Chapter 1

Category Theory

1.1 Categories, Functors and Natural Transformations

Definition 1.1.1 A *category* $\underline{\underline{C}}$ consists of:

E1) a class Obj $\underline{\underline{C}}$ — the *objects* of $\underline{\underline{C}}$ (which need not form a set);

E2) for each pair X, Y of objects of Obj $\underline{\underline{C}}$, a set denoted $\underline{\underline{C}}(X,Y)$ or $\mathrm{Hom}_{\underline{\underline{C}}}(X,Y)$ — *morphisms* in the category $\underline{\underline{C}}$ from X to Y;

E3) for each triple X, Y, Z of objects of Obj $\underline{\underline{C}}$, a set function $\circ : \underline{\underline{C}}(X,Y) \times \underline{\underline{C}}(Y,Z) \to \underline{\underline{C}}(X,Z)$ called *composition* (where $\circ(f,g)$ is usually written $g \circ f$ or simply gf);

E4) for each object X of Obj $\underline{\underline{C}}$, an element $1_X \in \underline{\underline{C}}(X,X)$ — *identity morphism* of X;

such that:

A1) $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f \in \underline{\underline{C}}(W,X)$, $g \in \underline{\underline{C}}(X,Y)$, and $h \in \underline{\underline{C}}(Y,Z)$;

A2) $1_Y \circ f = f$ and $f \circ 1_X = f$ for all $f \in \underline{\underline{C}}(X,Y)$.

A category whose objects form a set is called a *small category*.

Definition 1.1.2 A (*covariant*) *functor* $F : \underline{\underline{C}} \to \underline{\underline{D}}$ consists of:

E1) an object $F(X)$ of Obj $\underline{\underline{D}}$ for each object X of Obj $\underline{\underline{C}}$;

E2) a morphism $F(g) \in \underline{\underline{D}}\big(F(X), F(Y)\big)$ for each morphism $g \in \underline{\underline{C}}(X,Y)$;

such that:

A1) $F(h \circ g) = F(h) \circ F(g)$ for all $g \in \underline{\underline{C}}(X, Y)$, $h \in \underline{\underline{C}}(Y, Z)$;

A2) $F(1_X) = 1_{F(X)}$ for all objects X of $\operatorname{Obj} \underline{\underline{C}}$.

Examples:

1. sets; vector spaces over a given field; groups; rings; topological spaces. (These categories are not small categories)

   Note 1: Strictly speaking we should write "sets and set functions"; . . ., "topological spaces and continuous functions". However frequently we describe a category by specifying only the objects when, as above, it is understood what the morphisms are.

   Note 2: There is not universal agreement as to what is meant by the category of rings. My definition requires that there exist an identity element 1 for the multiplication and that $1 \neq 0$ and that a morphism of rings is required to take 1 to 1. Some people (e.g. Dummit and Foote), do not require a 1 and they refer to my definition of a ring as "ring with identity", whereas I refer to their definition as a "ring without 1", otherwise known as an "Rng". Still other people (e.g. Lang) insist on an identity, but do not require $1 \neq 0$.

2. The "homotopy category" $\mathcal{H}oTop$ is the category whose objects are topological spaces and whose morphisms $\mathcal{H}oTop(X, Y)$ are homotopy classes of continuous maps from X to Y. The "pointed homotopy category" $\mathcal{H}oTop_*$ is the category whose objects are topological spaces with chosen basepoint and morphisms are equivalences classes of basepoint-preserving continuous maps under basepoint-preserving homotopy.

3. Given a category $\underline{\underline{C}}$ and a subclass $\mathcal{S}$ of $\operatorname{Obj} \underline{\underline{C}}$, the *full subcategory* of $\underline{\underline{C}}$ generated by $\mathcal{S}$ is the category whose objects are $\mathcal{S}$ and morphisms $\mathcal{S}(X, Y)$ between objects X and Y in $\mathcal{S}$ is the same as $\underline{\underline{C}}(X, Y)$.

4. Functors can be composed in the obvious way yielding a "Category of small categories". (Objects are small categories, and morphisms are functors.)

   Note: There is no "Category of categories" since the condition that the morphisms between any pair object should form a set would not be satisfied.

5. Let $\underline{\underline{C}}$ be a category. The *opposite category* of $\underline{\underline{C}}$ is the category, written $\underline{\underline{C}}^{\mathrm{op}}$ given by $\operatorname{Obj} \underline{\underline{C}}^{\mathrm{op}} = \operatorname{Obj} \underline{\underline{C}}$ with $\underline{\underline{C}}^{\mathrm{op}}(X, Y) := \underline{\underline{C}}(Y, X)$, and the composition $g \circ f$ in $\underline{\underline{C}}^{\mathrm{op}}$ defined as $f \circ g$ in $\underline{\underline{C}}$. A *contravariant functor* from $\underline{\underline{C}}$ to $\underline{\underline{D}}$ is defined as a functor from $\underline{\underline{C}}^{\mathrm{op}}$ to $\underline{\underline{D}}$.

6. Let S be a partially ordered set. Define a category $\underline{\underline{C_S}}$ by $\mathrm{Obj}\ \underline{\underline{C_S}} = S$ and

$$\underline{\underline{C_S}}(x,y) = \begin{cases} \text{set with one element} & \text{if } x \le y; \\ \emptyset & \text{otherwise} \end{cases}$$

for x, $y \in S$. Since the relevant sets are either empty or singletons there is no choice as to the definition of $\circ$ or the element $1_x \in \underline{\underline{C_S}}(x,x)$, and the axioms for partially ordered sets guarantee that the axioms for a category are satisfied. (This category is a small category.)

7. Let $F : \underline{\underline{C}} \to \underline{\underline{D}}$ be a functor and let $D \in \mathrm{Obj}\,\underline{\underline{D}}$. Define a category denoted $F//D$ as follows. An object of $F//D$ is a pair (X,f) where $X \in \mathrm{Obj}\,\underline{\underline{C}}$ and $f \in \mathrm{Hom}_{\underline{\underline{D}}}(F(X),D)$. A morphism $(X,f) \to (Y,g)$ consists of a map $h : X \to Y$ in $\mathrm{Hom}_{\underline{\underline{C}}}(X,Y)$ such that

$$\begin{tikzcd} F(X) \arrow[rr, "F(h)"] \arrow[dr, "f"'] & & F(Y) \arrow[dl, "g"] \\ & D & \end{tikzcd}$$

commutes in $\underline{\underline{D}}$.

This is called a *comma category*. In the special case where F is the identity functor it is called the "category of objects over D".

Similarly there is a comma category $D\backslash\backslash F$ which has as a special case the "category of objects under D".

There is always a forgetful functor $F//D \to \underline{\underline{C}}$ given by $(X,f) \mapsto X$. Similarly there is a forgetful functor $D\backslash\backslash F \to \underline{\underline{C}}$.

We sometimes find it convenient to use the "Milnor-Moore" convention under which when one sees the name of a space in a place that one is expecting a self-map of that space, it stands for the identity map on that space.

Given categories $\underline{\underline{C}}$, $\underline{\underline{D}}$ the product category $\underline{\underline{C}} \times \underline{\underline{D}}$ is defined by

$$\mathrm{Obj}\,(\underline{\underline{C}} \times \underline{\underline{D}}) := (\mathrm{Obj}\ \underline{\underline{C}}) \times (\mathrm{Obj}\ \underline{\underline{D}}) \quad \text{with} \quad (\underline{\underline{C}} \times \underline{\underline{D}})\big((X,Y),(X',Y')\big) := \underline{\underline{C}}(X,X') \times \underline{\underline{D}}(Y,Y').$$

Definition 1.1.3 Let $F, G : \underline{\underline{C}} \to \underline{\underline{D}}$ be functors. A *natural transformation* $\eta : F \to G$ consists of a morphism $\eta_X \in \underline{\underline{D}}\big(F(X), G(X)\big)$ for each object X of Obj $\underline{\underline{C}}$ such that

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y) \end{tikzcd}$$

commutes in $\underline{\underline{D}}$ for all $f \in \underline{\underline{C}}(X, Y)$.

Let I denote the category coming (as described above) from the ordered set $\big\{\{0,1\} \mid 0 \prec 1\big\}$. The notation is motivated by the fact that in Section 5 we shall associate a topological space to a small category, and the space corresponding to the category I is the unit interval $I := [0,1]$.

It is easy to see

Proposition 1.1.4 *Let $\underline{\underline{C}}, \underline{\underline{D}}$ be categories. A natural transformation $\eta : F \to G$ determines, and is uniquely determined by, a functor $\underline{\underline{C}} \times I \to \underline{\underline{D}}$ whose restrictions to the subcategories whose objects are* (Obj $\underline{\underline{C}}$, 0) *and* (Obj $\underline{\underline{C}}$, 1) *are F and G respectively.* □

If $\exists$ morphisms $f : X \to Y$ and $g : Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$ then X and Y are called *isomorphic* (written $X \cong Y$) and f and g are called *isomorphisms*.

A morphism $f : X \to Y$ in $\underline{\underline{C}}$ which has the property that for any $\alpha, \beta : W \to X$, $f\alpha = f\beta$ only when $\alpha = \beta$ is called a *monomorphism* in $\underline{\underline{C}}$. A morphism $f : X \to Y$ in $\underline{\underline{C}}$ which has the property that for any $\alpha, \beta : Y \to Z$, $\alpha f = \beta f$ only when $\alpha = \beta$ is called an *epimorphism*. Note that while is isomorphism must be both a monomorphism and an epimorphism, a morphism which is both a monomophism and an epimorphism need not be an isomorphism.

A functor $F : \underline{\underline{C}} \to \underline{\underline{D}}$ for which there exists a functor $G : \underline{\underline{D}} \to \underline{\underline{C}}$ such that $GF = 1_{\underline{\underline{C}}}$ and $FG = 1_{\underline{\underline{D}}}$ is called an *isomorphism of categories*.

Let $F, G : \underline{\underline{C}} \to \underline{\underline{D}}$ be functors. A natural transformation $\eta : F \to G$ such that $\eta_X : F(X) \to G(X)$ is an isomorphism for all $X \in$ Obj $\underline{\underline{C}}$ is called a *natural equivalence*.

A functor $F : \underline{\underline{C}} \to \underline{\underline{D}}$ for which there exists a functor $G : \underline{\underline{D}} \to \underline{\underline{C}}$ such that there exists a natural equivalences $GF \to 1_{\underline{\underline{C}}}$ and $FG \to 1_{\underline{\underline{D}}}$ is called an *equivalence of categories*.

The statement that two categories are equivalent can be a nontrivial theorem.

Example 1.1.5

Theorem (Gelfand-Naimark) The functor

$$X \mapsto \{\text{continuous complex-valued function on } X\}$$

is an equivalence between the categories of compact topological spaces and the opposite category of the category of abelian C^*-algebras.

An object $P \in \mathrm{Obj}\, \underline{\underline{C}}$ is called a *projective object* in $\underline{\underline{C}}$ if given any epimorphism $g : B \twoheadrightarrow A$ and any momorphism $f : P \to A$ there exists a "lift" $f' : P \to B$ (not necessarily unique) such that $gf' = f$.

$$\begin{tikzcd}
P \arrow[d, dashed, "\exists f'"'] \arrow[dr, "f"] & \\
B \arrow[r, two heads, "g"] & A
\end{tikzcd}$$

Given an object $M \in \mathrm{Obj}\, \underline{\underline{C}}$, an epimorphism $P \twoheadrightarrow M$ in which P is projective is called a *projective presentation* of M.

If every object of a category has a projective presentation, the category is said to have *enough projectives*. Enough for what? We shall see when we come to discuss projective resolutions. (See Section 3.1)

Dually, we call $Q \in \mathrm{Obj}\, \underline{\underline{C}}$ an *injective object* in $\underline{\underline{C}}$ if given any monomorphism $g : A \rightarrowtail B$ and any morphism $A \to Q$ there exists an "extension" $f' : B \to Q$ (not necessarily unique) such that $f'g = f$. If every object of a category has a monomorphism to an injective object then the category is said to have *enough injectives*.

Examples:

1. If R is a ring then R^N is projective in the category of R-modules. A projective object of this form is called a *free R-module*. In the category R-modules, the following are equivalent:

   (a) P is projective

   (b) $\exists\, Q$ such that $P \oplus Q$ is free

   (c) If $g : B \longrightarrow P$ is surjective then g has a "right splitting"; that is, $\exists\ s : P \longrightarrow B$ such that $gs = 1_P$

For some rings, for example PID's, the only projective modules are free, but in general a ring can have projective modules which are not free.

2. In the category of abelian groups, $\mathbb{Z}$ is projective, however it is not injective. For example: the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is a monomorphism but the identity map $1_{\mathbb{Z}}$ does not admit an extension to $\mathbb{Q}$.

   $\mathbb{Q}$ is injective in the category of abelian groups. (See below.)

3. For any ring R, the category of R-modules has enough projectives. In detail, let M be an R-module. Choose a (possibly infinite) generating set $\{x_j\}_{j \in J}$. The map $R^{|J|} \to M$ whose restriction to the jth summand is given by $r \mapsto rx_j$ is a projective presentation of M.

4. An R-module is called *divisible* if for all $x \in M$ and $r \in R$, $\exists\, y \in M$ s.t. $x = ry$.

Proposition 1.1.6 *Let R be a integral domain and let M be an injective R-modules. Than M is divisible.*

Proof: Suppose $x \in M$ and $r \in M$. Let $A = B = R$ (as a module over itself) and define $g : A \to B$ by $g(a) = ra$. Since R is an integral domain, g is a monomorphism. Define $f : A \to M$ by $f(a) := ax$. Since M is injective, there exists an extension $f' : B \to M$. Set $y := f'(1)$. Then $ry = f'(r) = f(1) = x$. □

Proposition 1.1.7 *Let R be a PID. Then an R-module is injective iff it is divisible.*

Proof: As above, if M is injective then it is divisible.

Suppose now that M is divisible. Let $g : A \hookrightarrow B$ be a monomorphism and let $f : A \to M$. Consider the set $\mathcal{S}$ of pairs (X, f_X), where X is an R-submodule of B which contains A and $f_X : X \to Q$ is an extension of f. Partially order the set $\mathcal{S}$ by $(X, f_X) \prec (Y, f_Y)$ if $X \subset Y$ and $f_Y|_X = f_X$. The union of any totally ordered chain in $\mathcal{S}$ admits a map to M extending the maps in the chain, and so forms an upper bound for the chain. Thus by Zorn's lemma, $\mathcal{S}$ has a maximal element, say (N, F_N). It suffices to show $N = B$ since then we set $f' := f_N$. If $N \subsetneq B$, choose $b \in B - N$ and let $N' = N + \langle b \rangle \subset B$. Consider the ideal $I := \{r \in R \mid rb \in N\}$. If $I = \emptyset$ then $N' = N \oplus \langle b \rangle \subset B$, and we choose arbitrary $m \in M$. Otherwise, since R is a PID, write $I = (s)$ for some $s \in R$. Using divisibility find $m \in M$ such that $sm = f_N(sb)$. Define $f_{N'} : N' \to M$ by $f_{N'}|_N := f_N$ and $f_{N'}(b) := m$. This gives an extension of f_N to N' contradicting maximality of N'. □

Corollary 1.1.8 *$\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective in the category of abelian groups (equivalently $\mathbb{Z}$-modules).*

Proposition 1.1.9 *For any ring R, the category of R-modules has enough injectives.*

Proof: Consider first the case where $R = \mathbb{Z}$. Let A be an abelian group. For each nonzero $a \in A$, there exists a homomorphism $\phi_a : \langle a \rangle \to \mathbb{Q}/\mathbb{Z}$ given by $\phi(a) = x$ where $x \in \mathbb{Q}/\mathbb{Z}$ is an element whose order is divisible by the order of a, choosing arbitrary x if the order of a is infinite. Using the fact that $\mathbb{Q}/\mathbb{Z}$ is injective, choose an extension $\phi' : A \to \mathbb{Q}/\mathbb{Z}$ of ϕ_a. Define $\Phi' : A \to \prod_{a \in A} \mathbb{Q}/\mathbb{Z}$ to be the map whose projection onto the ath factor is ϕ_a. Then Φ' is injective, and since, in general, a product of injective modules is injective, Φ' is an injective presentation of A.

Now let R be arbitrary. Set $\bar{R} := \operatorname{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$, with r acting via $(rf)(x) := f(xr)$ (so that $s\big(rf)\big)(x) = (rf)(xs) = f(xsr) = \big((sr)\big)f(x)\big)$. Using the fact that $\mathbb{Q}/\mathbb{Z}$ is a divisible abelian group, we can check that $\bar{R}$ is an injective R-module. In detail, suppose $g : A \hookrightarrow B$ is an injection of R-modules, and let $f : A \to \bar{R}$ be an R-module homomorphism. Then f determines (and is determined by) the morphism $\bar{f} : A \to \mathbb{Q}/\mathbb{Z}$ of abelian groups given by $\bar{f}(a) = f(a)(1)$. (Given $\bar{f}$, we can recover f by $f(a)(r) = \bar{f}(ra)$.) Since $\mathbb{Q}/\mathbb{Z}$ is injective as a $\mathbb{Z}$-module, there exists an extension $\bar{f}' : B \to \mathbb{Q}/\mathbb{Z} \in \operatorname{Hom}_{\mathbb{Z}}(B, \mathbb{Q}\mathbb{Z})$. As above, $\bar{f}'$ determines an R-module homomorphism $f' : B \to \bar{R}$ and $\bar{f}'$ will be the desired extension of f.

Let M be an R-module and for $0 \neq m \in M$ let (m) be the R-submodule of M generated by (m). By the special case above, we get a nonzero homomorphism $\overline{\psi_m} : (m) \to \mathbb{Q}/\mathbb{Z}$ of abelian groups. As in the preceding paragraph, $\overline{\psi_m}$ determines a nonzero R-module homorphism $\psi_m : (m) \to \bar{R}$. Since $\bar{R}$ is an injective R-module there exists an extension $\phi_m : M \to \bar{R}$ of ψ_m. Define $\phi : M \to \prod_{0 \neq m \in M} \bar{R}$ to be the map whose projection to the mth factor is given by ψ_m. Then ψ is injective and $\prod_{0 \neq m \in M} \bar{R}$ in an injective R-module, so ψ forms an injective presentation of M.

□

1.2 Universal Constructions

Definition 1.2.1 Let $\underline{\underline{C}}$ be a small category and let $F : \underline{\underline{C}} \to \underline{\underline{X}}$ be a functor.

An object L of $\operatorname{Obj} \underline{\underline{X}}$ together with morphisms $\{\lambda_C : L \to F(C)\}_{C \in \operatorname{Obj} \underline{\underline{C}}}$ is called a *limit* of the functor F, written $\varprojlim_{\underline{\underline{C}}} F$, if $\lambda_D = F(f) \circ \lambda_C$ for every morphism $f : C \to D$ in $\underline{\underline{C}}$, and if given any object Y of Obj $\underline{\underline{X}}$ together with morphisms $\{y_C : Y \to F(C)\}_{C \in \text{Obj } \underline{\underline{C}}}$ satisfying $y_D = F(f) \circ y_C$ for every morphism $f : C \to D$ in $\underline{\underline{C_S}}$, there exists unique $\phi : Y \to L$ such that $\lambda_C \phi = y_C$ for all $C \in \text{Obj } \underline{\underline{C}}$.

Dually a *colimit* $\varinjlim_{\underline{\underline{C}}} F$ of the functor F is an object L of Obj $\underline{\underline{X}}$ together with morphisms $\{\lambda_C : F(C) \to L\}_{C \in \text{Obj } \underline{\underline{C}}}$ such that $\lambda_C = \lambda_D \circ F(f)$ for every morphism $f : C \to D$ in $\underline{\underline{C_S}}$, and given any object Y of Obj $\underline{\underline{X}}$ together with morphisms $\{y_C : F(C) \to Y\}_{C \in \text{Obj } \underline{\underline{C}}}$ satisfying $y_C = y_D \circ F(f)$ for every morphism $f : C \to D$ in $\underline{\underline{C_S}}$, there exists unique $\phi : L \to Y$ such that $\phi \lambda_C = y_C$ for all $C \in \text{Obj } \underline{\underline{C}}$.

As in calculus, limits (and colimits) do not necessarily exist, but if they exist they are unique, at least up to isomorphism, as recorded in the following proposition. As a consequence of which we usually refer to "the limit" (or colimit) rather than "a limit".

Proposition 1.2.2 *Limits and colimits of a given functor are unique (if one exists).*

Proof: Since a colimit is simply a limit over the "opposite category", it suffices to consider the case of limits.

Given limits L and L' of the same functor, use the existence statement of the universal property applied to L to produce a morphism $L' \to L$ and use the same property applied to L' to produce a morphism $L \to L'$. Then use the uniqueness statement in the universal property to show that the compositions $L \to L' \to L$ and $L' \to L \to L'$ are the identity. □

An important special case is where the functor is a diagram, defined as follows.

Definition 1.2.3 Let S be a partially ordered set with associated category $\underline{\underline{C_S}}$. A *diagram* in $\underline{\underline{X}}$ indexed by S consists of a functor from $\underline{\underline{C_S}}$ to $\underline{\underline{X}}$.

Given a diagram $D : \underline{\underline{C_S}} \to \underline{\underline{X}}$, we sometimes write X_j for $D(j)$.

Examples:

1. Let J be a set made into a partially ordered set in which no pair of distinct elements are comparable. A diagram in $\underline{\underline{X}}$ indexed by J is simply an object X_j for each $j \in J$. The limit of this diagram, if it exists, is written $\prod_{j \in J} X_j$ and called the product of the set of objects $\{X_j\}_{j \in J}$. Its universal property can be stated as saying that the product comes with a projection map to each factor and that a morphism into the product is uniquely determined by a morphism into each of the factors.

   Dually, the colimit of this diagram is called the coproduct of the set of objects $\{X_j\}_{j \in J}$, written $\coprod_{j \in J} X_j$.

2. Consider the special case of the preceding example in which the set J is the empty set. The existence portion of the universal property of limit is satisfied by every object in the category, but L satisfies the uniqueness property if and only if there is a unique morphism from X to L for each X in $\mathrm{Obj}\,\underline{\underline{X}}$. Such a limit (if it exists) is called *terminal object* of $\underline{\underline{X}}$. Similary a colimit over the empty set would be an *initial object*, an object with a unique morphism into each other object. An object of $\underline{\underline{X}}$ which is both an initial object and a terminal object is called a *zero object*. A zero object is often written as $*$, or, in the case of an additive category (see below) as 0.

   If a category has a zero object $*$ then for pair of objects X, Y the composition $X \to * \to Y$ of the unique maps into and out of the zero object is called the *zero morphism* from X to Y, and is often written as $*$ or 0.

   (a) In the category of sets, $\emptyset$ is an initial object, and a singleton set is a terminal object — there is no zero object.

   (b) In the category of vector spaces over some field, the vector space 0 is a zero object.

   (c) In the category of rings (which, recall, means "rings with identity" in my terminology), $\mathbb{Z}$ is an initial object and there is no terminal object (and thus no zero object).

   Example 1.2.4 Suppose $\underline{\underline{C}}$ is a category with an initial object X. Let $c_X : \underline{\underline{C}} \to \underline{\underline{C}}$ be the constant functor $c_X(Y) = X$ and $c_X(f) = 1_X$ for all objects Y and all morphisms $f : Y \to Y'$. Then there is a natural transformation $\eta : c_X \to 1_{\underline{\underline{C}}}$ given by $\eta_Y := \alpha_Y : X \to Y$, where α_Y is the unique morphism from X to Y.

   Similarly if $\underline{\underline{C}}$ has a terminal object X then there is natural transformation $1_{\underline{\underline{C}}} \to c_X$.

3. Let J be the partially ordered set x, y, z with $x \prec z$, $y \prec z$ (and x, y not comparable). A diagram $D : J \to \underline{\underline{X}}$ can be pictured as

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

   where $X = D(x)$, $Y = D(y)$ and $Z = D(z)$.

The limit of such a diagram would be an object P together with morphisms $P \to X$ and $P \to Y$ such that

$$\begin{tikzcd} P \arrow[r, "i_X"] \arrow[d, "i_Y"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

commutes and for which any object P' with morphisms i'_X, i'_Y into X and Y making the diagram commute, there is a unique morphism $\Phi : P' \to P$ such that $i'_X = i_X \circ \Phi$ and $i'_Y = i_Y \circ \Phi$. Such a limit is called the *pullback* of f and g.

The dual concept, called a *pushout* is the colimit over a diagram from the opposite category of $\underline{\underline{J}}$.

Example 1.2.5 Pushouts exist in the category of groups.

Let A, G, H be groups, and let $\alpha : A \to G$, $\beta : A \to H$ group homomorphisms, The pushout of α and β in the category of groups is given by the "amalgamated free product", $G *_A H$, defined as follows. Elements of $G_{*A}H$ are "words" $w_1 \dots w_n$ where for each j either $w_j \in G$ or $w_j \in H$, modulo relations generated by $\big(g\alpha(a)\big)h = g\big(\beta(a)h\big)$. (Thus every element can be written as a word alternating between elements of G and H.)

Example 1.2.6 Let R be a ring and let I and J be two-sided ideals in R.

The pullback of the inclusions $I \hookrightarrow R$, $J \hookrightarrow R$ exists and is given by the ideal $I \cap J$.

The pushout of the inclusions $I \cap J \hookrightarrow I$, $I \cap J \hookrightarrow J$ exists and is given by the ideal $I + J := \{x \in R \mid x = i + j \text{ for some } i \in I \text{ and some } j \in J\}$.

4. In the special case of a pullback of $f : X \to Z$, $g : Y \to Z$ in which X is a zero object (and f is the unique map from it to Z) the pullback is called the *kernel* of g. Strictly speaking, the kernel is a pair (K, i_Y) where $i_Y : K \to Y$. It has the property that $g \circ i_Y = *$ and that given any object K' with a map $i'_Y : K' \to Y$ such that $g \circ i'_Y = *$, there is a unique morphism $\Phi : K' \to K$ such that $i_Y \circ \Phi = i'_Y$.

   The dual concept of *cokernel* of g is the pushout of g with the unique map to the zero object.

5. A partially ordered set J having the property that $\forall i, j \in J\ \exists k$ s.t. $i \prec k$ and $j \prec k$ is called a *directed set*. A diagram indexed by a directed set is called a *direct system* and the colimit of such a diagram is called the *direct limit* of the system. We write $\varinjlim_J D(j)$ for the direct limit of the diagram D (if it exists).

   The dual notions are as follows. A diagram indexed by partially ordered set J which has the property that $\forall i, j \in J\ \exists k$ s.t. $k \prec i$ and $k \prec j$ is called an *inverse system* and the limit of such a diagram is called the *inverse limit* of the system, written $\varprojlim_J D(j)$.

   An important special case is where the partially ordered set is the integers, and which case we sometimes write $\varinjlim_n X_n$ and $\varprojlim_n X_n$ for the direct and inverse limits.

   The following properties, familiar from calculus, hold:

   (a) If the system stablilizes (that is, $\exists j$ such that $X_j \to X_{j'}$ is an isomorphism for all $j' \succ j$) then the direct limit is isomorphic to X_j.

   (b) Changing the value of finitely many terms (or any initial portion) of a sequence does not affect the limit. More precisely, for any j, the direct limit of the entire system is isomorphic to the direct limit of the subsystem indexed by $\{n \mid n \succ j\}$.

   More generally, the direct limit is isomorphic to the direct limit over any cofinal subsystem where a cofinal subsystem S is one having the property that $\forall j \in J\ \exists x \in S$ such that $j \prec s$. Thus, as in calculus, a limit over the even integers equals the limit of the entire sequence, assuming the limits exists.

   The analogous statements hold for inverse limits.

   In the special case where the diagram has values in a category of sets, if the maps $X_j \to X_{j'}$ are injections then $\varinjlim_J D(j) \cong \cup_J X_j$ and $\varprojlim_J D(j) \cong \cap_J X_j$.

1.3 Additive and Abelian Categories

Definition 1.3.1 *A category $\underline{\underline{C}}$ is called an* additive category *if:*

1) it has a zero object;

2) the product exists for every pair of objects of Obj $\underline{\underline{C}}$*;*

3) for every pair of objects X, Y of Obj $\underline{\underline{C}}$*, $\underline{\underline{C}}(X, Y)$ has an abelian group structure under which for all objects W, X, Y of* Obj $\underline{\underline{C}}$ *composition $\underline{\underline{C}}(W, X) \times \underline{\underline{C}}(X, Y) \to \underline{\underline{C}}(W, Y)$ is bilinear.*

A functor $F : \underline{\underline{C}} \to \underline{\underline{D}}$ *between additive categories is called an* additive functor *if* $F : \underline{\underline{C}}(X, Y) \to \underline{\underline{D}}\big(F(X), F(Y)\big)$ *is a group homomorphism for all objects* X, Y *of* Obj $\underline{\underline{C}}$.

Definition 1.3.2 *A category* $\underline{\underline{C}}$ *is called an* abelian category *if:*

1) *it is additive;*

2) *every morphism has a kernel and a cokernel;*

3) *every monomorphism is the kernel of its cokernel;*

4) *every epimorphism is the cokernel of its kernel;*

5) *every morphism can be factored as the composition of an epimorphism and a monomorphism.*

Notice that the "1st Isomorphism Theorem" is built into the definition.

Theorem 1.3.3 *(Freyd) Every small abelian category can be embedded as a subcategory of the category of modules over some commutative ring.*

Let R be a commutative ring. Then $\varinjlim_J D(j)$ and $\varprojlim_J D(j)$ exist for any diagram D in the category of R-modules. Explicit constructions are as follows.

If $j \prec j'$, let $\phi_{j,j'} : X_j \to X_{j'}$ denote the morphism $D(j \prec j')$ in the diagram. Then

$$\varinjlim_J D(j) \cong \oplus_J X_j / \sim$$

where $x \sim \phi_{j,j'} x$ and

$$\varprojlim_J D(j) \cong \{(x_j) \in \prod_J X_j\} \mid \phi_{j,j'}(x_j) = x_{j'}\}.$$

Exercise 1.3.4 Verify that the right hand sides of the two constructions above do indeed have the universal properties required to satisfy the definition of colimit and limit respectively.

Proposition 1.3.5 *The composition of pullback squares is a pullback square. That is,*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & E \arrow[d] \\ C \arrow[r] & D \arrow[r] & F \end{tikzcd}$$

if the left and right squares are pullbacks then the outer square is a pullback.

Proof: Given $W \to E$ and $W \to C$ whose compositions into F are equal, right diagram gives a unique map $W \to B$ such that $W \to B \to D$ equals $W \to C \to D$ and then the left diagram gives a unique map $W \to A$ with the desired properties. $\square$

Proposition 1.3.6 *In an abelian category, a map is a monomorphism if and only if its kernel is* 0*. Similarly, a map is an epimorphism if and only if its cokernel is* 0*.*

Proof: Let $f : X \to Y$ be a monomorphism. Given $j : A \to X$ such that $fj = 0$, since we also have $f0 = 0$ the definition of monomorphism gives $j = 0$, so j factors through $0 \to X$ (necessarily uniquely since there is only one morphism from A to 0). Thus 0 is the kernel of f.

Conversely, if 0 is the kernel of f, given $i, j : A \to X$ such that $if = jf$ we have $(i-j)f = 0$ so $i - j$ factors through the kernel 0, meaning that $i = j = 0$. Thus f is a monomorphism.

Dualizing gives the proof of the final statement. $\square$

Proposition 1.3.7 *In an abelian category, a morphism which is simultaneously a monomorphism and an epimorphism is an isomorphism.*

Proof: Let $f : X \to Y$ such that f is a monomorphism and an epimorphism. Since f is an epimorphism, its cokernel is 0, as shown above. But f is a monomorphism, and so, since each monomorphism is the kernel of its cokernel, $f : X \to Y$ is the kernel of $Y \to 0$. Since the composition $Y \xrightarrow{1_Y} Y \to 0$ is 0, there is a unique lift $g : Y \to X$ of 1_Y to the kernel $f : X \to Y$ of $Y \to 0$. Using $fg = 1_Y$, we have $fgf = f$ which, using the fact that f is a monomorphism, gives $gf = 1_X$. Thus f and g are inverse isomorphisms. $\square$

Proposition 1.3.8 *Let*

$$\begin{tikzcd} P \arrow[r, "i"] \arrow[d, "j"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

be a pullback diagram in an abelian category. Then the induced map $\operatorname{Ker} j \to \operatorname{Ker} g$ *is an isomorphism (and by symmetry the induced map* $\operatorname{Ker} i \to \operatorname{Ker} f$ *is an isomorphism).*

Proof: Let $\alpha : K \to P$ and $\alpha' : K' \to Y$ be the kernels of j and g respectively, and let $\phi : K \to K'$ be the induced map of kernels. The paiir of map $0 : K' \to X$ and $\alpha' : Y \to Z$

satisfy $g\alpha' = f0$ and so induce a unique map $h : K' \to P$ such that $ih = \alpha'$ and $jh = 0$. Since $jh = 0$ there is a unique $\psi : K' \to K$ such that $\psi\alpha = h$. Then $\alpha'\phi\psi = i\alpha\psi = ih = \alpha'$, giving $\phi\psi = 1_{K'}$ since α' is a monomorphism. Also $\alpha\psi\phi = h\phi$. Since $ih\phi = \alpha'\phi = i\alpha$ and $jh\phi = 0 = j\alpha$, the uniqueness property of the pullback P gives $h\phi = \alpha$. Thus $\alpha\psi\phi = h\phi = \alpha$, given $\psi\phi = 1_K$ since α is a monomorphism.

Similarly pushouts preserve cokernels. □

Corollary 1.3.9 *If*

$$\begin{tikzcd} P \arrow[r, "i_X"] \arrow[d, "i_Y"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

is a pullback in an abelian category and g is an monomorphism then i_X is an monomorphism.

Similarly the pushout of an epimorphism is an epimorphism. □

Proposition 1.3.10 *Let*

$$\begin{tikzcd} P \arrow[r, "i"] \arrow[d, "j"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

be a commutative diagram in an abelian category. Then the diagram is a pullback if and only if the kernel of $X \oplus Y \xrightarrow{f \perp (-g)} Z$ is $P \xrightarrow{(i,j)} X \oplus Y$.

