while $\mathcal{H}^i(\underline{\Omega}_Z^{n-p})_x = 0$ for $0 < i < p-2$).*

Regarding the statement, it is worth noting that, as before, $\mathcal{H}^{p-1}$ is the top possible nonzero cohomology of $\underline{\Omega}_Z^{n-p}$; see Section 2.6. Though the starting point is similar, the proof is somewhat different from that of the vanishing results, in that it appeals to the $V$-filtration (and its connection with the minimal exponent), as well as to duality for nearby and vanishing cycles.

The non-vanishing result has some interesting consequences. The first stems from the fact that if $Y$ is a variety with quotient or toroidal singularities, then $\mathcal{H}^i(\underline{\Omega}_Y^p) = 0$ for all $i \geq 1$ and all $p$ (for quotient singularities, see [DB81, Section 5], and for toroidal singularities, see [GNAPGP88, Chapter V.4]). Thanks to Theorem 1.5, we deduce that in these cases minimal exponents are surprisingly rather small:

**Corollary 1.6.** *If a singular hypersurface $Z$ has quotient or toroidal singularities, then $1 < \widetilde{\alpha}(Z) \leq 2$.*

[3] In an open subset with algebraic coordinates $x_1, \ldots, x_n$, the ideal $J_f$ is generated by $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$.

Note that the upper bound is sharp: the hypersurface defined by $x_1x_2 - x_3x_4$ in $\mathbf{C}^4$ is toric and its minimal exponent is 2; see for instance the paragraph after Theorem 1.1. The lower bound is due to the fact that these are rational singularities.

The second consequence is that the cohomology sheaves $\mathcal{H}^i\underline{\Omega}_Z^p$, with $p, i \geq 1$, are not upper semicontinuous in families. This should be contrasted with a result for $p = 0$ (when $Z$ is not necessarily a hypersurface) due to Kovács and Schwede [KS16], who have shown that nearby deformations of Du Bois singularities are again Du Bois.

**Example 1.7.** Let $f, g \in \mathbf{C}[X_1, \ldots, X_n]$ with $n \geq 5$, be chosen so that $f$ defines a hypersurface with quotient singularities, with a singular point at 0 (hence $\widetilde{\alpha}_0(f) \leq 2$ by Corollary 1.6) while $g$ defines a hypersurface with a singular point at 0 and such that $\widetilde{\alpha}_0(g) > 2$. Consider the family of hypersurfaces parametrized by $\mathbf{A}^1$, defined by $h_t := tf + (1-t)g$. For $t = 1$ we have a hypersurface with quotient singularities, hence $\mathcal{H}^i(\underline{\Omega}_{Z(h_1)}^p) = 0$ for all $i \geq 1$ and all $p$. On the other hand, the minimal exponent is lower semicontinuous in families (see [MP20a, Theorem E(2)]), hence for general $t$ we have $\widetilde{\alpha}_0(h_t) > 2$ (and the hypersurface $Z(h_t)$ has a singular point at 0). Theorem 1.5 then implies that $\mathcal{H}^1(\underline{\Omega}_{Z(h_t)}^{n-2}) \neq 0$.

The paper is organized as follows: we begin by reviewing in the next section some basic facts about the minimal exponent, the Hodge filtration on the local cohomology $\mathcal{H}_Z^1(\mathcal{O}_X)$, and the graded pieces of the Du Bois complex. In particular, we recall the description of these graded pieces in terms of the de Rham complex of $\mathcal{H}_Z^1(\mathcal{O}_X)$. The proofs of Theorems 1.1 and 1.4 are given in Section 3, while the proof of Theorem 1.5 is the content of Section 4.

## 2. Review of the Du Bois complex, Hodge filtration, and minimal exponent

In this section we review some basic facts about the objects in its title that we will need for the proofs of our main results. By a *variety* we mean a reduced, separated scheme of finite type over $\mathbf{C}$.

### 2.1. **Du Bois complex.**

For an introduction to the *Du Bois complex* (sometimes called the *filtered De Rham complex*) and its basic properties, we refer to [GNAPGP88, Chapter V.3], [PS08, Chapter 7.3], [Ste85], and to the original paper of Du Bois [DB81]. A useful list of properties is also collected together in [KS11, Theorem 4.2].

Recall that for a variety $Z$, this is a filtered complex denoted $(\underline{\Omega}_Z^\bullet, F^\bullet)$. We will only be interested in its graded pieces, suitably shifted:

$$\underline{\Omega}_Z^p := \mathrm{Gr}_F^p \underline{\Omega}_Z^\bullet[p].$$

For every $p$ this is an element in the bounded derived category of coherent sheaves on $Z$, which can be nonzero only when $0 \leq p \leq \dim Z$; moreover, there is a canonical morphism

$$\Omega_Z^p \to \underline{\Omega}_Z^p,$$

which is an isomorphism if $Z$ is smooth. The variety $Z$ is said to have *Du Bois singularities* if $\mathcal{O}_Z \to \underline{\Omega}_Z^0$ is an isomorphism.

Suppose now that $Z$ is a closed subvariety of a smooth irreducible variety $X$, with $\dim X = n$. We consider a proper morphism $f\colon Y \to X$ that is an isomorphism over $X \smallsetminus Z$, with $Y$ smooth, and such that the inverse image of $Z$ (with the reduced scheme structure) is a simple normal crossing divisor $E$. A key fact, due to Steenbrink [Ste85, Proposition 3.3], is that for every $p$, we have an exact triangle in the derived category

$$\mathbf{R}f_*\Omega_Y^p(\log E)(-E) \to \Omega_X^p \longrightarrow \underline{\Omega}_Z^p \xrightarrow{+1} . \tag{2}$$

We briefly recall the argument, for the benefit of the reader. Since $f$ is an isomorphism over $X \smallsetminus Z$ and $X$ and $Y$ are smooth, we have an exact triangle

$$\Omega_X^p \longrightarrow \mathbf{R}f_*\Omega_Y^p \oplus \underline{\Omega}_Z^p \longrightarrow \mathbf{R}f_*\underline{\Omega}_E^p \xrightarrow{+1} \tag{3}$$

(see [DB81, Proposition 3.9]). We apply the octahedral axiom for the composition

$$\mathbf{R}f_*\Omega_Y^p \xrightarrow{\alpha} \mathbf{R}f_*\Omega_Y^p \oplus \underline{\Omega}_Z^p \xrightarrow{\beta} \mathbf{R}f_*\underline{\Omega}_E^p,$$

where $\alpha = (\mathrm{id}, 0)$ and $\beta$ is given by the sum of the obvious morphisms. If $Q = \mathrm{cone}(\beta \circ \alpha)$, then we deduce using (3) that we have an exact triangle

$$\underline{\Omega}_Z^p \longrightarrow Q \longrightarrow \Omega_X^p[1] \xrightarrow{+1} .$$

On the other hand, recall that $\underline{\Omega}_E^p = \Omega_Y^p/\Omega_Y^p(\log E)(-E)$ (see [PS08, Example 7.25]), which immediately implies that $Q[-1] \simeq \mathbf{R}f_*\Omega_Y^p(\log E)(-E)$. We thus obtain (2).