Proof: Suppose that the diagram is a pullback. Given $(p, q) : A \to X \oplus Y$ such that $\big(f \perp (-g)\big) \circ (p, q) = 0$, we have $fp - gq = 0$ so the definition of pullback gives a unique map $h : A \to P$ such that $jh = p$ and $ih = q$. That is, $(i, j) \circ h = (p, q)$, so $(p, q) : A \to X \oplus Y$ is the kernel of $\big(f \perp (-g)\big) \circ (p, q) = 0$.

Conversely, suppose that the kernel of $X \oplus Y \xrightarrow{f \perp (-g)} Z$ is $P \xrightarrow{(i,j)} X \oplus Y$. Given $p : A \to X$ and $q : A \to Y$ such that $fp = gq$,we have $\big(f \perp (-g)\big) \circ (p, q) = 0$, so the definition of kernel gives a unique map $h : A \to P$ such that $(i, j) \circ h = (p, q)$, or equivalently $jh = p$ and $ih = q$. So the diagram is a pullback. □

Lemma 1.3.11 *In the commutive diagram*

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "\cong"] & B \arrow[r, two heads] \arrow[d] & C \arrow[d, "\cong"] \\ A' \arrow[r, tail] & B' \arrow[r, two heads] & C' \end{tikzcd}$$

suppose that A, A' are the kernels of the epimorphism $B \twoheadrightarrow C$ and $B' \twoheadrightarrow C'$ respectively. Then $B \to B'$ is an isomorphism.

Proof: Each epimorphism is the cokernel of its kernel, so $B \to C$ and $B' \to C'$ are the cokernels of $A \to B$ and $A' \to B$ respectively. Let $B' \to Q$ be the cokernel of $B' \to B$. The map $A' \to B' \to Q$ is zero since its composition with the isomorphism $A \to A'$ is 0. Therefore the universal property of cokernel gives a unique map $C' \to Q$ such that $B' \to C' \to Q$ is the cokernel map $B' \to Q$. Since $B \to C \xrightarrow{\cong} C' \to Q$ is 0, the uniquenesss property of the cokernel says that $C \xrightarrow{\cong} C' \to Q$ is 0, and so $C' \to Q$ is zero because $C \to C'$ is an isomorphism. Thus the cokernel map $B' \to Q$ is 0. It follows that 0 satisfies the universal property of the cokernel of $B \to B'$ and so $Q = 0$. Thus $B \to B'$ is an epimorphism. A dual argument shows that it is a monomorphism. □

Remark 1.3.12 This lemma is a special case of the 5-Lemma, discussed in the next chapter.

Proposition 1.3.13

If

$$\begin{tikzcd} P \arrow[r, "i_X"] \arrow[d, "i_Y"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

is a pullback in an abelian category and g is an epimorphism then i_X is an epimorphism.

Similarly, if

$$\begin{array}{ccc} P & \xrightarrow{i_X} & X \\ {\scriptstyle i_Y}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & Z \end{array}$$

is a pushout in an abelian category and g is a monomorphism then i_X is a monomorphism.

□

Proof: Let $P \xrightarrow{h} R \rightarrowtail^{k} X$ be the factorization of i_X into the composition of an epimorphism and a monomorphism. Define P' as the pullback

$$\begin{array}{ccc} P' & \longrightarrow & R \\ \downarrow & & \downarrow \\ P & \xrightarrow{i_X} & X \end{array}$$

By Corollary 1.3.9, $P' \to P$ is a monomorphism. According to Proposition 1.3.5, P' is also the pullback of $R \rightarrowtail^{k} X \to q$ and $Y \xrightarrow{g} Z$. Since $fkh = fi_X = gi_Y$, the universal property of the pullback gives a unique map $\phi : P \to P'$ such that $P \xrightarrow{\phi} P' \to R$ is h and $P \xrightarrow{\phi} P' \to P$ is 1_P. In particular, $P' \to P$ is an epimorphism, since its composition with ϕ is an epimorphism (namely, 1_P). Thus $P' \to P$ and $\phi : P' \to P$ are inverse isomorphisms.

Proposition 1.3.13 gives a commutative diagram

$$\begin{array}{ccccc} P' & \longrightarrow & R \oplus Y & \longrightarrow & Z \\ \downarrow{\scriptstyle \cong} & & \downarrow & & \| \\ P & \longrightarrow & X \oplus Y & \longrightarrow & Z \end{array}$$

Applying the preceding lemma, gives that the monomorphism $R \oplus Y \xrightarrow{k \oplus 1_Y} X \oplus Y$ is an isomorphism. But the cokernel of $k \oplus 1_Y$ is isomorphic to the cokernel of k. Thus k is an isomorphism and so the composition $i_X = hk$ is the composition of two epimorphisms and is thus an epimorphism. □

Example 1.3.14 The preceding proposition fails in an arbitrary category. For example, consider the category with objects a, b and c where aside from the identities, the only morphisms are $f : a \to b$, $g, h : b \to c$ and $j : a \to b$ where $gf = hf = j$. Then the pull back of g and h is the square

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "f"'] & b \arrow[d, "h"] \\ b \arrow[r, "g"] & c. \end{tikzcd}$$

However g is an epimorphism and f is not.

Proposition 1.3.15 *In an abelian category, let*

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "p"'] & B \arrow[d, "q"] \\ C \arrow[r, tail, "j"] & D \end{tikzcd}$$

be a commutative diagram in which i, j are injections and p, q are surjections. Then the following are equivalent

1. *The induced map* $\ker p \to \ker q$ *is an isomorphism*
2. *The induced map* $\operatorname{coker} i \to \operatorname{coker} j$ *is an isomorphism*
3. *The diagram is a pullback.*
4. *The diagram is a pushout.*

Proof: The proof is left as an exercise. □

A diagram with the properties in the previous proposition is sometimes called *bicartesian.*

1.4 Representable Functors and Yoneda's Lemma

For any pair of objects A, B of a category $\underline{\underline{C}}$, the definition of a category requires that $\underline{\underline{C}}(A,B)$ form a set, but this set might have greater structure. For example, in an additive category, it forms an abelian group. In the category of R-modules over a commutative ring R it has even more structure: it forms an R-module with the scalar multiplication given by $(rf)(a) := r\big(f(a)\big)$.

Given $A \in \text{Obj}\,\underline{\underline{C}}$, there is a functor on $\underline{\underline{C}}$ given by the association $X \mapsto \underline{\underline{C}}(A,X)$. In general this functor goes from $\underline{\underline{C}}$ to $\mathcal{S}ets$ although, as above, it might take values in a category with greater structure. A contravariant functor of the form $X \mapsto \underline{\underline{C}}(A,X)$ is called *representable* and X is called the representing object.

The fact that a functor is representable can be a nontrivial theorem.

Example 1.4.1

Theorem (Brown Representability Theorem) Let E^* be a generalized cohomology theory which is defined on the category of CW-complexes and satisfies the "Milnor Wedge Axiom". Then for each n, the functor

$$E^n(\ ) : CW\text{-complexes} \to \mathcal{A}b$$

is representable.

In the case where E is ordinary cohomology, the representing object, written $K(\mathbb{Z},n)$, is called an Eilenberg-Mac Lane space, characterized (up to homotopy equivalence) by the property that

$$\pi_q\big(K(\mathbb{Z},n)\big) = \begin{cases} \mathbb{Z} & \text{if } q = n; \\ 0 & \text{if } q \neq n. \end{cases}$$

In the case where E is K-theory, the representing object is

$$\begin{cases} BU & \text{if } q \text{ is even;} \\ U & \text{if } q \text{ is odd,} \end{cases}$$

where $U := \varinjlim U(n)$ is the unitary group.

Let $F : \underline{\underline{C}} \to \mathcal{S}ets$ be a representable functor, represented by $A \in \underline{\underline{C}}$ and let $G : \underline{\underline{C}} \to \mathcal{S}ets$ be any functor. Let $\eta : F \to G$ be a natural transformation. Then $\eta_A : \underline{\underline{C}}(A,A) = F(A) \to G(A)$. Taking the image of $1_A \in \underline{\underline{C}}(A,A)$ gives an element of $G(A)$. Conversely, given F and G as above, let y be an element of $G(A)$. Use y to define a natural transformation $\eta(y) : F \to G$ as follows. For any element $f : A \to X$ in $F(X) = \underline{\underline{C}}(A,X)$, applying G gives $G(f) : G(A) \to G(X)$ and we define $\eta_X := G(f)(y) \in G(X)$. The resulting maps $\eta(y)_X$ create a natural transformation from F to G. (Exercise: check).

Lemma 1.4.2 *(Yoneda)*

Let $F : \underline{\underline{C}} \to \mathcal{S}ets$ be a representable functor, represented by $A \in \underline{\underline{C}}$ and let $G : \underline{\underline{C}} \to \mathcal{S}ets$ be any functor. Then the associations $\eta \mapsto \eta_A(1_A)$ and $y \mapsto \eta(y)$ sets up a bijection between natural transformations $\eta := F \to G$ and the elements of the set $G(A)$.

Proof: Exercise. $\square$

Example 1.4.3 A natural transformation from $H^n(\ ) \to H^q(\ )$ is called a *cohomology operation*. By Yoneda's Lemma, the set of cohomology operations from $H^n(\ ) \to H^q(\ )$ corresponds bijectively to the elements of $H^q\big(K(\mathbb{Z}, n)\big)$.

1.5 Adjoint Functors

Let $\underline{\underline{C}}$ and $\underline{\underline{D}}$ be categories and let $F : \underline{\underline{C}} \to \underline{\underline{D}}$ and $G : \underline{\underline{D}} \to \underline{\underline{C}}$ be functors. If there is a natural bijection of sets $\underline{\underline{D}}(FX, Y) \to \underline{\underline{C}}(X, GY)$ for every object X of $\operatorname{Obj} \underline{\underline{C}}$ and Y of $\operatorname{Obj} \underline{\underline{D}}$ then F and G are called *adjoint functors*, written $F \longrightarrow\!\!| \ G$. (Natural means that it commutes with morphisms in both variables.) F is called the *left adjoint* or *coadjoint* and G is called the *right adjoint* or *adjoint*. Let $I_{\underline{\underline{C}}}$ and $I_{\underline{\underline{D}}}$ denote the identity functors on $\underline{\underline{C}}$ and $\underline{\underline{D}}$. If $F \longrightarrow\!\!| \ G$ then there are induced natural transformations $\alpha : I_{\underline{\underline{C}}} \to GF$ and $\beta : FG \to I_{\underline{\underline{D}}}$ where α_X corresponds to 1_{FX} and β_Y to 1_{GY} under the bijections giving the adjointness.

Examples

1. Let R be a commmutive ring and let M be an R-module. Define F : R-modules $\to$ R-modules by $F(N) := M \otimes_R N$. (See section 2.2 for the definition of tensor product $M \otimes_R N$.) Let G : R-modules $\to$ R-modules by $G(N) := \operatorname{Hom}_R(M, N)$. Define $\Phi : \operatorname{Hom}_R\big(F(X), Y\big) \to \operatorname{Hom}_R\big(X, G(Y)\big)$ by $\Big(\big(\Phi(f)\big)(x)\Big)(m) := f(m \otimes x) \in Y$. Then Φ is natural in both varibles and the map $\Psi : \operatorname{Hom}_R\big(X, G(Y)\big) \to \operatorname{Hom}_R\big(F(X), Y\big)$ determined by $\big(\Psi(f)\big)(m \otimes x) := f(x)(m) \in Y$ is a natural inverse to Φ. Thus $F \longrightarrow\!\!| \ G$.

2. Define the (reduced) suspension

$$S : \text{pointed topological spaces} \to \text{pointed topological spaces}$$

by $S(X) := X \times [0,1]/\!\sim$ where $(x, 0) \sim (x', 0)$, $(x, 1) \sim (x', 1)$ and $(*, t) \sim (*, t')$ for all x, x', t, t'. For a pointed topological space X, define its loop space ΩX by $\Omega X : \{f : [0,1] \to X \mid f(0) = f(1) = *\}$ with the "compact-open" topology. Then $S \longrightarrow\!\!| \ \Omega$.

Proposition 1.5.1 *Let $F : \underline{\underline{C}} \to \underline{\underline{D}}$ and $G : \underline{\underline{D}} \to \underline{\underline{C}}$ with $F \dashv G$.*

a) $G(\beta_Y) \circ \alpha_{GY} = 1_{GY}$ for all objects Y of $\mathrm{Obj}\,\underline{\underline{D}}$.

b) $\beta_{FX} \circ F(\alpha_X) = 1_{FX}$ for all objects X of $\mathrm{Obj}\,\underline{\underline{C}}$.

c) The adjoint of a map $p : X \to GY$ in $\underline{\underline{C}}$ is $\beta_Y \circ F(p)$.

d) The adjoint of a map $q : FX \to Y$ in $\underline{\underline{D}}$ is $G(q) \circ \alpha_X$.

Proof: Let $\Phi_{A,B} : \underline{\underline{D}}(FA, B) \cong \underline{\underline{C}}(A, GB)$ denote the adjunction bijection. Let $q : FX \to Y$. Naturality gives

$$\begin{tikzcd}
\underline{\underline{D}}(FX, FX) \arrow[r, "\cong", "\Phi_{X,FX}"'] \arrow[d, "q_*"] & \underline{\underline{C}}(X, GFX) \arrow[d, "(Gq)_*"] \\
\underline{\underline{D}}(FX, Y) \arrow[r, "\cong", "\Phi_{X,FX}"'] & \underline{\underline{C}}(X, GY)
\end{tikzcd}$$

Chasing 1_{FX} around the diagram: the left-bottom path gives the adjoint of $q_*(1_{FX})$ which is q, whereas the top-right path gives (by definition of α) $(Gq)_*(\alpha_X)$ which is $Gq \circ \alpha_X$. This proves part (d).

Applying part (d) with $X := GY$ and $q := \beta_Y : FX = FGY \to Y$ gives that $G(\beta_Y) \circ \alpha_G Y$ is the adjoint of β_Y. But 1_{GY} is the adjoint of β_Y by definition of β. Thus $G(\beta_Y) \circ \alpha_{GY} = 1_{GY}$, proving part (a).

The proofs of parts (c) and (b) are similar.

$\square$

The motivation for the terminology "adjoint" and "coadjoint" as an alternative to "right adjoint" and "left adjoint" is provided by the following proposition.

Proposition 1.5.2 *Let $F : \underline{\underline{C}} \to \underline{\underline{D}}$ and $G : \underline{\underline{D}} \to \underline{\underline{C}}$ with $F \dashv G$. Then G preserves limits (including pullbacks and products as special cases) F preserves colimits (pushouts and coproducts).*

Proof: To illustrate the idea, we will give the proof that F preserves pullbacks. The proof of the rest is similar. Let

$$\begin{tikzcd} P \arrow[r, "i_X"] \arrow[d, "i_Y"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

be a pullback diagram in $\underline{\underline{D}}$. Then

$$\begin{tikzcd} G(P) \arrow[r, "G(i_X)"] \arrow[d, "G(i_Y)"'] & G(X) \arrow[d, "G(f)"] \\ G(Y) \arrow[r, "G(g)"] & G(Z) \end{tikzcd}$$

commutes in $\underline{\underline{C}}$. If

$$\begin{tikzcd} P' \arrow[r, "i'_X"] \arrow[d, "i'_Y"'] & G(X) \arrow[d, "G(f)"] \\ G(Y) \arrow[r, "G(g)"] & G(Z) \end{tikzcd}$$

also commutes in $\underline{\underline{C}}$, then applying F gives the commutative diagram

$$\begin{tikzcd} F(P') \arrow[r, "F(i'_X)"] \arrow[d, "F(i'_Y)"'] & FG(X) \arrow[d, "FG(f)"] \\ FG(Y) \arrow[r, "FG(g)"] & FG(Z) \end{tikzcd}$$

in $\underline{\underline{D}}$. Applying the natural transformation $\beta : FG \to I_{\underline{\underline{D}}}$ gives

$$\begin{tikzcd}
F(P') \arrow[r, "\beta_{FG(X)} \circ F(i'_X)"] \arrow[d, "\beta_{FG(Y)} \circ F(i'_Y)"'] & X \arrow[d, "f"] \\
Y \arrow[r, "g"] & Z
\end{tikzcd}$$

The universal property of the original pullback square then gives a unique map $\phi : F(P') \to P$ such that $\beta_{FG(X)} \circ F(i'_X) \circ \phi = i_X$ and $\beta_{FG(Y)} \circ F(i'_Y) \circ \phi = i_Y$. The adjoint ψ of ϕ is the desired map $P' \to G(P)$ having the property that $\psi \circ G(i_X) = i'_X$ and $\psi \circ G(i_Y) = i'_Y$. Furthermore, if $\psi' : P' \to G(P)$ also satisfies these equations then taking the adjoint would give a map $\phi' : F(P') \to P$ satisfying the same questions as ϕ. The uniqueness part of the universal property of pullback would guarantee that $\phi' = \phi$ and so $\psi' = \psi$. Thus the diagram obtained by applying G to the original pullback in $\underline{\underline{D}}$ becomes a pullback in $\underline{\underline{C}}$.

$\square$

Chapter 2

Chain Complexes

2.1 Basic Concepts

In this section we will be working in some abelian category. In a few places (e.g. 5-Lemma) it is convenient to assume that the category comes with a "forget functor" to $Sets$, since that allows us to give proofs which involve "diagram chasing". However by Freyd's Theorem, any of the results which can be reduced to statements over a finite diagram are valid in any abelian category. Giving the proofs from the universal properties rather than diagram chasing would be possible, although sometimes awkward.

A *differential object* in an abelian category consists of an object X together with a self-map $d : X \to X$ such that $d^2 = 0$. A morphism of differential objects is a map $f : X \to X'$ such that $d'f = fd$.

The kernel of d is denoted $Z(X)$, called the *cycles* of X.

The image of d is denoted $B(X)$, called the *boundaries* of X.

The condition $d^2 = 0$ guarantees that $\operatorname{Im} d \subset \ker d$, so to the differential object (X, d) we can associate its homology $H(X, d) := \operatorname{Ker} d / \operatorname{Im} d$. Often G has extra structure and we require d to satisfy some compatibility condition in order that $H(G, d)$ should also have this structure. For example a differential Lie algebra (L, d) is defined as a Lie algebra L together with a differential d which, in addition to $d^2 = 0$ satisfies the condition $d[x, y] = [dx, y] + [x, dy]$. These conditions yield an induced Lie algebra structure on $H(L)$.

A *graded object* (indexed by the integers) in a category $\underline{\underline{C}}$ consists of an object M together with objects $\{M_n\}_{n\in\mathbb{Z}}$ such that $M = \oplus_{n\in\mathbb{Z}} M_n$.

When each object M_n has the underlying structure of a set, elements of M which lie in M_p for some p are called *homogeneous*. If $x \in M_p$ we sometimes write $|x| = p$ and refer to p as the *degree* of x.

Definition 2.1.1 Let $\underline{\underline{C}}$ be an abelian category. A *chain complex* (C, d) in the category $\underline{\underline{C}}$ consists of a graded object C together with morphisms $d_n : C_n \to C_{n-1}$ such that $d_{n-1} \circ d_n = 0$. The maps d_n are called *boundary operators* or *differentials*.

The cycles and boundaries of C become graded sub-objects of C under the grading $Z_p(C) := Z(C) \cap C_p$ and $B_p(C) := B(C) \cap C_p$.

The homology of C also inherits the structure of a graded object given by $H_p(C) := Z_p(C)/B_p(C)$.

Definition 2.1.2 A *chain map* $f : C \to D$ consists of monorphisms f_p $\forall p$ s.t.

$$\begin{tikzcd} C_p \arrow[r, "d_p"] \arrow[d, "f_p"] & C_{p-1} \arrow[d, "f_{p-1}"] \\ D_p \arrow[r, "d_p"] & D_{p-1} \end{tikzcd}$$

Notation:

1. The subscripts are often omitted, so we might write $d^2 = 0$ or $fd = df$. Thus (C, d) forms a differential object in which the differential has degree -1 and a chain map becomes a morphism of differential objects.

2. We often write $H_*(C)$ to refer to the graded object $H_*(C) := \oplus_{p\in\mathbb{Z}} H_p(C)$.

The composition of chain maps is a chain map so chain complexes and chain maps form a category. Notice that the category of chain complexes (and chain maps) is an abelian category.

A chain map $f : C \to D$ induces a morphism $f_* : H_p(C) \to H_p(D)$ for all p, defined as the induced map of cokernels from the diagram

$$\begin{tikzcd} 0 \arrow[r] & B_p(C) \arrow[r] \arrow[d, "f"] & Z_p(C) \arrow[r] \arrow[d, "f"] & H_p(C) \arrow[r] \arrow[d, dotted, "f_*"] & 0 \\ 0 \arrow[r] & B_p(D) \arrow[r] & Z_p(D) \arrow[r] & H_p(D) \arrow[r] & 0 \end{tikzcd}$$

If our objects have the structure of sets, this can be expressed as follows.

Let $x \in Z_p(C)$ represent an element $[x] \in H_p(C)$.
Then $df(x) = fd(x) = f(0) = 0$ so $f(x) \in Z_p(D)$.
Define $f_*([x]) := [f(x)]$.
If x, x' represent the same element of $H_p(C)$ then $x - x' = dy$ for some $y \in C_{p+1}(C)$. Therefore $fx - fx' = fdy = d(fy)$ which implies $f(x), f(x')$ represent the same element of $H_p(D)$. So f_* is well defined.

The *suspension* SC of a chain complex C is defined by shifting dimensions according to the rule $(SC)_n := C_{n-1}$. Thus $H_n(SC) = H_{n-1}(C)$.

A *cochain complex* is defined in the same way as a chain complex except that the differential has degree $+1$ instead of -1. The terminology *cochain map*, *cocycles*, *coboundaries*, *cohomology* is used for the cohomology versions of the similar concepts in homology. In a cochain complex C, it is customary to write the pth gradation as C^p rather than C_p.

Given a chain complex C there are two obvious ways of producing a cochain complex D from C:

1. Reindexing: $D^p := C_{-p}$
2. Duality: $D^p := \mathrm{Hom}_R(C_p, R)$, if C_* is a chain complex of R-modules.

It is clear in the first example, where D is obtained from C by reindexing, that $H^p(D) = H_{-p}(C)$. The relationship between $H^*\big(\mathrm{Hom}(C, R)\big)$ and $H_*(C)$ is less obvious and is given by the Universal Coefficient Theorem, discussed later.

Definition 2.1.3 A composition of morphisms (in an abelian category)

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is called *exact* at Y if $\mathrm{Ker}\, g = \mathrm{Im}\, f$. A sequence

$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \dots \longrightarrow X_1 \xrightarrow{f_1} X_0$$

is called *exact* if it is exact at X_i for all $i = 1, \dots, n-1$.

Remark 2.1.4 An exact sequence can be thought of as a chain complex whose homology is zero. More generally, homology can be thought of as the deviation from exactness.

A nonnegatively graded chain complex whose homology groups $H_n(\ )$ are zero for $n > 0$ is called *acyclic*.

Definition 2.1.5 A 5-term exact sequence of the form

$$0 \longrightarrow A \xrightarrow{\quad f \quad} B \xrightarrow{\quad g \quad} C \longrightarrow 0$$

is called a *short exact sequence.*

Proposition 2.1.6 *Let*

$$0 \longrightarrow A \xrightarrow{\quad f \quad} B \xrightarrow{\quad g \quad} C \longrightarrow 0$$

be a short exact sequence. Then f is injective, g is surjective and $B/A \cong C$.

Proof: Exactness at $A \Rightarrow$ Ker f = Im $(0 \to A) = 0 \Rightarrow f$ injective.
Exactness at $C \Rightarrow$ Im g = Ker $(C \to 0) = C \Rightarrow g$ surjective.
Exactness at $B \Rightarrow B/\ker g \cong \operatorname{Im} g = C \Rightarrow B/\operatorname{Im} f \cong B/A$. □

Corollary 2.1.7
(a) $0 \to A \xrightarrow{f} B \to 0$ *exact* $\Rightarrow f$ *is an isomorphism.*
(b) $0 \to A \to 0$ *exact* $\Rightarrow A = 0$.

Definition 2.1.8
A map $i : A \to B$ is called a *split monomorphism* if $\exists s : B \to A$ s.t. $si = 1_A$.
A map $p : A \to B$ is called a *split epimorphism* if $\exists s : B \to A$ s.t. $ps = 1_B$.

Note: The splitting s (should it exist) is not unique.

It is trivial to check:
(1) A split monomorphism is a monomorphism
(2) A split epimorphism is an epimorphism

Proposition 2.1.9 *The following three conditions (1a, 1b, and 2) are equivalent:*

1. $\exists$ *a short exact sequence* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *s.t.*

    1a) i is a split monomorphism

    1b) p is a split epimorphism

2. $B \cong A \oplus C$.

Remark 2.1.10 The isomorphism in 2 will depend upon the choice of splitting s in 1a (respectively 1b).

2.2 Double Complexes and Tensor Products

Definition 2.2.1 *A* double complex *in* $\underline{\underline{C}}$ *consists of an object* $C_{p,q}$ *of* $\mathrm{Obj}\,\underline{\underline{C}}$ *for each pair of integers, together with morphisms* $d' : C_{p,q} \to C_{p-1,q}$ *and* $d'' : C_{p,q} \to C_{p,q-1}$ *such that:*

1) $d'^2 = 0 : C_{p,q} \to C_{p-2,q}$;

2) $d''^2 = 0 : C_{p,q} \to C_{p,q-2}$;

3) $d'd'' = d''d' : C_{p,q} \to C_{p-1,q-1}$.

Thus $C_{p,*}$ is a chain complex for each p, $C_{*,q}$ is a chain complex for each q, and $d' : C_{p,*} \to C_{p-1,*}$ and $d'' : C_{*,q} \to C_{*,q-1}$ are chain maps. That is, for each p, $C_{p,*}$ forms a chain complex in the category of chain complexes and similarly $C_{*,q}$ is a chain complex of chain complexes for each q.

Given a double complex C, form the total complex, $\mathrm{Tot}\, C$, by $(\mathrm{Tot}\, C)_n = \bigoplus_{p+q=n} C_{p,q}$ with $d : (\mathrm{Tot}\, C)_n \to (\mathrm{Tot}\, C)_{n-1}$ defined as the map whose restriction to $C_{p,q}$ is

$$\big(d' + (-1)^p d''\big) : C_{p,q} \to C_{p-1,q} \oplus C_{p,q-1} \hookrightarrow (\mathrm{Tot}\, C)_{n-1}.$$

It is easy to verify that $d^2 = 0$. (The sign in the definition of d was included to make this true.)

From a double complex C, we can also form a "product" total complex $\mathrm{Tot}^\pi C$ by

$$(\mathrm{Tot}^\pi C)_n = \prod_{p+q=n} C_{p,q}$$

with $d : (\mathrm{Tot}^\pi C)_{n+1} \to (\mathrm{Tot}^\pi C)_n$ defined as the map whose projection to $C_{p,q}$ is

$$\big(d' + (-1)^p d''\big) : (\mathrm{Tot}^\pi C)_{n+1} \twoheadrightarrow C_{p+1,q} \times C_{p,q+1} \to C_{p,q}.$$

Let R be a ring. In the rest of this section we will assume that our objects have the structure of R-modules.

Let M be a right R-module and let N be a left R-module. The *tensor product* $M \otimes_R N$ is the abelian group with the universal property

$$\begin{tikzcd} M \times N \arrow[rr, "R \text{ bilinear}"] \arrow[dr] & & X \\ & M \otimes_R N \arrow[ur, dotted, "\exists!"] & \end{tikzcd}$$

Explicity,

$$M \otimes_R N = \big(\text{Free Abelian Group on } (M \times N)\big) / \sim$$

where

1. $(m, n_1 + n_2) \sim (m, n_1) + (m, n_2)$
2. $(m_1 + m_2, n) \sim (m_1, n) + (m_2, n)$
3. $(mr, n) \sim (m, rn)$

$[(m, n)]$ in $M \otimes_R N$ is written $m \otimes n$.
Arbitrary elements of $M \otimes_R N$ have the form $\sum_{i=1}^k m_i \otimes n_i$.

Remark 2.2.2 In general $M \otimes N$ has no natural R-module structure, although it would have one if either M or N has the extra structure of an R-bimodule. In particular, if the ring R is commutative then $M \otimes_R N$ becomes a left R-module via

$$r \cdot (m \otimes n) := (r \cdot m) \otimes n.$$

Notice that R is an R-bimodule even if R is not commutative. (ie. left multiplication commutes with right multiplication – R is associative) so the "extended module" $R \otimes N$ always has the structure of an R-module with R acting via left multiplication on the left factor.

Having defined the tensor product of R-modules, we next define the tensor product of chain complexes of R-modules. As per the discussion above, in general this will be a chain complex of abelian groups, although if we have extra conditions, such as a commutative ring R, then it will inherit an R-module structure.

Suppose that (C, d_1) and (D, d_2) are chain complexes of right and left R-modules respectively. Set $(C \otimes D)_n := \oplus_{i+j}^n C_i \otimes D_j$. Define $d : (C \otimes D)_n \to (C \otimes D)_{n-1}$ to be the homomorphism induced by $d(x \otimes y) := d_1 x \otimes y + (-1)^{|x|} x \otimes d_2 y$, where x is homogeneous of degree $|x|$. Equivalently, $(C \otimes D, d)$ is the total complex of the double complex $C_p \otimes D_q$. The chain complex $(C \otimes D, d)$ is called the *tensor product* of the chain complexes (C, d_1) and (D, d_2).

Later we will find a formula (Künneth Theorem) which gives the homology of the two tensor product of chain complexes in terms of the homology of the factors.

2.3 Chain Homotopy

Definition 2.3.1 Let $f, g : C \to D$ be chain maps. A collection of maps $s_p : C_p \to D_{p+1}$ is called a *chain homotopy* from f to g if the relation $ds + sd = f - g : C_p \to D_p$ is satisfied for each p. If there exists a chain homotopy from f to g, then f and g are called *chain homotopic.*

Remark 2.3.2 The intuition motivating the preceding definition is as follows.

Consider the CW-structure on the unit interval $I := [0,1]$ consisting of 0-cells for the points $\{0\}$ and $\{1\}$ and a 1-cell joining them. The cellular chain complex $C(I)$ for this CW-complex is given by $C(I)_0 = R \oplus R$, $C(I)_1 = R$ and $C(I)_p = 0$ for $p \neq 0, 1$. The differential is given by $d(c) = b - a$, where a and b are generators of $C(I)_0$ and c is a genarator of $C(I)_1$. If X and Y are CW-complexes, then $C(X \times Y) = C(X) \otimes C(Y)$ and in particular, $C(I \times X) = C(I) \otimes C(X)$.

If $F : I \times X \to Y$ is a (geometric) homotopy from f to g, then $F_*(a \otimes x) = f_*(a)$ and $F_*(b \otimes x) = g_*(x)$ are forced by the fact that the restriction of F to $X \times \{0\}$ and $X \times \{1\}$ are f and g respectively. Thus the (additional) information provided by F_* is the value of $F(c \otimes x) \in C(I)_{|x|+1}$, which corresponds to $s(x)$. The formula $ds + sd = f - g$ follows from the formula for $d(c)$ in $C(I)$. Here the sign in $ds + sd = \pm(f - g)$ comes from the fact that I chose to write $I \times X$ rather than $X \times I$ (for this very reason!). If we were instead to model our chain homotopy by maps from $X \times I$ we would get an equivalent concept but our sign convention would come out differently.

Proposition 2.3.3 *Chain homotopy is an equivalence relation.*

Proof: Exercise $\square$

Proposition 2.3.4 $f \simeq f'$, $g \simeq g' \Rightarrow gf \simeq g'f'$.

Proof: $\underline{C} \overset{f}{\underset{f'}{\rightrightarrows}} \underline{D} \overset{g}{\underset{g'}{\rightrightarrows}} \underline{E}$

Show $gf \simeq gf'$:

Let $s : f \simeq f'$. $s : C_p \to D_{p+1}$ s.t. $ds + sd = f' - f$.

$g \circ s : C_p \to E_{p+1}$ satisfies $dgs + gsd = gds + gsd = g(ds + sd) = g(f' - g) = gf' - gf$.

Similarly $g'f \simeq g'f'$.

$\square$

Definition 2.3.5 A map $f : C \to D$ is a *chain (homotopy) equivalence* if $\exists\ g : D \to C$ s.t. $gf \simeq 1_C$, $fg \simeq 1_D$.

Proposition 2.3.6 $f \simeq g \Rightarrow f_* = g_* : H_*(C) \to H_*(D)$.

Proof: Let $[x] \in H_p(C)$ be represented by $x \in Z_p(C)$. Let $s : f \simeq g$.

Then $fx - gx = sdx + dsx = dsx \in B_p(C)$. So $[fx] = [gx] \in H_p(D)$. $\square$

Corollary 2.3.7 *$f : C \to D$ is a chain equivalence $\Rightarrow f_* : H_*(C) \to H_*(D)$ is an isomorphism.*

$\square$

2.4 Snake Lemma; 5-Lemma; Algebraic Mayer-Vietoris

Lemma 2.4.1 (Snake Lemma) *Let*

$$\begin{tikzcd}
 & A' \arrow[r, "i'"] \arrow[d, "f'"] & A \arrow[r, "i''"] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & B' \arrow[r, "j'"] & B \arrow[r, "j''"] & B'' &
\end{tikzcd}$$

be a commutative diagram in which the rows are exact. Then $\exists$ *an exact sequence*

$$\ker f' \to \ker f \to \ker f'' \xrightarrow{\partial} \operatorname{coker} f' \to \operatorname{coker} f \to \operatorname{coker} f''.$$

If i' *is a monomorphism then* $\ker f' \to \ker f$ *is a monomorphism and if* j'' *is an epimorphism then* $\operatorname{coker} f \to \operatorname{coker} f''$ *is an epimorphism.*

Proof:

Step 1. Construction of the map ∂ (called the "connecting homomorphism"):

Let $x \in \ker f''$. Choose $y \in A$ s.t. $i''(y) = x$. Since $j'' fy = f'' i'' y = f'' x = 0$, $fy \in \ker j'' = \mathrm{Im} j'$ so $fy = j'(z)$ for some $z \in B'$. Define $\partial x = [z]$ in coker f'.