2.2. **A consequence of Grothendieck duality.** Let us keep the same notation as in the previous paragraph. Note that since $\big(\Omega_Y^p(\log E)(-E)\big)^\vee = \Omega_Y^{n-p}(\log E) \otimes \omega_Y^{-1}$, it follows from Grothendieck duality that

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathbf{R}f_*\Omega_Y^p(\log E)(-E), \omega_X\big) \simeq \mathbf{R}f_*\Omega_Y^{n-p}(\log E). \tag{4}$$

Of course, since $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(-, \omega_X\big)$ is a duality, we deduce from (4) that we also have

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathbf{R}f_*\Omega_Y^{n-p}(\log E), \omega_X\big) \simeq \mathbf{R}f_*\Omega_Y^p(\log E)(-E). \tag{5}$$

**Assumption:** From now on we assume in addition that $Z$ is a (nonempty) reduced hypersurface in a smooth, irreducible variety $X$ of dimension $n$.

2.3. **Filtered $\mathcal{D}_X$-modules and duality.** Let $\mathcal{D}_X$ be the sheaf of differential operators on $X$. Recall that if $\mathcal{M}$ is a left $\mathcal{D}_X$-module on $X$, then the de Rham complex of $\mathcal{M}$ is the complex $\mathrm{DR}_X(\mathcal{M})$:

$$0 \to \mathcal{M} \to \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{M} \to \cdots \to \Omega_X^n \otimes_{\mathcal{O}_X} \mathcal{M} \to 0,$$

placed in cohomological degrees $-n, \ldots, 0$, with the differentials defined using the usual de Rham differential and the integrable connection on $\mathcal{M}$. If $(\mathcal{M}, F)$ is a filtered $\mathcal{D}_X$-module (so that the filtration is compatible with the order filtration on $\mathcal{D}_X$), then $\mathrm{DR}_X(\mathcal{M})$ carries an induced filtration, with $F_p\mathrm{DR}_X(\mathcal{M})$ being the subcomplex:

$$0 \to F_p\mathcal{M} \to \Omega_X^1 \otimes_{\mathcal{O}_X} F_{p+1}\mathcal{M} \to \ldots \to \Omega_X^n \otimes_{\mathcal{O}_X} F_{p+n}\mathcal{M} \to 0.$$

We will be interested in the filtered $\mathcal{D}_X$-modules associated to certain mixed Hodge modules in the sense of M. Saito's theory, see [Sai88], [Sai90]. For such filtered $\mathcal{D}_X$-modules there is a duality functor $\mathbf{D}$, satisfying the following compatibility with the Grothendieck dual of the de Rham complex:

$$\mathrm{Gr}_p^F\mathrm{DR}_X\big(\mathbf{D}(\mathcal{M})\big) \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{Gr}_{-p}^F\mathrm{DR}_X(\mathcal{M}), \omega_X[n]\big) \tag{6}$$

for every $p$ (see [Sai88, Section 2.4] and also [Sai94, Remark 2.6]).

2.4. **Localization, Hodge filtration, and minimal exponent.** The $\mathcal{D}_X$-module we are interested in is $\mathcal{O}_X(*Z)$, the sheaf of rational functions on $X$ with poles along $Z$. This underlies a mixed Hodge module, hence in particular carries a Hodge filtration; for a detailed study of this filtration, see [MP19]. It is known that the Hodge filtration is contained in the pole order filtration, i.e. for every $p \geq 0$ we have $F_p\mathcal{O}_X(*Z) \subseteq \mathcal{O}_X\big((p+1)Z\big)$, which leads to the definition of the $p$-th Hodge ideal $I_p(Z)$ by the formula

$$F_p\mathcal{O}_X(*Z) = \mathcal{O}_X\big((p+1)Z\big) \otimes I_p(Z).$$

Note also that we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(*Z) \longrightarrow \mathcal{H}^1_Z(\mathcal{O}_X) := \mathcal{O}_X(*Z)/\mathcal{O}_X \longrightarrow 0$$

of filtered $\mathcal{D}_X$-modules, where $\mathcal{O}_X$ underlies the trivial mixed Hodge module $\mathbf{Q}^H_X[n]$ and its filtration satisfies $\mathrm{Gr}^F_q\mathcal{O}_X = 0$ for all $q \neq 0$, while $\mathcal{H}^1_Z(\mathcal{O}_X)$ coincides with the first local cohomology sheaf of $\mathcal{O}_X$ along $Z$, and its Hodge filtration is induced by that on $\mathcal{O}_X(*Z)$.

We now turn to the *minimal exponent* $\widetilde{\alpha}(Z)$ of $Z$, which was originally defined by Saito in [Sai94] as the negative of the greatest root of the reduced Bernstein-Sato polynomial $b_Z(s)/(s+1)$; it is therefore a refinement of the log canonical threshold $\mathrm{lct}(Z)$, which satisfies

$$\mathrm{lct}(Z) = \min\{\widetilde{\alpha}(Z), 1\}.$$

By convention, we have $\widetilde{\alpha}(Z) = \infty$ if and only if $b_Z(s) = s+1$, which is the case if and only if $Z$ is smooth. There is also a local version $\widetilde{\alpha}_x(Z)$ of this invariant around each point $x \in Z$, such that $\widetilde{\alpha}(Z) = \min_{x\in Z} \widetilde{\alpha}_x(Z)$. See [MP20a, Section 6] for a general discussion and study of the minimal exponent.

It turns out that the minimal exponent governs the complexity of the Hodge filtration in various ways. For instance, it determines how far the Hodge filtration agrees with the pole order filtration: for a nonnegative integer $p$, we have

$$\widetilde{\alpha}(Z) \geq p+1 \iff F_k\mathcal{O}_X(*Z) = P_k\mathcal{O}_X(*Z) \quad \text{for} \quad k \leq p \iff I_k(Z) = \mathcal{O}_X \quad \text{for} \quad k \leq p$$

(see [Sai16, Corollary 1], and also [MP20a, Corollary C]). Under these equivalent conditions we also say that the pair $(X, D)$ is *p-log-canonical*, as the case $p = 0$ is precisely the case of log-canonical pairs. It is this interpretation of the minimal exponent that will be used in this paper.

We have the following numerical criteria for minimal exponents, which can be applied in the context of the results in the Introduction:

- $\widetilde{\alpha}(Z) \geq 1 \iff Z$ has du Bois singularities $\iff (X, Z)$ is log-canonical. See [Sai09, Theorem 0.5] for the first equivalence, and [KS11, Corollary 6.6] for the second.
- $\widetilde{\alpha}(Z) > 1 \iff Z$ has rational singularities; see [Sai93, Theorem 0.4].
- If a point $x \in Z$ has multiplicity $m \geq 2$, while the singular locus of its projectivized tangent cone $\mathbf{P}(C_xZ)$ has dimension $r$ (with $r = -1$ if $\mathbf{P}(C_xZ)$ is smooth), then

$$\frac{n-r-1}{m} \leq \widetilde{\alpha}_x(Z) \leq \frac{n}{m}; \tag{7}$$

  see [MP20a, Theorem E]. (The inequality $\widetilde{\alpha}_x(Z) \leq \frac{n}{2}$ also follows from [Sai94, Theorem 0.4].) In particular $\widetilde{\alpha}_x(Z) = \frac{n}{m}$ if $x$ is an ordinary singular point.
- If $Z$ has a weighted homogeneous isolated singularity, where the variable $x_i$ has weight $w_i$, then $\widetilde{\alpha}(Z) = \sum w_i$; see [Sai09, 4.1.5].

- Let $\mu\colon Y \to X$ be a log resolution of the pair $(X, Z)$, chosen such that it is an isomorphism over $X \smallsetminus Z$ and such that the strict transform $\widetilde{Z}$ of $Z$ is smooth. Define integers $a_i$ and $b_i$ by the expressions
$$\mu^* Z = \widetilde{Z} + \sum_{i=1}^m a_i F_i \quad \text{and} \quad K_{Y/X} = \sum_{i=1}^m b_i F_i,$$
where $F_1, \ldots, F_m$ are the prime exceptional divisors, and set
$$\gamma := \min_{i=1,\ldots,m} \left\{ \frac{b_i + 1}{a_i} \right\}.$$
Then we have $\widetilde{\alpha}(Z) \geq \gamma$; see [MP20a, Corollary D], cf. also [DM20, Corollary 1.5].
- The minimal exponent also provides a bound for the generation level of the Hodge filtration $F_\bullet \mathcal{O}_X(*Z)$, shown in [MP20b, Theorem A] to be at most $n - 1 - \lceil \widetilde{\alpha}(Z) \rceil$.

It is shown in [MP20b, Proposition 7.4] that if $Z$ is singular and $\widetilde{\alpha}(Z) > p$, for a nonnegative integer $p$, then the codimension in $Z$ of the singular locus $Z_{\mathrm{sing}}$ is at least $2p$. We will need the following variant, that can be proved along the same lines:

**Lemma 2.1.** *If $Z$ is a singular effective divisor on the smooth variety $X$ and if $\widetilde{\alpha}(Z) \geq p+1$ for a nonnegative integer $p$, then*
$$\mathrm{codim}_Z(Z_{\mathrm{sing}}) \geq 2p + 1.$$

*Proof.* We may and will assume that $Z$ is affine. If $q = \dim(Z_{\mathrm{sing}})$, then after successively cutting $X$ with $q$ general hyperplane sections, we obtain a smooth closed subvariety $Y$ of $X$, of codimension $q$, such that the divisor $Z|_Y$ is singular. Moreover, we have $\widetilde{\alpha}(Z|_Y) \geq \widetilde{\alpha}(Z) \geq p + 1$ by [MP20b, Lemma 7.5]. In this case, it follows from (7) that $p + 1 \leq \frac{n-q}{2}$, hence $r = n - 1 - q \geq 2p + 1$. □

2.5. **The graded pieces of the Du Bois complex via the De Rham complex of $\mathcal{H}^1_Z(\mathcal{O}_X)$.** The connection between the graded pieces of the Du Bois complex and the Hodge filtration on $\mathcal{O}_X(*Z)$ is provided by the following result:

**Lemma 2.2.** *For every $p$, there is an isomorphism*
$$\underline{\Omega}_Z^p \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{Gr}^F_{p-n}\mathrm{DR}_X(\mathcal{H}^1_Z(\mathcal{O}_X)), \omega_X\big)[p+1]. \tag{8}$$

*Proof.* Let $f\colon Y \to X$ be a morphism as in Section 2.1 assumed, in addition, to be projective. The explicit filtered resolution of the right $\mathcal{D}_Y$-module $\omega_Y(*E)$ corresponding to $\mathcal{O}_Y(*E)$ given in [MP19, Proposition 3.1] implies that we have
$$\mathrm{Gr}^F_{p-n}\mathrm{DR}_Y\big(\mathcal{O}_Y(*E)\big) \simeq \Omega_Y^{n-p}(\log E)[p];$$
cf. [MP19, Theorem 6.1]. Since $\mathcal{O}_X(*Z)$ is the push-forward of $\mathcal{O}_Y(*E)$ (in the category of mixed Hodge modules), we obtain using Saito's Strictness Theorem (see [Sai88, Section 2.3.7], cf. [MP19, Section C.4]) that
$$\mathrm{Gr}^F_{p-n}\mathrm{DR}_X\big(\mathcal{O}_X(*Z)\big) \simeq \mathbf{R}f_*\Omega_Y^{n-p}(\log E)[p].$$
Moreover, the canonical morphism
$$\mathrm{Gr}^F_{p-n}\mathrm{DR}_X(\mathcal{O}_X) \to \mathrm{Gr}^F_{p-n}\mathrm{DR}_X\big(\mathcal{O}_X(*Z)\big)$$
gets identified with the canonical morphism
$$\alpha\colon \Omega_X^{n-p}[p] \to \mathbf{R}f_*\Omega_Y^{n-p}(\log E)[p].$$

Since $\mathrm{Gr}^F_{p-n}\mathrm{DR}_X(-)$ is an exact functor, we have an exact sequence of complexes

$$0 \to \mathrm{Gr}^F_{p-n}\mathrm{DR}_X(\mathcal{O}_X) \to \mathrm{Gr}^F_{p-n}\mathrm{DR}_X\big(\mathcal{O}_X(*Z)\big) \to \mathrm{Gr}^F_{p-n}\mathrm{DR}_X(\mathcal{H}^1_Z(\mathcal{O}_X)) \to 0,$$

which induces an exact triangle

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{Gr}^F_{p-n}\mathrm{DR}_X(\mathcal{H}^1_Z(\mathcal{O}_X)), \omega_X\big) \longrightarrow \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{Gr}^F_{p-n}\mathrm{DR}_X(\mathcal{O}_X(*Z)), \omega_X\big) \xrightarrow{\beta}$$

$$\xrightarrow{\beta} \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{Gr}^F_{p-n}\mathrm{DR}_X(\mathcal{O}_X), \omega_X\big) \xrightarrow{+1}.$$

In addition, using the previous discussion, we see that $\beta$ gets identified with the Grothendieck dual of $\alpha$, which in turn is identified via (5) with the morphism

$$\mathbf{R}f_*\Omega^p_Y(\log E)(-E)[-p] \longrightarrow \Omega^p_X[-p].$$

The isomorphism in (8) thus follows from the exact triangle (2). □

We finally note that because of the compatibility between duality for mixed Hodge modules and duality for the corresponding De Rham complexes in (6), the isomorphism (8) is equivalent to the isomorphism

$$\underline{\Omega}^p_Z \simeq \mathrm{Gr}^F_{n-p}\mathrm{DR}_X\big(\mathbf{D}(\mathcal{H}^1_Z(\mathcal{O}_X))\big)[p+1-n]. \tag{9}$$

This result was originally obtained by Saito, see [Sai09, Section 2].

2.6. **Steenbrink's vanishing theorem.** Let $f\colon Y \to X$ be a proper morphism that is an isomorphism over $X \smallsetminus Z$, with $Y$ smooth and $E = f^*(D)_{\mathrm{red}}$ a simple normal crossing divisor. Recall that $n = \dim X$. Since the de Rham complex of any filtered $\mathcal{D}_X$-module is supported in nonpositive degrees, it follows from (9) that

$$\mathcal{H}^q(\underline{\Omega}^p_Z) = 0 \quad \text{for all} \quad q \geq n-p.$$

(This is a special case of a vanishing result that holds for arbitrary varieties $Z$; see [PS08, Theorem 7.29].) This in turn implies via the exact triangle (2) the fact that

$$R^q f_*\Omega^p_Y(\log E)(-E) = 0 \quad \text{for} \quad q > n-p,$$

the assertion of Steenbrink's vanishing theorem in our setting (see [Ste85, Theorem 2]).