Show ∂ well defined:

Suppose $y, y' \in A$ s.t. $i''y = x = i''y'$.

$i''(y - y') = 0 \Rightarrow y - y' = i'(w)$ for some $w \in A'$. Hence $fy - fy' = fi'w = j'f'w$.

Therefore if we let $fy = j'z$ and $fy' = j'z'$ then $j'(z - z') = j'f'w \Rightarrow z - z' = f'w$ (since j is an injection). So $[z] = [z']$ in Coker f'. $\surd$

Step 2: Exactness at Ker f'':

Show the composition $\ker f \xrightarrow{i''} \ker f'' \xrightarrow{\partial}$ Coker f' is trivial.

Let $k \in$ Ker f. Then $\partial(i''k) = [z]$ where $j'(z) = f(k) = 0$. So $z = 0$.

So $\partial \circ i'' = 0$. Hence Im $(i'') \subset$ Ker ∂.

Conversely let $x \in$ Ker ∂. Let $y \in A$ s.t. $i''y = x$. We wish to show that we can replace y by a $y' \in \ker f$ which satisfies $i''y' = x$.

Since commutativity of the diagram gives $j''y = 0$ there exist $z \in B'$ s.t. $j'z = fy$. So $\partial x = [z]$. $\partial x = 0 \Rightarrow z \in \mathrm{Im}\, f'$.

Hence $z = f'w$ for some $w \in A'$.

Set $y' := y - i'w$. Then $i''y = iy - i''i'w = iy = x$ and $fy' = fy - fi'w = fy - j'f'w = fy - j'z = 0$.

Hence $y' \in$ Ker f.

The rest of the proof is left as an exercise. $\square$

Proof: Proof from the universal properties

To define ∂:

Set $K' := \ker f'$, $K := \ker f$, $K'' := \ker f''$.

Let P be the pullback of $A \to A''$ and $K'' \to A''$. Then the kernel of $P \twoheadrightarrow A''$ equals the kernel A' of $A \twoheadrightarrow A''$.

Since $P \to A \to B \to B''$ is the zero map, there exists a unique lift $P \to B'$ of $P \to A \to B$. Since $A' \to P \to B' \rightarrowtail B$ is zero and $B' \to B$ is a monomorphism, it follows that $A' \to P \to B$ is zero and therefore lifts uniquely to a map $\partial : A'' = \operatorname{coker}(A' \to P) \to B'$.

To show that the sequence is exact at ∂:

For simplicity, we reduce to the special case where i' is a monomomorphism and j'' an epimorphism. Exactness gives a factorization of i' as $A' \xrightarrow{k} \ker i'' \xrightarrow{\bar{i'}}$ and then the fact that j' is a monomorphism gives a factorization of f' as $A' \xrightarrow{k} \ker i'' \xrightarrow{\bar{f}'} B'$. Replacing A', i' and f' by $\ker i''$, $\bar{i'}$ and $\bar{f}'$ respectively gives a similar diagram in which the top left map is a monomorphism without affecting the map ∂. Similarly we may assume that j'' is an epimorphism.

Sub-Lemma 2.4.2 *Let*

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r, "i'"] \arrow[d, "f'"] & X \arrow[r, "i''"] \arrow[d, "f"] & X'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & Y' \arrow[r, "j'"] & Y \arrow[r, "j''"] & Y'' \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram in which the rows are exact. Let Q be the pullback of i'' with the $\ker f'' \to X''$ *and let $\theta : Q \to B'$ be the map induced from the zero composition $Q \to X \to Y \to Y''$. Then the induced maps from* $\ker(Q \to Y') \to \ker(Q \to Y) \to \ker f$ *are both isomorphism.*

Proof: The fact that $\ker(Q \to Y') \to \ker(Q \to Y)$ is an isomorphism is immediate, since j' is a monomorphism.

Since $X \to X''$ is an epimorphism, Proposition 1.3.13 and Lemma 1.3.15 imply that the pullback square defining Q is bicartesian. Thus the induced map $\ker(Q \to Y') \to \ker f$ is an isomorphism by Lemma 1.3.15. $\square$

Proof of Lemma: (cont.) Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, equal] & P' \arrow[r] \arrow[d] & \ker\partial \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & K'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A'/K' \arrow[r] & B' \arrow[r] & \operatorname{coker} f' \arrow[r] & 0
\end{tikzcd}$$

where P' is defined as the pullback of $P \twoheadrightarrow K''$ and $(\ker\partial) \rightarrowtail K''$. Thus the top right square is bicartesian. By the Lemma, $\ker(P \to B') \cong \ker f$. The diagram

$$\begin{tikzcd}
K' \arrow[r] \arrow[d] & K \arrow[d] \\
A' \arrow[r] & P'
\end{tikzcd}$$

is bicartesian since the induced map on cokernels vertically is the identity map of B'. Therefore the induced map on cokernels horizonal is an isomorphism $K/K' \to \ker\partial$. That is, there is a short exact sequence $0 \to K \to K' \to \ker\partial \to 0$ which is equivalent to exactness at $\ker f$ in the Snake Lemma. $\square$

Lemma 2.4.3 (5-Lemma)
Let

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & D \arrow[r] \arrow[d, "i"] & E \arrow[d, "j"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

be a commutative diagram with exact rows. If f, g, i, j are isomorphisms then h is also an isomorphism.

(Actually , we need only f mono and j epi with g and i iso.)
Proof: Exercise $\square$

A sequence
$$0 \to \underline{C} \xrightarrow{f} \underline{D} \xrightarrow{g} \underline{E} \to 0$$
of chain complexes and chain maps is a short exact sequence of chain complexes if
$$0 \to C_p \xrightarrow{f_p} D_p \xrightarrow{g_p} E_p \to 0$$
is a short exact sequence (of R-modules) for each p.

Theorem 2.4.4 *Let*
$$0 \to P \xrightarrow{f} Q \xrightarrow{g} R \to 0$$
be a short exact sequence of chain complexes. Then there is an induced natural (long) exact sequence
$$\cdots \to H_n(P) \xrightarrow{f_*} H_n(Q) \xrightarrow{g_*} H_n(R) \xrightarrow{\partial} H_{n-1}(P) \xrightarrow{f_*} H_{n-1}(Q) \to \ldots$$

Remark 2.4.5 Natural means:
$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P' \arrow[r] & Q' \arrow[r] & R' \arrow[r] & 0
\end{tikzcd}$$
implies
$$\begin{tikzcd}
\ldots \arrow[r] & H_n(P) \arrow[r] \arrow[d] & H_n(Q) \arrow[r] \arrow[d] & H_n(R) \arrow[r, "\partial"] \arrow[d] & H_{n-1}(P) \arrow[r] \arrow[d] & \ldots \\
\ldots \arrow[r] & H_n(P') \arrow[r] & H_n(Q') \arrow[r] & H_n(R') \arrow[r, "\partial"] & H_{n-1}(P') \arrow[r] & \ldots
\end{tikzcd}$$

Proof:
1. Definition of ∂:
   Let $[r] \in H_n(R)$, $r \in Z_n(R)$. Find $q \in Q_n$ s.t. $g(q) = r$.

$g(dq) = d(qd) = dr = 0$ (since $r \in Z_n(R)$), which implies $dg = fp$ for some $p \in P_{n-1}$.
$f(dp) = dfp = d^2q = 0 \Rightarrow dp = 0$ (as f injective).
So $p \in Z_{n-1}(p)$. Define $\partial[r] = [p]$.

2. ∂ is well defined:
   (a) Result is independent of choice of q:
   Suppose $g(q) = g(q') = r$.
   $g(q - q') = 0 \Rightarrow q - q' = f(p'')$ for some $p'' \in P_n$.
   Find p' s.t. $dq' = fp'$.
   $f(p - p') = d(q - q') = dfp'' = fdp'' \Rightarrow p - p' = dp'' \in B_{n-1}(P)$.
   So $[p] = [p']$ in $H_{n-1}(P)$.

   (b) Result is independent of the choice of representative for $[r]$:
   Suppose $r' \in Z_n(R)$ s.t. $[r'] = [r]$.
   $r - r' = dr''$ for some $r'' \in R_{n+1}$.
   Find $q'' \in Q_{n+1}$ s.t. $gq'' = r''$.
   $gdq'' = dgq'' = dr'' = r - r' = g(q) - r' \Rightarrow r' = g(q - dq'')$.
   Set $q' := q - dq'' \in Q_n$.
   $gq' = r'$ so we can use q' to compute $\partial[r']$.
   $dq' = dq - d^2q'' = dq$ so the definition of $\partial[r']$ agrees with the definition of $\partial[r]$.

3. Sequence is exact at $H_{n-1}(P)$.
   To show that the composition $H_n(R) \xrightarrow{\partial} H_{n-1}(P) \xrightarrow{f_*} H_{n-1}(Q)$ is trivial:
   Let $[r] \in H_n(R)$. Find $q \in Q_n$ s.t. $gq = r$.
   Then $\partial[r] = [p]$ where $fp = dq$.
   So $f_*\partial[r] = [fp] = [dq] = 0$ since $dq \in B_{n-1}(Q)$.
   Hence Im $\partial \subset$ Ker f_*.
   Conversely let $[p] \in$ Ker f_*.
   Since $[fp] = 0$, $fp = dq$ for some $q \in Q_n$.
   Let $r = gq$. Then $\partial[r] = [p]$.
   So Ker $f_* \subset$ Im ∂.

   The proof of exactness at the other places is left as an exercise. $\square$

Proposition 2.4.6 (Algebraic Mayer-Vietoris) *Let*

$$\begin{array}{ccccccccccccccc}
\longrightarrow & A_n & \xrightarrow{i} & B_n & \xrightarrow{j} & C_n & \xrightarrow{\partial} & A_{n-1} & \xrightarrow{i} & B_{n-1} & \xrightarrow{j} & C_{n-1} & \longrightarrow \\
& \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & \\
\longrightarrow & A'_n & \xrightarrow{i'} & B'_n & \xrightarrow{j'} & C'_n & \xrightarrow{\partial} & A'_{n-1} & \xrightarrow{i'} & B'_{n-1} & \xrightarrow{j'} & C'_{n-1} & \longrightarrow
\end{array}$$

be a commutative diagram with exact rows. Suppose $\gamma : C_n \to C'_n$ is an isomorphism $\forall n$. Then there is an induced long exact sequence

$$\ldots \longrightarrow A_n \xrightarrow{\rho} B_n \oplus A'_n \xrightarrow{q} B'_n \xrightarrow{\Delta} A_{n-1} \longrightarrow B_{n-1} \oplus A'_{n-1} \longrightarrow B'_{n-1}$$

where

$\rho(a) = (ia, \alpha a)$
$q(b, a') = \beta b - i'a'$
$\Delta = \partial \gamma^{-1} j'$

Proof: Exercise □

2.5 Algebraic Mapping Cylinders and Cones

Theorem 2.5.1 *(Algebraic Mapping Cylinder) Let $f : A \to X$ be a chain map. Then there exists a factorization $f = \phi i$ where $i : A \to X'$ is a monomorphism and $\phi : X' \to X$ is a chain homotopy equivalence.*

Proof: Set $X'_n = A_n \oplus X_n \oplus A_{n-1}$. Define $d : X'_n \to X'_{n-1}$ by $d(a, x, b) = (da - b, dx + fb, -db)$ and check that $d^2 = 0$. Define chain maps $j : X \to X'$ and $\phi : X' \to X$ by $j(x) = (0, x, 0)$ and $\phi(a, x, b) = fa + x$. Then $\phi j = 1_X$ and a chain homotopy $s : j\phi \simeq 1_{X'}$ is given by $s(a, x, b) = (0, 0, a)$. The inclusion $i : A \to X'$ given by $i(a) = (a, 0, 0)$ is a chain map and satisfies $f = \phi i$. □

The chain complex X' is called the *algebraic mapping cylinder* of f and the quotient complex X'/A is called the *algebraic mapping cone* of f. These definitions are motivated by their geometric counterparts. For a continous function $f : A \to X$, the (geometric) mapping cylinder M_f is defined by $M_f := \big(X \cup (A \times I)\big)/\sim$ where $(a, 0) \sim f(a)$ and the (geometric) mapping cone is defined as $M_f/(A \times 1)$. In both the geometric and the algebraic case, the usefulness of the mapping cylinder/cone constructions is that they give a 3rd term which fits naturally with f_* to give long exact sequence without insisting that f_* be an injection.

Chapter 3

Derived Functors

3.1 Projective and Injective Resolutions

Let R be a ring.

Definition 3.1.1 *Let M be an R-module. A nonnegatively graded chain complex P_* of R-modules is called a* projective resolution *of M if P_n is a projective R-module for all n and*

$$H_n(P_*) = \begin{cases} M & \text{if } n = 0; \\ 0 & \text{if } n \neq 0. \end{cases}$$

Proposition 3.1.2 *Every R-module has a projective resolution.*

Proof: Let M be an R-module. Since the category of R-modules has enough projectives, we can choose a projective presentation $q_0 : P_0 \twoheadrightarrow M$ of M. Let $j_0 : K_0 \to P_0$ be the kernel of q_0 and let $q_1 : P_1 \to K_0$ be a projective resolution of K_0. Continue inductively to define P_n, q_n, K_n, and j_n for all n. Define $d_n := j_{n-1} \circ q_n : P_n \to P_{n-1}$ for $n \geq 1$. Then $d^2 = 0$ and (P_*, d) forms a projective resolution of M. □

Similarly, since the category of R-modules has enough injectives, every R-module has an injective resolution.

Throughout this chapter, we will be working in an abelian category $\underline{\underline{C}}$ which has enough projectives (respectively injectives in the cases where we are using injective resolutions).

Theorem 3.1.3 *Let P_* and Q_* be nonnegatively graded chain complexes in the abelian category $\underline{\underline{C}}$. such that P_n is projective for all n and Q_* is acyclic. Then*

a) Given any homomorphism $f : H_0(P_*) \to H_0(Q_*)$ there exists a chain map $\phi : P_* \to Q_*$ such that $\phi_* = f$ on $H_0(\ )$.

b) If ϕ, $\psi : P_* \to Q_*$ are two chain maps inducing the same homomorphism on $H_0(\ )$ then $\phi \simeq \psi$. $\square$

Proof: Let $f : H_0(P_*) \to H_0(Q_*)$. The quotient map $Q_0 = Z_0(Q) \twoheadrightarrow Z_0(Q)/B_0(Q) = H_0(Q)$ is surjective so the projectivity of P_0 gives a lift of the composition $P_0 = Z_0(P) \to Z_0(P)/B_0(P) = H_0(P) \xrightarrow{f} H_0(Q)$ to some map $\phi_0 : P_0 \to Q_0$.

$$\begin{tikzcd}
P_0 \arrow[r, two heads] \arrow[d, dotted, "\phi_0"] & H_0(P) \arrow[d, "f"] \\
Q_0 \arrow[r, two heads] & H_0(Q)
\end{tikzcd}$$

Using that Q is acyclic, the map $Q_1 \twoheadrightarrow B_0(Q) = \ker\big(Q_0 \to H_0(Q)\big)$ is a surjection and the commutativity of the above diagram shows that $\mathrm{Im}(\phi_0 \circ d) \subset \ker\big(Q_0 \to H_0(Q)\big)$. Thus the projectivity of P_1 gives a lift of the composition $\phi_0 \circ d : P_1 \to B_0(Q)$ to some map $\phi_1 : P_1 \to Q_1$.

$$\begin{tikzcd}
P_1 \arrow[r, equal] \arrow[d, dotted, "\phi_1"] & P_1 \arrow[r, "d"] \arrow[d, "\phi_0 \circ d"] & P_0 \arrow[d, "\phi_0"] \\
Q_1 \arrow[r, two heads, "d"] & B_0 \arrow[r, hook] & Q_0
\end{tikzcd}$$

Using that Q is acyclic, there is a surjection $Q_2 \twoheadrightarrow B_1(Q) = Z_1(Q)$ and the commutativity of the above diagram shows that $\mathrm{Im}(\phi_1 \circ d) \subset Z_1 = B_1$ Thus the projectivity of P_2 gives a lift of the composition $\phi_1 \circ d : P_2 \to B_1(Q)$ to some map $\phi_2 : P_2 \to Q_2$.

$$\begin{tikzcd}
P_2 \arrow[r, equal] \arrow[d, dotted, "\phi_2"] & P_2 \arrow[r, "d"] \arrow[d, "\phi_1 \circ d"] & P_1 \arrow[d, "\phi_1"] \\
Q_2 \arrow[r, two heads, "d"] & B_1 \arrow[r, hook] & Q_1
\end{tikzcd}$$

Continuing, we get a map $\phi_n : P_n \to Q_n$ for all n, giving a chain map $\phi : P_* \to Q_*$ having the property that $\phi_* = f$ on $H_0(\ )$.

Suppose now that $\phi, \psi : P_* \to Q_*$ are two chain maps inducing the same homomorphism on $H_0()$.

To begin, set $s_{-1} := 0 : P_{-1} \to Q_0$. Since ϕ_*, ψ_* agree on $H_0(\ )$ and Q_* is acyclic, the image of the difference $(\phi_* - \psi_*)(P_0)$ is contained in $B_0(Q)$. Therefore the projectivity of P_0 gives a lift of the $\phi_0\psi_0$ to some map $s_0 : P_0 \to Q_1$.

$$\begin{tikzcd}
P_0 \arrow[r, equal] \arrow[d, dotted, "s_0"] & P_0 \arrow[r, equal] \arrow[d, "\phi_0 - \psi_0"] & P_0 \arrow[d, "\phi_0 - \psi_0"] \\
Q_1 \arrow[r, two heads, "d"] & B_0 \arrow[r, hook] & Q_0
\end{tikzcd}$$

Then $ds_0 + s_{-1}d = ds_0 = \phi_0 - \psi_0$.

Suppose inductively that $s_n : P_n \to Q_{n+1}$ has been constructed so that $ds_n + s_{n-1}d = \phi_n - \psi_n$. Using that ϕ_* and ψ_* are chain maps we get

$$d(\phi_{n+1} - \psi_{n+1}) = (\phi_n - \psi_n)d = (ds_n + s_{n-1}d)d = ds_n d$$

and so $d(\phi_n - \psi_n - s_n d) = 0 : P_{n+1} \to Q_n$. Thus $\mathrm{Im}(\phi_n - \phi_n - s_n d) \subset Z_n Q = B_n Q$, since Q_* is acyclic. Therefore the projectivity of P_{n+1} gives a lift of $\phi_n - \psi_n - s_n d$ to some map $s_{n+1} : P_{n+1} \to Q_{n+2}$.

$$\begin{tikzcd}
P_{n+1} \arrow[r, equal] \arrow[d, dotted, "s_{n+1}"] & P_{n+1} \arrow[r, equal] \arrow[d, "\phi_{n+1} - \psi_{n+1} - s_n d"] & P_{n+1} \arrow[d, "\phi_{n+1} - \psi_{n+1} - s_n d"] \\
Q_{n+2} \arrow[r, two heads, "d"] & B_{n+1} \arrow[r, hook] & Q_{n+1}
\end{tikzcd}$$

Then $ds_{n+1} + s_n d = \phi_{n+1} - \phi_{n+1}$. □

Let $F : \underline{\underline{C}} \to \underline{\underline{D}}$ be an additive functor between abelian categories. Let M be an object in $\underline{\underline{C}}$. Pick a projective resolution P_* of M. Then $F(P_*)$ is a chain complex in $\underline{\underline{D}}$. If F preserves exactness (image under F of an exact sequence is exact) then $F(P_*)$ will be acyclic with $H_0\big(F(P_*)\big) = F(M)$, however this will not be true for arbitrary F. Set $(L_nF)(M) := H_n\big(F(P_*)\big)$, the nth *left derived functor* of F. It is easy to see that L_nF is indeed a functor from $\underline{\underline{C}}$ to $\underline{\underline{D}}$ and it follows from Theorem 3.1.3 that $(L_nF)(M)$ is well defined in the sense of being independent (up to isomorphism) of the choice of the projective resolution P_*.

Similarly, we can define the right derived functors, R_nF of a contravariant additive functor F by means of an injective resolution.

Intuitively L_nF measures the deviation of F from preserving exactness. A functor which preserves exactness is called *exact*. F is exact if and only if F preserves kernels and cokernels. A functor which preserves cokernels is called *right exact*. That is, if $f : X \to Y$ then the induced map $F(Y)/\operatorname{Im} F(f) \to F(Y/\operatorname{Im} f)$ is an isomorphism. Similarly, a covariant functor which preservers kernels (the induced $F(\operatorname{Ker} f) \to \operatorname{Ker} F(f)$ is an isomorphism) is called *right exact*.

Proposition 3.1.4 *If* $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ *is a short exact sequence and* F *is right exact then* $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$ *is exact.*

Proof: Applying the definition to $C/\operatorname{Im} g$ gives $F(C)/\operatorname{Im} F(g) = F(0)$. Since F is an additive functor, $F(0) = 0$ so $F(g)$ is an epimorphism.

Applying the definition to $B/\operatorname{Im} f$ gives that the induced map $F(B)/\operatorname{Im} F(f) \to F(C)$ is an isomorphism. That is, $\ker F(g) = \operatorname{Im} F(f)$. $\square$

Left derived functors are most useful in the case where F is right exact, since this is sufficient to conclude that $L_0F = F$.

Proposition 3.1.5 *If* F *is right exact then* $L_0F = F$.

Proof: Let $F : \underline{\underline{C}} \to \underline{\underline{D}}$ be right exact. Let X belong to $\operatorname{Obj}(\underline{\underline{C}})$ and let P_* be a projective resolution of X. Applying the definition of right exactness to $d : P_1 \to P_0$ gives

$$\big(L_0(F)\big)(X) = H_0\big(F(P_*)\big) = F(P_0)/\operatorname{Im} F(d) = F\big(P_0/d(P_1)\big) = F(X).$$

$\square$

Similarly a right exact contravariant functor F has the property that $R_0(F) = F$.

Theorem 3.1.6 *Let* $F : \underline{\underline{C}} \to \underline{\underline{D}}$ *be an additive functor. Given a short exact sequence* $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ *in* $\underline{\underline{C}}$, *there is a natural induced long exact sequence*

$$\ldots \xrightarrow{\partial} L_n(A) \xrightarrow{L_n(i)} L_n(B) \xrightarrow{L_n(j)} L_n(C) \longrightarrow \ldots \xrightarrow{\partial} L_0(A) \xrightarrow{L_0(i)} L_0(B) \xrightarrow{L_0(j)} L_0(C)$$

in $\underline{\underline{D}}$.

Proof: Let $\theta_A : P_0^A \to A$ and $\theta_0^C : P_0^C \to C$ be projective presentations of A and C respectively. Choose a lift $\theta_0' : P_0^C \to B$ of θ_0^C. Set $P_0^B := P_0^A \oplus P_0^C$ and let $\theta_0^B : P^B \to B$ be the map whose restrictions to P_0^A and P_0^C are given by $i \circ \theta_0^A$ and θ_0' respectively. Then

$$\begin{tikzcd} 0 \arrow[r] & P_0^A \arrow[r] \arrow[d, "\theta_0^A"] & P_0^B \arrow[r] \arrow[d, "\theta_0^B"] & P_0^C \arrow[r] \arrow[d, "\theta_0^C"] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r, "j"] & C \arrow[r] & 0 \end{tikzcd}$$

commutes and so θ_0^B is surjective by the Snake Lemma. That is, $\theta_0^B : P_0^B \to B$ is a projective presentation of B.

Let K_0^A, K_0^B and K_0^C be the kernels of θ_0^A, θ_0^B and θ_0^C respectively. By the Snake Lemma there is an induced exact sequence $0 \to K_0^A \to K_0^B \to K_0^C \to 0$. Repeat the above procedure to get compatible projective presentations $\theta_1^A : P_1^A \to K_0^A$, $\theta_1^B : P_1^B \to K_0^B$, $\theta_1^C : P_1^C \to K_0^C$. Let d_1^A be the composite $P_1^A \twoheadrightarrow K_0^A \rightarrowtail P_0^A$ and define $d_1^B : P_1^B \to P_0^B$ and $d_1^C : P_1^C \to P_0^C$ similarly. Continuing, get projectives P_n^A, P_n^B, P_n^C for each n where (by construction) $P_n^B \cong P_n^A \oplus P_n^C$. Then $(P^A)_*$, $(P^B)_*$ and $(P^C)_*$ are projective resolutions of A, B and C respectively and we have a short exact sequence of chain complexes $0 \to (P^A)_* \to (P^B)_* \to (P^C)_* \to 0$ which splits as graded $\underline{\underline{C}}$-objects (but not necessarily as chain complexes). Since F is an additive functor, $F(P_n^B) \cong F(P^A)_n \oplus F(P_C)^n$ for each n. Thus $0 \to \big(F(P^A)\big)_* \to \big(F(P^B)\big)_* \to \big(F(P^C)\big)_* \to 0$ is a short exact sequence of chain complexes. The long exact homology sequence associated to this short exact sequence is the one stated in the proposition. $\square$

Proposition 3.1.7 *Let T be an additive right exact functor. Let $P_{q-1} \xrightarrow{d_{q-1}} P_{q-2} \to \ldots \to P_0 \twoheadrightarrow A$ be exact with $P_{q-1}, \ldots, P_0$ projective, and let $K \xrightarrow{j} P_{q-1}$ be the kernel of $P_{q-1} \to P_{q-2}$. Then the induced sequence $0 \to L_qTA \to TK \xrightarrow{Tj} TP_{q-1}$ is exact.*

Proof: Let $\ldots \to P_n \ldots \to P_{q+1} \xrightarrow{d_{q+1}} P_q$ be a (re-indexed) projective resolution of K. Then

$$\ldots \to P_n \ldots \to P_{q+1} \xrightarrow{d_{q+1}} P_q \xrightarrow{d_q} P_{q-1} \to \ldots \to P_0$$

is a projective resolution of A, where d_q is the compositive $P_q \twoheadrightarrow K \overset{j}{\rightarrowtail} P_{q-1}$. Using the right exactness of T, we have a commutative diagram with exact rows

$$\begin{tikzcd} TP_{q+1} \arrow[r, "T(d_{q+1})"] \arrow[d] & TP_q \arrow[r] \arrow[d, "T(d_q)"] & TK \arrow[r] \arrow[d, "T(j)"] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & TP_{q-1} \arrow[r, equal] & TP_{q-1} \end{tikzcd}$$

where exactness at TP_q uses $\operatorname{Im} T(d_{q+1}) = T(\operatorname{Im} d_{q+1})$, obtained from the right exactness of T. The Snake Lemma yields the exact sequence $TP_{q+1} \xrightarrow{T(d_{q+1})} \ker\bigl(T(d_q)\bigr) \to \ker\bigl(T(j)\bigr) \to 0$. Thus $\ker T(j) \cong \ker T(d_q)/\operatorname{Im} T(d_{q+1}) = H_q(TP_*) = LqTA$ as claimed. □

Similarly

Proposition 3.1.8 *Let S be an additive left exact covariant functor. Let $P_{q-1} \xrightarrow{d_{q-1}} P_{q-2} \to \ldots \to P_0 \twoheadrightarrow A$ be exact with $P_{q-1}, \ldots, P_0$ projective, and let $K \xrightarrow{j} P_{q-1}$ be the kernel of $P_{q-1} \to P_{q-2}$. Then the induced sequence $0 \to SP_{q-1} \to SK \to R^qSA$ is exact.*

Example 3.1.9 Let M be a right R-module over a ring R. Define $F : R\text{-modules} \to \mathcal{AB}$ by $F(N) := M \otimes_R N$. The left derived functors of F are called $\operatorname{Tor}^R_n(M, \ )$. That is, $\operatorname{Tor}^R_n(M,N) := L_n(F)(N)$. Since F has a (right) adjoint it is left exact by Prop. 1.5.2. Thus $\operatorname{Tor}^R_0(M,N) := M \otimes N$.

Using the preceding definition, the definition of $\operatorname{Tor}^R_n(M,N)$ would involve tensoring a projective resolution of N with M. Given a left R-module N, we could define a similar functor F' : R-modules $\to \mathcal{AB}$ by $F'(M) := M \otimes_R N$, yielding derived functors ${\operatorname{Tor}'}^R_n(M,N) := L_n(F')(M)$ whose definition involves tensoring a projective resolution of M with N. It turns out that ${\operatorname{Tor}'}^R_n(M,N) \cong \operatorname{Tor}^R_n(M,N)$. It is not hard to give a direct proof of this, but we will do it instead later as an illustration of the use of spectral sequences.

A module M which has the property that tensoring with M preserves exactness is called *flat*. Thus, if M is flat then $\operatorname{Tor}^R_n(M,N) = 0$ for any N when $n > 0$. Of course a projective module is automatically flat, but it is possible for a module to be flat without being free. For example, $\mathbb{Q}$ is a flat $\mathbb{Z}$-module which is not projective.

The right derived functors of the adjoint G : R-modules $\to \mathcal{AB}$ given by $G(M) := \operatorname{Hom}_R(M,N)$ (for fixed N) are written $\operatorname{Ext}^n_R(M,N)$ and defined by means of an injective resolution of M. As the right adjoint in the pair $F \dashv G$ is left exact we get $\operatorname{Ext}^0_R(M,N) = \operatorname{Hom}_R(M,N)$. We could consider instead the left derived functors of $G'(N) := \operatorname{Hom}_R(M,N)$ (defined through a projective resolution of N), but it turns out that $L_n(G')(N) = R_n(G)(M)$.

The notation $\operatorname{Tor}^R(M,N)$ is often used for $\operatorname{Tor}^R_1(M,N)$ and similarly $\operatorname{Ext}_R(M,N)$ stands for $\operatorname{Ext}^1_R(M,N)$.

If R is a Principal Ideal Domain then any submodule of a projective module is projective (and in fact is a free R-module). (See MAT1100–1101 notes, Page 92.) Thus if $\theta : Q \to M$ is a projective presentation of M then $P_* := \to 0 \to \ldots \to 0 \to \ldots 0 \to \ker\theta \hookrightarrow Q$ forms a projective resolution of M in which $P_n = 0$ for $n > 1$. It follows that if R is a PID then $\operatorname{Tor}^R_n(M,N) = 0$ for any M and N when $n > 1$.

Example 3.1.10 According to the following theorem, for diagrams indexed by the integers the derived functors of $\varinjlim$ are 0 although, as we will see, the situation for $\varprojlim$ is different.

Theorem 3.1.11 *("Homology commutes with direct limits")*

Let J be a directed set and let $\{C_j\}_{j\in J}$ be a direct system of chain complexes indexed by J $H\left(\varinjlim_J (C_j)_*\right) = \varinjlim_J H_*(C_j)$.

Proof:

Lemma 3.1.12 *Let $G = \varprojlim_{i\in J} G_j$. Then any element of G has a representative of the form $\phi_k(g)$ for some $g \in G_k$, where $\phi_k : G_k \to G$ is the canonical map.*

Proof: Let $X = (g_j)_{j\in J}$ represent an element of G. Since x has only finitely many nonzero components, the definition of direct system implies that $\exists k \in J$ s.t. $j \leq k\ \forall j$ s.t. $g_j \neq 0$. Then adding $\phi_{k,j}(g_j) - g_j$ to x for all j s.t. $g_j \neq 0$ gives a new representative for x with only one nonzero component. (i.e. for some k, $x = \phi_k(g)$ with $g \in G_k$.) $\square$

Lemma 3.1.13 *If $g \in G_k$ s.t. $\phi)k(g) = 0$ then $\phi_{m,k}(g) = 0$ for some m.*

Proof:
Notation: For "homogeneous" elements of $\oplus_{\alpha\in J} G_\alpha$ (i.e. elements with just 1 nonzero component) write $|h| = \alpha$ to mean that $h \in G_\alpha$, or more precisely that the only nonzero component of h lies in G_α.

$\phi_k(g) = 0 \Rightarrow g \in H' \Rightarrow$

$$g = \sum_{t=1}^{n} \phi_{j_t,i_t} g_t - g_t \qquad \text{where } g_t \in G_{i_t} \tag{3.1}$$

Find m s.t. $k \leq m$ and $i_r \leq m$ and $j_r \leq m\ \forall r$. Set $g' = \phi_{m,k} g$.

Adding $g' - g = \phi_{m.k} g - g$ to equation 3.1 gives

$$g' = \sum_{t+0}^{n} \phi_{j_t,i_t} g_t - g_t \qquad \text{where } g_0 = g \tag{3.2}$$

Note that for any $\alpha < m$, collecting terms on RHS in G_α gives 0, since LHS is 0 in degree α.

Among $S := \{i_0, \ldots, i_n, j_0, \ldots, j_n, m\}$ find α which is minimal. (i.e. each other index occuring is either greater or not comparable) Since $j_t\ i_t$, α is one of the i's so this means $J - t \neq \alpha$ for any t.

For each t with $|g_t| = \alpha$, add $g_t - \phi_{m,|g_t|} g_t$ to both sides of equation 3.2.

As noted above, $\sum_{\{t||g_t|=\alpha\}} g_t = 0$ so $\sum_{\{t||g_t|=\alpha\}} \phi_{m,|g_t|} g_t$ is also 0 and so we are actually adding 0 to the equation. However we can rewrite it using:

$\phi_{j_t,i_t}g_t - g_t + g_t - \phi_{m,|g_t|}g_t = \phi_{j_t,i_t}g_t - \phi_{m,|g_t|}g_t \overset{(|g_t|=i_t)}{=\!=\!=} \phi_{j_t,i_t}g_t - \phi_{m,j_t}\phi_{j_t,i_t}g_t = \phi_{m,j_t}\tilde{g}_t$ where $\tilde{g}_t = -\phi_{j_t,i_t}g_t$. Therefore we now have a new expression of the form $g' = \sum \phi_{j_t,i_t}g)t - g_t$; however the new2 set S is smaller than before since it no longer contains α (and no new index was added).