## 3. Proof of the vanishing results

In this section we fix a smooth, irreducible variety $X$ of dimension $n$ and a reduced hypersurface $Z$ in $X$. Before proving Theorem 1.1 and related results, we make some preliminary considerations.

We denote by $P_\bullet$ the pole order filtration on $\mathcal{O}_X(*Z)$, that is

$$P_k\mathcal{O}_X(*Z) = \mathcal{O}_X\big((k+1)Z\big) \quad \text{for} \quad k \geq 0$$

and $P_k\mathcal{O}_X(*Z) = 0$ for $k < 0$. We also denote by $P_\bullet$ the induced filtration on $\mathcal{H}^1_Z(\mathcal{O}_X) = \mathcal{O}_X(*Z)/\mathcal{O}_X$. For every nonnegative integer $k$, with $k \leq n$, consider the complex

$$C^\bullet_k = \mathrm{Gr}_{k-n}\mathrm{DR}_X\big(\mathcal{H}^1_Z(\mathcal{O}_X), P\big).$$

In other words, $C^\bullet_k$ is the following complex, placed in cohomological degrees $-k, \ldots, 0$:

$$0 \to \Omega^{n-k}_X \otimes_{\mathcal{O}_X} \mathcal{O}_Z(Z) \to \Omega^{n-k+1}_X \otimes_{\mathcal{O}_X} \mathcal{O}_Z(2Z) \to \ldots \to \omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_Z\big((k+1)Z\big) \to 0.$$

If $f$ is a local equation defining $Z$, then the differential of the complex at $\Omega_X^{n-k+i}\otimes\mathcal{O}_Z\big((i+1)Z\big)$ acts as

$$\eta\otimes[1/f^{i+1}] \quad\mapsto\quad -(i+1)(\eta\wedge df)\otimes[1/f^{i+2}].$$

Since the Hodge filtration $F_\bullet$ on $\mathcal{O}_X(*Z)$ satisfies $F_i\mathcal{O}_X(*Z)\subseteq P_i\mathcal{O}_X(*Z)$ for all $i$, we have a canonical morphism

$$\varphi_k\colon \mathrm{Gr}_{k-n}\mathrm{DR}_X\big(\mathcal{H}^1_Z(\mathcal{O}_X),F\big)\to C_k^\bullet.$$

Moreover, as explained in Section 2.4, we have $\widetilde{\alpha}(Z)\geq p+1$ if and only if $F_k\mathcal{O}_X(*Z)=P_k\mathcal{O}_X(*Z)$ for all $k\leq p$, or equivalently, $F_k\mathcal{H}^1_Z(\mathcal{O}_X)=P_k\mathcal{H}^1_Z(\mathcal{O}_X)$ for all $k\leq p$. We thus see that if $\widetilde{\alpha}(Z)\geq p$, then $\varphi_p^i$ is an isomorphism for all $i\neq 0$ and $\varphi_p^0$ is injective, and in addition this last map is an isomorphism if and only if $\widetilde{\alpha}(Z)\geq p+1$. We thus obtain the following:

**Lemma 3.1.** *For every $p\geq 0$, the following assertions are equivalent:*

i) $\widetilde{\alpha}(Z)\geq p+1$.
ii) *$\varphi_k$ is an isomorphism of complexes for all $0\leq k\leq p$.*
iii) *$\varphi_k$ is an isomorphism in the derived category for all $0\leq k\leq p$.*

We will use the following consequence:

**Proposition 3.2.** *For every $p\geq 0$, we have $\widetilde{\alpha}(Z)\geq p+1$ if and only if for all $0\leq k\leq p$ the morphism $\varphi_k$ induces an isomorphism*

$$\psi_k\colon \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(C_k^\bullet,\omega_X[n]\big)\to \underline{\Omega}_Z^k[n-k-1].$$

*Proof.* Since $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(-,\omega_X[n]\big)$ is a duality, it follows that for every $k$, the morphism $\varphi_k$ is an isomorphism if and only if $\psi_k=\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\varphi_k,\omega_X[n]\big)$ is an isomorphism. The assertion in the proposition follows from Lemma 3.1 and the fact that for every $k$, we have an isomorphism

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{Gr}_{k-n}\mathrm{DR}_X(\mathcal{H}^1_Z(\mathcal{O}_X)),\omega_X[n]\big)\simeq\underline{\Omega}_Z^k[n-k-1]$$

provided by Lemma 2.2. ☐

We can now prove the first result stated in the Introduction.

*Proof of Theorem 1.1.* We begin by showing that if $\widetilde{\alpha}(Z)\geq p+1$, then $\mathcal{H}^j(\underline{\Omega}_Z^k)=0$ for $j\geq 1$ and $0\leq k\leq p$. By Proposition 3.2, it is enough to show that

$$\mathcal{E}xt^j_{\mathcal{O}_X}(C_k^\bullet,\omega_X)=0\quad\text{for all}\quad 0\leq k\leq p\quad\text{and}\quad j>k+1. \tag{10}$$

In order to prove this we may work locally, and thus assume that $Z$ is defined in $X$ by some $f\in\mathcal{O}_X(X)$. In this case, it follows from our description of the complex $C_k^\bullet$ that if $A^\bullet$ is the complex

$$0\longrightarrow\mathcal{O}_X\xrightarrow{df}\Omega_X^1\xrightarrow{-\wedge df}\cdots\xrightarrow{-\wedge df}\Omega_X^n\longrightarrow 0,$$

placed in cohomological degrees $-n,\ldots,0$, then $C_k^\bullet$ is isomorphic to the "stupid" truncation $\sigma^{\geq -k}(A^\bullet|_Z)$.

In particular, we see that for every $k$ there is a short exact sequence of complexes

$$0\longrightarrow C_{k-1}^\bullet\longrightarrow C_k^\bullet\longrightarrow\Omega_X^{n-k}|_Z[k]\longrightarrow 0. \tag{11}$$

For every vector bundle $\mathcal{E}$ on $X$, we have

$$\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{E}|_Z,\omega_X)\simeq\mathcal{E}^\vee\otimes_{\mathcal{O}_X}\omega_Z\quad\text{and}\quad\mathcal{E}xt^j_{\mathcal{O}_X}(\mathcal{E}|_Z,\omega_X)=0\ \ \text{for}\ \ j\neq 1. \tag{12}$$

For future reference, we first note that since $Z = \mathrm{div}(f)$, we have $\omega_Z \simeq \omega_X|_Z$ and thus the first isomorphism in (12) gives

$$\mathcal{E}xt^1_{\mathcal{O}_X}(\Omega_X^{n-k}|_Z, \omega_X) \simeq \Omega_X^k \quad \text{for every} \quad k. \tag{13}$$

Using the vanishings in (12), we deduce from (11) that

$$\mathcal{E}xt^j_{\mathcal{O}_X}(C^\bullet_{k-1}, \omega_X) \simeq \mathcal{E}xt^j_{\mathcal{O}_X}(C^\bullet_k, \omega_X)$$

for all $j \neq k, k+1$. The assertion in (10) now follows by an easy induction on $k$.

We next turn to the description of $\mathcal{H}^0(\underline{\Omega}_Z^k)$ for $0 \leq k \leq p$. The assertion is trivial if $Z$ is smooth, hence from now on we assume that $Z$ is singular. In this case, we have seen in Section 2.4 that $\widetilde{\alpha}(Z) \leq \frac{n}{2}$, hence $p+1 \leq \frac{n}{2}$. As before, we may and will assume that $X$ is affine and $Z$ is defined by $f \in \mathcal{O}_X(X)$.