Repeat this process until the set S consists of just $\{m\}$. Then no i's are left in S (since $i_t < m\ \forall t$) which means that there are no terms left in the sum. That is, Equation 3.1 reads $g' = 0$, as required. $\square$

Proof of Theorem

Let $\psi_{j,i} : (C_i)_* \to (C_j)_*$ be the maps in the direct system $\varinjlim_J C_j$. Definition of maps $\phi_{j,i} : H(C_{i*}) \to H(C_{j*})$ is $\phi_{j,i} = (\psi_{j,i})_*$.

$$\begin{tikzcd}
H(C_{i*}) \arrow[rr, "\phi_{j,i} = (\psi_{j,i})_*"] \arrow[dr, "\phi_i"] \arrow[ddr, "(\psi_i)_*"'] & & H(C_{j*}) \arrow[dl, "\phi_j"'] \arrow[ddl, "(\psi_j)_*"] \\
& \varinjlim_J H(C_j) \arrow[d, dashed, "\theta"] & \\
& H(C) &
\end{tikzcd}$$

Claim θ is onto:

Given $[x] \in H(C)$, where $x \in C$, find a representative $x_k \in C_{k*}$ for x. (That is, $x = \psi_k x_k$).

Since x represents a homology class, $\partial x = 0$. Hence $\psi_k \partial x_k = \partial \psi_k x_k = \partial x = 0$. Replacing x_k by $x_m = \phi_{m,k}x_k$ for some m, get a new representative for x s.t. $\partial x_m = 0$. Therefore x_m represents a homology class $[x_m] \in H(C_{m*})$ and

$$\begin{tikzcd}
{[x_m]} \arrow[dr] & H(C_{m*}) \arrow[rr] \arrow[dr] & & \varinjlim_J H(C_j) \arrow[dl, "\theta"] \\
& {[x]} & H(C) &
\end{tikzcd}$$

shows $\in \operatorname{Im}\theta$.

Claim θ is $1-1$:

Let $y \in \lim_J H(C_j)$ s.t. $\theta(y) = 0$.

Find a representative $[x_k] \in H(C_{k*})$ for y, where $x_k \in X_{k*}$. (That is, $y = \phi_k(x_k)$.)

$$\begin{tikzcd}
{[x_k]} \arrow[rr] & & {[y]} \\
H(C_{k*}) \arrow[rr] \arrow[dr, "\psi_{k*}"'] & & \varinjlim_J H(C_j) \arrow[dl, "\theta"] \\
& H(C) &
\end{tikzcd}$$

Since $\theta y = 0$, $[\psi_k x_k] = 0$ in $H(C)$. That is, $\exists v \in C$ s.t. $\partial v = \psi_k x_k$.

May choose l s.t. $v = \psi_{l*}(w_l)$.

Find m s.t. $k, l \leq m$. Then replacing x_k, w_l by their images in $(C_m)_*$ we get that $x - \partial w_m$ stabilizes to 0 so that $\exists m' \geq m$ s.t. $[x_{m'}] = [\partial w_{m'}] = 0$. Hence $y = 0$.

$\square$

Remark 3.1.14 Note that homology does not commute with arbitrary colimits.

Corollary 3.1.15 *Suppose $X_1 \subset X_2 \subset \ldots \subset X_n \subset \ldots$ are inclusions of CW-complexes and $X = \cup_{n=1}^{\infty} X_n$. Then $H_*(X) \cong \varinjlim_n H_*(X_n)$.*

We next consider the derived functors of the inverse limit functor. As we shall see later, failure of homology to commute with inverse limits is a source of great complication in working with spectral sequences.

Let $\mathcal{AB}$ denote the category of abelian groups and let $\mathcal{I}nv$ be the category of inverse systems of abelian groups indexed over the nonpositive integers. Then $\varprojlim$ is a functor from $\mathcal{I}nv$ to $\mathcal{AB}$. Unlike $\varinjlim$, this functor fails to preserve exactness.

Let $\underline{G}$ be an inverse system $\ldots \longrightarrow G^{n+1} \xrightarrow{\phi^n} G^n \longrightarrow \ldots \xrightarrow{\phi^0} G^0$ where we use the convention $G^n = G_{-n}$. Define $\phi : \prod_k G^k \to \prod_k G^k$ to be the map whose projection onto G^n is the composite, $\prod_k G^k \twoheadrightarrow G^{n+1} \xrightarrow{\phi^n} G^n$. Then

$$\varprojlim(\underline{G}) = \ker\left((1-\phi) : \prod_k G^k \to \prod_k G^k\right).$$

Set $\varprojlim^1(\underline{G}) = \operatorname{coker}(1-\phi)$. From the definition it is easy to see that if ϕ^n is onto for all n then $1-\phi$ is onto. In other words,

Proposition 3.1.16 *Let $\underline{G} = \{G^n\}$ be an inverse system in which $\phi^n : G^{n+1} \to G^n$ is onto for all n. Then $\varprojlim^1(\underline{G}) = 0$.* $\square$

The Snake Lemma implies

Proposition 3.1.17 *Let* $0 \to \underline{G}' \to \underline{G} \to \underline{G}'' \to 0$ *be a short exact sequence of inverse systems. Then there is an induced exact sequence*

$$0 \to \varprojlim (\underline{G}') \to \varprojlim (\underline{G}) \to \varprojlim (\underline{G}'') \to \varprojlim{}^1(\underline{G}') \to \varprojlim{}^1(\underline{G}) \to \varprojlim{}^1(\underline{G}'') \to 0.$$

□

Proposition 3.1.18 *Let* $I = \{I^n\}$ *be an injective in* $\mathcal{I}nv$. *Then* $\phi^n : I^{n+1} \to I^n$ *is surjective for all* n. *In particular,* $\varprojlim^1(\underline{I}) = 0$.

Proof: Construct an injection from I to a system J whose structure maps are surjective. For example, set $J_n := I_n \oplus J_{n-1}$. Since I is injective, there is a retraction $J \to I$ and this implies that the structure maps in I are surjective too. □

The dual of Theorem 3.1.6 combined with Proposition 3.1.17 and Proposition 3.1.18 yield the following proposition, which justifies the notation.

Proposition 3.1.19 $\varprojlim^1$ *is the first right derived functor of* $\varprojlim$.

Proof: Let I be an injective presentation of an inverse system A and set $B := I/A$. Proposition 3.1.18 says that $\varprojlim^1 I = 0$, while $(R\varprojlim)(I) = 0$ since I is injective. Therefore the long exact sequences from the dual of Theorem 3.1.6 and Proposition 3.1.17 give that both $\varprojlim^1 A$ and $(R\varprojlim)(A)$ are isomorphic to the cokernel of $\varprojlim I \to \varprojlim B$ □

When working with $\varprojlim_n^1$ the following sufficient condition for its vanishing, known as the Mittag-Leffler condition, which generalizes Prop. 3.1.16, is often useful.

Theorem 3.1.20 *Suppose* A *is an inverse system in which for each* n *there exists* $k(n) \leq n$ *such that* $\operatorname{Im}(A_i \to A_n)$ *equals* $\operatorname{Im}(A_{k(n)} \to A_n)$ *for all* $i \leq k(n)$. *Then* $\varprojlim_n^1 A = 0$. *In particular, if all of the system maps in the inverse system* $\{A_n\}$ *are epimorphisms then* $\varprojlim_n^1 A = 0$.

Theorem 3.1.21 *Let* $\ldots \to (C^{n+1})_* \to (C^n)_* \to (C^{n-1})_* \to \ldots$ *be an inverse system of chain complexes indexed by the integers. Let* $C = \varprojlim_n^1 C^n$. *Suppose that* $\varprojlim_n^1 (C^n)_* = 0$. *Then for each* q *there is a (Milnor) short exact sequence*

$$0 \to \varprojlim_n{}^1 H_{q+1}(C^n) \to H_q(C) \to \varprojlim_n H_q(C^n) \to 0.$$

Proof: Regard $B(C)$ and $Z(C)$ as subcomplexes of C where the restriction of the differential on C to either of these subcomplexes is 0. Since $\varprojlim$ preserves injections, $Z_q(C) = \varprojlim_n Z_q(C_n)$.

Applying $\varprojlim_n$ to the short exact sequence of inverse systems

$$0 \to Z_i(C^n) \to (C^n)_i \to B_{i-1}(C^n) \to 0$$

gives

$$0 \to \varprojlim_n Z_i(C^n) \to C_i \to \varprojlim_n B_{i-1}(C^n) \to \varprojlim_n{}^1 Z_i(C^n) \to \varprojlim_n{}^1 C_i \to \varprojlim_n{}^1 B_{i-1}(C^n) \to 0$$

Using our assumption that $\varprojlim_n{}^1 C_i = 0$ for all i we get $\varprojlim_n{}^1 B_i(C^n) = 0$ for all i and the sequence becomes

$$0 \to Z_i(C) \to C_i \to \varprojlim_n B_{i-1}(C^n) \to \varprojlim_n{}^1 Z_i(C^n) \to 0.$$

Equivalently, using $C_i/Z_i(C) \cong B_{i-1}(C)$, the sequence, applied with $i := q+1$, can be written as

$$0 \to B_q(C) \to \varprojlim_n B_q(C^n) \to \varprojlim_n{}^1 Z_{q+1}(C^n) \to 0.$$

Applying $\varprojlim_n$ to the short exact sequence of inverse systems

$$0 \to B_q(C^n) \to Z_q(C^n) \to H_q(C^n) \to 0$$

gives

$$0 \to \varprojlim_n B_q(C^n) \to \varprojlim_n Z_q(C^n) \to \varprojlim_n H_q(C^n) \to \varprojlim_n{}^1 B_q(C^n) \to \varprojlim_n{}^1 Z_q(C^n) \to \varprojlim_n{}^1 H_q(C_n) \to 0.$$

Using our previous deduction that $\varprojlim_n{}^1 B_q(C^n) = 0$, we conclude that $\varprojlim_n{}^1 Z_q(C^n) \cong \varprojlim_n{}^1 H_q(C_n)$ and the sequence becomes

$$0 \to \varprojlim_n B_q(C^n) \to Z_q(C) \to \varprojlim_n H_q(C^n) \to 0.$$

Applying the Snake Lemma to

$$\begin{tikzcd}
B_q(C_n) \arrow[r, equal] \arrow[d] & B_q(C) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\varprojlim_n B_q(C^n) \arrow[r] & Z_q(C) \arrow[r] & \varprojlim_n H_q(C^n)
\end{tikzcd}$$

gives the short exact sequence

$$0 \to \big(\varprojlim_n B_q(C^n)\big)/B_q(C) \to Z_q(C)/B_q(C) \to \varprojlim_n H_q(C^n) \to 0.$$

However $Z_q(C)/B_q(C) = H_q(C)$ and from above we have

$$\varprojlim_n B_q(C^n)/B_q(C) \cong \varprojlim_n{}^1 Z_{q+1}(C^n) \cong \varprojlim_n{}^1 H_{q+1}(C_n).$$

□

A common application is when $X_1 \subset X_2 \subset \ldots \subset X_n \subset \ldots$ are inclusions of CW-complexes and $X = \cup_{n=1}^{\infty} X_n$ and we wish to calculate $H^*(X)$. We set C^n equal to the cellular cochain complex of X_n and use that the $C^n \to C^{n-1}$ is onto, which by the Mittag-Leffler condition (Theorem 3.1.20) guarantees that our hypothesis $\varprojlim_n{}^1 (C^n)_* = 0$ is satisfied.

This yields

Theorem 3.1.22 (Milnor) *Suppose $X_1 \subset X_2 \subset \ldots \subset X_n \subset \ldots$ are inclusions of CW-complexes and $X = \cup_{n=1}^{\infty} X_n$. Then there is a natural short exact sequence*

$$0 \to \varprojlim_n{}^1 H^{q-1}(X_n) \to H^q(X) \to \varprojlim_n H^q(X_n) \to 0$$

□

originally proved by Milnor by applying Mayer-Vietoris to the "infinite mapping telescope" as in [16, Theorem 13.1.3].

3.2 Künneth Theorem and Universal Coefficient Theorems

Theorem 3.2.1 (Künneth Theorem) *Let C and D be chain complexes of R-modules. Suppose that C_n and the boundaries $B_n(C)$ are flat R-modules for all n. Then for all n there is a natural short exact sequence of R-modules*

$$0 \to \big(H_*(C) \otimes_R H_*(D)\big)_n \to H_n(C \otimes_R D) \to \mathrm{Tor}^R\big(H_*(C), H_*(D)\big)_{n-1} \to 0.$$

If $B_n(C)$ and $B_n(D)$ are both projective for all n or if R is a PID then the sequence is split (although not split naturally).

Proof: Regard $B(C)$ and $Z(C)$ as subcomplexes of C where the restriction of the differential on C to either of these subcomplexes is zero. Then $0 \to Z(C) \hookrightarrow C \overset{d}{\twoheadrightarrow} SB(C) \to 0$ becomes a short exact sequence of chain complexes. Tensoring with D gives a long exact sequence of chain complexes

$$\ldots \to \operatorname{Tor}\bigl(SB(C), D\bigr) \to Z(C) \otimes D \longrightarrow C \otimes D \to S(B) \otimes C \to 0.$$

Since $B_n(C)$ is flat for all n, $\operatorname{Tor}\bigl(SB(C), D\bigr) = 0$ so the above sequence reduces to a short exact sequence. The associated long exact homology sequence is

$$\to H_n\bigl(B(C)\otimes D\bigr) \xrightarrow{\iota} H_n\bigl(Z(C)\otimes D\bigr) \to H_n(C\otimes D) \to H_{n-1}\bigl(B(C)\otimes D\bigr) \xrightarrow{\iota} H_{n-1}\bigl(Z(C)\otimes D\bigr) \to$$

where ι is induced by the inclusion $B_n(C) \hookrightarrow Z_n(C)$ and we have made the replacement $H_q(SX) = H_{q-1}(X)$. Therefore there is a short exact sequence

$$0 \to \operatorname{coker} \iota_n \to H_n(C \otimes D) \to \ker \iota_{n-1} \to 0. \tag{3.3}$$

Tensoring the short exact sequence $0 \to B_n(C) \to Z_n(C) \to H_n(C) \to 0$ with $H(D)$ gives

$$\to \operatorname{Tor}_R\bigl(C, H(D)\bigr) \to \operatorname{Tor}_R\bigl(H(C), H(D)\bigr) \to \bigl(B(C) \otimes H(D)\bigr) \xrightarrow{\iota} \bigl(Z(C) \otimes H(D)\bigr) \to \bigl(H(C) \otimes H(D)\bigr) \to 0, \tag{3.4}$$

where for graded objects X, Y, we set

$$\bigl(\operatorname{Tor}(X, Y)\bigr)_n := \oplus_{i+j=n} \operatorname{Tor}(X_i, Y_j).$$

Given a short exact sequence $0 \to T \to U \to V$ of R-modules, such that V is flat, the long exact Tor sequence shows that $\operatorname{Tor}_n^R(T, M) \cong \operatorname{Tor}_n^R(U, M)$ for $n > 0$ and any M. Thus T is flat if and only if U is flat. In particular, our hypothesis that C is flat is equivalent to assuming that $Z(C)$ is flat. Thus $\operatorname{Tor}_R\bigl(Z(C), H(D)\bigr) = 0$ and so (3.4) gives $\ker \iota_q \cong \operatorname{Tor}_R\bigl(H(C), H(D)\bigr)_q$ and $\operatorname{coker} \iota_q \cong \bigl(H(C) \otimes H(D)\bigr)_q$. Therefore (3.3) becomes

$$0 \to \bigl(H(C) \otimes H(D)\bigr)_n \to H_n(C \otimes D) \to \operatorname{Tor}_R\bigl(H(C) \otimes H(D)\bigr)_{n-1} \to 0$$

as desired.

If $B_{n-1}(C)$ is projective then the sequence $0 \to Z_n(C) \to C_n \to B_{n-1}(C) \to 0$ splits and so there is a retraction $C_n \to Z_n(C)$. Similarly $B_{n-1}(D)$ projective implies the existence of a retraction $D_n \to Z_n(D)$. Thus if $B_{n-1}(C)$ and $B_{n-1}(D)$ are projective we get a map $\theta : C \otimes D \to Z(C) \otimes Z(D)$ of graded R-modules. Since

$$\theta\big(B(C \otimes D) \subset B(C) \otimes Z(D) + Z(C) \otimes B(D)$$

it induces a map $H(C \otimes D) \to H(C) \otimes H(D)$ which splits the natural map $H(C) \otimes H(D) \to H(C \otimes D)$.

Next, consider the special case where R is a PID.

Lemma 3.2.2 *Let R be a PID and let H be a graded R-module. Then there exists a free chain complex C (i.e. C_n is a free R-module for each n) such that $H_n(C) = H$.*

Proof: Let $q_p : F_p \twoheadrightarrow H_p$ be a free presentation of H_p. Since a submodule of a free module over a PID is free (See MAT1100–1101 notes, Page 92) $K_p := \ker q_p$ is free. Set $C_p := F_p \oplus K_{p-1}$ and define $d : C_p \to C_{p-1}$ by $d(x, y) := (y, 0)$. Then $d^2 = 0$, $Z_p(C) = F_p$ and $B_p(C) = K_p$ so that $H_p(C) = F_p/K_p \cong H_p$. □

Lemma 3.2.3 *Let C and D be chain complexes over a PID R such that C is free. Let $\psi : H_*(C) \to H_*(D)$ be a morphism of graded R-modules. Then there exists a chain map $\phi : C \to D$ such that $\phi_* = \psi$.*

Proof: Since R is a PID, the submodule $B_p(C)$ of C_p is free and thus $C_p \cong Z_p(C) \oplus B_{p-1}(C)$. Since $Z_p(C) \subset C_p$ is free (thus projective) there exists a lift $f_p : Z_p(C) \to Z_{p-1}(D)$ making the right square below commute.

$$\begin{tikzcd}
0 \arrow[r] & B_p(C) \arrow[r] \arrow[d, dotted, "f'_p"] & Z_p(C) \arrow[r] \arrow[d, dotted, "f_p"] & H_p(C) \arrow[r] \arrow[d, "\psi_p"] & 0 \\
0 \arrow[r] & B_p(D) \arrow[r] & Z_p(D) \arrow[r] & H_p(D) \arrow[r] & 0
\end{tikzcd}$$

Let $f'_p : B_p(C) \to B_p(D)$ be the induced map on kernels. Since $B_p(C)$ is free (thus projective) there exists $f''_{p-1} : B_{p-1}(C) \to D_p$ such that $d \circ f''_{p-1} = f'_{p-1}$. Define

$$\psi_p : C_p \cong Z_p(C) \oplus B_{p-1}(C) \to D_p$$

by $\phi_p(x, y) := f'_p(x) + f''_{p-1}(y)$. Then ϕ_* forms a chain map which has the desired property. □

Proof of Künneth Theorem (concluded)

To show that the sequence splits in the case where R is a PID:

By the lemmas there exist free chain complexes F and G together with chain maps $F \to C$ and $G \to D$ which induce isomorphisms on homology. By the naturality of the exact Künneth sequence we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \big(H_*(F) \otimes_R H_*(G)\big)_n \arrow[r] \arrow[d, "\cong"] & H_n(F \otimes_R G) \arrow[r] \arrow[d, "\cong"] & \mathrm{Tor}^R\big(H_*(F), H_*(G)\big)_{n-1} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \big(H_*(C) \otimes_R H_*(D)\big)_n \arrow[r] & H_n(C \otimes_R D) \arrow[r] & \mathrm{Tor}^R\big(H_*(C), H_*(D)\big)_{n-1} \arrow[r] & 0
\end{tikzcd}$$

Since F and G are free chain complexes, the top sequence splits and so the bottom does also. $\square$

Example 3.2.4 Let C be the chain complex

$$\ldots \to 0 \to \ldots \to 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0 \to \ldots.$$

let C be the chain complex

$$\ldots \to 0 \to \ldots \to 0 \to \mathbb{Z} \longrightarrow 0 \to 0 \to \ldots$$

and let D and D' be the chain complex

$$\ldots \to 0 \to \ldots \to 0 \to 0 \to \mathbb{Z}/(2\mathbb{Z}) \to 0 \to \ldots$$

where the nonzero terms are in degrees 1 and 0. Let $\phi : C \to D$ and $\phi' : C' \to D'$ be the obvious maps. Then $\phi'_* = 0$. If the splittings were natural, it would follow that $(\phi \otimes \phi')_* = 0$, but it is the identity on $H_1(C \otimes D) \cong H_1(C' \otimes D') \cong \mathbb{Z}/(2\mathbb{Z})$.

For an R-module G, let $\underline{G}$ denote the chain complex

$$\underline{G}_n = \begin{cases} G & \text{if } n = 0; \\ 0 & \text{if } n \neq 0 \end{cases}$$

with (perforce) the zero differential. Then for any chain complex C of R-modules, $(C \otimes_R \underline{G})_n = C_n \otimes_R G$ for all n. The homology of this chain complex is called the *homology of C with coefficients in G*, written $H_*(C; \underline{G})$. As a special case of the Künneth Theorem we get

Theorem 3.2.5 (Universal Coefficient Theorem) *Let C be a chain complex of R-modules and let G be an R-module. Suppose that C_n and $B_n(C)$ are flat R-modules for all n. Then for all n there is a natural short exact sequence of R-modules*

$$0 \to H_n(C) \otimes G \to H_n(C;G) \to \mathrm{Tor}^R\big(H_{n-1}(C), G\big) \to 0.$$

If G is free or if $B_n(C)$ is free for all n or if R is a PID then the sequence is split (although not split naturally). $\square$

3.3 Relative Derived Functors

Let $\mathcal{E}$ be a class of epimorphisms in a category $\underline{\underline{C}}$. An object P of $\underline{\underline{C}}$ is called *projective relative to* $\mathcal{E}$ if for every epimorphism $g : B \to A$ of $\mathcal{E}$ and homomorphism $f : P \to A$ there exists a "lift" $f' : P \to B$ (not necessarily unique) such that $gf' = f$. The *closure* $C(\mathcal{E})$ of a class $\mathcal{E}$ of epimorphisms consists of those epimorphisms $g : B \to A$ such that for every projective P relative to $\mathcal{E}$ and every map $f : P \to A$ there exists a lift $f' : P \to B$ such that $gf' = f$. It follows directly that, as with subspaces of topological spaces, closure is an idempotent operation ("the closure of the closure equals the closure") and a class of epimorphisms will be called closed if it equals its closure. It is easy to see that a closed class of epimorphism must be closed under the operations of compositions and taking direct sums.

The category $\underline{\underline{C}}$ is said to have *enough projectives* relative to the class $\mathcal{E}$ if for every object M of $\underline{\underline{C}}$ there exists an epimorphism $\epsilon : P \to M$ of $\mathcal{E}$ with P projective relative to $\mathcal{E}$.

If $\mathcal{E}$ is a closed class of epimorphisms such that $\underline{\underline{C}}$ has enough projectives relative to $\mathcal{E}$ then we can form projective resolutions relative to $\mathcal{E}$ for every object of $\underline{\underline{C}}$, define derived functors $L_q^{\mathcal{E}}(T)$ relative to $\mathcal{E}$ for any additive functor $T : \underline{\underline{C}} \to \underline{\underline{A}}$ to an abelian category $\underline{\underline{A}}$, and all the standard properties will carry over to this generalization.

The theorem that the left adjoint of an adjoint pair preserves projectives carries over to relative projectives.

Proposition 3.3.1 *Let $F : \underline{\underline{C}} \to \underline{\underline{D}}$ and $G : \underline{\underline{D}} \to \underline{\underline{C}}$ be functors between abelian categories $\underline{\underline{C}}$, $\underline{\underline{D}}$ such that $F \dashv G$. Suppose that $G : \underline{\underline{D}}(A, B) \to \underline{\underline{C}}(GA, GB)$ is a set injection for every pair of objects A, B of $\underline{\underline{D}}$. Let $\mathcal{E}$ be a closed class of epimorphisms in $\underline{\underline{C}}$ such that $\underline{\underline{C}}$ has enough projectives relative to $\mathcal{E}$. Then $G^{-1}(\mathcal{E})$ is a closed class of epimorphisms in $\underline{\underline{D}}$ such that $\underline{\underline{D}}$ has enough projectives relative to $G^{-1}(\mathcal{E})$, and the projectives of $\underline{\underline{D}}$ relative to $G^{-1}(\mathcal{E})$ are the direct summands of FP, where P is a projective in $\underline{\underline{C}}$ relative to $\mathcal{E}$.*

Proof: Exercise. $\square$

Note that the injectivity condition of G on morphisms guarantees that e is an epimorphism whenever $G(e)$ is.

Similarly we can define injectives relative to a class of monomorphisms, relative injective resolutions, and use an adjoint pair $F \dashv G$ in which F is an injection on morphisms to transport these from one category to another.

Chapter 4

Spectral Sequences

A *spectral sequence* is defined as a sequence $\big((E^r, d^r)\big)_{r=n_0,n_0+1,\ldots,}$ of differential abelian groups such that $E^{r+1} = H(E^r, d^r)$. By reindexing we could always arrange that $n_0 = 1$, but sometimes it is more natural to begin with some other integer. If all terms (E^r, d^r) of the spectral sequence have the appropriate additional structure, we might refer, for example, to a spectral sequence of R-modules, Hopf algebras, or whatever. If there exists N such that $E^r = E^N$ for all $r \geq N$ (equivalently $d^r = 0$ for all $r \geq N$) the spectral sequence is said to *collapse* at E^N.

Notice that, in an intuitive sense, the terms of a spectral sequence get smaller as $r \to \infty$ as each is a subquotient of its predecessor.

The definition of spectral sequence is so broad that we can say almost nothing of interest about them without putting on some additional conditions. We will begin by considering the most common type of spectral sequence, historically the one that formed the motivating example: the spectral sequence of a filtered chain complex.

4.1 Filtered Objects

To study a complicated object X, it often helps to filter X and study it one filtration at a time. A filtration $\mathcal{F}_X$ of a group X is a nested collection of subgroups

$$\mathcal{F}_X := \ldots F_n X \subset F_{n+1} X \subset \ldots \subset X \qquad -\infty < n < \infty.$$

A morphism $f : \mathcal{F}_X \to \mathcal{F}_Y$ of filtered groups is a homomorphism $f : X \to Y$ such that $f\big(F_n(X)\big) \subset F_n(Y)$. The groups $F_n X/F_{n-1} X$ are called the "filtration quotients" and their direct sum $\mathrm{Gr}(\mathcal{F}_X) := \bigoplus_n F_n X/F_{n-1} X$ is called the *associated graded group* of the filtered group $\mathcal{F}_X$. In cases where X has additional structure we might define special types of filtrations satisfying some compatibility conditions so that $\mathrm{Gr}(\mathcal{F}_X)$ inherits the additional structure. For example, an *algebra filtration* of an algebra X is defined as one for which $(F_nX)(F_kX) \subset F_{n+k}X$.

Since our plan is to study X by computing $\mathrm{Gr}(\mathcal{F}_X)$, the first question we need to consider is what conditions we need to place on our filtration so that $\mathrm{Gr}(\mathcal{F}_X)$ retains enough information to recover X. Our experience from the 5-Lemma suggests that the appropriate way to phrase the requirement is to ask for conditions on the filtrations which are sufficient to conclude that $f : X \to Y$ is an isomorphism whenever $f : \mathcal{F}_X \to \mathcal{F}_Y$ is a morphism of filtered groups for which the induced $\mathrm{Gr}(f) : \mathrm{Gr}(X) \to \mathrm{Gr}(Y)$ is an isomorphism.

It is clear that $\mathrm{Gr}(\mathcal{F}_X)$ can tell us nothing about $X - (\cup X_n)$ so to have a chance to recover X from $\mathrm{Gr}(\mathcal{F}_X)$ we need that $X = \cup X_n$. Similarly we need that $\cap X_n = 0$. However the latter condition is insufficient as can be seen from the following example.

Example 4.1.1 Let $X := \bigoplus_{k=1}^{\infty} \mathbb{Z}$ and $Y := \prod_{k=1}^{\infty} \mathbb{Z}$. Set

$$F_nX := \begin{cases} X & \text{if } n \geq 0; \\ \displaystyle\bigoplus_{k=-n}^{\infty} \mathbb{Z} & \text{if } n < 0, \end{cases} \qquad F_nY := \begin{cases} Y & \text{if } n \geq 0; \\ \displaystyle\prod_{k=-n}^{\infty} \mathbb{Z} & \text{if } n < 0 \end{cases}$$

and let $f : X \to Y$ be the inclusion. Then $\mathrm{Gr}(f)$ is an isomorphism but f is not.

$\mathcal{F}_X$ is called *cocomplete* if the canonical map $X \to \varinjlim_n F_nX$ is an isomorphism and $\mathcal{F}_X$ is called *complete* if $X \to \varprojlim_n (X/F_nX)$ is an isomorphism. $\mathcal{F}_X$ is called *bicomplete* if it both complete and cocomplete.

Note that "$\mathcal{F}_X$ cocomplete" is equivalent to "$\cup F_nX = X$" but "$\mathcal{F}_X$ complete" is stronger than "$\cap F_nX = 0$". More precisely, let $\mathcal{F}_X$ be a filtered abelian group. Applying Propostion 3.1.17 to the short exact sequence $0 \to F_nX \to X \to X/F_nX \to 0$ of inverse systems gives an exact sequence

$$0 \to \varprojlim_n F_nX \to \varprojlim_n X \to \varprojlim_n X/F_n \to {\varprojlim_n}^1 F_nX \to {\varprojlim_n}^1 X.$$

Since $\varprojlim_n X = X$ and ${\varprojlim_n}^1 X = 0$, we get

Theorem 4.1.2 *$\mathcal{F}_X$ is complete if and only if $\varprojlim_n F_nX = 0$ and ${\varprojlim_n}^1 F_nX = 0$.*

Theorem 4.1.3 (Comparison Theorem) *Let $\mathcal{F}_X$ be bicomplete and let $\mathcal{F}_Y$ be cocomplete with $\cap F_nY = 0$. Suppose that $f : \mathcal{F}_X \to \mathcal{F}_Y$ is a morphism such that $\mathrm{Gr}(f) : \mathrm{Gr}(X) \to \mathrm{Gr}(Y)$ is an isomorphism. Then $f : X \to Y$ is an isomorphism.*

Proof: Consider first the special case where the filtrations are bounded below. That is, suppose that there exists N such that $F_nX = F_nY = 0$ for $n < N$. Then $f_n : F_nX \to F_nY$ is an isomorphism for all n by induction and the 5-Lemma and so taking $\lim_{n\to\infty}$ gives $f_\infty : X \cong Y$.

Turning now to the general case, for each integer k define an induced filtration on X/F_kX by

$$F_n(X/F_kX) = \begin{cases} F_nX/F_kX & \text{if } n \geq k; \\ 0 & \text{if } n < k, \end{cases}$$

and similarly filter Y/F_kY. These filtrations are bounded below, so for each k the induced map $X/F_kX \to Y/F_kY$ is an isomorphism by the special case above. Suppose $f(x) = 0$. If $x \neq 0$, by cocompleteness of $\mathcal{F}_X$, there exists unique n such that $x \in F_n - F_{n-1}$. However since $F_nX/F_{n-1}X \to F_nY/F_{n-1}Y$ is an isomorphism, in particular an injection, from $f([x]) = 0$ we get $[x] = 0$ in $F_nX/F_{n-1}X$. This says $x \in F_{n-1}X$, contradicting the choice of n.

Now suppose $y \in Y$. Since $\mathcal{F}_Y$ is cocomplete, $y \in F_nY$ for some n. Set $y_n := y$. Since $F_nX/F_{n-1}X \to F_nY/F_{n-1}Y$ is an isomorphism, in particular an epimorphism, there exists $x_n \in F_nX$ such that $f([x_n]) = [y_n]$. That is, $f(x_n) \cong y_n$ modulo $F_{n-1}Y$. Set $y_{n-1} := y - f(x_n) \in F_{n-1}$. Repeating the argument, there exists $x_{n-1} \in F_{n-1}X$ such that $y_{n-1} - f(x_{n-1}) \in F_{n-2}Y$. Continuing, for all k we find $x_{n-k} \in F_{n-k}X$ such that $y - f(x_n) - f(x_{n-1}) - \ldots - f(x_{n-k}) \in F_{n-k-1}Y$. Set $w_{n-k} := x_n + x_{n-1} + \ldots + x_{n-k}$. The projection map $X/F_{n-k-1}X \twoheadrightarrow X/F_{n-k}X$ takes w_{n-k-1} to w_{n-k} since it takes $[x_{n-k-1}]$ to 0. Thus $w := (\ldots, w_{n-k-1}, w_{n-k}, \ldots, w_{n-1}, w_n, 0, \ldots)$ is a "consistent sequence" describing an element of $\varprojlim_k (X/F_kX) \cong X$. By construction $f(w) - y \in F_{n-k}Y$ for all k. Thus $f(w) = y$ since $\cap_m F_m(Y) = 0$. $\square$

4.2 Filtered Chain Complexes

Let $\mathcal{F}_C$ be a filtered chain complex. In many applications our goal is to compute $H_*(C)$ from knowledge of $H_*(F_nC/F_{n-1}C)$ for all n. The overall plan, which is not guaranteed to be successful in general, would be:

1) Use the given filtration on C to define a filtration on $H_*(C)$;

2) Use our knowledge of $H_*(\operatorname{Gr} C)$ to compute $\operatorname{Gr} H_*(C)$;

3) Reconstruct $H_*(C)$ from $\operatorname{Gr} H_*(C)$.

To begin, set $F_n(H_*C) := \operatorname{Im}(s_n)_*$, where $s_n : F_n(C) \to C$ is the inclusion (chain) map from the filtration. The spectral sequence which we will define for this situation can be regarded as a method of keeping track of the information contained in the infinite collection of long exact homology sequences coming from the short exact sequences $0 \to F_{n-1}C \to F_nC \to F_nC/F_{n-1}C \to 0$. When working with a long exact sequence, knowledge of two of every three terms gives a handle on computing the remaining terms but does not, in general, completely determine those terms, which explains intuitively why we have some reason to hope that a spectral sequence *might* be useful and also why it is not *guaranteed* to solve our problem.

Before proceeding with our motivating example, we digress to discuss spectral sequences formed from "exact couples".

4.3 Exact Couples

In this section we will define exact couples, show how to associate a spectral sequence to an exact couple, and discuss some properties of spectral sequences coming from exact couples. As we shall see, a filtered chain complex gives rise to an exact couple and we will examine this spectral sequence in greater detail.

Exact couples were invented by Massey and many books use them as a convenient method of constructing spectral sequences. Other books bypass discussion of exact couples and define the spectral sequence coming from a filtered chain complex directly.

Definition 4.3.1 *An* exact couple *consists of a triangle*

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

containing abelian groups D, E, and together with homomorphisms i, j, k such that the diagram is exact at each vertex.

In the following, to avoid conflicting notation considering the many superscripts and subscripts which will be needed we use the convention that an n-fold composition will be written $f^{\circ n}$ rather than the usual f^n.

Given an exact couple, set $d := jk : E \to E$. By exactness, $kj = 0$, so $d^{\circ 2} = jkjk = 0$ and therefore (E, d) forms a differential group. To the exact couple we can associate another exact couple, called its derived couple, as follows. Set $D' := \operatorname{Im} i \subset D$ and $E' := H(E, d)$. Define $i' := i|_{D'}$ and let $j' : D' \to E'$ be given by $j'(iy) := \overline{j(y)}$, where $\overline{x}$ denotes the equivalence class of x. The map $k' : E' \to D'$ is defined by $k'(\overline{z}) := kz$. One checks that the maps j' and k' are well defined and that (D', E', i', j', k') forms an exact couple. Therefore from our original exact couple we can inductively form a sequence of exact couples $(D^r, E^r, i^r, j^r, k^r)_{r=1}^{\infty}$ with $D^1 := D$, $E^1 := E$, $D^r := (D^{r-1})'$ and $E^r := (E^{r-1})'$. This gives a spectral sequence $(E^r, d^r)_{r=1}^{\infty}$ with $d^r = j^r k^r$.

Example 4.3.2 In our motivating example, to a filtered differential abelian group $\mathcal{F}_C$ we associate an exact couple as follows. Set $D := \bigoplus_p D_{p,q}$ where $D_{p,q} = H_{p+q}(F_pC)$ and $E := \bigoplus_{p,q} E_{p,q}$ where $E_{p,q} = H_{p+q}(F_pC/F_{p-1}C)$.

The long exact homology sequences coming from the sequences

$$0 \to F_{p-1}C \xrightarrow{a} F_pC \xrightarrow{b} F_pC/F_{p-1}C \to 0$$

give rise, for each p, q, to maps $a_* : D_{p-1,q+1} \to D_{p,q}$, $b_* : D_{p,q} \to E_{p,q}$, and $\partial : E_{p,q} \to D_{p-1,q}$. Define $i : D \to D$ to be the map whose restriction to $D_{p-1,q+1}$ is the composition of a_* with the canonical inclusion $D_{p-1,q+1} \hookrightarrow D$. Similarly define $j : D \to E$ and $k : E \to D$ to be the maps whose restrictions to each summand are the compositions of b_* and ∂ with the inclusions.

The indexing scheme for the bigradations is motivated by the fact that in many applications it causes all of the nonzero terms to appear in the first quadrant so it is the most common choice, although one sometimes sees other conventions.

In our motivating example, the terms of the initial exact couple came with a bigrading $D = \bigoplus D_{p,q}$ and $E = \bigoplus E_{p,q}$ and writing $|f|$ for the bidegree of a morphism f we had: $|i| = (1, -1)$; $|j| = (0, 0)$; $|k| = (-1, 0)$; $d = (-1, 0)$. It follows that $|i^r| = (1, -1)$; $|j^r| = (-r + 1, r - 1)$; $|k^r| = (-1, 0)$; $|d^r| = (-r, r - 1)$ which is considered the standard bigrading for a bigraded exact couple. The standard bigrading for a bigraded spectral sequence is one such that $|d^r| = (-r, r - 1)$, which is consistent with the grading arising from a bigraded exact couple. Ignoring the second gradation gives $|i^r| = 1$; $|j^r| = -r + 1$; $|k^r| = -1$, which is the standard grading on a graded exact couple.

We observed earlier that terms of a spectral sequence get smaller as $r \to \infty$ as each is a subquotient of its predecessor. Note that the bigrading is such that this applies to each pair of coordinates individually (e.g. $E^{r+1}_{p,q}$ is a subquotient of $E^r_{p,q}$) and so in particular if the p, q-position ever becomes 0 that position remains 0 forevermore.

There is actually a second exact couple we could associate to $\mathcal{F}_C$, which yields the same spectral sequence: use the same E as above but replace D by $\bigoplus D_{p,q}$ with $D_{p,q} = H_{p+q+1}(C/F_pC)$, and define i, j, and k in a manner similar to that above.

When dealing with cohomology rather than homology the usual starting point would be a system of inclusions of cochain complexes $\ldots \subset F^{n+1}C \subset F^nC \subset F^{n-1} \ldots \subset C$. This can be reduced to the previous case by replacing the cochain complex C by a chain complex C_* using the convention $C_p := C^{-p}$ and filtering the result by $F_nC_* := F^{-n}C$. The usual practice, equivalent to the above followed by a rotation of 180°, is to leave the original indices and instead reverse the arrows in the exact couple. In this case it is customary to write $D_r^{p,q}$ and $E_r^{p,q}$ for the terms in the exact couple and spectral sequence.

In applications it is often the case that E^1 is known and that our goal includes computing D^1. The example of the filtered chain complex with the assumption that we know $H_*(F_pC/F_{p-1}C)$ for all p is fairly typical.

Since each D^r is contained in D^{r-1} and each E^r is a sub-quotient of E^{r-1} the terms of these exact couples get smaller as we progress. To get properties of the spectral sequence we need to examine this process and in particular to analyze that which remains in the spectral sequence as we let r go to infinity.

For $x \in E$, if $dx = 0$ then $\overline{x}$ belongs to E^2 and so $d^2(\overline{x})$ is defined. In the following we shall usually simplify the notation by writing simply x in place of $\overline{x}$ and writing $d^r x = 0$ to mean "$d^r x$ is defined and equals 0".

If $dx = 0$, …, $d^{r-1}x = 0$ then x represents an element of E^r and $d^r x$ is defined. Set $Z^r := \{x \in E \mid d^m x = 0 \ \forall m \leq r\}$, where by convention $Z^0 := E^1$. Then $E^{r+1} \cong Z^r/{\sim}$ where $x \sim y$ if there exists $z \in E$ such that for some $t \leq r$ we have $d^m z = 0$ for $m < t$ (thus $d^t z$ is defined) and $d^t z = x - y$. With this as motivation we set $Z^\infty := \cap_r Z^r = \{x \in E \mid d^m x = 0 \ \forall m\}$ (known as the "infinite cycles") and define $E^\infty := Z^\infty/{\sim}$ where $x \sim y$ if there exists $z \in E$ such that for some t we have $d^m z = 0$ for $m < t$ and $d^t z = x - y$.

Next we relate E^∞ to quantities obtained from the exact couple.

Notice that $D^{r+1} = \operatorname{Im} i^{\circ r} \cong D/\ker i^{\circ r}$. There is no analogy of this statement for $r = \infty$. Instead we have separate concepts so we set $D^\infty := D/\cup_r \ker i^{\circ r}$ and ${}^\infty D := \cap_r \operatorname{Im} i^{\circ r}$.

Let $D \overset{i}{\twoheadrightarrow} D' = \operatorname{Im} i \overset{\alpha}{\hookrightarrow} D$ be the factorization of $i : D \to D$ as the composition of an epimorphism and a monomorphism. Then ${}^\infty D$ is the inverse limit of the system

$$\ldots \overset{\alpha}{\hookrightarrow} D^{n+1} \overset{\alpha}{\hookrightarrow} D^n \overset{\alpha}{\hookrightarrow} \ldots \overset{\alpha}{\hookrightarrow} D^1.$$

Explicitly, ${}^\infty D = \bigcap_n i^n D$ equals elements of D infinitely "divisible" by i. If x is infinitely divisible by i then so is ix and so i induces a map ${}^\infty\imath : {}^\infty D \to {}^\infty D$. At the other end, D^∞ is the direct limit of

$$D^1 \overset{i}{\twoheadrightarrow} \ldots \overset{i}{\twoheadrightarrow} D^n \overset{i}{\twoheadrightarrow} D^{n+1} \overset{i}{\twoheadrightarrow} \ldots.$$

Again i induces a canonical map $\imath : D^\infty \to D^\infty$.

For $y \in D$, using the fact that $kj = 0$ and induction we see that $d^n(jy)$ is defined and equal to zero for all n. Thus jy is an infinite cycle, so we get an induced map $\tilde{\jmath} : D \to Z^\infty(E)$. Applying the same considerations to the derived exact couples gives an induced map $\tilde{\jmath} : D^n \to Z_\infty(E^n)$

for each n. The diagram

$$\begin{tikzcd}
D \arrow[r, two heads, "i"] \arrow[d, "\jmath"] & \dots \arrow[r, two heads, "i"] & D^n \arrow[r, two heads, "i"] \arrow[d, "\jmath"] & D^{n+1} \arrow[r, two heads, "i"] \arrow[d, "\jmath"] & \dots \\
Z^\infty(E) \arrow[r, two heads] & \dots \arrow[r, two heads] & Z^\infty(E^n) \arrow[r, two heads] & Z^\infty(E^{n+1}) \arrow[r, two heads] & \dots
\end{tikzcd}$$

commutes since $\tilde{\jmath} : D^n \to Z^\infty(E^n)$ is induced by $j^n = ji^{(n-1)^{-1}}$. Therefore there is an induced map of direct limits $\tilde{\jmath} : D^\infty \to E^\infty$.

Turning to the other end, the commutative square on the right of the diagram

$$\begin{tikzcd}
Z^n(E) \arrow[r] \arrow[d, "\tilde{k}"] & Z^{n-1} \arrow[r, "d^n"] \arrow[d, "k"] & E^n \arrow[d, equal] \\
Z(E^n) \arrow[r] \arrow[d, "\tilde{k}"] & E^n \arrow[r, "d^n"] \arrow[d, "k"] & E^n \arrow[d, equal] \\
D^{n+1} \arrow[r] & D^n \arrow[r, "j^n"] & E^n
\end{tikzcd}$$

yields the induced map $\tilde{k}$ of kernels. Due to the equality on the far right, the left squares are a pullbacks. Taking inverse limits in the diagram of pullbacks

$$\begin{tikzcd}
\dots \arrow[r] & Z^{n+1}(E) \arrow[r] \arrow[d, "\tilde{k}"] & Z^n(E) \arrow[r] \arrow[d, "\tilde{k}"] & \dots \arrow[r] & \dots & E \arrow[d, "k"] \\
\dots & & & & & \\
\dots \arrow[r, tail, "\alpha"] & D^{n+2} \arrow[r, tail, "\alpha"] & D^{n+1} \arrow[r, tail, "\alpha"] & \dots \arrow[r, tail, "\alpha"] & \dots & D
\end{tikzcd}$$

yields an induced map $\tilde{k} : Z^\infty(E) \to {}^\infty D$ which fits into the square

$$\begin{tikzcd}
Z^\infty(E) \arrow[r] \arrow[d, "\tilde{k}"] & E \arrow[d, "k"] \\
{}^\infty D \arrow[r] & D.
\end{tikzcd}$$

which, using that $\varprojlim$ preserves injections, is seen to be a pullback. Applying the same considerations to the derived exact couples gives an induced map $\tilde{k} : Z^\infty(E^n) \to {}^\infty D^n$ for each n. However ${}^\infty D^n = \bigcap_m i^m D^n = \bigcap_m i^{m+n} D = {}^\infty D$. The maps $\tilde{k} : Z^\infty(E^n) \to {}^\infty D^n = {}^\infty D$ are compatible and so induce a map $\tilde{k} : E^\infty \to {}^\infty D$ from the direct limit.

The analogue of the rth derived exact couple when $r = \infty$ is the following exact sequence.

Theorem 4.3.3 *There is an exact sequence*

$$0 \to D^\infty \xrightarrow{i} D^\infty \xrightarrow{j} E^\infty \xrightarrow{k} {}^\infty D \xrightarrow{i} {}^\infty D \xrightarrow{j} \varprojlim_r{}^1 Z^r \xrightarrow{k} \varprojlim_r{}^1 D^r \xrightarrow{i} \varprojlim_r{}^1 D^r \to 0.$$

Proof: The fact that $i^\infty : D^\infty \to D^\infty$ is a monomorphism can be seen from the explicit description of i^∞. Consider the diagram

$$\begin{tikzcd}
D \arrow[r, "i"] \arrow[d, equal] & D \arrow[r, "\tilde{j}"] \arrow[d, equal] & Z^\infty(E) \arrow[r, "\tilde{k}"] \arrow[d, hook] & {}^\infty D \arrow[r, "i"] \arrow[d, hook] & {}^\infty D \arrow[d, hook] \\
D \arrow[r, "i"] & D \arrow[r, "j"] & E \arrow[r, "\tilde{k}"] & D \arrow[r, "i"] & D
\end{tikzcd} \tag{4.1}$$

in which the bottom row is known to be exact and the second last square is a pullback. In general, given a diagram

$$\begin{tikzcd}
A' \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & C' \arrow[d, hook] \\
A \arrow[r] & B \arrow[r] & C
\end{tikzcd}$$

in which the bottom row is exact and the left square is a pullback, the top row must be exact. It follows that the top row of $(*)$ is exact at the first ${}^\infty D$. Also, $\operatorname{Ker} \tilde{j} = \operatorname{Ker} j = \operatorname{Im} i$, and since the second last square is a pullback, $\operatorname{Ker} \tilde{k} = \operatorname{Ker} k = \operatorname{Im} j = \operatorname{Im} \tilde{j}$. Therefore the top row of $(*)$ is exact everywhere. Applying the same considerations to the derived exact couples and using

${}^{\infty}D^n = {}^{\infty}D$ gives a commutative diagram

$$\begin{tikzcd}
D \arrow[r, two heads, "i"] \arrow[d, "i"'] & \dots \arrow[r, two heads, "i"] & D^n \arrow[r, two heads, "i"] \arrow[d, "i"'] & D^{n+1} \arrow[r, two heads, "i"] \arrow[d, "i"'] & \dots \\
D \arrow[r, two heads, "i"] \arrow[d, "j"'] & \dots \arrow[r, two heads, "i"] & D^n \arrow[r, two heads, "i"] \arrow[d, "j"'] & D^{n+1} \arrow[r, two heads, "i"] \arrow[d, "j"'] & \dots \\
Z^{\infty}(E) \arrow[r, two heads] \arrow[d, "\tilde{k}"'] & \dots \arrow[r, two heads] & Z^{\infty}(E^n) \arrow[r, two heads] \arrow[d, "\tilde{k}"'] & Z^{\infty}(E^{n+1}) \arrow[r, two heads] \arrow[d, "\tilde{k}"'] & \dots \\
{}^{\infty}D \arrow[r, equal] \arrow[d, "i"'] & \dots \arrow[r, equal] & {}^{\infty}D \arrow[r, equal] \arrow[d, "i"'] & {}^{\infty}D \arrow[r, equal] \arrow[d, "i"'] & \dots \\
{}^{\infty}D \arrow[r, equal] & \dots \arrow[r, equal] & {}^{\infty}D \arrow[r, equal] & {}^{\infty}D \arrow[r, equal] & \dots
\end{tikzcd}$$

with exact columns. Passing to direct limits, which preserves exactness (Theorem 3.1.11), gives the exact sequence

$$0 \to D^{\infty} \xrightarrow{i} D^{\infty} \xrightarrow{j} E^{\infty} \xrightarrow{k} {}^{\infty}D \xrightarrow{i} {}^{\infty}D \tag{4.2}$$

The original exact couple gives a short exact sequence

$$0 \to E^r/\operatorname{Ker} k \to D^r \xrightarrow{i} D^{r+1} \to 0 \tag{4.3}$$

Since the boundaries $d(E^r)$ are contained in $\operatorname{Ker} k$, the projection map $Z^{r-1} \twoheadrightarrow E^r$ induces an isomorphism $Z^{r-1}/\operatorname{Ker} k \cong E^r/\operatorname{Ker} k$. Since $\operatorname{Ker} k$ is a constant sequence, $\varprojlim_r{}^1 \operatorname{Ker} k = 0$ (by Mittag-Leffler). Thus Proposition 3.1.17 gives $\varprojlim_r{}^1 Z^r/\operatorname{Ker} k \cong \varprojlim_r{}^1 Z^r$. Applying proposition 3.1.17 to the sequence 4.3 and making this substitution gives the exact sequence

$$0 \to \varprojlim_r Z^r/\operatorname{Ker} k \xrightarrow{i} {}^{\infty}D \xrightarrow{i} {}^{\infty}D \xrightarrow{j} \varprojlim_r{}^1 Z^r \xrightarrow{k} \varprojlim_r{}^1 D^r \xrightarrow{i} \varprojlim_r{}^1 D^r \to 0.$$

Splicing this together with the sequence 4.2 gives the theorem. □

4.4 Convergence of Spectral Sequences

As noted earlier, the definition of spectral sequence is so broad that we need to put some conditions on our spectral sequences to make them useful as a computational tool. From now on we will restrict attention to spectral sequences arising from exact couples. To see how to proceed we examine the motivating example more closely.

Consider our motivating example. For a filtered chain complex $\mathcal{F}_C$ with structure maps $s_p : F_pC \to C$ we defined $F_p\big(H_*(C)\big) = \operatorname{Im} s_{p_*}$. Let $D_{*,*}$, $E_{*,*}$ be the associated exact couple. Examining the definitions, $D^\infty = \oplus_p D^\infty_p$ where $D^\infty_p = H(C_p)/\cup_m i^{-m}(0)$. Since $(s_p)_*$ is a composition with i^m for every m, we have $(s_p)_*(\cup_m i^{-m}(0) = 0$ and so $(s_p)_*$ induces $\bar{s}_p : D^\infty_p \to \operatorname{Im}(s_p)_* = F_p\big(H(C)\big)$. Observe that $\bar{s}_p \circ i = \bar{s}_{p-1}$ so the image of the restriction of $\bar{s}_p \circ i =$ to $i(D^\infty_{p-1})$ is contained in $F_{p-1}\big(H(C)\big)$. Therefore there is an induced map $\bar{s} : D^\infty/iD^\infty \to \operatorname{Gr}\big(H(C)\big)$.

Note that if C is cocomplete then the fact that homology commutes with direct limits implies that $H(C)$ is cocomplete, so the hypothesis of the next Lemma is always satisfied in that case.

Lemma 4.4.1 *If $\mathcal{F}_{H(C)}$ is cocomplete then $\bar{s}$ is a monomorphism.*

Proof: Suppose $\bar{s}([z]) = 0$ for $z \in D_p$. Since $\mathcal{F}_{H(C)}$ is cocomplete, $H(C) = \varinjlim_p D_p$ so there exists m such that $i^m(z) = 0$, by a property of $\varinjlim$. (If an element becomes 0 in the direct limit, then it becomes 0 at some "finite stage" of the system according to Lemma 3.1.13.) Thus $[z] = 0$ in D^∞_p. □

Theorem 4.4.2 *Let C be a filtered differential abelian group. If $\mathcal{F}_H(C)$ is cocomplete then*

i) $\bar{s}_p : D^\infty_p \xrightarrow{\cong} F_p\big(H(C)\big)$.

ii) $\bar{s} : D^\infty/i^\infty(D^\infty) \to \operatorname{Gr}\big(H(C)\big)$ *is an isomorphism;*

iii) There is an exact sequence $0 \to \operatorname{Gr}\big(H(C)\big) \xrightarrow{j^\infty} E^\infty \xrightarrow{\infty k} {}^\infty D \xrightarrow{\infty i} {}^\infty D$.

Proof: $\bar{s}_p : D^\infty_p \to F_p\big(H(C)\big)$ is surjective by definition of $F_p\big(H(C)\big)$. However in the case where $\mathcal{F}_{H(C)}$ is cocomplete, it is also injective by the preceding Lemma, giving (i).

Part (iii) follows from part (ii) and Theorem 4.3.3.

Suppose $y \in F_p\big(H_*(X)\big)$ represents an element of $\operatorname{Gr}_p\big(H_*(X)\big)$. Then y belongs to $\operatorname{Im}(s_p)_* : H_*(C_p) \to H_*(C)$ and so $[y] = \bar{s}[y_p]$ where $(s_p)_*(y_p) = y$. Therefore $\bar{s}$ is onto. Suppose now that

x, belonging to D_p^∞, satisfies $\bar{s}([x]) = 0$ in $\mathrm{Gr}_p\big(H_*(C)\big)$. Then $\bar{s}_p(x)$ belongs to $F_{p-1}\big(H_*(C)\big)$ so there exists $y \in D_{p-1}^\infty$ such that $\bar{s}_p(x) = \bar{s}_{p-1}(y) = \bar{s}_p(iy)$. By the preceding Lemma, $\bar{s}_p$ is a monomorphism, so $x = iy$ and thus $[x] = 0$ in D^∞/iD^∞, which proves (ii).

□

Motivated by this example, we say that the spectral sequence (E^r) *abuts* to $\mathcal{F}_L$ if there is an isomorphism $\mathrm{Gr}\, L \to E^\infty$. Here we mean an isomorphism of graded abelian groups, which makes sense since under our assumptions E^r inherits a grading from E^1 for each r. If in addition the filtration on L is cocomplete, we say that (E^r) *weakly converges* to $\mathcal{F}_L$ and if it is bicomplete we say that (E^r) *converges* (or *strongly converges*) to $\mathcal{F}_L$. The notation $(E^r) \Rightarrow \mathcal{F}_L$ (or simply $(E^r) \Rightarrow L$ when the filtration on L is either understood or unimportant) is often used in connection with convergence but there is no universal agreement as to which of the three concepts (abuts, weakly converges, or converges) it refers to! We will use it to mean "abuts". We will also use the expression (E^r) *quasi-converges* to $\mathcal{F}_L$ to mean that the spectral sequence weakly converges to $\mathcal{F}_L$ with $\cap_n F_n L = 0$. (Note: The terminology "quasi-converges" is nonstandard although the concept has appeared in the literature, sometimes under the name "converges".)

While it would be overstating things to claim that convergence of the spectral sequence shows that E^∞ determines $H(C)$, it is clear that convergence is what we need in order to expect that E^∞ contains enough information to possibly reconstruct $H(C)$. The sense in which this is true is stated more precisely in the following theorem.

Theorem 4.4.3 (Spectral Sequence Comparison Theorem) *Let $f = (f^r) : (E^r) \to \tilde{E}^r$ be a morphism of spectral sequences.*

1) *If $f^N : E^N \to \tilde{E}^N$ is an isomorphism for some N then f^r is an isomorphism for all $r \geq N$ (including $r = \infty$).*

2) *Suppose in addition that (E^r) converges to $\mathcal{F}_X$ and $(\tilde{E}^r)$ quasi-converges to $\mathcal{F}_{\tilde{X}}$. Let $\phi : \mathcal{F}_X \to \mathcal{F}_{\tilde{X}}$ be a morphism of filtered abelian groups which is compatible with f. (That is, there is exist isomorphisms $\eta : \mathrm{Gr}\, X \cong E^\infty$ and $\tilde{\eta} : \mathrm{Gr}\, \tilde{X} \cong \tilde{E}^\infty$ such that $f^\infty \circ \eta = \tilde{\eta} \circ \mathrm{Gr}(f)$.) Then $f : X \to \tilde{X}$ is an isomorphism.*

Proof: The commutative diagram

$$\begin{tikzcd}
E^N \arrow[r, "f^N", "\cong"'] \arrow[d, "d^r"] & \tilde{E}^N \arrow[d, "d^r"] \\
E^N \arrow[r, "f^N", "\cong"'] & \tilde{E}^N
\end{tikzcd}$$

shows that $f^{N+1} : E^{N+1} \to \tilde{E}^{N+1}$ and (taking kernels) that $f^{N+1} : Z^{N+1}(E) \to Z^{N+1}(\tilde{E})$ are isomorphisms. It follows by induction that these statements hold for all $r \geq N$. Thus taking the limit as r goes to infinity, gives that $f : Z^\infty(E) \to Z^\infty(\tilde{E})$ is an isomorphihsm and then passing to the quotient gives that $f : E^\infty \to \tilde{E}^\infty$ is an isomorphism. Therefore the theorem follows from Theorem 4.1.3 and the definitions of convergence/quasiconvergence. $\square$

Example 4.4.4 Within the constraints provided by Theorem 4.4.3, a spectral sequence might have many limits.

Let $0 \to A \to B \to C \to 0$ and $0 \to A \to B' \to C \to 0$ be short exact sequences with $B \ncong B'$. Let E be the spectral sequence with $(E^r, d^r) = (A \oplus C, 0)$ for all r. Let $L = B$ with filtration $0 \subset A \subset B$, and let $L' = B'$ with filtration $0 \subset A \subset B'$. Then $\mathrm{Gr}(L) \cong \mathrm{Gr}(L') \cong A \oplus C = E^\infty$, and both $\mathcal{F}_L$ and $\mathcal{F}_{L'}$ are bicomplete. Thus the spectral sequence converges (strongly) to both L and L'.

A typical calculation of some group Y by means of spectral sequences might proceed as an application of Theorem 4.4.3 along the lines of the following plan.

1) Subgroups F_nY forming a filtration of Y are defined, although usually not computable at this point. The subgroups are chosen in a manner that seems natural bearing in mind that to be useful it will be necessary to show convergence properties.

2) Directly or by means of an exact couple, a spectral sequence is defined in a manner that seems to be related to the filtration.

3) Some early term of the spectral sequence (usually E^1 or E^2) is calculated explicitly and the differentials d^r are calculated successively resulting in a computation of E^∞.

4) With the aid of the knowledge of E^∞, a conjecture $Y = G$ is formulated for some G.

5) A suitable filtration on G and a map of filtrations $\mathcal{F}_G \to \mathcal{F}_Y$ or $\mathcal{F}_Y \to \mathcal{F}_G$ is defined.

6) The spectral sequence is demonstrated to converge to G.

7) The spectral sequence is demonstrated to converge to Y and Theorem 4.4.3 is applied.

It is step (5) that shows that Y is the "correct" object to which the spectral sequence converges. The hardest steps are usually (3) and (7). For step (3), in most cases the calculations require knowledge which cannot be obtained from the spectral sequence itself, although the spectral sequence machinery plays its role in distilling the information and pointing the way to exactly what needs to be calculated. Steps (4-6) are frequently very easy, and often not stated explicitly, with "by construction of G" being the most common justification of (6). We now discuss the types of considerations involved in step (7).

Convergence of a spectral sequence to a desired L can be difficult to verify in general partly because the conditions are stated in terms of some filtration (usually understood only in a theoretical sense) on an initially unknown L rather than in terms of properties of the spectral sequence itself or an exact couple from which it arose. Theorems 3.1.11 and 4.4.2(ii) give us the following extremely important special case in which we can conclude convergence to $H(C)$ of the spectral sequence for $\mathcal{F}_C$ based on conditions that are often easily checked.

Theorem 4.4.5 *If $\mathcal{F}_C$ is a filtered differential abelian group such that $\mathcal{F}_C$ is cocomplete and there exists M such that $H(F_nC) = 0$ for $n < M$. Then the spectral sequence for $\mathcal{F}_C$ converges to $H(C)$.* □

Although the second hypothesis, which implies that ${}^{\infty}D = 0$ in the associated exact couple, is very strong it handles the large numbers of commonly used filtrations which are 0 in negative degrees.

Under the conditions of Theorem 4.4.5, in the standard case where C is a chain complex, inserting the bigradings into Theorem 4.4.2 gives a short exact sequence $0 \to D^{\infty}_{p-1,q+1} \to D^{\infty}_{p,q} \to E^{\infty}_{p,q} \to 0$ with $D^{\infty}_{p,q} \cong F_p\big(H_{p+q}(C)\big)$; equivalently $F_k\big(H_n(C)\big)/F_{k-1}\big(H_n(C)\big) \cong E^{\infty}_{k,n-k}$. Thus the only E^{∞} terms relevant to the computation to $H_n(C)$ are those on the diagonal $p + q = n$. In the important case of a first quadrant spectral sequence ($E^r_{p,q} = 0$ if $p < 0$ or $q < 0$), the number of nonzero terms on any diagonal is finite so the E^{∞}-terms on the diagonal $p + q = n$ give a finite composition series for each $H_n(C)$.

Here is an elementary example of an application of a spectral sequence.

Example 4.4.6 Let $S_*(\ )$ denote the singular chain complex, let $H_*(\ ) := H_*\big(S_*(\ )\big)$ denote singular homology, and let $H^{\text{cell}}_*(\ )$ denote cellular homology. Let X be a CW-complex with n-skeleton $X^{(n)}$. The inclusions $S_*\big(X^{(n)}\big) \to S_*(X)$ yield a filtration on $S_*(X)$. In the associated spectral sequence,

$$E^1_{p,q} = H_{p+q}\left(X^{(p)}/X^{(p-1)}\right) \cong \begin{cases} \text{free abelian group on the p-cells of } X & \text{if } q = 0; \\ 0 & \text{if } q \neq 0. \end{cases}$$

The differential $d^1_{p,0} : H_p\left(X^{(p)}/X^{(p-1)}\right) \to H_{p-1}\left(X^{(p-1)}/X^{(p-2)}\right)$ is the definition of the differential in cellular homology. Therefore

$$E^2_{p,q} = \begin{cases} H^{\text{cell}}(X) & \text{if } q = 0; \\ 0 & \text{if } q \neq 0. \end{cases}$$

Looking at the bidegrees, the domain or range of $d^2_{p,q}$ is zero for each p and q so $d^2 = 0$, and similarly $d^r = 0$ for all $r > 2$. Therefore the spectral sequence collapses with $E^2 = E^\infty$. The spectral sequence converges to $H_*(X)$ so the terms on the diagonal $p + q = n$ form a composition series for $H_n(X)$. Since the $(n, 0)$ term is the only nonzero term on this diagonal, $H_n(X) \cong H_n^{\text{cell}}(X)$. That is, "cellular homology equals singular homology".

Consider a graded exact couple with standard gradation. Although the definition of convergence gives little idea of where to look for an object to which the spectral sequence might converge, there are two obvious candidates, $L_{-\infty}$ and L_∞, defined as follows. Set $L_\infty := \varinjlim_n D_n$ and $L_{-\infty} := \varprojlim_n D_n$. That is, $L_{-\infty}$ and L_∞ are the direct and inverse limits of the system

$$\dots \xrightarrow{i} D_{n-1} \xrightarrow{i} D_n \xrightarrow{i} D_{n+1} \xrightarrow{i} \dots$$

where $D = \oplus_n D_n$. Filter L_∞ by $F_n L_\infty := \operatorname{Im}(D_n \to L_\infty)$ and filter $L_{-\infty}$ by $F_n L_{-\infty} := \ker(L_{-\infty} \to D_n)$. The gradation on D induces a gradation on D^∞ and it follows from the definitions that $F_n L_\infty = (D_n)^\infty$. Thus $(D_n)^\infty / i(D_{n-1}{}^\infty) = \operatorname{Gr}_n L_\infty$. At the other end, $L_{-\infty}$ consists of sequences (x_n) satisfying $x_n = i(x_{n-1})$ for all n. In particular, each x_n lies in the set ${}^\infty D$ of infinitely divisible elements. Thus the canonical map $L_{-\infty} \to D_n$ lifts to ${}^\infty D_n$ yielding an injection $L_{-\infty}/F_n L_{-\infty} \to {}^\infty D_n$. Therefore for each n there is an injection $\operatorname{Gr}_n L_{-\infty} \to K_n$ where $K_n = \ker({}^\infty D_{n-1} \to {}^\infty D_n)$. In general the map $L_{-\infty} \to {}^\infty D_n$ need not be surjective (an element could be in the image of $i^{\circ r}$ for each finite r without being part of a consistent infinite sequence) although it is surjective in the special case when ${}^\infty D_s \to {}^\infty D_{s+1}$ is surjective for each s. In the latter case we would have $\operatorname{Gr} L_{-\infty} \cong K$.

Theorem 4.3.3 gives $\varprojlim_r{}^1 Z^r = 0$ as a sufficient condition that ${}^\infty D_s \to {}^\infty D_{s+1}$ be surjective for each s, where (Z^r) refers to the system of inclusions $\dots \subset Z^{r+1} \subset Z^r \subset Z^{r-1} \subset \dots$. Thus $\varprojlim_r{}^1 Z^r = 0$ is a sufficient condition for $\operatorname{Gr} L_{-\infty} \cong K$.

Taking into account the short exact sequence $0 \to D^\infty/i(D^\infty) \to E^\infty \to K \to 0$ coming from Theorem 4.3.3, the preceding discussion yields two obvious candidates for a suitable $\mathcal{F}_L$: $\mathcal{F}_{L_\infty}$ or $\mathcal{F}_{L_{-\infty}}$. In theory there are other possibilities, but in practice one of these two cases usually occurs. We examine them individually and see what additional conditions are required for convergence. Remember however that both L_∞ and $L_{-\infty}$ are defined in terms of D and that usually D is unknown; in fact, computation of D is usually the object of the exercise. Nevertheless, D is usually known in a theoretical sense — typically the exact couple was created from an abstractly defined filtration, and its properties may be known even if the exact object is unknown. In the most common examples, one has a filtration which is bounded below (there exists N below which the filtration is 0) in which case one hopes to prove convergence to $\mathcal{F}_{L_\infty}$

or the filtration is bounded above (there exists N such that $F_nX = X$ for $n \geq N$) in which case one hopes to prove convergence to $\mathcal{F}_{L_{-\infty}}$. The first case arises frequently when dealing with (generalized) homology, while the second frequently arises when dealing with (generalized) cohomology, assuming one treats cohomology as "negatively graded homology".