Since $\mathcal{H}^i(\underline{\Omega}_Z^k) = 0$ for all $i < 0$, it follows from Proposition 3.2 and what we have already proved above that for $0 \leq k \leq p$ we have

$$\mathcal{E}xt^{k+1}_{\mathcal{O}_X}(C^\bullet_k, \omega_X) \simeq \mathcal{H}^0(\underline{\Omega}_Z^k) \quad \text{and} \quad \mathcal{E}xt^j_{\mathcal{O}_X}(C^\bullet_k, \omega_X) = 0 \quad \text{for} \quad j \neq k+1. \tag{14}$$

For $k = 0$ we deduce using (13):

$$\mathcal{E}xt^1_{\mathcal{O}_X}(C^\bullet_0, \omega_X) = \mathcal{E}xt^1_{\mathcal{O}_X}(\omega_X|_Z, \omega_X) \simeq \mathcal{O}_Z.$$

From now on, we assume that $p \geq 1$ and consider $k$ such that $1 \leq k \leq p$. Since the canonical morphism $\Omega_Z^k \to \mathcal{H}^0(\underline{\Omega}_Z^k)$ is an isomorphism over the smooth locus of $Z$, in order to conclude that it is an isomorphism everywhere it is enough to show that if $\mathcal{J}$ is the ideal defining the singular locus $Z_{\mathrm{sing}}$ in $Z$, then

$$\mathrm{depth}\big(\mathcal{J}, \mathcal{H}^0(\underline{\Omega}_Z^k)\big) \geq 2 \quad \text{and} \quad \mathrm{depth}(\mathcal{J}, \Omega_Z^k) \geq 2. \tag{15}$$

Indeed, recall that for every coherent sheaf $\mathcal{F}$ on $Z$, we have an exact sequence

$$0 \to H^0_{\mathcal{J}}(\mathcal{F}) \to \mathcal{F}(Z) \to \mathcal{F}(Z \smallsetminus Z_{\mathrm{sing}}) \to H^1_{\mathcal{J}}(\mathcal{F}),$$

and if $\mathrm{depth}(\mathcal{J}, F) \geq 2$, then $H^0_{\mathcal{J}}(\mathcal{F}) = 0 = H^1_{\mathcal{J}}(\mathcal{F})$ (see [Har67, Theorem 3.8]).

Note that if $r = \mathrm{codim}_Z(Z_{\mathrm{sing}})$, since $Z$ is Cohen-Macaulay, we have $\mathrm{depth}(\mathcal{J}, \mathcal{E}) = r$ for every locally free sheaf $\mathcal{E}$ on $Z$. The first inequality in (15) actually holds for all varieties with rational singularities; see Remark 3.3 below. In our case however there is also a quick argument: the exact sequences (11), together with the isomorphisms in (14) give an exact sequence

$$0 \to \mathcal{H}^0(\underline{\Omega}_Z^{k-1}) \to \Omega_X^k|_Z \to \mathcal{H}^0(\underline{\Omega}_Z^k) \to 0,$$

where the description of the term in the middle follows from (13). We thus conclude that

$$\mathrm{depth}\big(\mathcal{J}, \mathcal{H}^0(\underline{\Omega}_Z^k)\big) \geq \min\big\{\mathrm{depth}(\mathcal{J}, \Omega_X^k|_Z), \mathrm{depth}\big(\mathcal{J}, \mathcal{H}^0(\underline{\Omega}_Z^{k-1})\big) - 1\big\}$$
$$= \min\big\{r, \mathrm{depth}\big(\mathcal{J}, \mathcal{H}^0(\underline{\Omega}_Z^{k-1})\big) - 1\big\}$$

(see, for example, [BH93, Proposition 1.2.9]). Since we have already seen that $\mathcal{H}^0(\underline{\Omega}_Z^0) \simeq \mathcal{O}_Z$, iterating this argument $k$ times gives

$$\mathrm{depth}\big(\mathcal{J}, \mathcal{H}^0(\underline{\Omega}_Z^k)\big) \geq r - k \geq r - p.$$

On the other hand, since $\widetilde{\alpha}(Z) \geq p+1$, it follows from Lemma 2.1 that $r - p \geq p + 1 \geq 2$, hence we get the first inequality in (15).

The second inequality in (15) follows similarly: note first that we have the presentation

$$\mathcal{O}_Z \xrightarrow{df} \Omega^1_X|_Z \longrightarrow \Omega^1_Z \longrightarrow 0$$

and that the zero-locus of $df$ is $Z_{\rm sing}$, whose codimension in $Z$ is $r \geq 2p+1 \geq k$. Since $Z$ is Cohen-Macaulay, it follows from the description of depth via Koszul homology (see [Mat89, Theorem 16.8]) that we have an exact complex

$$0 \longrightarrow \mathcal{O}_Z \xrightarrow{df} \Omega^1_X|_Z \xrightarrow{-\wedge df} \ldots \xrightarrow{-\wedge df} \Omega^k_X|_Z \longrightarrow \Omega^k_Z \to 0$$

(note that exactness at the last two terms holds in general). Breaking this into short exact sequences and using again Lemma 2.1, we see that

$$\mathrm{depth}(\mathcal{J}, \Omega^k_Z) \geq r-k \geq r-p \geq p+1 \geq 2.$$

This completes the proof of the theorem. □

*Remark* 3.3. Under the assumptions of Theorem 1.1, if $p \geq 1$, then one can describe in fact the sheaves $\mathcal{H}^0(\underline{\Omega}^q_Z)$ for *all* $q$ with $0 \leq q \leq n$. Indeed, we have seen that the condition $\widetilde{\alpha}(Z) \geq 2$ implies that $Z$ has rational singularities by [Sai93, Theorem 0.4]. Since $Z$ is a hypersurface, this is equivalent to $Z$ having klt singularities by [Kol97, Corollary 11.13]. In this case, it was shown in [HJ14, Theorem 5.4] that if $j\colon Z_{\rm sm} \hookrightarrow Z$ is the inclusion of the smooth locus, then

$$\mathcal{H}^0(\underline{\Omega}^q_Z) \simeq j_*\Omega^q_{Z_{\rm sm}} \simeq (\Omega^q_Z)^{\vee\vee}$$

for all $q$, via an identification with the sheaves of $h$-differentials. More recently, this was shown to hold for all varieties with rational singularities by Kebekus-Schnell [KS18, Corollary 1.12].

While the statement and argument for the vanishing in Theorem 1.1 are particularly transparent, a stronger statement can be made about the vanishing of individual cohomologies, in terms of the size of the loci in $Z$ where the minimal exponent is small.