Case I: Conditions for convergence to $\mathcal{F}_{L_\infty}$

A direct limit is always cocomplete in its "canonical filtration", so $\mathcal{F}_{L_\infty}$ is always cocomplete. Therefore besides $\operatorname{Gr} L_\infty \cong E^\infty$ (equivalently $K = 0$) it is required to verify that $\mathcal{F}_{L_\infty}$ is complete. As we saw earlier, the completeness condition can be restated as $\cap_n F_n L_\infty = 0$ and $\varprojlim_n{}^1 F_n L_\infty = 0$. Assuming convergence to $\mathcal{F}_{L_\infty}$, the condition $K = 0$ shows that any nonzero element x of ${}^\infty D$ survives to give a nontrivial element of L_∞. However, being infinitely divisible, x will lie in $F_n L_\infty$ for all n, which, since, $\cap_n F_n L_\infty = 0$, contradicts $x \neq 0$. Thus convergence to $\mathcal{F}_{L_\infty}$ implies ${}^\infty D = 0$, so this is a necessary condition for convergence. Conversely, if ${}^\infty D = 0$ then, $K = 0$ and so the spectral sequence converges weakly to $\mathcal{F}_{L_\infty}$. The spectral sequence might not converge to $\mathcal{F}_{L_\infty}$. For example, suppose there is a nonzero element $[x] \in L_\infty$ having the property that for all t there exists y_t such that $x - i^t y$ lies in $\ker i^m$ for some m. Then $[x] = [i^t y_t]$ in L_∞ and so $[x] \in \cup_n F_n L_\infty$. However, if the reason why ${}^\infty D = 0$ is that $D_n = 0$ for all sufficiently small n (as in our motivating example), then $F_n L_\infty = 0$ for all sufficiently small n, and so the filtration is complete, and in particular the weak convergence of the spectral sequence becomes convergence.

Case II: Conditions for convergence to $\mathcal{F}_{L_{-\infty}}$

Any inverse limit is complete in its canonical filtration so $\mathcal{F}_{L_{-\infty}}$ is always complete and the issues are whether $\operatorname{Gr} L_{-\infty} \cong E^\infty$ and whether $\mathcal{F}_{L_{-\infty}}$ is cocomplete. $\mathcal{F}_{L_{-\infty}}$ is cocomplete if and only if every element of $L_{-\infty}$ lies in $\ker(L_{-\infty} \to D_n)$ for some n, for which a sufficient condition is that $L_\infty = 0$ or equivalently $E^\infty \cong K$. Therefore if the reason for the isomorphism $\operatorname{Gr} L_{-\infty} \cong E^\infty$ is that the maps $E^\infty \twoheadrightarrow K$ and $\operatorname{Gr} L_{-\infty} \rightarrowtail K$ are isomorphisms then the rest of the convergence conditions are automatic. In particular, to deduce convergence to $\mathcal{F}_{L_{-\infty}}$ it suffices to know that $L_\infty = 0$ and $\varprojlim_r{}^1 Z^r = 0$.

4.5 Multiplicative Spectral Sequences

Definition 4.5.1 *A* differential graded algebra *(DGA) is a graded algebra A together with a map $d : A_n \to A_{n-1}$ such that $d^2 = 0$ and*

$$d(xy) = d(x)y + (-1)^{|x|} x d(y).$$

It is easy to check that if (A, d) is a differential graded algebra then the multiplication on A induces a well defined multiplication on $H(A)$.

In many spectral sequences of interest, each E^r carries the additional structure of a differential algebra such that the multiplication on E^{r+1} is induced from that on E^r. In this case we say that the spectral sequence is *multiplicative*. If (E^r) is a multiplicative spectral sequence then there is an induced multiplication on E^∞.

If Q is an algebra, we say that a filtration $\mathcal{F}_Q$ on Q is *multiplicative* or an *algebra filtration* if $F_n(Q)F_k(Q) \subset F_{n+k}(Q)$ for all n and k. In this case there is an induced algebra structure on the associated graded object $\mathrm{Gr}_*(Q)$. To say that a multiplicative spectral sequence converges multiplicatively to a filtered algebra Q we impose the additional condition that the isomorphism $Gr_*(Q) \cong E^\infty$ be a ring homomorphism.

Massey has given the following necessary and sufficient condition, whose proof is straightforward, for the spectral sequence arising from an exact couple to be multiplicative.

Theorem 4.5.2 (Massey) *Let*

$$\begin{tikzcd}
D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\
& E \arrow[ul, "k"] &
\end{tikzcd}$$

be a graded exact couple in which E is a graded algebra. Suppose whenever elements $x,\ y \in E$ and elements $a,\ b \in D$ satisfy $k(x) = i^n(a)$ and $k(y) = i^n(b)$ for some integer n, that there is an element $c \in D$ such that $k(xy) = i^n(c)$ and $j(c) = j(a)y + (-1)^{|x|}xj(b)$. Then the differential on E turns E into a differential graded algebra, and so the multiplication on E induces one on E'. Furthermore the derived couple satisfies the preceding condition also and thus the resulting spectral sequence is multiplicative. □

The following is an example where Massey's conditions are easy to check. Using Massey's condition, we can check that a the spectral sequence associated to a filtered differential graded algebra is multiplicative.

Proposition 4.5.3 *Let $\mathcal{F}_A$ be a filtered differential graded algebra. Then the associated spectral sequence (abutting to $H_*(A)$) is multiplicative.*

Proof: Let x be an element of $E_{p+*} = H_*\big(F_p(A)/F_{p-1}(A)\big)$ and let y be an element of $E_{p'+*} = H_*\big(F_{p'}(A)/F_{p'-1}(A)\big)$. Suppose $k(x) = i^n(a)$ and $k(y) = i^n(b)$ where a belongs to $H_*\big(F_{p-1-n}(A)\big)$ and b belongs to $H_*\big(F_{p'-1-n}(A)\big)$. Pick representatives $\tilde{x} \in F_p(A)$, $\tilde{y} \in$

$F_{p'}(A)$, $\hat{a} \in F_{p-1-n}(A)$, $\hat{b} \in F_{p'-1-n}(A)$ in A for x, y, a, and b. Set $\tilde{a} := i^n(\hat{a})$ and $\tilde{b} := i^n(\hat{b})$. Then $d(\tilde{x}) = \tilde{a} + d(w)$ and $d(\tilde{y}) = \tilde{b} + d(w')$ for some $w \in F_{p-1}(A)$ and $w' \in F_{p'-1}(A)$, where d denotes the differential in A. In $F_{p+p'-1}(A)$ we have

$$\begin{aligned} d(\tilde{x}\tilde{y}) &= \tilde{a}\tilde{y} + (-1)^{|x|}\tilde{x}\tilde{b} + d(w)\tilde{y} + (-1)^{|x|}\tilde{x}d(w') \\ &= \tilde{a}\tilde{y} + (-1)^{|x|}\tilde{x}\tilde{b} + d(w\tilde{y}) + d(\tilde{x}w') - d(ww') + (-1)^{|x|}w\tilde{b} - \tilde{a}w'. \end{aligned}$$

Set $c = [\hat{a}\tilde{y} + (-1)^{|x|}\tilde{x}\hat{b} + (-1)^{|x|}w\hat{b} - \hat{a}w'] \in H_*\big(F_{p+p'-1-n}(A)\big)$. Then $k(xy) = i^n(c)$ and since $(-1)^{|x|}w\hat{b} - \hat{a}w'$ lies in filtration $p + p' - 2 - n < p + p' - 1$, $j(c) = j(a)y + (-1)^{|x|}xj(b)$. $\square$

4.6 Some examples of standard spectral sequences and their use

To this point we have considered the general theory of spectral sequences. The properties of the spectral sequences arising in many specific situations have been well studied. Usually the spectral sequence would be defined either directly, through an exact couple, or by giving some filtration on a chain complex. This defines the E^1-term. Typically a theorem would then be proved giving some formula for the resulting E^2-term. In many cases conditions under which the spectral sequence converges may also be well known.

In this section we shall take a brief look at the Serre Spectral Sequence, Atiyah-Hirzebruch Spectral Sequence, Spectral Sequence of a Double Complex, Grothendieck Spectral Sequence, Change of Ring Spectral Sequence, Bockstein Spectral Sequence and Eilenberg-Moore Spectral Sequence, and do a few sample calculations.

4.6.1 Serre Spectral Sequence

Let $F \to X \xrightarrow{\pi} B$ be a fibre bundle (or more generally a fibration) in which the base B is a CW-complex. Define a filtration on the total space by $F_nX := \pi^{-1}B^{(n)}$. This yields a filtration on $H_*(X)$ by setting $F_nH_*(X) := \operatorname{Im}\big(H_*(F_nX) \to H_*(X)\big)$. The spectral sequence coming from the exact couple in which $D^1_{p,q} := H_{p+q}(F_pX)$ and $E^1_{p,q} := H_{p+q}(F_pX, F_{p-1}X)$ is called the "Serre Spectral Sequence" of the fibration. Theorems from topology guarantee that this filtration is cocomplete and that $E^1_{p,q} = 0$ if either $p < 0$ or $q < 0$. Therefore the Serre Spectral Sequence is always a first quadrant spectral sequence converging to $H_*(X)$.

Theorem 4.6.1.1 (Serre) *In the Serre Spectral Sequence of the fibration $F \to E \to B$ there is an isomorphism $E^2_{p,q} \cong H_p\big(B; {}^tH_q(F)\big)$.*

Here ${}^tH_*(F)$ denotes a "twisted" or "local" coefficient system in which the differential is modified to take into account the action, coming from the fibration, of the fundamental groupoid of the base B on the fibre F. In the special case where B is simply connected and $\operatorname{Tor}\bigl(H_*(B), H_*(F)\bigr) = 0$ the Universal Coefficient Theorem says that the E^2-term reduces to $E^2_{p,q} \cong H_p(B) \otimes H_q(F)$.

The Serre Spectral Sequence for cohomology, $E_2^{p,q} \cong H^p\bigl(B; {}^tH^q(F)\bigr) \Rightarrow H^{p+q}(X)$, has the advantage that it is a spectral sequence of algebras which both greatly simplifies calculation of the differentials d_r which are restricted by the requirement that they satisfy the Leibniz rule with respect to the cup product on $H^*(B)$ and $H^*(F)$, and also allows the computation of the cup product on $H^*(X)$. Since it is a first quadrant spectral sequence, convergence is not an issue.

Frequently in applications of the Serre Spectral Sequence instead of using the spectral sequence to calculate $H_*(X)$ from knowledge of $H_*(F)$ and $H_*(B)$ it is instead $H_*(X)$ and one of the other two homologies which is known, and one is working backwards from the spectral sequence to find the homology of the third space.

Example 4.6.1.2 The universal S^1-bundle is the bundle $S^1 \to S^\infty \to \mathbb{C}P^\infty$ where S^∞ is contractible. We will calculate $H^*(\mathbb{C}P^\infty)$ from the Serre Spectral Sequence of this bundle, taking $H^*(S^1)$ and $H^*(S^\infty)$ as known. We also take as known that $\mathbb{C}P^\infty$ is path connected, so $H^0(\mathbb{C}P^\infty) \cong \mathbb{Z}$.

$$E_2^{p,q} \cong H^p(\mathbb{C}P^\infty) \otimes H^q(S^1) \cong \begin{cases} H^p(\mathbb{C}P^\infty) & \text{if } q = 0 \text{ or } 1; \\ 0 & \text{otherwise.} \end{cases}$$

E_∞ terms on the diagonal $p + q = n$ form a composition series for $H^n(S^\infty)$ which is zero for $n \neq 0$. Therefore $E_\infty^{p,q} = 0$ unless $p = 0$ and $q = 0$, with $E_\infty^{0,0} \cong \mathbb{Z}$. Because all nonzero terms lie in the first quadrant, the bidegrees of the differentials show that $d_r(E_2^{1,0}) = 0$ for all $r \geq 2$, so $0 = E_\infty^{1,0} = E_2^{1,0} = H^1(\mathbb{C}P^\infty)$. Since $E_2^{1,q} \cong E_2^{1,0} \otimes E_2^{0,q}$ it follows that $E_2^{1,q} = 0$ for all q. Taking into the account the known zero terms, the bidegrees of the differentials show that $E_3^{0,1} \cong \ker(d_2 : E_2^{0,1} \to E_2^{2,0})$ and $E_\infty^{0,1} = E_3^{0,1}$. Similarly $E_\infty^{2,0} = E_3^{2,0} \cong \operatorname{coker}(d_2 : E_2^{0,1} \to E_2^{2,0})$. Therefore the vanishing of these E_∞ terms shows that $d_2 : E_2^{0,1} \cong E_2^{2,0}$ and in particular $H^2(\mathbb{C}P^\infty) \cong E_2^{0,1} = H^1(S^1) \cong \mathbb{Z}$. It follows that $E_2^{2,q} \cong \mathbb{Z} \otimes E_2^{0,q} \cong E_2^{0,q}$ for all q. With the aid of the fact that we showed $E_2^{1,1} = 0$ we can repeat the argument used to show $E_2^{1,q} = 0$ for all q to conclude that $E_2^{3,q} = 0$ for all q. Repeating the procedure we inductively find that $E_2^{p,q} \cong E_2^{p-2,q}$ for all $p > 0$ and all q and in particular $H^n(\mathbb{C}P^\infty) \cong \begin{cases} \mathbb{Z} & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$

The cup products in $H^*(\mathbb{C}P^\infty)$ can also be determined by taking advantage of the fact that the spectral sequence is a spectral sequence of algebras. Let $a \in E_2^{2,0} \cong \mathbb{Z}$ be a generator and set

$x := d_2 a$. By the preceding calculation, d_2 is an isomorphism so x is a generator of $H^2(\mathbb{C}P^\infty)$. Therefore $x \otimes a$ is a generator of $E_2^{2,2}$ and the isomorphism d_2 gives that $d_2(x \otimes a)$ is a generator of $H^4(\mathbb{C}P^\infty)$. However $d_2(x \otimes a) = d_2(x \otimes 1)(1 \otimes a) = 0 \otimes 1 + (-1)^2(x \otimes 1)d_2 a = x^2 \otimes 1$ and thus x^2 is a generator of $H^4(\mathbb{C}P^\infty)$. Inductively it follows that x^n is a generator of $H^{2n}(\mathbb{C}P^\infty)$ for all n and so $H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[x]$.

Exercise 4.6.1 Use the fibration $\Omega S^n \to PS^n \to S^n$ to calculate $H^*(\Omega S^n)$ including its ring structure.

Exercise 4.6.2 Up to homotopy, the space $\mathbb{C}P^\infty$ deloops to give an "Eilenberg-MacLane space" $K(\mathbb{Z}, 3)$ defined by the property that

$$\pi_n\big(K(\mathbb{Z},3)\big) = \begin{cases} \mathbb{Z} & \text{if } n = 3; \\ 0 & \text{otherwise.} \end{cases}$$

Let $j : S^3 \to K(\mathbb{Z}, 3)$ be a generator of $\pi_3\big(K(\mathbb{Z},3)\big)$ and let $S^3\langle 3\rangle$ denote the homotopy-theoretic fibre of j. The long exact homotopy sequence of a fibration shows that

$$\pi_q(S^3\langle 3\rangle) = \begin{cases} \pi_q(S^3) & \text{if } q \neq 3 \\ 0 & \text{if } q = 3 \end{cases}$$

Use the Serre spectral sequence for the induced fibration $\mathbb{C}P^\infty \to S^3\langle 3\rangle \to S^3$ to calculate $H_*(S^3\langle 3\rangle)$. Applying the Hurewicz theorem to the results shows that $\pi_4(S^3) \cong \mathbb{Z}/(2\mathbb{Z})$.

When working backwards from the Serre or other first quadrant spectral sequences in which $E^2_{p,q} \cong E^2_{p,0} \otimes E^2_{0,q}$ the following analogue of the comparison theorem (4.4.3) is often useful.

Theorem 4.6.1.3 (Zeeman Comparison Theorem) *Let E and E' be first quadrant spectral sequences such that $E^2_{p,q} = E^2_{p,0} \otimes E^2_{0,q}$ and ${E'}^2_{p,q} = {E'}^2_{p,0} \otimes {E'}^2_{0,q}$. Let $f : E \to E'$ be a homomorphism of spectral sequences such that $f^2_{p,q} = f^2_{p,0} \otimes f^2_{0,q}$. Suppose that $f^\infty_{p,q} : E^\infty_{p,q} \to {E'}^\infty_{p,q}$ is an isomorphism for all p and q. Then the following are equivalent:*

1) $f^2_{p,0} : E^2_{p,0} \to {E'}^2_{p,0}$ is an isomorphism for $p \leq n-1$;

2) $f^2_{0,q} : E^2_{0,q} \to {E'}^2_{0,q}$ is an isomorphism for $q \leq n$.

There is a version of the Serre Spectral Sequence for generalized homology theories coming from the exact couple obtained by applying the generalized homology theory to the Serre filtration of X.

Theorem 4.6.1.4 (Serre Spectral Sequence for Generalized Homology)
Let $F \to X \to B$ be a fibration and let Y be an (unreduced) homology theory satisfying the Milnor wedge axiom. Then there is a (right half-plane) spectral sequence with $E^2_{p,q} \cong H_p\big(B; {}^tY_q(F)\big)$ converging to $Y_{p+q}(X)$.

Cocompleteness of the filtration follows from the properties of generalized homology theories satisfying the wedge axiom [14], and the rest of the convergence conditions are trivial since the filtration is 0 in negative degrees. Here, unlike the Serre Spectral Sequence for ordinary homology, the existence of terms in the 4th quadrant opens the possibility for composition series of infinite length, although in the case where B is a finite dimensional complex all the nonzero terms of the spectral sequence will live in the strip between $p = 0$ and $p = \dim B$ and so the filtrations will be finite.

The special case of the fibration $* \to X \to X$ yields what is known as the "Atiyah-Hirzebruch" Spectral Sequence.

Theorem 4.6.1.5 (Atiyah-Hirzebruch Spectral Sequence) *Let X be a CW-complex and let Y be an (unreduced) homology theory satisfying the Milnor wedge axiom. Then there is a (right half-plane) spectral sequence with $E^2_{p,q} \cong H_p\big(X; Y_q(*)\big)$ converging to $Y_{p+q}(X)$.*

Applying the spectral sequence comparison theorem (4.4.3) gives

Corollary 4.6.3 *Let Y and Y' be generalized cohomology theories. Suppose that there is a natural transformation $\eta : Y_* \to Y'_*$ such that $\eta_* : Y_*(*) \to Y_*(*)$ is an isomorphism. Then $\eta_X : Y_*(X) \to Y'_*(X)$ is an isomorphism for any CW-complex X.* □

In the cohomology Serre Spectral Sequence for generalized cohomology (including the cohomology Atiyah-Hirzebruch Spectral Sequence) convergence of the spectral sequence to $Y^*(X)$ is not guaranteed. Convergence to $\varprojlim_n Y^*(F_nX)$, should that occur, would be of the type discussed in Case II of Section 4.4. Since $X_n = \emptyset$ for $n < 0$, the system defining L_∞ stabilizes to 0. Therefore $L_\infty = 0$ and by the discussion of Section 4.4, $\varprojlim_r^1 Z_rX = 0$ becomes a sufficient condition for convergence to $\varprojlim_n Y^*(F_nX)$. However since the real object of study is usually $Y^*(X)$, the spectral sequence is most useful when one is also able to show $\varprojlim_n^1 Y^*(F_nX) = 0$ in which case the Milnor exact sequence [14]

$$0 \to \varprojlim_n{}^1 Y^*(F_n X) \to Y^*(X) \to \varprojlim_n Y^*(F_n X) \to 0$$

gives $Y^*(X) \cong \varprojlim_n Y^*(F_n X)$.

If $Y^*(\ )$ has cup products then the spectral sequence has the extra structure of a spectral sequence of $Y^*(*)$-algebras. In the case where B is finite dimensional all convergence problems disappear since the spectral sequence lives in a strip and the filtrations are finite.

Example 4.6.1.6 Let $K^*(\ )$ be complex K-theory. Since $K^*(*) \cong \mathbb{Z}[z, z^{-1}]$, with $|z| = 2$, in the Atiyah-Hirzebruch Spectral Sequence for $K^*(\mathbb{C}P^n)$ we have

$$E_2^{p,q} = \begin{cases} \mathbb{Z} & \text{if } q \text{ is even and } p \text{ is even with } 0 \le p \le 2n; \\ 0 & \text{otherwise.} \end{cases}$$

Because $\mathbb{C}P^n$ is a finite complex, the spectral sequence converges to $K^*(\mathbb{C}P^n)$. Since all the nonzero terms have even total degree and all the differentials have total degree $+1$, the spectral sequence collapses at E_2 and we conclude that $K^q(\mathbb{C}P^n) = 0$ if q is odd and that it has a composition series consisting of $(n+1)$ copies of $\mathbb{Z}$ when q is even. Since $\mathbb{Z}$ is a free abelian group, this uniquely identifies the group structure of $K^{\text{even}}(\mathbb{C}P^n)$ as $\mathbb{Z}^{n+1}$. To find the ring structure we can make use of the fact that this is a spectral sequence of $K^*(*)$-algebras. The result is $K^*(\mathbb{C}P^n) \cong K^*(*)[x]/(x^{n+1})$, where $|x| = 2$.

In the Atiyah-Hirzebruch Spectral Sequence for $K^*(\mathbb{C}P^\infty)$ again all the terms have even total degree so the spectral sequence collapses at E_2. We noted earlier that collapse of the spectral sequence implies that $\varprojlim_r{}^1 Z_r X = 0$ and so the spectral sequence convergences to $\varprojlim_n K^*(\mathbb{C}P^n)$, where we used $F_{2n}\mathbb{C}P^\infty = \mathbb{C}P^n$. Since our preceding calculation shows that $K^*(\mathbb{C}P^n) \to K^*(\mathbb{C}P^{n-1})$ is onto, Proposition 3.1.16 implies that $\varprojlim_n{}^1 K^*(\mathbb{C}P^n) = 0$. Therefore the spectral sequence converges to $K^*(\mathbb{C}P^\infty)$ and we find that $K^*(\mathbb{C}P^\infty) \cong \varprojlim_n K^*(\mathbb{C}P^n)$, which is isomorphic to the power series ring $K^*(*)[[x]]$, where $|x| = 2$.

In topology one might be interested in the Atiyah-Hirzebruch Spectral Sequence in the case where X is a spectrum rather than a space (a spectrum being a generalization in which cells in negative degrees are allowed including the possibility that the dimensions of the cells are not bounded below). In such cases the spectral sequence is no longer constrained to lie in the right half-plane and convergence criteria are not well understood for either the homology or cohomology version.

4.6.2 Spectral Sequence of a Double Complex

A double complex is a chain complex of chain complexes. That is, it is a bigraded abelian group $C_{p,q}$ together with two differentials $d' : C_{p,q} \to C_{p-1,q}$ and $d'' : C_{p,q} \to C_{p,q-1}$ satisfying $d' \circ d' = 0$, $d'' \circ d'' = 0$, and $d'd'' = d''d'$. Given a double complex C its total complex $\operatorname{Tot} C$ is defined by $(\operatorname{Tot} C)_n := \bigoplus_{p+q=n} C_{p,q}$ with differential defined by $d|_{C_{p,q}} := d' + (-1)^p d'' : C_{p,q} \to C_{p-1,q} \oplus C_{p,q-1} \subset \operatorname{Tot}_{n-1} C$.

There are two natural filtrations, $\mathcal{F}'_{\operatorname{Tot} C}$ and $\mathcal{F}''_{\operatorname{Tot} C}$ on $\operatorname{Tot} C$ given by

$$\big(F'_p(\operatorname{Tot} C)\big)_n = \bigoplus_{\substack{s+t=n \\ s\le p}} C_{s,t} \qquad \text{and} \qquad \big(F''_p(\operatorname{Tot} C)\big)_n = \bigoplus_{\substack{s+t=n \\ t\le p}} C_{s,t}$$

yielding two spectral sequences abutting to $H_*(\operatorname{Tot} C)$. In $E'_{p,q}$ we have

$$\big(F'_p(\operatorname{Tot} C)/F'_{p-1}(\operatorname{Tot} C)\big)_n = C_{p,n-p}$$

with the differential given by $d(x) = (-1)^p d''(x)$. So $F'_p(\operatorname{Tot} C)/F'_{p-1}(\operatorname{Tot} C)$ is isomorphic to $S^p C_{p,*}$, where S denotes the suspension functor on chain complexes defined by $(SC)_n = C_{n-1}$. Thus

$$E'^1_{p,q} = H_{p+q}\big(F_p(\ )/F_{p-1}(\ )\big) = H_{p+q}(S^p C_{p,*}) = H_q(C_{p,*}).$$

The map $d^1 = j^1 k^1$ is induced by d', so $E'^2_{p,q} = H_p\big(H_q(C_{*,*})\big)$. Similarly $E''^2_{p,q} = H_q\big(H_p(C_{*,*})\big)$.

Convergence of these spectral sequences is not guaranteed, although the first will always converge if there exists N such that $C_{p,q} = 0$ for $p < N$ and the second will converge if there exists N such that $C_{p,q} = 0$ for $q < N$.

The following is easy to check.

Proposition 4.6.4 *If one of the two differentials in a double complex is trivial, then the spectral sequence collapses at E^2.* □

In the important special case of a first quadrant double complex both spectral sequences converge and information is often obtained by playing one off against the other.

From the double complex C one could instead form the product total complex $(\operatorname{Tot}^\pi C)_n := \prod_{p+q=n} C_{p,q}$ and proceed in a similar manner to construct the same spectral sequences with different convergence problems.

Example 4.6.2.1 Let M and N be R-modules. Let $\operatorname{Tor}'^R_*(M,N)$ and $\operatorname{Tor}''^R_*(M,N)$ be the derived functors of $(\_) \otimes N$ and $M \otimes (\_)$ respectively. Let P_* be a projective resolution of M and let Q_* be a projective resolution of N. Define a first quadrant double complex by $C_{p,q} := P_p \otimes Q_q$. Since P_p is projective,

$$H_q(C_{p,*}) = P_p \otimes H_q(Q) = \begin{cases} 0 & \text{if } q \neq 0; \\ P_p \otimes N & \text{if } q = 0, \end{cases}$$

and so in the first spectral sequence of the double complex,

$${E'}^2_{p,q} = \begin{cases} 0 & \text{if } q \neq 0; \\ \operatorname{Tor}'^R_p(M,N) & \text{if } q = 0. \end{cases}$$

Therefore the spectral sequence collapses to give $H_n(\operatorname{Tot} C) \cong \operatorname{Tor}'^R_n(M,N)$. Similarly the second spectral sequence shows that $H_n(\operatorname{Tot} C) \cong \operatorname{Tor}''^R_n(M,N)$. Thus $\operatorname{Tor}^R_*(M,N)$ can be computed equally well from a projective resolution of either variable.

The technique of using a double complex in which one spectral sequence yields the homology the total complex to which both converge can be used to prove

Theorem 4.6.2.2 (Grothendieck Spectral Sequence) *Let* $\underline{\underline{C}} \xrightarrow{F} \underline{\underline{B}} \xrightarrow{G} \underline{\underline{A}}$ *be a composition of additive functors, where* $\underline{\underline{C}}$, $\underline{\underline{B}}$, *and* $\underline{\underline{A}}$ *are abelian categories. Assume that all objects in* $\underline{\underline{C}}$ *and* $\underline{\underline{B}}$ *have projective resolutions. Suppose that* F *takes projectives to projectives. Then for all objects* C *of* $\underline{\underline{C}}$ *there exists a (first quadrant) spectral sequence with* $E^2_{p,q} = (L_pG)\big((L_qF)(C)\big)$ *converging to* $\big(L_{p+q}(GF)\big)(C)$.

Proof: Since F takes projectives to projectives,

$$(L_nG)(FP) = \begin{cases} 0 & \text{if } n > 0; \\ G(FP) & \text{if } n = 0 \end{cases}$$

for any projective P of $\underline{\underline{C}}$.

Let P_* be a projective resolution of C in $\underline{\underline{C}}$. Define a double complex T in $\underline{\underline{B}}$ by letting $T_{*,q}$ be a projective resolution in $\underline{\underline{B}}$ of $F(P_q)$, with the d'-differential coming from this resolution. By Theorem 3.1.3 the maps $F(P_q) \to F(P_{q-1})$ yield maps $T_{0,q} \to T_{0,q-1}$ and any choices can be extended to chain maps $T_{*,q} \to T_{*,q-1}$ yielding the d''-differential. In fact, by the method of Theorem 3.1.6 it is always possible to construct the resolutions and maps in such a way that for each p the maps in the vertical direction are each the composition of a split epimorphism and a split monomorphism and so the homology in the vertical direction, $H_q(T_{p,*})$, forms a projective resolution of $H_q\big(F(P_*)\big) = (L_qF)(C)$; we will assume that such a choice has been made. In particular, any additive functor commutes with homology in the vertical direction.

The double complex $G(T)_{**}$ in $\underline{\underline{A}}$ is contained in the first quadrant, so both of its spectral sequences converge to $H_*\big(\operatorname{Tot} G(T)_{**}\big)$. In one spectral sequence we have

$$E^2_{p,q} = H_q\big(H_p(GT_{**})\big) = H_q\big((L_pG)T_{**}\big)$$
$$= H_q\left(\begin{cases} 0 & \text{if } p > 0; \\ G(T)_{0q} & \text{if } p = 0 \end{cases}\right) = \begin{cases} 0 & \text{if } p > 0; \\ L_q(GF)(C) & \text{if } p = 0. \end{cases}$$

Therefore $H_*\big(\operatorname{Tot} G(T)_{**}\big) = L_*(GF)(C)$. In the other spectral sequence, our choice of the maps $T_{*,q} \to T_{*,q-1}$ gives

$$E^2_{p,q} = H_p\big(H_q(GT_{**})\big) = H_p\big(G(H_qT_{**})\big) = (L_pG)\big((L_qF)(C)\big).$$

□

Naturally there is a corresponding version for right derived functors.

Example 4.6.5 An application of the Grothendieck spectral sequence is the following "Change of Rings Spectral Sequence".

Let $f : R \to S$ be a ring homomorphism, let M be a right S-module and let N be a left R-module. Let $F(A) = S \otimes_R A$ and $G(B) = M \otimes_S B$, and note that $GF(A) = M \otimes_R A$. Applying the Grothendieck spectral sequence to the composition

$$\text{Left } R\text{-Modules} \xrightarrow{F} \text{Left } S\text{-Modules} \xrightarrow{G} \text{Abelian Groups}$$

yields a convergent spectral sequence $E^2_{p,q} \cong \operatorname{Tor}^S_p\big(M, \operatorname{Tor}^R_q(S,N)\big) \Rightarrow \operatorname{Tor}^R_{p+q}(M,N)$.

Corollary 4.6.6 *Let $f : R \to S$ be a ring homomorphism and suppose that S is flat as an abelian group. Let M be a right S-module and let N be a left R-module. Then $\operatorname{Tor}^S_p\big(M, S \otimes_R N\big) \cong \operatorname{Tor}^R_p(M,N)$.*

Proof: Since S is flat

$$\operatorname{Tor}^R_q(S,N) = \begin{cases} 0 & \text{if } q > 0 \\ S \otimes_R N & \text{if } q = 0 \end{cases}$$

Thus the spectral sequences collapses to give the result.

It is also easy to give a direct proof. In general $(\ ) \otimes_R S$ takes projective modules to projective modules (because, for example, it has a right adjoint). Thus, since S is flat, tensoring a projective R-resolution with S yields a projective S-resolution. Combined with $M \otimes_S (S \otimes_R N) \cong M \otimes_R N)$, this gives the result. □

Similarly $\operatorname{Tor}^S_p\big(M \otimes_R S, N\big) \cong \operatorname{Tor}^R_p(M,N)$.

4.6.3 Bockstein Spectral Sequence

Let C be a chain complex of free abelian groups and let p be a positive integer. The short exact sequence of groups $0 \to \mathbb{Z}/(p\mathbb{Z}) \to \mathbb{Z}/(p^2\mathbb{Z}) \to \mathbb{Z}/(p\mathbb{Z}) \to 0$ yields a short exact sequence of chain complexes $0 \to C \otimes \mathbb{Z}/(p\mathbb{Z}) \to C \otimes \mathbb{Z}/(p^2\mathbb{Z}) \to C \otimes \mathbb{Z}/(p\mathbb{Z}) \to 0$. The boundary map $\beta_p : H_n(C; \mathbb{Z}/(p\mathbb{Z})) \to H_{n-1}(C; \mathbb{Z}/(p\mathbb{Z}))$ from the corresponding long exact homology sequence is called the $\mod p$ Bockstein. When p is clear from the context, it is dropped in the notation. The commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "p"] \arrow[d] & \mathbb{Z} \arrow[r, "\theta_p"] \arrow[d] & \mathbb{Z}/(p\mathbb{Z}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{Z}/(p\mathbb{Z}) \arrow[r, "p"] & \mathbb{Z}/(p^2\mathbb{Z}) \arrow[r] & \mathbb{Z}/(p\mathbb{Z}) \arrow[r] & 0 \end{tikzcd}$$

shows that β_p factors as the composition $H_*(C; \mathbb{Z}/(p\mathbb{Z})) \xrightarrow{\partial} H_{*-1}(C; \mathbb{Z}) \to H_{*-1}(C; \mathbb{Z}/(p\mathbb{Z}))$. There is a graded exact couple in which $D_s = H_s(C; \mathbb{Z})$ and $E_s = H_s(C; \mathbb{Z}/(p\mathbb{Z}))$ with the maps induced from the short exact sequence of groups $0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \longrightarrow \mathbb{Z}/(p\mathbb{Z}) \to 0$. The corresponding spectral sequence is called the $\mod p$ *Bockstein spectral sequence* of C. It is clear from the definitions that β is the d^1-differential of the Bockstein spectral sequence. The d^r-differential is written $\beta^{(r)}$ and is called the rth Bockstein modulo p.

From the definitions in Section 4.3, in the mod p Bockstein spectral sequence i^∞ is multiplication by p, $D^\infty = H_*(C)/(p\text{-torsion})$, $D^\infty/(i^\infty D^\infty) = (H_*(C)/\text{torsion}) \otimes \mathbb{Z}/(p\mathbb{Z})$, and ${}^\infty D = \{x \in H_*(C) \mid x \text{ is infinitely divisible by } p\}$. Therefore Theorem 4.3.3 yields,

Proposition 4.6.7 *If $H_*(C)$ has no infinitely p-divisible elements then*

$$E^\infty \cong (H_*(C)/\text{torsion}) \otimes \mathbb{Z}/(p\mathbb{Z})$$

in the mod p *Bockstein spectral sequence for $H_*(C)$. In particular this holds if $H_*(C)$ has finite type.* $\square$

From the definition of the differentials we get

Proposition 4.6.8

a) Let $x \in C_s \otimes \mathbb{Z}/(p\mathbb{Z})$ represent a homology class and let $y \in C_s \otimes \mathbb{Z}$ be any element such that $\theta_p(y) = x$, where θ_p denotes reduction modulo p. Then $\beta^{(j)}[x] = 0$ for $j < r$ if and only if y is divisible by p^r. If $\beta^{(j)}[x] = 0$ for $j < r$ then $\beta^{(r)}[x]$ is defined and is given by $[\theta_p(dy/p^r)]$.

b) The Bockstein spectral sequence of C collapses at E^r if and only if $H_*(C)$ has no elements of order p^m for $m \geq r$. □

For the rest of this subsection we will assume that p is a prime and let $(\ )_{(p)}$ denote localization at p. If $H_*(C)$ is finitely generated then $H_*(C)_{(p)}$ can be reconstructed from its Bockstein spectral sequence. Specifically, by the structure theorem for finitely generated modules over a PID,

$$H_n(C)_{(p)} = Z^s_{(p)} \oplus \mathbb{Z}/(p^{t_1}\mathbb{Z}) \oplus \mathbb{Z}/(p^{t_2}\mathbb{Z}) \oplus \ldots \oplus \mathbb{Z}/(p^{t_k}\mathbb{Z})$$

for some integers s, t_1, t_2, …, t_k. By the universal coefficient theorem and the preceding proposition, corresponding to a summand $Z_{(p)}$ there will be a basis element $x \in H_n(C;\mathbb{Z}/(p\mathbb{Z})) = E^1_n$ such that $\beta^{(r)}x = 0$ for all r, and corresponding to a summand $\mathbb{Z}/(p^t\mathbb{Z})$ there will be a pair of basis elements $x \in E^1_n$, $y \in E^1_{n+1}$ such that $\beta^{(r)}x = 0$ for all r, $\beta^{(r)}y = 0$ for $r < t$ and $\beta^{(t)}y = x$.

If $H_*(X)$ is finitely generated, it is common practice to describe $H_*(X)$ indirectly by exhibiting $H_*(X;\mathbb{Z}/(p\mathbb{Z}))$ together with a description of the mod p Bockstein spectral sequence of $H_*(X)$ for all p. As seen above, this completely determines $H_*(X)$ but often has the advantage that $H_*(X;\mathbb{Z}/(p\mathbb{Z}))$ is simpler and/or has more structure than $H_*(X)$. For example, one could determine $H_*(\mathbb{R}P^\infty)$ from the description $H_*(\mathbb{R}P^\infty;\mathbb{Z}/(p\mathbb{Z})) = 0$ for $p > 2$, $H_*(\mathbb{R}P^\infty;\mathbb{Z}/(2\mathbb{Z})) = \Gamma[x]$ where $|x| = 1$, and $\beta\big(\gamma_{2n}(x)\big) = \gamma_{2n-1}(x)$ for all n. (This describes the E^1-term and d^1-differential of the mod 2 Bockstein spectral sequence; normally one would have to go on to describe the higher Bocksteins but in this case $E^2 = H(E^1, d^1) = 0$ so the description is complete.) The advantage of this indirect description of $H_*(\mathbb{R}P^\infty)$ is that the H-space structure on $\mathbb{R}P^\infty$ induces a Hopf algebra structure on $H_*(\mathbb{R}P^\infty;F)$ for any field F so this description actually contains more information than $H_*(\mathbb{R}P^\infty)$.

The Bockstein spectral sequence is unusual in two respects:

1) it is the only spectral sequence in common use which is not naturally bigraded;

2) its primary use is not in computing some object to which it "converges" but rather in studying the p-torsion in $H_*(C)$ by considering the level of the spectral sequence at which various classes disappear. Although, as noted, convergence of the Bockstein spectral sequence is usually not if interest, one object to which it converges is $H_*(C)[1/p]$ filtered by the subgroup $\mathcal{F}_n = \{x \mid p^n x \in H_*(C)\}$. Here, $A[1/p]$ denotes the localization of A away from p, given by $A \otimes \mathbb{Z}[1/p]$ where $\mathbb{Z}[1/p] = \{x \in \mathbb{Q} \mid p^n x \in \mathbb{Z} \text{ for some } n\}$

Proposition 4.6.9 *If C is a differential graded algebra of free abelian groups then the Bockstein spectral sequence of C is multiplicative.*

Proof: The proof is similar to that of Proposition 4.5.3 with $c = [\tilde{a}\tilde{y} + (-1)^{|x|}\tilde{x}\tilde{b} + (-1)^{|x|}pw\tilde{b} - p\tilde{a}w']$ where $d\tilde{x} = p^n\tilde{a} + dw$ and $d\tilde{y} = p^n\tilde{b} + dw'$, for $[\tilde{x}]$, $[\tilde{y}] \in H_*(C)$. □

4.6.4 Eilenberg-Moore Spectral Sequence

For a topological group G, Milnor showed how to construct a universal G-bundle $G \to EG \to BG$ in which EG is the infinite join $G^{*\infty}$ with diagonal G-action. There is a natural filtration $F_n BG := G^{*(n+1)}/G$ on BG and therefore an induced filtration on the base of any principal G-bundle. This filtration yields a spectral sequence including as a special case a tool for calculating $H_*(BG)$ from knowledge of $H_*(G)$.

Theorem 4.6.4.1 *Let $G \to X \to B$ be a principal G-bundle and let $H_*(\ )$ denote homology with coefficients in a field. Then there is a first quadrant spectral sequence with $E^2_{p,q} = \mathrm{Tor}^{H_*(G)}_{pq}\big(H_*(X), H_*(*)\big)$ converging to $H_{p+q}(BG)$.*

Here the group structure makes $H_*(G)$ into an algebra and $\mathrm{Tor}^A_{pq}(M,N)$ denotes degree q of the graded object formed as the pth derived functor of the tensor product of the graded modules M and N over the graded ring A.

There is also a version [6] which, like the Serre Spectral Sequence, is suitable for computing $H^*(G)$ from $H^*(BG)$.

Theorem 4.6.4.2 *Let*

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d, "\pi"] \\ X \arrow[r, "f"] & B \end{tikzcd}$$

be a pullback square in which π is a fibration and X and B are simply connected. Suppose that $H^*(X)$, $H^*(Y)$, and $H^*(B)$ are flat R-modules of finite type, where $H^*(\ )$ denotes cohomology with coefficients in the Noetherian ring R. Then there is a (second quadrant) spectral sequence with $E_2^{p,q} \cong \mathrm{Tor}^{H^*(B)}_{pq}\big(H^*(X), H^*(Y)\big)$ converging to $H^{p+q}(W)$.

The cohomological version of the Eilenberg-Moore Spectral Sequence, stated above, contains the more familiar Tor for modules over an algebra. For the homological version one must dualize these notions appropriately to define the cotensor product of comodules over a coalgebra, and its derived functors cotor.

Provided the action of the fundamental group of B is sufficiently nice there are extensions of the Eilenberg-Moore Spectral Sequence to the case where B is not simply connected, although they do not always converge, and extensions to generalized (co)homology theories have also been studied.

Chapter 5

Simplicial Objects

5.1 Definitions

Definition 5.1.1 *A* simplicial $\underline{\underline{C}}$-object K *consists of a collection of objects* $\{K_n\}_{n\geq 0}$ *together with* $\underline{\underline{C}}$ *morphisms* $d_j : K_n \to K_{n-1}$ *for* $j = 0, \ldots, n$ *and* $s_j : K_n \to K_{n+1}$ *for* $j = 0, \ldots, n$ *satisfying the simplicial identities:*

$$\begin{array}{ll} d_i d_j = d_{j-1} d_i & \textit{for } \ i < j; \\ s_i s_j = s_{j+1} s_i & \textit{for } \ i \leq j; \\ d_i s_j = s_{j-1} d_i & \textit{for } \ i < j; \\ d_i s_j = 1_{K_n} & \textit{for } \ i = j, j+1; \\ d_i s_j = s_j d_{i-1} & \textit{for } \ i > j+1. \end{array}$$

A simplicial map $f : K \to L$ *is a collection of* $\underline{\underline{C}}$ *morphisms* $f_n : K_n \to L_n$ *which commutes with all the* d_j*'s and* s_j*'s. The maps* d_j *and* s_j *are called* boundary maps *and* degeneracy maps *respectively.*

Example 5.1.2 Let X be an object of a category $\underline{\underline{C}}$. Define the corresponding "constant simplicial object" in $\underline{\underline{C}}$ by $K_n = X$ for all n, and all the boundary and degeneracy maps are the identity 1_X.

Example 5.1.3 Let X be a set and let $K_n = \prod_{i=0}^{n} X = X^{n+1}$. Define $d_j : K_n \to K_{n-1}$ by

$$d_j\big((x_0, x_1, \ldots, x_n)\big) = (x_0, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$$

and $s_j : K_n \to K_{n+1}$ by

$$s_j\big((x_0, x_1, \ldots, x_n)\big) = (x_0, \ldots, x_{j-1}, x_j, x_j, x_{j+1}, \ldots, x_n).$$

Then the simplicial identities are satisfied, so we have a simplicial set (i.e. a simplicial object in the category of sets).

Let $\Delta^n \subset \mathbb{R}^n$ denote the "standard n-simplex", defined as the convex hull of the $n+1$ points $\epsilon_0 := (0,0,\ldots,0)$, $\epsilon_1 := (1,0,\ldots 0)$, $\epsilon_2 := (0,1,\ldots,0)$, $\ldots$, $\epsilon_n := (0,0,\ldots 1)$. For $0 \le j \le n$, let $\delta^j : \Delta^{n-1} \to \Delta^n$ be the inclusion of the jth face and let $\sigma_j : \Delta^{n+1} \to \Delta^n$ be the affine map determined by

$$\sigma_j(\epsilon_i) = \begin{cases} \epsilon_i & \text{if } i \le j; \\ \epsilon_{i-1} & \text{if } i > j. \end{cases}$$

(Thus σ_j collapses the jth face.) Given a simplicial set K, we can form a topological space $|K|$, called the *geometric realization* of K by $|K| = \coprod (K_n \times \Delta^n)/\!\sim$ where $(d_j x, t) \sim (x, \delta^j t)$ and $(s_j x, t) \sim (x, \sigma_j t)$. From the construction, $|K|$ has a natural CW-structure.

Conversely, to a topological space X we can associate a simplicial set $\operatorname{Sing}(X)$, called the "singular complex of X" by

$$\operatorname{Sing}(X)_n = \{\text{continuous functions from } \Delta^n \text{ to } X\}$$

with $d_j f = f \circ \delta^j$ and $s_j f = f \circ \sigma_j$.

Theorem 5.1.4 $\operatorname{Hom}_{\mathit{SimplicialSets}}(K, \operatorname{Sing} X) = \operatorname{Hom}_{\mathcal{T}op}(|K|, X)$. *That is, we have adjoint functors* $|\ | \dashv \operatorname{Sing}(\ )$. □

Example 5.1.5 Let $\underline{\underline{C}}$ be a small category. Define a simplicial set $\mathcal{N}\underline{\underline{C}}$, called the *nerve* of $\underline{\underline{C}}$, as follows. An element of $(\mathcal{N}\underline{\underline{C}})_n$ consists of a composition $C_n \xrightarrow{f_n} C_{n-1} \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_2} \to C_1 \xrightarrow{f_1} C_0$ of n morphisms in $\underline{\underline{C}}$. The boundary maps d_j delete the object C_j and compose in the missing spot to get the composition $C_n \xrightarrow{f_n} \ldots \to C_{j+1} \xrightarrow{f_{j-1} \circ f_j} C_{j+1} \ldots \xrightarrow{f_1} C_0$ in $(\mathcal{N}\underline{\underline{C}})_{n-1}$ for $1 \le j \le n-1$, while for $j = 0$ or $j = n$ we simply delete the object at the relevant end. The degeneracy s_j creates a longer composition by adding a duplicate copy of C_j with the identity map 1_{C_j} between the two copies. The simplicial identies are satisfied, so $\mathcal{N}\underline{\underline{C}}$ is a simplicial set.

It is easy to see that if I is the category coming from the ordered set $\big\{\{0,1\} \mid 0 \prec 1\big\}$ then $|\mathcal{N} I|$ is homeomorphic to the unit interval $I := [0,1]$.

We shall sometimes write $B\underline{\underline{C}}$ for $|\mathcal{N}\underline{\underline{C}}|$ and similarly if $F : \underline{\underline{C}} \to \underline{\underline{D}}$ is a functor between small categories we might write BF for $|\mathcal{N} F|$. The explanation for this notation will appear in Chapter 6.

Products exist in the category of simplicial sets. Let K and L be simplicial sets. Then the product simplicial set is given by $(K \times L)_n = K_n \times L_n$ with d_j and s_j acting componentwise.

Proposition 5.1.6 *If at least one of K, L has finite type (i.e. finitely many nondegenerate elements in each degree) then $|K \times L| \cong |K| \times |L|$.*

Proposition 5.1.7 *Let $F, G : \underline{\underline{C}} \to \underline{\underline{D}}$ be functors between small categories. A natural transformation $\eta : F \to G$ induces a homotopy $|F| \simeq |G|$.*

Proof: According to Proposition 1.1.4, the natural transformation corresponds to a functor $\underline{\underline{C}} \times I \to \underline{\underline{D}}$ and the realization of this functor gives the homotopy. □

Remark 5.1.8 An interesting feature of the preceding proposition is that although the homotopy relation is symmetric, the existence of a natural transformation is not. In other words, given $\eta : F \to G$ we get a homotopy $|F| \simeq |G|$. This homotopy can, of course, be reversed to give a homotopy $|G| \simeq |F|$ even though the latter homotopy is not necessarily the geometric realization of any natural transformation $G \to F$.

Corollary 5.1.9 *If the small category $\underline{\underline{C}}$ has either an initial object or a terminal object then $|\mathcal{N}\,\underline{\underline{C}}|$ is contractible.*

Proof: Example 1.2.4 gives the contracting homotopy. □

Let A be a simplicial abelian group. Define $d\colon A_n \to A_{n-1}$ by $d = \sum_{j=0}^{n}(-1)^j d_j$. It is easy to check that $d^2 = 0$, so (A, d) forms a chain complex.

Given a simplical set K, let $\mathbb{Z}K$ be the simplicial abelian group in which $(\mathbb{Z}K)_n$ is the free abelian group on the set K_n with the face and degeneracy maps induced from those in K. We define the homology of the simplicial set K by $H_*(K) := H_*(\mathbb{Z}K, d)$.

It is clear from the definition of singular homology that $\mathbb{Z}\,\mathrm{Sing}(X)$ is the singular chain complex of X.

Theorem 5.1.10 *The natural transformation $K \to \mathrm{Sing}\,|K|$ coming from the adjunction induces a homology isomorphism $H_*(\mathbb{Z}K, d) \cong H_*(\mathrm{Sing}\,|K|) = H_*(|K|)$ for all K.*

Definition 5.1.11 *Let K be a simplicial $\underline{\underline{C}}$-object, let C belong to* Obj $\underline{\underline{C}}$. *Let $\epsilon : K_0 \to C$ be an "augmentation" of K satisfying $\epsilon \circ d_0 = \epsilon \circ d_1 : K_1 \to C$ (equivalent to giving a morphisms of simplicial sets from K to the constant simplicial set on C). A* conical contraction *of K to C is defined as follows. Conventionally set $K_{-1} := C$ and $d_0 := \epsilon : K_0 \to K_{-1}$. A conical contraction consists of an extra degeneracy $s_{-1} : K_{n-1} \to K_n$ for $n \geq 0$ satisfying the following extension of the simplicial identities:*

1. *$s_{-1}s_j = s_{j+1}s_{-1}$ for $-1 \leq j \leq n$,*

2. $d_0 s_{-1} = 1$,

3. $d_j s_{-1} = s_{-1} d_{j-1}$ *for* $j > 0$.

Example 5.1.12 Suppose $\underline{\underline{X}}$ is a small category with a terminal object Z. Define an augmentation $\epsilon : \mathcal{N}\underline{\underline{C}} \to \{Z\}$ by setting $\epsilon(C_0) := Z$. Define $s_1 : \mathcal{N} C_n \to \mathcal{N} C_{n+1}$ by

$$s_{-1}(C_n \to \ldots \to C_1 \to C_0) := C_n \to \ldots \to C_1 \to C_0 \to Z$$

where $C_0 \to Z$ is the unique morphism from C_0 to Z. Then s_{-1} is a conical contraction of $\mathcal{N}\underline{\underline{C}}$ to the point $\{Z\}$. This gives another proof of the statement in Corollary 5.1.9 that the realization of the nerve of a category with a terminal object is contractible.

Lemma 5.1.13 *Let A be a simplicial abelian group with an augmentation $\epsilon : A_0 \to C$ and a conical contraction to C. Let $\underline{C}$ be the chain complex consisting of C in degree 0 and 0 in all other degrees. Then ϵ induces a chain homotopy equivalence $A \simeq \underline{C}$.*

Proof: Define $\underline{\epsilon} : (A, d) \to \underline{C}$ by $\underline{\epsilon_0} = \epsilon : A_0 \to C$ and, perforce, $\underline{\epsilon_n} = 0$ for $n \neq 0$. Since $\epsilon_0 \circ d = \epsilon \circ d_0 - \epsilon \circ d_1 = 0 : A_1 \to \underline{C}_0$, and the other squares trivially commute, $\underline{\epsilon}$ is a chain map. Define $\sigma : \underline{C} \to (A, d)$ by $\sigma_0 := s_{-1}C \to A_0$ and, perforce, $\sigma_n = 0$ for $n \neq 0$. Trivially, σ is a chain map.

Our assumptions on s_{-1} include $\epsilon \circ s_{-1} = d_0 s_{-1} = s_{-1} = 1_C$, so $\underline{\epsilon} \circ \sigma = 1_{\underline{C}}$.

For $n > 0$, set $s := s_{-1} : A_n \to A_{n+1}$. For $n > 0$,

$$\begin{aligned}
ds + sd &= \sum_{j=0}^{n+1} (-1)^j d_j s^{-1} + \sum_{j=0}^{n} (-1)^j s_{-1} d_j \\
&= 1_{A_n} + \sum_{j=1}^{n+1} (-1)^j d_j s^{-1} + \sum_{j=0^n} (-1)^j s_{-1} d_j \\
&= 1_{A_n} + \sum_{j=1}^{n+1} (-1)^j s_{-1} d_{j-1} + \sum_{j=0}^{n} (-1)^j s_{-1} d_j \\
&= 1_{A_n} - \sum_{j=0}^{n} (-1)^j s_{-1} d_j + \sum_{j=0}^{n} (-1)^j s_{-1} d_j \\
&= 1_{A_n}
\end{aligned}$$

For $n = 0$, the chain complex (A, d) does not include our conventional $d_0 : A_0 \to C$ and we get

$$\begin{aligned} ds + sd &= d_0 s^{-1} - d_1 s^{-1} + 0 \\ &= 1_{A_0} - s_{-1} d_0 = 1_{A_0} - \sigma\epsilon \end{aligned}$$

Thus s is a chain homotopy $1_A \simeq \sigma \circ \underline{\epsilon}$. □

Remark 5.1.14 Let K be a simplicial set with a conical contraction to a point. Applying the preceding to the singular complex of $|K|$ shows that $H_*(|K|) = 0$, but in fact $|K| \simeq *$ and if K is a "Kan complex" then the stronger statement that $K \simeq *$ as simplicial sets holds. In particular, this applies to any simplical group, including non-abelian ones, since a simplicial group is always a Kan complex.

5.2 Triples and Cotriples

Let $\underline{\underline{C}}$ and $\underline{\underline{D}}$ be categories and let $F : \underline{\underline{C}} \to \underline{\underline{D}}$ and $G : \underline{\underline{D}} \to \underline{\underline{C}}$ be functors such that $F \dashv G$. Set $S = G \circ F : \underline{\underline{C}} \to \underline{\underline{C}}$ and let $T = F \circ G : \underline{\underline{D}} \to \underline{\underline{D}}$. Define $\mu : S \circ S \to S$ be the natural transformation $\mu_X = G(\beta_{F(X)})$, where $\beta : FG \to I_{\underline{\underline{D}}}$ is the natural transformation defined in Section 1.5. Then letting $\alpha : I_{\underline{\underline{C}}} \to GF$ be as in Section 1.5, $\mu_X \circ \mu_{S(X)} = \mu_X \circ S(\mu_X) : S^3(X) \to S(X)$, $\alpha_{S(X)} \circ \alpha_X = S(\alpha_X) \circ \alpha_X : X \to S^2(X)$, $\mu_X \circ \alpha_{S(X)} = 1_{S(X)}$, and $\mu_X \circ S\alpha_X = 1_{S(X)}$, for every object X of $\underline{\underline{C}}$.

Definition 5.2.1 *Let $\underline{\underline{A}}$ be a category. A* triple *or* monad *(S, μ, η) on $\underline{\underline{A}}$ consists of a functor $S : \underline{\underline{A}} \to \underline{\underline{A}}$ together with natural transformations $\mu : S^2 \to S$ and $\eta : I_{\underline{\underline{A}}} \to S$ such that $\mu \circ \mu S = \mu \circ S\mu : S^3 \to S$, $\eta S \circ \eta = S\eta \circ \eta : I_{\underline{\underline{A}}} \to S^2$, and $\mu \circ \eta S = \mu \circ S\eta = 1_S : S \to S$.*

A cotriple *(T, ψ, ϵ) on $\underline{\underline{A}}$ consists of a functor $T : \underline{\underline{A}} \to \underline{\underline{A}}$ together with natural transformations $\psi : T \to T^2$ and $\epsilon : T \to I_{\underline{\underline{A}}}$, such that $\psi T \circ \psi = T\psi \circ \psi : T \to T^3$, $\epsilon \circ \epsilon T = \epsilon \circ T\epsilon : T^2 \to I_{\underline{\underline{A}}}$, and $T\epsilon \circ \psi = \epsilon T \circ \psi = 1_T : T \to T$.*

An adjoint pair $F \dashv G$ gives rise to a triple $S := GF$ on $\underline{\underline{C}}$ and a cotriple $T := FG$ on $\underline{\underline{D}}$.

Let T be a cotriple on a category $\underline{\underline{D}}$. For each object X of $\underline{\underline{D}}$ we construct a simplicial set $T_\bullet X$ as follows. Set $(T_\bullet X)_n = T^{n+1} X$ and write $T_n X$ for $(T_\bullet X)_n$. For $j = 0, \ldots, n$, define maps $d_j : T_n X \to T_{n-1} X$ by $d_j = T^{n-j}(\epsilon_{T^j X})$ and $s_j : T_n X \to T_{n+1} X$ by $s_j = T^{n-j}\psi_{T^j X}$. These maps satisfy the simplicial identities, so $T_\bullet X$ becomes a simplicial object in the category $\underline{\underline{D}}$ which comes with a natural augmentation $\epsilon_X : T_0 X \to X$. In particular, if $\underline{\underline{D}}$ is an abelian category, we get a chain complex $(T_\bullet X, d)$ for each object X of $\underline{\underline{D}}$.

Suppose now that the cotriple comes with a natural transformation $\eta : I_{\underline{\underline{D}}} \to T$ having the property that $\epsilon \circ \eta = 1_{\underline{\underline{D}}}$. Set $s_{-1} := T^{n+1}\eta : T_n \to T_{n+1}$. Then s_{-1} forms a conical contraction to X on $T_\bullet X$ and in thus $(T_\bullet X, d) \simeq \underline{X}$ in this case, where $\underline{X}$ denotes the chain complex consisting of X concentratred in degree 0. In particular, if the cotriple T arises from an adjoint pair $F \dashv G$, then on any object $X = F(A)$ in the image of F, the composition $F \circ \alpha_A : X = F(A) \to FGF(A) = T(X)$ gives a natural transformation $\eta : I_{\operatorname{Im} F} \to T|_{\operatorname{Im} F}$, and so $(T_\bullet X, d) \simeq \underline{X}$ wheneven $X = F(A)$ for some A.

In the case where $\underline{\underline{D}}$ is an abelian category we plan to use this process to produce projective resolutions in $\underline{\underline{D}}$. Note that since the coadjoint F always preserves projectives, if G happens to take values in projective objects of $\underline{\underline{C}}$, then $(T_\bullet X, d) \simeq \underline{X}$ will be a projective resolution of $H_0(T_\bullet X, d) = H_0(\underline{X}) = X$ in the category $\underline{\underline{D}}$.

Remark 5.2.2 This section contains the motivation for attempts to generalize the key notions of homological algebra, such as derived functors, to nonabelian categories. If T takes values among $\underline{\underline{D}}$-projectives, one replaces the notion of projective resolution by the simplicial $\underline{\underline{D}}$-object $T_\bullet X$. An interesting special case is where we wish to consider derived funtors of some functor $Q : \underline{\underline{D}} \to \underline{\underline{E}}$ where $\underline{\underline{E}}$ is an abelian category, but $\underline{\underline{D}}$ is not. Although our analogue of the projective resolution produces a simplicial set in the nonabelian category $\underline{\underline{D}}$, taking derived functors of Q requires applying Q to the resolution which gives a simplicial object in the abelian category $\underline{\underline{E}}$ from which we get a chain complex in the standard sense.

Furthermore, even if T does not take values among projectives, we can define a "projective class" (a collection of objects satisfying the lifting property with respect to a class of morphisms satisfying certain axioms analogous to those satisfied by the collection of epimorphism) and under suitable conditions the image of T might form a projective class in which case we can do "relative homological algebra" relative to this projective class.

Chapter 6

Group Homology and Cohomology

6.1 Representations

Let G be a group and let K be a commutative ring.

A (linear) *representation* of G consists of a K-module V and an action $G \times V \mapsto V$ satisfying

$$g \cdot (av + bw) = a\, g \cdot v + b\, g \cdot w \quad \forall g \in G,\ a, b \in K,\ v, w \in W.$$

Equivalently, a rep. is a group homomorphism $G \mapsto \mathrm{Aut}_K(V)$.

Another formulation: Define a ring $K[G]$, called the *group ring*, as follows. As an abelian group,

$$K[G] = \{\text{free } K\text{-module with basis } G\}.$$

Multiplication is determined by $g \cdot h = gh$ (the left defines multiplication in $K[G]$; the right is multiplication in G). Then a rep. of G on V is a ring homomorphism $K[G] \mapsto \mathrm{End}_K(V)$. This makes V a left $K[G]$-module.

Note that as rings,

$$K[G \times H] = K[G] \otimes_{\mathbb{Z}} K[H].$$

$K[G]$ is commutative if and only if G is abelian.

Yet another formulation of represention:

To the group G we associate a category $\underline{\underline{\underline{BG}}}$ as follows. $\underline{\underline{\underline{BG}}}$ has only one object labelled $*$. The set of morphisms $\underline{\underline{\underline{BG}}}(*, *)$ is defined to be G with composition of morphisms given by multiplication within G. (The explanation of the notation $\underline{\underline{\underline{BG}}}$ will appear in section 6.5.)

The preceding paragraph contains an alternative approach to the definition of a group. That is, instead of defining a group as a set with a binary operation satisfying certain axioms, a group can be defined as a category with one object in which every morphisms is an isomorphism.

(More generally, a category in which every morphism is an isomorphism is called a *groupoid*.) A representation of G corresponds to a functor $\underline{\underline{BG}} \to K$-modules.

The equivalence of the various approaches above can be summarized by saying that there are equivalences of categories:

$$\begin{aligned} G\text{-representations on } K\text{-vector-spaces} &\simeq K[G]\text{-modules} \\ &\simeq \text{the category of functors from } \underline{\underline{BG}} \text{ to } K\text{-modules.} \end{aligned}$$

Two extreme cases of group actions are as follows

1. Trivial action:

   $g \cdot v = v$ for all $g \in G$ and $v \in V$.

2. Free action:

   $g \cdot v = v$ only if g is the identity e of G.

Let ρ be a representation of G on a K-module A. Define the *invariant submodule* A^G of A by $A^G := \{a \in A \mid g \cdot a = a \text{ for all } g \in G \text{ and } a \in A\}$. Define the *coinvariant quotient* A_G of A by $A_G := A/\sim$ where $g \cdot a \sim a$ for all $g \in G$ and $a \in A$. The association $A \mapsto A^G$ is a contravariant functor from $K[G]$-modules to K-modules, and the association $A \mapsto A^G$ is a covariant functor from $K[G]$-modules to K-modules.

Proposition 6.1.1 *Let $T : K$-modules $\to K[G]$-modules be the "trivial module" functor which associates to the module A the representation consisting of A with trivial G-action.*

The invariant subgroup functor $A \mapsto A^G$ is right adjoint to the trivial module functor.

The coinvariant quotient functor $A \mapsto A^G$ is left adjoint to the trivial module functor.

Proof: Let A be a $K[G]$-module and let X be a K-module.

1. Given a $K[G]$-module map $f : TX \to A$, since the action on TX is trivial, the image of f lands in A^G, so f can be regarded as an element of $\mathrm{Hom}_K(X, A^G)$. Conversely, if $f \in \mathrm{Hom}_K(X, A^G)$, the composition $X \xrightarrow{f} A^G \hookrightarrow A$ gives a corresponding element of $\mathrm{Hom}_{K[G]}(TX, A)$. These are inverse processes.

2. Given a $K[G]$-module map $f : A \to TX$, the trivial action on TX forces $f(g \cdot a) = f(a)$ for all $a \in A$ and $g \in G$. This produces an induced map $\bar{f} : A_G \to X$ such that f factors as the composition $A \twoheadrightarrow A_G \xrightarrow{\bar{f}} TX$. Conversely, given $\bar{f} : A_G \to X$, defining f as the composition $A \twoheadrightarrow A_G \xrightarrow{\bar{f}} TX$ gives a corresponding element of $\mathrm{Hom}_{K[G]}(A, TX)$. These are inverse processes.

$\square$

It follows from the preceding proposition that $A \mapsto A^G$ is a left exact functor and that $A \mapsto A_G$ is a right exact functor. We define $H_n(G;A)$, the *group homology of G with coefficients in A*, by setting $H_n(G;A)$ to be the nth left derived funtor of $A \mapsto A_G$. We define $H^n(G;A)$, the *group homology of G with coefficients in A*, by setting $H^n(G;A)$ to be the nth right derived funtor of $A \mapsto A^G$.

Write simply K for the $K[G]$-module TK.

Proposition 6.1.2

1. $A_G \cong K \otimes_{K[G]} A$
2. $A^G \cong \mathrm{Hom}_{K[G]}(K, A)$

Proof: The second statement is precisely the adjointness in the second statement of Proposition 6.1. The first follows from the first statement of Proposition 6.1 since it says that both functors have the same left adjoint. (It is also trivial to check the statment directly from the definitions.) □

Corollary 6.1.3

1. $H_*(G;A) \cong \mathrm{Tor}_*^{K[G]}(K, A)$.
2. $H^*(G;A) \cong \mathrm{Ext}^*_{K[G]}(K, A)$.

□

The equations in this corollary are sometimes treated as the definitions of group homology and cohomology.

The right and left exactness of $A \mapsto A_G$ and $A \mapsto A^G$ give $H_0(G;A) = A_G$ and $H^0(G;A) = A^G$.

The preceding Corollary tells us that $H_*(G;A)$ and $H^*(G;A)$ can be computed using a projective $K[G]$-resolution of the trivial $K[G]$-module K. One way to get a projective $K[G]$-resolution of the trivial $K[G]$-module K is to begin with a projective $\mathbb{Z}[G]$-resolution of the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$ and apply the change of rings functor $K \otimes_{\mathbb{Z}} (\ )$. This means that the relationship between $\mathrm{Tor}_*^{K[G]}(K, A)$ and $\mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}, A)$ is given by the universal coefficient theorem, and $\mathrm{Ext}_*^{K[G]}(K, A)$ is similarly related to $\mathrm{Ext}_*^{\mathbb{Z}[G]}(\mathbb{Z}, A)$. Since this tells us how to obtain the general case from the special case $K = \mathbb{Z}$, for the rest of this chapter we will assume that $K = \mathbb{Z}$.

Remark 6.1.4 The preceding should not be interpreted as saying that the coefficient ring K plays no role in representation theory — this is far from true. For example, if G is a finite group then the represenation theory in the case where the characteristic of K divides the order of G looks significantly different, and cannot readily be obtained, from that in the case where the characteristic of K does not divide the order of the group. If K is a field of characteristic 0, then $K[G]$ is a semisimple ring and the representation theory of finite groups in this case is well understood. In characteristic p there are not as many tools, and, for example, there are many basic problems concerning the representation theory of the symmetric group which were solved long ago when the coefficient ring K is a field of characteristic zero but are major unsolved problems when K is a field of characteristic p. Perhaps one way to look at this is that there is a lot more to representation theory than the information contained in group (co)homology.

Proposition 6.1.5 *Let $\phi : G \to L$ be a group homomorphism and let A be a $\mathbb{Z}[L]$-module. Then* $\operatorname{Tor}_*^{\mathbb{Z}[L]}(\mathbb{Z}[L] \otimes_{\mathbb{Z}[G]} \mathbb{Z}; A) \cong H_*(G; A)$

Proof: Since $\mathbb{Z}[L]$ is a free, thus flat, abelian group for any L, the result follows immediately form 4.6.6.

□

6.2 G-bundles

To put the preceding section in perspective we discuss the topological approach to group homology and cohomology. The section is intended for intuition and motivation. The material will be surveyed without proofs.

In this section, let $K = \mathbb{Z}$ and, to avoid confusion with the topological versions, we will write $H_*^{\text{alg}}(G)$ and $H^*_{\text{alg}}(G)$ for $H_*(G; \mathbb{Z})$ and $H^*(G; \mathbb{Z})$ as defined in the preceding section.