**Theorem 3.4.** *If $Z$ is a reduced hypersurface in a smooth variety $X$, and for some integer $p > 0$ the locus $Z_p$ consisting of the points of $Z$ where $\widetilde{\alpha}_x(Z) < p+1$ (equivalently, the closed subscheme defined by the Hodge ideal $I_p(Z)$) satisfies* $\mathrm{codim}_X Z_p > i+p+2$ *for some $i \geq 1$, then $\mathcal{H}^i(\underline{\Omega}^p_Z) = 0$.*

*Proof.* We use freely the notation and arguments from the proof of Theorem 1.1. Note first that the morphism $\varphi_p$ in Lemma 3.1 has the property that both $A^\bullet = \ker(\varphi_p)$ and $B^\bullet = \mathrm{coker}(\varphi_p)$ have all terms supported on $Z_p$. Moreover, these complexes are concentrated in cohomological degrees $\leq 0$. We have seen in the proof of Theorem 1.1 that $\mathcal{E}xt^{i+p+1}_{\mathcal{O}_X}(C^\bullet_p, \omega_X) = 0$. The short exact sequences

$$0 = \mathcal{E}xt^{i+p+1}_{\mathcal{O}_X}(C^\bullet_p, \omega_X) \to \mathcal{E}xt^{i+p+1}_{\mathcal{O}_X}\big(\mathrm{im}(\varphi_p), \omega_X\big) \to \mathcal{E}xt^{i+p+2}_{\mathcal{O}_X}(B^\bullet, \omega_X)$$

and

$$\mathcal{E}xt^{i+p+1}_{\mathcal{O}_X}(\mathrm{im}(\varphi_p), \omega_X) \to \mathcal{E}xt^{i+p+1}_{\mathcal{O}_X}\big(\mathrm{Gr}_{p-n}\mathrm{DR}_X(\mathcal{H}^1_Z(\mathcal{O}_X)), \omega_X\big) = \mathcal{H}^i(\underline{\Omega}^p_Z) \to \mathcal{E}xt^{i+p+1}_{\mathcal{O}_X}(A^\bullet, \omega_X)$$

imply that in order to conclude it suffices to show that

$$\mathcal{E}xt^{i+p+1}_{\mathcal{O}_X}(A^\bullet, \omega_X) = \mathcal{E}xt^{i+p+2}_{\mathcal{O}_X}(B^\bullet, \omega_X) = 0.$$

It is thus enough to show that if $\mathcal{F}^\bullet$ is a complex on $X$ concentrated in degrees $\leq 0$ and $\mathrm{codim}_X \mathrm{Supp}(\mathcal{F}^q) > m$ for all $q$, then $\mathcal{E}xt^m_{\mathcal{O}_X}(\mathcal{F}^\bullet, \omega_X) = 0$. This follows from the hypercohomology spectral sequence

$$E_1^{i,j} = \mathcal{E}xt^j_{\mathcal{O}_X}(\mathcal{F}^{-i}, \omega_X) \Rightarrow \mathcal{E}xt^{i+j}_{\mathcal{O}_X}(\mathcal{F}^\bullet, \omega_X),$$

since when $i + j = m$, we have $E_1^{i,j} = 0$: indeed, we may assume that $i \geq 0$, hence $j = m - i \leq m$ and then $\mathcal{E}xt^j_{\mathcal{O}_X}(\mathcal{F}^{-i}, \omega_X) = 0$ as $\mathrm{codim}_X \mathrm{Supp}(\mathcal{F}^{-i}) > m \geq j$. □

An immediate consequence is a range of automatic vanishing in terms of the size of the singular locus of $Z$.

**Corollary 3.5.** *If the singular locus of the hypersurface $Z$ has dimension $s$, then for all $p \geq 0$ we have*

$$\mathcal{H}^i(\underline{\Omega}_Z^p) = 0 \quad \text{for} \quad 1 \leq i < n - s - p - 2.$$

*Remark* 3.6. The results above are of course relevant even in the non-Du Bois case, or equivalently when we have $\widetilde{\alpha}_x(Z) < 1$ at some points, as Theorem 3.4 implies that $\mathcal{H}^i(\underline{\Omega}_Z^0) = 0$ for $i < n - s - 2$ if $s$ is the dimension of the non-Du Bois locus. For instance, if $Z$ has isolated singularities, then $\mathcal{H}^i \underline{\Omega}_Z^0 \neq 0$ can happen only for $i = 0$ and $i = n - 2$, and it does happen for both if $Z$ is not Du Bois.[4]

We next deduce the two corollaries of Theorem 1.1.

*Proof of Corollary 1.2.* Since the morphism $\Omega_X^k \to \Omega_Z^k$ is surjective for every $k \geq 0$, the assertion follows directly from Theorem 1.1 via the exact triangle (2). It is worth noting that Corollary 1.2 is in fact equivalent to the vanishings $\mathcal{H}^i \underline{\Omega}_Z^q = 0$ for $i > 0$, plus the surjectivity of the natural map $\Omega_X^q \to \mathcal{H}^0 \underline{\Omega}_Z^q$, for all $0 \leq q \leq p$. □

*Proof of Corollary 1.3.* The assertion follows directly from Theorem 1.1 and the version of the Akizuki-Nakano vanishing theorem for the graded pieces of the Du Bois complex:

$$H^i(Z, \underline{\Omega}_Z^j \otimes L) = 0 \quad \text{for} \quad i + j > \dim Z$$

(see [PS08, Theorem 7.29]). □

Using a similar approach to that in the proof of Theorem 1.1, we obtain the vanishing result in Theorem 1.4, as follows.

*Proof of Theorem 1.4.* We may and will assume that $Z$ is singular, in which case our hypothesis implies $q \leq n/2$; in particular, we have $q \neq n$. It is enough to prove that $\mathcal{H}^{n-q-1}(\underline{\Omega}_Z^q) = 0$: indeed, the vanishing of $R^{n-q} f_* \Omega_X^q(\log E)(-E)$ then follows from the exact triangle (2) since $q \neq n$ gives $\mathcal{H}^{n-q}(\Omega_X^q) = 0$.

Furthermore, we may assume that $q \leq n - 2$: if $q = n - 1$, the fact that $q \leq n/2$ implies $n = 2$. Moreover, the hypothesis that $\widetilde{\alpha}(Z) \geq q = 1$ gives that $(X, Z)$ is log canonical and it is well known that this can only happen for nodal curves (note that in this case we clearly have $\mathcal{H}^0(\underline{\Omega}_Z^1) \neq 0$).

Using Lemma 2.2, we see that

$$\mathcal{H}^{n-q-1}(\underline{\Omega}_Z^q) \simeq \mathcal{E}xt^n_{\mathcal{O}_X}\big(\mathrm{Gr}_{q-n} \mathrm{DR}_X(\mathcal{H}_Z^1(\mathcal{O}_X)), \omega_X\big).$$

[4] Note that $\mathcal{H}^{n-1}(\underline{\Omega}_Z^0) = 0$ by Theorem 1.4.

Note now that since $\widetilde{\alpha}(Z)\geq q$, the morphism of complexes

$$\varphi_q\colon \mathrm{Gr}_{q-n}\mathrm{DR}_X(\mathcal{H}^1_Z(\mathcal{O}_X))\to C_q^{\bullet}$$

is injective (see the discussion before Lemma 3.1), and its cokernel is a sheaf $\mathcal{F}$ (supported in cohomological degree 0). We thus have an exact sequence

$$\mathcal{E}xt^n_{\mathcal{O}_X}(C_q^{\bullet},\omega_X)\to \mathcal{E}xt^n_{\mathcal{O}_X}\big(\mathrm{Gr}_{q-n}\mathrm{DR}_X(\mathcal{H}^1_Z(\mathcal{O}_X)),\omega_X\big)\to \mathcal{E}xt^{n+1}_{\mathcal{O}_X}(\mathcal{F},\omega_X)=0,$$

hence it is enough to show that $\mathcal{E}xt^n_{\mathcal{O}_X}(C_q^{\bullet},\omega_X)=0$.

Arguing as in the proof of Theorem 1.1, working locally we may assume that $Z$ is defined in $X$ by a global equation. In this case we have a short exact sequence of complexes

$$0\to C_{q-1}^{\bullet}\to C_q^{\bullet}\to \Omega_X^{n-q}|_Z[q]\to 0.$$

We deduce the exact sequence

$$\mathcal{E}xt^{n-q}_{\mathcal{O}_X}(\Omega_X^{n-q}|_Z,\omega_X)\to \mathcal{E}xt^n_{\mathcal{O}_X}(C_q^{\bullet},\omega_X)\to \mathcal{E}xt^n_{\mathcal{O}_X}(C_{q-1}^{\bullet},\omega_X).$$

The first term vanishes since $q\neq n-1$ and the third term vanishes since $q\neq n$, as we have seen in (14). We thus have $\mathcal{E}xt^n_{\mathcal{O}_X}(C_q^{\bullet},\omega_X)=0$, completing the proof of the theorem. ☐

## 4. Proof of the non-vanishing result

The proof of Theorem 1.5 makes use of the $V$-filtration, so we begin with a very brief review of this notion. For more details, we refer for example to [Sai88, Section 3.1] or [MP20a, Section 2]. Suppose that $X$ is a smooth, irreducible, complex algebraic variety of dimension $n$, and $Z$ is a hypersurface in $X$ defined by $f\in\mathcal{O}_X(X)$. It is common to denote by $B_f$ the $\mathcal{D}$-module push-forward $\iota_+\mathcal{O}_X$, where $\iota\colon X\hookrightarrow W=X\times\mathbf{A}^1$ is the graph embedding $\iota(x)=\big(x,f(x)\big)$. If $t$ denotes the coordinate on $\mathbf{A}^1$, then there is an isomorphism

$$B_f\simeq \mathcal{O}_X[t]_{f-t}/\mathcal{O}_X[t]\simeq\bigoplus_{i\geq 0}\mathcal{O}_X\cdot\partial_t^i\delta,$$

where $\delta$ denotes the class of $\frac{1}{f-t}$, and the actions of $t$ and of a derivation $P\in\mathrm{Der}_{\mathbf{C}}(\mathcal{O}_X)$ are given by

$$t\cdot h\partial_t^i\delta=fh\partial_t^i\delta-ih\partial_t^{i-1}\delta\quad\text{and}\quad P\cdot h\partial_t^i\delta=P(h)\partial_t^i\delta-P(f)h\partial_t^{i+1}\delta.$$

The $\mathcal{D}_W$-module $B_f$ carries a (Hodge) filtration given by

$$F_{p+1}B_f=\bigoplus_{0\leq i\leq p}\mathcal{O}_X\cdot\partial_t^i\delta.$$

This filtered $\mathcal{D}_X$-module underlies a pure Hodge module of weight $n$.

When dealing with duality, it is more common to use right $\mathcal{D}_X$-modules. In order to avoid confusion when citing various results, we will follow this tradition. Recall that there is a canonical equivalence of categories between left and right $\mathcal{D}_X$-modules such that if $\mathcal{M}^r$ is the right $\mathcal{D}_X$-module corresponding to the left $\mathcal{D}_X$-module $\mathcal{M}$, then we have an isomorphism of $\mathcal{O}_X$-modules

$$\mathcal{M}^r\simeq\omega_X\otimes_{\mathcal{O}_X}\mathcal{M}. \tag{16}$$

For example, the right $\mathcal{D}_X$-module corresponding to $\mathcal{O}_X$ is $\omega_X$ and the right $\mathcal{D}_X$-module corresponding to $\mathcal{H}^1_Z(\mathcal{O}_X)$ is $\mathcal{H}^1_Z(\omega_X)$.

We similarly have an equivalence between filtered left and right $\mathcal{D}_X$-modules and the standard convention is that if $(\mathcal{M}^r,F)$ corresponds to $(\mathcal{M},F)$, then the isomorphism (16) identifies $F_{p-n}\mathcal{M}^r$ with $\omega_X \otimes_{\mathcal{O}_X} F_p\mathcal{M}$. Similar considerations apply of course on $X \times \mathbf{A}^1$. Various $\mathcal{D}$-module operations (including duality) are compatible with these equivalences. For example, the right $\mathcal{D}$-module corresponding to $B_f$ is

$$B_f^r := \iota_+\omega_X = \omega_X \otimes_{\mathcal{O}_X} B_f.$$

We also note that the filtered De Rham complex associated to $(\mathcal{M}^r, F)$ can be taken to be the filtered De Rham complex of $(\mathcal{M}, F)$; in particular, we have

$$\mathrm{Gr}^F_\bullet \mathrm{DR}_X\big(\mathcal{H}^1_Z(\omega_X)\big) = \mathrm{Gr}^F_\bullet \mathrm{DR}_X\big(\mathcal{H}^1_Z(\mathcal{O}_X)\big).$$

The $V$-filtration on $B_f$ is a decreasing, exhaustive, discrete, and left continuous filtration $(V^\alpha B_f)_{\alpha\in\mathbf{Q}}$ parametrized by rational numbers. It is characterized uniquely by a number of properties listed for instance in [Sai88, Section 3.1]. The Hodge filtration on $B_f$ induces a filtration on each $V^\alpha B_f$ and thus on $\mathrm{Gr}_V^\alpha B_f = V^\alpha B_f / V^{>\alpha} B_f$ as well. We have a corresponding $V$-filtration on $B_f^r$ given by $V^\alpha B_f^r = \omega_X \otimes_{\mathcal{O}_X} V^\alpha B_f$. Note that since the Hodge filtrations on $\mathrm{Gr}_V^\alpha B_f$ and $\mathrm{Gr}_V^\alpha B_f^r$ are induced by those on $B_f$ and $B_f^r$, respectively, these satisfy

$$F_{p-n-1}\mathrm{Gr}_V^\alpha B_f^r = \omega_X \otimes_{\mathcal{O}_X} F_p \mathrm{Gr}_V^\alpha B_f. \tag{17}$$

An important fact is a result of Saito, see [Sai16, (1.3.8)], describing the minimal exponent via the $V$-filtration: if $q$ is a non-negative integer and $\alpha \in (0,1]$ is a rational number, then

$$\widetilde{\alpha}(Z) \geq q + \alpha \iff \partial_t^q \delta \in V^\alpha B_f. \tag{18}$$

This setting is relevant for us since the filtered right $\mathcal{D}_X$-module $\mathcal{H}^1_Z(\omega_X)$, corresponding to the $\mathcal{D}_X$-module appearing in Lemma 2.2, is isomorphic to the cokernel of the morphism of filtered $\mathcal{D}_X$-modules

$$\mathrm{Gr}_V^0 B_f^r \xrightarrow{\cdot t} \mathrm{Gr}_V^1 B_f^r$$

between the vanishing cycles and (a Tate twist of) the nearby cycles of $f$ (see [Sai90, Section 2.24]). It follows that $\mathbf{D}\big(\mathcal{H}^1_Z(\omega_X)\big)$ is isomorphic to the kernel of the dual morphism

$$\mathbf{D}(\mathrm{Gr}_V^1 B_f^r) \to \mathbf{D}(\mathrm{Gr}_V^0 B_f^r), \tag{19}$$

where $\mathbf{D}$ is the duality functor on filtered $\mathcal{D}$-modules; see [Sai88, Section 2.4]. Since $B_f^r$ underlies a pure polarizable Hodge module of weight $n$, we have an isomorphism $\mathbf{D}(B_f^r) \simeq B_f^r(n)$. Here, for a filtered $\mathcal{D}$-module $(\mathcal{M}, F)$, we use the notation $(\mathcal{M}, F)(q)$ for the filtered $\mathcal{D}$-module $(\mathcal{M}, F[q])$, where $F[q]_i\mathcal{M} = F_{i-q}\mathcal{M}$. Using the compatibility between duality and vanishing/nearby cycles proved by Saito in [Sai89, Theorem 1.6], we also have isomorphisms of filtered (right) $\mathcal{D}_X$-modules

$$\mathbf{D}(\mathrm{Gr}_V^1 B_f^r) \simeq \mathrm{Gr}_V^1 B_f^r(n+1) \quad \text{and} \quad \mathbf{D}(\mathrm{Gr}_V^0 B_f^r) \simeq \mathrm{Gr}_V^0 B_f^r(n).$$

Moreover, the morphism (19) gets identified with the morphism

$$\mathrm{Gr}_V^1 B_f^r(n+1) \xrightarrow{\cdot \partial_t} \mathrm{Gr}_V^0 B_f^r(n). \tag{20}$$

After this preparation, we can prove the result stated in the Introduction.

*Proof of Theorem 1.5.* It follows from the formula (9) for the graded pieces of the Du Bois complex that for every $i$, we have

$$\mathcal{H}^i(\underline{\Omega}_Z^{n-p}) \simeq \mathcal{H}^{i-p+1}\big(\mathrm{Gr}_p^F \mathrm{DR}_X \mathbf{D}(\mathcal{H}^1_Z(\omega_X))\big).$$

On the other hand, it follows from the previous discussion that ${\rm Gr}_p^F{\rm DR}_X{\bf D}\big(\mathcal{H}_Z^1(\omega_X)\big)$ is isomorphic to the kernel of the morphism

$$
{\rm Gr}_{p-n-1}^F{\rm DR}_X\big({\rm Gr}_V^1(B_f^r)\big)\to {\rm Gr}_{p-n}^F{\rm DR}_X\big({\rm Gr}_V^0(B_f^r)\big)
$$

induced by right multiplication with $\partial_t$. If we write these complexes explicitly in terms of left $\mathcal{D}$-modules, using the identification in (17), we see that ${\rm Gr}_p^F{\rm DR}_X{\bf D}\big(\mathcal{H}_Z^1(\omega_X)\big)$ is the kernel of the morphism of complexes

$$
\begin{array}{ccccccccccc}
0 \longrightarrow & \Omega_X^{n-p+1}\otimes {\rm Gr}_1^F{\rm Gr}_V^1B_f & \longrightarrow \cdots \longrightarrow & \Omega_X^{n-1}\otimes {\rm Gr}_{p-1}^F{\rm Gr}_V^1B_f & \longrightarrow & \omega_X\otimes {\rm Gr}_p^F{\rm Gr}_V^1B_f & \longrightarrow 0\\
& \downarrow & & \downarrow & & \downarrow & \\
0 \longrightarrow & \Omega_X^{n-p+1}\otimes {\rm Gr}_2^F{\rm Gr}_V^0B_f & \longrightarrow \cdots \longrightarrow & \Omega_X^{n-1}\otimes {\rm Gr}_{p}^F{\rm Gr}_V^0B_f & \longrightarrow & \omega_X\otimes {\rm Gr}_{p+1}^F{\rm Gr}_V^0B_f & \longrightarrow 0
\end{array}
$$

placed in cohomological degrees $-(p-1),\ldots,0$ and in which the vertical maps are given by left multiplication by $\partial_t$. Under the assumption of the theorem, we will identify the top complex and show that in the bottom complex all terms are 0.

By (18), the condition $\widetilde{\alpha}(Z)>p$ is equivalent to the fact that $\partial_t^p\delta\in V^{>0}B_f$, and in fact $\partial_t^j\delta\in V^{>0}B_f$ for all $j\leq p$. We thus see that $F_{p+1}B_f\subseteq V^{>0}B_f$, hence ${\rm Gr}_j^F{\rm Gr}_V^0B_f=0$ for all $j\leq p+1$. Therefore the bottom complex in the above diagram is 0 and we conclude that

$$
\mathcal{H}^i(\underline{\Omega}_Z^{n-p})\simeq \mathcal{H}^{i-p+1}\big({\rm Gr}_{p-n-1}^F{\rm DR}_X\big({\rm Gr}_V^1(B_f^r)\big)\big).
$$

Again using (18), since $\widetilde{\alpha}(Z)\geq p$ we deduce that $\partial_t^j\delta\in V^1B_f$ for $j\leq p-1$. We conclude that for $1\leq j\leq p$, we have $F_jV^1B_f=\bigoplus_{i\leq j-1}\mathcal{O}_X\cdot\partial_t^i\delta$. Note also that $F_jV^{>1}B_f=t\cdot F_jV^{>0}B_f$; this is a general property of filtered $\mathcal{D}$-modules underlying mixed Hodge modules, see [Sai88, (3.2.1.2)]. This implies that for $j\leq p$, we have

$$
F_jV^{>1}B_f+F_{j-1}V^1B_f=\mathcal{O}_X\cdot\delta\oplus\cdots\oplus\mathcal{O}_X\cdot\partial_t^{j-2}\delta\oplus(f)\cdot\partial_t^{j-1}\delta.
$$

We thus conclude that the morphism

$$
\mathcal{O}_X/(f)\to {\rm Gr}_j^F{\rm Gr}_V^1B_f=F_jV^1B_f/(F_jV^{>1}B_f+F_{j-1}V^1B_f)
$$

that maps the class of $h$ to the class of $h\partial_t^{j-1}\delta$, is an isomorphism.

Suppose now that we have algebraic local coordinates $x_1,\ldots,x_n$ in a neighborhood of $x$. A straightforward computation then shows that $\mathcal{H}^i(\underline{\Omega}_Z^{n-p})$ is the cohomology in degree $i-p+1$ of the "stupid" truncation $\sigma^{\geq -p+1}$ of the Koszul complex on $\mathcal{O}_X/(f)$ associated to the sequence $\partial f/\partial x_1,\ldots,\partial f/\partial x_n$. This immediately gives the formula for $\mathcal{H}^{p-1}(\underline{\Omega}_Z^{n-p})$ in i).

Suppose now that $p\geq 3$ and $f$ has an isolated singularity at $x$. In this case, by Generic Smoothness, around $x$ the zero-locus of $J_f$ is contained in the hypersurface defined by $f$, hence it is equal to $\{x\}$. Therefore the elements $\partial f/\partial x_1,\ldots,\partial f/\partial x_n$ form a regular sequence in $\mathcal{O}_{X,x}$, so that

$$
\mathcal{H}^i(\underline{\Omega}_Z^{n-p})_x\simeq {\rm Tor}_{p-1-i}^{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/J_f,\mathcal{O}_{Z,x})
$$

for $1\leq i\leq p-1$. The assertions in ii) are immediate consequences. (The vanishing statement also follows from Corollary 3.5, and holds for an arbitrary isolated singularity.) □

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