One might hope that to a group G we could associate some topological space X having the property that $H_*(X) = H_*^{\text{alg}}(G)$ and $H^*(X) = H^*_{\text{alg}}(G)$. The Eilenberg-Moore spectral sequence suggests that X should have the property that $\Omega X \simeq G$. Such a space X exists. It is written BG and goes by the name of "the classifying space of G". To see what it classifies, we have to consider G-bundles. Since this discussion holds for any topological group, not just those with the discrete topology, in this section we will let G denote an arbitrary topological group.

A *numerable cover* of a topological space X is one which possesses a partition of unity.

Definition 6.2.1 Let B be a topological space with chosen basepoint $*$. A (*locally trivial*) *fibre bundle* over B consists of a map $p : E \to B$ such that for all $b \in B$ there exists an open neighbourhood U of b for which there is a homeomorphism $\phi : p^{-1}(U) \to p^{-1}(*) \times U$ satisfying $\pi'' \circ \phi = p|_U$, where π'' denotes projection onto the second factor. If there is a numerable open cover of B by open sets with homeomorphisms as above then the bundle is said to be *numerable.*

For any spaces F and B, there is a product bundle $\pi_2 : F \times B \to B$. A bundle which is isomorphic to a such a product is called a *trivial bundle.*

Definition 6.2.2 Let G be a topological group and B a topological space. A *principal G-bundle* over B consists of a fibre bundle $p : E \to B$ together with an action $G \times E \to E$ such that:

1) the "shearing map" $G \times E \to E \times E$ given by $(g, x) \mapsto (x, g \cdot x)$ maps $G \times E$ homeomorphically to its image;

2) $B = E/G$ and $p : E \to E/G$ is the quotient map;

3) for all $b \in B$ there exists an open neighbourhood U of b such that $p : p^{-1}(U) \to U$ is G-bundle isomorphic to the trivial bundle $\pi'' : G \times U \to U$. That is, there exists a homeomorphism $\phi : p^{-1}(U) \to G \times U$ satisfying $p = \pi'' \circ \phi$ and $\phi(g \cdot x) = g \cdot \phi(x)$, where $g \cdot (g', u) = (gg', u)$.

The shearing map is injective if and only if the action is free, so by condition (1), the action of G on the total space of a principal bundle is always free. If G and E are compact, then of course a free action suffices to satisfy condition (1). In general, a free action produces a well defined "translation map" $\tau : Q \to G$, where $Q = \{(x, g \cdot x) \in X \times X\}$ is the image of the shearing function. Condition (1) is equivalent to requiring a free action with a continuous translation function.

Given a bundle $\xi = \pi : E \to B$ and a map $f : B' \to B$, the map from the pullback (in the category of topological spaces) of π and f to B' forms a bundle over B', denoted $f^*(\xi)$.

Theorem 6.2.3 *Let G be a topological group and let ξ be a numerable principal G-bundle over a space B. Suppose $f \simeq h : B' \to B$. Then $f^*(\xi) \cong h^*(\xi)$.*

Corollary 6.2.4 *Any bundle over a contractible base is trivial.*

Definition 6.2.5 A numerable principal G-bundle γ over a pointed space $\tilde{B}$ is called a *universal G-bundle* if:

1) for any numerable principal G-bundle ξ there exists a map $f : B \to \tilde{B}$ from the base space B of ξ to the base space $\tilde{B}$ of γ such that $\xi = f^*(\gamma)$;

2) whenever f, h are two pointed maps from some space B into the base space $\tilde{B}$ of γ such that $f^*(\gamma) \cong h^*(\gamma)$ then $f \simeq h$.

In other words, a numerable principal G-bundle γ with base space $\tilde{B}$ is a universal G-bundle if, for any pointed space B, pullback induces a bijection from the homotopy classes of maps $[B, \tilde{B}]$ to isomorphism classes of numerable principal bundles over B. If $p : E \to B$ and $p' : E \to B'$ are both universal G-bundles for the same group G then the properties of universal bundles produce maps $\phi : B \to B'$ and $\psi : B' \to B$ such that $\psi \circ \phi \simeq 1_B$ and $\phi \circ \psi = 1_{B'}$. It follows that the universal G-bundle (should it exist) is unique up to homotopy equivalence.

The first construction of universal G-bundle was given by Milnor, and is known as the Milnor construction. We will later be describing a different construction of a universal G-bundle. It will give a different topological space from Milnor as the base, but of course, they must be homotopy equivalent, according to the preceding discussion.

Theorem 6.2.6

1. *If $EG \to EG/G$ is a universal G-bundle then EG must be a contractible space on which G acts freely.*

2. *Let EG be a contractible space with a numerable cover and a free action of G. Then the quotient map $EG \to EG/G$ is a universal G-bundle.*

That is, a principal bundle $EG \to EG/G$ is a universal bundle if and only if G is contractible.

Theorem 6.2.7 (Milnor) *For every topological group G, there exists a contractible space EG with a numerable cover and a free action of G.*

Given a universal bundle $EG \to EG/G$, set $BG := EG/G$. According to the preceding discussion this base space always exists and depends, up to homotopy equivalence, only on G and not upon the choice of contractible total space EG. It is called the "classifying space" of the group G since it classifies principal G-bundles in the sense of the following theorem:

Theorem 6.2.8 *Given any topological group, there exists a classifying space BG, unique up to homotopy equivalence, and a universal bundle $EG \to BG$ having the property that for any space B, pullback sets up a bijection between $[B, BG]$ and isomorphism classes of principal G-bundles over B.*

Henceforth we shall use the notation BG to denote any space which is the base space of some universal G bundle, Thus BG denotes a homotopy type. Any particular topological space which has the homotopy type BG could be called a model for the classifying space, but this distinction will not usually be important to us.

Proposition 6.2.9 $\Omega BG \simeq G$

Theorem 6.2.10 *In the case where G is a discrete group (i.e an object in the category of groups to be treated as an object in the category of topological groups by assigning it the discrete topology) $H_*(BG) \cong H_*^{\mathrm{alg}}(G)$ and $H^*(BG) \cong H^*_{\mathrm{alg}}(G)$.*

Note that there is a slight conflict in notation. In order to discuss BG we have to treat G as a topological group even though its assigned topology is trivial. As a topological space, the topological $H_*(G)$ and $H^*(G)$ already have a meaning, although they are 0 above degree 0 since as a topological space, G is an isolated collection of points. Thus when one writes $H_*(G)$ and $H^*(G)$, one generally means what we have been writing in this section as $H_*^{\mathrm{alg}}(G)$ and $H^*_{\mathrm{alg}}(G)$, since the other meaning of the notation is not a very interesting object to consider.

The preceding discussion makes sense for any topological group and in general BG can be a complicated and interesting space. Returning the original case where G is a discrete group, the resulting space BG becomes the Eilenberg-Mac Lane space $K(G,1)$ characterized by the fact that it has a unique nonzero homotopy group consisting of G in degree 1.

Example 6.2.11

1. $B\mathbb{Z} = S^1$
2. $B\big(\mathbb{Z}/(2\mathbb{Z})\big) = \mathbb{R}P^\infty$

6.3 Homotopy Theory of Nerves of Categories

In this section we continue our motivational discussion (without proofs) by considering some applications of the theory of fibrations to the nerves of categories.

One of the features of a fibre bundle $p : X \to B$ is that the "fibres", $F_b = p^{-1}(b)$, are homeomorphic for all points b in a common path component of B and one of its properties is the long exact homotopy sequence:

Theorem 6.3.1 *Let $p : X \to B$ be a fibre bundle with fibre F. Then there is a long exact homotopy sequence*

$$\ldots \to \pi_{n+1}(B) \to \pi_n(F) \to \pi_n(X) \to \pi_n(B) \to \pi_{n-1}(F) \to \ldots$$

But from a homotopy viewpoint, having homeomorphic fibres and the rest of the rigid structure of a fibre bundle is overkill. In homotopy theory it is more natural to generalize fibre bundles to "fibrations". We will not go into the precise definition of fibration (see [16] for those details) but merely state the relevant properties.

If $p : X \to B$ is a fibration then the fibres over different points need not be homeomorphic, but they are homotopy equivalent so we have a well defined notion of the homotopy type F of the fibre and we still have the exact sequence

$$\ldots \to \pi_{n+1}(B) \to \pi_n(F) \to \pi_n(X) \to \pi_n(B) \to \pi_{n-1}(F) \to \ldots.$$

Theorem 6.3.2 *(Hurewicz)*
A fibre bundle is a fibration.

Theorem 6.3.3 *Let $f : X \to Y$ be a map of pointed spaces which induces a surjective map on path components. Then there exists a factorization $f = p\phi$ where $\phi : X \to X'$ is a homotopy equivalence and $p : X' \to Y$ is a fibration.*

The fibre of p is called the "homotopy fibre" of f. As a homotopy type it is well defined in the sense that not only is $p^{-1}(y) \simeq p^{-1}(y')$ for every $y, y' \in Y$ but any factorization $f = \bar{\phi} \circ \bar{p}$ in which $\bar{\phi} : X \to \bar{X}$ is a homotopy equivalence and $\bar{p} : \bar{X} \to Y$ is a fibration, yields the same homotopy type $\bar{p}^{-1}(y)$.

As in the case of kernels, strictly speaking the fibre is the inclusion map $F \hookrightarrow E$ rather than F itself. We sometimes write "$F \to E \to B$" is a fibration to mean that $E \to B$ is a fibration whose fibre is $F \to E$. Similarly the phrase "$F \to E \to B$" is a "homotopy fibration" will be understood to mean that $F \to E$ is the homotopy fibre of $E \to B$.

Theorem 6.3.4 *(Quillen Theorem B)*
Let $G : \underline{\underline{C}} \to \underline{\underline{D}}$ be a functor between small categories. Suppose that for every morphism $f : D \to D'$ in $\underline{\underline{D}}$ the realization of the induced functor $\underline{\underline{C}}//D \to \underline{\underline{C}}//D'$ between the comma categories is a homotopy equivalence. Then $F \to |\mathcal{N}\,\underline{\underline{C}}| \xrightarrow{|\mathcal{N}\,G|} |\mathcal{N}\,\underline{\underline{D}}|$ is a homotopy fibration, where $F := |\mathcal{N}\,\underline{\underline{C}}//D|$ (which, up to homotopy equivalence, is independent of D).

Let $\underline{\underline{C}}$ be a small category and let $A : \underline{\underline{CX}}$ be a functor from $\underline{\underline{C}}$ to an abelian category $\underline{\underline{X}}$. Define a simplicial $\underline{\underline{X}}$ object $K(\underline{\underline{C}}; A)$ by

$$K(\underline{\underline{C}}; A)_p := \coprod_{C_0 \to \ldots \to C_p \in \mathcal{N}\,C_p} A(C_0).$$

For $j > 0$, the boundary d_j is defined as the map whose restriction to the summand indexed by $C_0 \to \ldots \to C_p$ maps to $K(\underline{\underline{C}}, A)_{p-1}$ through the summand $A(C_0)$ indexed by $d_j(C_0 \to \ldots \to C_p)$, while d_0 is defined as the map whose restriction to the summand $C_0 \to \ldots \to C_p$ maps

$A(C_0)$ into the summand $A(C_1)$ indexed by $C_1 \to \ldots \to C_p$ using the map $A(C_0 \to C_1)$. The degeneracies s_j is defined in a similar fashion, using the identity map $A(C_0) \to A(C_0)$ entering $K(\underline{\underline{C}}, A)_{p+1}$ through the summand indexed by $s_j(C_0 \to \ldots \to C_p)$. We write $H_*(\underline{\underline{C}}; A)$ for the homology $H_*\big(K(\underline{\underline{C}}, A), d\big)$ of the chain complex associated to this simplicial abelian group. In the case where A is a constant functor $A(C) = X$ for all $C \in \operatorname{Obj}\underline{\underline{C}}$ with $A(f) = 1_X$ for all morphisms f, $H_*(\underline{\underline{C}}; A) = H_*(B\underline{\underline{C}}; X)$, where, as before, $B\underline{\underline{C}} := |\mathcal{N}\,\underline{\underline{C}}|$ In general $H_*(\underline{\underline{C}}; A) = H_*(B\underline{\underline{C}}; X)$ represents the homology of a local (sometimes called "twisted") coefficient system on $(B\underline{\underline{C}}; X)$.

Theorem 6.3.5

$H_p(\underline{\underline{C}}; A)$ is the p the pth derived functor $\varinjlim_{\underline{\underline{C}}}{}_p(A)$ of the colimit functor $\varinjlim_{\underline{\underline{C}}} A \to \underline{\underline{X}}$.

The outline of the proof is as follows. Examination of the definitions shows that $H_0(\underline{\underline{C}}; A) = \varinjlim_{\underline{\underline{C}}} A$. A short exact sequence $A \to B \to C$ of functors from $\underline{\underline{C}}$ to $\underline{\underline{X}}$ gives rise to a corresponding short exact sequence of chain complexes yielding a long exact homology sequence

$$\to H_p(\underline{\underline{C}}; A) \to H_p(\underline{\underline{C}}; B) \to H_p(\underline{\underline{C}}; C) \to H_{p-1}(\underline{\underline{C}}; A) \to \ldots \to H_0(\underline{\underline{C}}; A) \to H_0(\underline{\underline{C}}; B) \to H_0(\underline{\underline{C}}; C).$$

The general properties of derived functors yields a long exact homology sequence

$$\to \varinjlim_{\underline{\underline{C}}}{}_p(A) \to \varinjlim_{\underline{\underline{C}}}{}_p(B) \to \varinjlim_{\underline{\underline{C}}}{}_p(C) \to \varinjlim_{\underline{\underline{C}}}{}_{p-1}(A) \to \ldots \varinjlim_{\underline{\underline{C}}}{}_0(A) \to \varinjlim_{\underline{\underline{C}}}{}_0(B) \to \varinjlim_{\underline{\underline{C}}}{}_0(C).$$

The key step is to show that if $P : \underline{\underline{C}} \to X$ is projective in the category of functors from $\underline{\underline{C}} \to X$ then $H_q(\underline{\underline{C}}; P) = 0$ for $q > 0$. Applying this to a projective presentation $R \to P \to A$ of A shows that

$$H_p(\underline{\underline{C}}, A) = H_{p-1}(\underline{\underline{C}}, R) \overset{\text{(induction)}}{=\!=\!=} \varinjlim_{\underline{\underline{C}}}{}_{p-1}(R) = \varinjlim_{\underline{\underline{C}}}{}_p(A).$$

6.4 (Co)Homology of a Pushout

Let

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"] & B \arrow[d, "f"] \\ C \arrow[r, "h"] & G \end{tikzcd}$$

be a pushout in the category of groups. Then

Lemma 6.4.1

$$\begin{tikzcd}
\mathbb{Z}[G] \otimes_{\mathbb{Z}[A]} IA \arrow[r, "i_*"] \arrow[d, "j_*"] & \mathbb{Z}[G] \otimes_{\mathbb{Z}[B]} IB \arrow[d, "f_*"] \\
\mathbb{Z}[G] \otimes_{\mathbb{Z}[C]} IC \arrow[r, "h_*"] & IG
\end{tikzcd}$$

is a pushout in the category of $\mathbb{Z}[G]$ *modules, where* i_* *denotes* $1_{\mathbb{Z}[G]} \otimes i$*, and* j_**,* f_**,* h_* *are defined similarly.*

Proof: Let K be the two sided ideal in $\mathbb{Z}[G]$ generated by the ideals $\operatorname{Im} f_*$ and $\operatorname{Im} h_*$. The intersection $\operatorname{Im} f_* \cap \operatorname{Im} h_*$ is $\operatorname{Im}(f \circ i)_* = \operatorname{Im}(h \circ j)_*$. As in Example 1.2.6, K is the pushout of the given diagram.

We show that the inclusion $K \subset IG$ is an equality. Suppose $g - 1$ is a generator of IG. Writing g as a word in $\operatorname{Im} f$ and $\operatorname{Im} h$, we have $g = wx$ where either $x \in B' := f(B)$ or $x \in C' := h(C)$, and w is a word with one fewer letter than g. For words y of length 1 in G, either $y - 1 \in \operatorname{Im} f_*$ or $y - 1 \in \operatorname{Im} h_*$ so by induction we may assume that $w - 1 \in K$, and, since x has length 1, we also have $x - 1 \in K$. Since $x - 1 \in K$ we have $w(x - 1) \cong 0$ modulo K so $g \cong w$ modulo K, and thus $g - 1 \cong w - 1 \cong 0$ modulo K. Therefore every generator of IG lies in K and so $IG = K$. □

Let Q denote the cokernel of h_*, which since the diagram is a pushout, is isomorphic to the cokernel of i_*. Applying Theorem 3.1.6 to the top row gives a long exact sequence for $\operatorname{Tor}_*(\ )$, and applying it to the bottom row gives a similar long exact $\operatorname{Tor}_*(\ )$ sequence from the bottom row. Since the terms coming from the cokernel are isomorphic, Algebraic Mayer-Vietoris gives the long exact sequence

$$\ldots \to \operatorname{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}[G] \otimes_{\mathbb{Z}[A]} IA, \mathbb{Z}) \to \operatorname{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}[G] \otimes_{\mathbb{Z}[B]} IB, \mathbb{Z}) \oplus \operatorname{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}[G] \otimes_{\mathbb{Z}[C]} IC, \mathbb{Z})$$
$$\to \operatorname{Tor}_n^{\mathbb{Z}[G]}(IG, \mathbb{Z}) \to \operatorname{Tor}_{n-1}^{\mathbb{Z}[G]}(\mathbb{Z}[G] \otimes_{\mathbb{Z}[A]} IA, \mathbb{Z}) \to \ldots$$

For any K, Proposition 6.1.5 gives $\operatorname{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}[G] \otimes_{\mathbb{Z}[K]} IK, \mathbb{Z}) \cong \operatorname{Tor}_n^{\mathbb{Z}[K]}(IK, \mathbb{Z})$ and the exact sequence of $\mathbb{Z}[K]$ modules $0 \to IK \to \mathbb{Z}[K] \to \mathbb{Z} \to 0$ gives (by Theorem 3.1.6) $\operatorname{Tor}_n^{\mathbb{Z}[K]}(IK, \mathbb{Z}) \cong \operatorname{Tor}_{n+1}^{\mathbb{Z}[K]}(\mathbb{Z}, \mathbb{Z}) = H_{n+1}(K)$. Thus our pushout of groups yields a long exact Mayer-Vietoris sequence

$$\ldots \to H_q(A) \to H_q(B) \oplus H_q(C) \to H_q(G) \to H_{q-1}(A) \to$$
$$\ldots \to H_1(A) \to H_1(B) \oplus H_1(C) \to H_1(G)$$

6.5 Bar Construction

Let G be a group and let $\underline{\underline{BG}}$ be the associated category (one object $*$ with morphisms given by G). Let $\underline{\underline{EG}}$ denote the category of objects over $*$, which is the same as the comma category $1_{\underline{\underline{BG}}}//*$ associated to the identity functor $1_{\underline{\underline{BG}}}$. Then $|\mathcal{N}\, EG|$ is contractible, since EG has the terminal object $1_* : * \to *$.

Every comma category comes with a forgetful functor $\underline{\underline{C}}//D \to \underline{\underline{C}}$. Write $P : \underline{\underline{EG}} \to \underline{\underline{BG}}$ for this forgetful functor (which drops the target $*$).

In $\underline{\underline{BG}}$, a composition of length n is simply a sequence of n elements of the group G. Thus $\mathcal{N}\,\underline{\underline{BG}}$ is the simplicial set in which $(\mathcal{N}\,\underline{\underline{BG}})_n = G^n$ with boundary $d_j\big((g_n, \ldots, g_1)\big) = (g_1, \ldots, g_j g_{j-1}, \ldots, g_1)$ given by multiplying adjacent elements for $1 \le j \le n-1$, while d_0 and d_n drop the end elements. For $1 \le j \le n-1$, the degeneracy s_j inserts the identity between g_j and g_{j+1} while s_0 and s_n insert it at the ends. Similarly $(\mathcal{N}\,\underline{\underline{EG}})_n = G^{n+1}$ where $d_j\big((g_n, \ldots, g_1, g_0)\big)$ and $s_j\big((g_n, \ldots, g_1, g_0)\big)$ are given by acting on $(g_n \ldots, g_1)$ as above, and P drops g_0. The group G acts on $\mathcal{N}\,\underline{\underline{EG}}$ by $g \cdot (g_n, \ldots, g_1, g_0) := (g_n, \ldots, g_1, gg_0)$ and it is clear that this action of G is a free action. Since the boundaries and degeneracy operate on the leftmost n elements, this action commutes with them (whereas acting with g on each component would not) giving $\mathcal{N}\,\underline{\underline{EG}}$ the structure of a simplicial G-set. In the quotient $(\mathcal{N}\,\underline{\underline{EG}})/G$, $(g_n, \ldots, g_1, g_0) \sim (g_n, \ldots, g_1, e)$. Comparing this with $\mathcal{N}\,\underline{\underline{BG}}$ shows that we have a simplicial isomorphism $\Theta : (\mathcal{N}\,\underline{\underline{EG}})/G \cong \mathcal{N}\,\underline{\underline{BG}}$ such that the composition $\mathcal{N}\,\underline{\underline{EG}} \twoheadrightarrow (\mathcal{N}\,\underline{\underline{EG}})/G \cong \mathcal{N}\,\underline{\underline{BG}} \xrightarrow{\Theta}$ is $\mathcal{N}\,P$. Since $|\mathcal{N}\,\underline{\underline{EG}}|$ is a contractible G-space on which G acts freely, we see that $|\underline{\underline{BG}}| \xrightarrow[\Theta]{\cong} |(\mathcal{N}\,\underline{\underline{EG}})/G| \simeq BG$, justifying our notation.

Let R be a ring. (We will be applying the discussion below to the ring $\mathbb{Z}[G]$.) There is a canonical ring homomorphism $\eta : \mathbb{Z} \to R$. Define the "extended module functor" $E : \mathcal{AB} \to$ R-modules by $E(M) := R \otimes_{\mathbb{Z}} M$ (as in the Change of Rings section). If $J : R$-modules $\to \mathcal{AB}$ is the forgetful functor, then $\mathrm{Hom}_R\big((E(M), N\big) \cong \mathrm{Hom}_{\mathcal{AB}}\big(M, J(N)\big)$. That is $E \longrightarrow\!\!| \; J$. As in section 5.2 the adjunction produces a simplicial R-module $T_\bullet N$ for each abelian group N. Since the left adjoint E always preserves projectives (and in this case takes free modules to free modules), if N is a free abelian group then $(T_\bullet N)_n$ will be a projective (in fact free) R-module for all n, and the chain complex $(T_\bullet N, d)$ will be a complex of R-projectives. Also, if we start with an R-module X and let $N = J(X)$, then as in Section 5.2, $(T_\bullet N, d)$ will have a conical contraction to X. Thus, if we begin with an R-module X whose underlying abelian group is a free abelian group then $(T_\bullet X, d)$ forms a projective resolution of X as an R-module. This is called the *bar resolution* of X.

Remark 6.5.1 Eilenberg and Mac Lane found it faster to write '$a|b$' than '$a \otimes b$' which was the origin of the name "bar resolution".

An *augmentation* of a ring R is a ring homomorphism $\epsilon : R \to \mathbb{Z}$ such that $\epsilon \circ \eta = 1_{\mathbb{Z}}$. In the case of $\mathbb{Z}[G]$, an augmentation is given by $\epsilon(\sum n_i g_i) := \sum n_i$.

For an augmented ring we can define a "reduced bar construction" as follows.

Let R be a ring with augmentation $\epsilon : R \to \mathbb{Z}$. Set $IR := \ker \epsilon$, known as the "augmentation ideal". Let X be an R-module and let $B_* X$ be the resolution of X described above. Then $(T_\bullet)X = R^{\otimes(n+1)} \otimes X$ with differentials given by

$$d_j(r_n \otimes \cdots \otimes a_{j+1} \otimes a_j \otimes a_{j-1} \otimes \cdots \otimes r_0 \otimes x) = r_n \otimes \cdots \otimes r_{j+1} \otimes r_j r_{j-1} \otimes \cdots \otimes r_0 \otimes x)$$

for $j > 0$ and

$$d_0(r_n \otimes \cdots \otimes r_0 \otimes x) = \epsilon(r_0)(r_n \otimes \cdots \otimes a_1 \otimes x).$$

IR is an ideal in R, so the subspace $(IR)_\bullet$ defined by $(IR)_n := R \otimes (J)^{\otimes n}$ (with R-module structure given by acting on the left factor) forms an R-free subcomplex of $T_\bullet X$. Since

$$(d_{j+1} - d_j)(r_n \otimes \cdots \otimes r_{j+1} \otimes 1 \otimes r_{j-1} \otimes \cdots \otimes r_0 \otimes x) = 0$$

for $j < n$, the projection $R \otimes (1-\epsilon) \otimes \cdots \otimes (1-\epsilon) : R^{\otimes(n+1)} \otimes X \to R \otimes (IR)^{\otimes n} \otimes X$ is a chain map. Thus $((IR)_\bullet X, d)$ is a retract (as a chain complex) of the acyclic chain complex $(T_\bullet X, d)$. In particular $H_q\big(((IR)_\bullet X, d)\big) = 0$ for $q > 0$. Since it is easy to check that $H_0\big(((IR)_\bullet X, d)\big) = X$ the complex $((IR)_\bullet X, d)$ is a projective R-module resolution of X. It is called the "reduced bar complex" of X.

Let G be a group. Consider the special case where $R = \mathbb{Z}[G]$ and X is the trivial $\mathbb{Z}[G]$-module $\mathbb{Z}$. Then $(\mathbb{Z}T_\bullet \mathbb{Z}, d)$ and $((I\mathbb{Z}[G])_\bullet X, d)$ form projective resolutions of $\mathbb{Z}$ as a $\mathbb{Z}[G]$-module and is thus are suitable for computing $H_*(G)$ and $H^*(G)$. Explicitly, the *bar resolution* in degree n looks like $\mathbb{Z}[G]^{\otimes(n+1)} \otimes \mathbb{Z}$ which is same as $\mathbb{Z}[G]^{\otimes(n+1)}$. The reduced bar resolution in degree looks like $\mathbb{Z}[G] \otimes I^{\otimes(n)}$ in degree n, where I denotes the augmentation ideal.

6.6 Interpretations of $H_1(\ )$, $H^1(\ )$, $H_2(\ )$, $H^2(\ )$

Let $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ be the augmentation and write $IG := \ker \epsilon$ for the augmentation ideal.

Proposition 6.6.1 *IG is the free abelian group on $\{g - 1 \mid g \in G\}$*

Proof: It is clear that as an abelian group, $\mathbb{Z}[G]$ is the free $\mathbb{Z}$-module on G. Suppose $x = \sum n_i g_i \in IG$, where $\sum n_i = 0$. Then $x = \sum n_i g_i - \sum n_i = 0 = \sum n_i(g_i - 1)$ showing that $\{g - 1 \mid g \in G\}$ generates IG. Conversely, if $\sum n_i(g_i - 1) = 0$ in IG regarding this as an equation in the free group $\mathbb{Z}[G]$ gives $n_i = 0$ for all i. □

Let A be a $\mathbb{Z}[G]$-module.

Since $\mathrm{Tor}^n_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) = 0$ for $n > 0$, from the short exact sequence of $\mathbb{Z}[G]$ modules $0 \to IG \xrightarrow{i} \mathbb{Z}[G] \to \mathbb{Z} \to 0$ we get

Proposition 6.6.2

1. *$H_n(G; A) \cong \mathrm{Tor}^{\mathbb{Z}[G]}_{n-1}(IG; A)$ for $n > 1$*

2. *there is a long exact sequence*

$$0 \to H_1(G; A) = \mathrm{Tor}_{\mathbb{Z}[G]}(\mathbb{Z}, A) \to IG \otimes_{\mathbb{Z}G} A \xrightarrow{i_*} \mathbb{Z}G \otimes_{\mathbb{Z}G} A = A \to \mathbb{Z} \otimes_{\mathbb{Z}G} A \to 0$$

and so $H_1(G; A) = \ker(i_* : IG \otimes_{\mathbb{Z}G} A \xrightarrow{i_*} A)$ where i_* is given by $i_*\big((g-1)\otimes a\big) = ga - a$.

□

In general there does not seem to any useful reinterpretation of the second part of this proposition, but in the special case where A has trivial G-action we can say more. Thus suppose that A has trivial G-action. Then $ga = a$ and so $i_* = 0$. This gives

$$H_1(G; A) \cong IG \otimes_{\mathbb{Z}[G]} A = \big(IG/(IG)^2\big) \otimes A$$

Lemma 6.6.3 *$IG/(IG)^2 \cong G_{\mathrm{ab}}$ where G_{ab} denotes the "abelianization of G, defined the quotient $G/[G, G]$ of G by its commutator subgroup.*

Proof: IG is the free abelian group on $\{g - 1 \mid g \in G\}$. Define $\phi : IG \to G_{ab}$ by $\phi(g-1) := [g]$. Notice that

$$\begin{aligned}\phi\big((x-1)(y-1)\big) = \phi(xy - x - y + 1) &= \phi\big((xy-1) - (x-1) - (y-1)\big) \\ &= \phi(xy-1) - \phi(x-1) - \phi(y-1) = [xy][x^{-1}][y^{-1}] = [xyx^{-1}y^{-1}] = [e]\end{aligned}$$

Thus the restriction of ϕ to $(IG)^2$ is trivial so there is an induced morphism $\bar{\phi} : IG/(IG)^2 \to G_{\mathrm{ab}}$ of abelian groups. Conversely, the map $\psi : G \to IG/(IG)^2$ given by $\psi(g) := [g - 1]$ restricts trivially to G_{ab} by the same calculation so it induces a homomorphism $\bar{\psi} : G_{ab} \to IG/(IG)^2$. It is clear that $\bar{\phi}$ and $\bar{\psi}$ are inverses. □

Summarizing the precding we have:

Proposition 6.6.4 *Let A be a trivial G-module. Then $H_1(G; A) \cong G_{\mathrm{ab}} \otimes A$.*

This result is familiar from topology. Recall that in the case of trivial action, $H_1(G;\mathbb{Z}) = H_1^{\mathcal{T}opologicalSpaces}(BG)$ and that BG is the Eilenberg-Mac Lane space $K(G,1)$. According to the Hurewicz Theorem $H_1\big(K(G,1)\big) = \Big(\pi_1\big(K(G,1)\big)\Big)_{\mathrm{ab}} = G_{\mathrm{ab}}$

We now look at $H^1(G;A)$.

Again using $0 \to IG \xrightarrow{i} \mathbb{Z}[G] \to \mathbb{Z}$ we get the exact sequence

$$0 \to H^0(G;A) \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \xrightarrow{i^*} \mathrm{Hom}_{\mathbb{Z}G}(IG, A) \to H^1(G,A) \to 0.$$

Thus $H^1(G,A) \cong \operatorname{coker} i^* : \big(A \to \mathrm{Hom}_{\mathbb{Z}G}(IG,A)\big)$.

Set $\mathrm{Der}(G,A) := \{f : G \to A \mid f(gh) = f(g) + g \cdot f(h)\}$. An element of $\mathrm{Der}(G,A)$ is called a "crossed homomorphism". The intuition behind the notation is that if A is made into a G-bimodule by giving it a trivial right action of G, then elements of $\mathrm{Der}(G,A)$ are "derivations" (satisfy the Leibniz rule).

Lemma 6.6.5 $\mathrm{Hom}_{\mathbb{Z}G}(IG,A)\big) \cong \mathrm{Der}(G.A)$

Proof: Given $f \in \mathrm{Hom}_{\mathbb{Z}G}(IG,A)\big)$ define $\bar{f} : G \to A$ by $\bar{f}(g) := f(g-1)$. Then $\bar{f}(gh) = f(gh-1) = f\big((g(h-1)+(g-1)\big) = g \cdot f(h-1) + f(g-1) = g \cdot \bar{f} + \bar{f}(g)$ so $\bar{f}$ lies in $\mathrm{Der}(G,A)$.

Conversely, given $\bar{f} \in \mathrm{Der}(G,A)$ define $f : IG \to A$ by $f(g-1) := \bar{f}(g)$. The same calculation shows that f is a $\mathbb{Z}[G]$-module homomorphism so lies in $\mathrm{Hom}_{\mathbb{Z}G}(IG,A)\big)$. It is clear that $f \mapsto \bar{f}$ and $\bar{f} \mapsto f$ are inverse operations. □

Under the identification $\mathrm{Hom}_{\mathbb{Z}G}(IG,A)\big) \cong \mathrm{Der}(G.A)$ the map $i^* : A \to \mathrm{Hom}_{\mathbb{Z}G}(IG,A)$ corresponds to the one taking a to the "derivation" f_a where $f_a(g) = (g-1)a$, called the "inner derivation" corresponding to A. Thus setting $I\,\mathrm{Der}(G,A) = \{\text{inner derivations}\}$ we get

Proposition 6.6.6 $H^1(G,A) \cong \mathrm{Der}(G,A)/I\,\mathrm{Der}(G.A)$

In the case where A has trivial G-action, an element of $\mathrm{Der}(G,A)$ becomes simply a group homomorphism $G \to A$ and $I\,\mathrm{Der}(G,A)$ becomes the trivial subgroup. Since A is an abelian group, a group homomorphism $G \to A$ corresponds to a homomorphism from G_{ab} to A. Thus in the case of trivial G-action we get $H^1(G;A) = \mathrm{Hom}(G_{\mathrm{ab}}, A) = \mathrm{Hom}\big(H_1(G), A\big)$ which is a special case of the universal coefficient theorem.

Lemma 6.6.7 *Let $\phi : B \twoheadrightarrow C$ be a surjective group homomorphism with kernel A. Then*

1. $\mathbb{Z} \otimes_{\mathbb{Z}[A]} \otimes \mathbb{Z}[B] \cong \mathbb{Z}[C]$ *as right $\mathbb{Z}[B]$-modules.*

2. $\mathrm{Tor}_n^{\mathbb{Z}[A]}(\mathbb{Z}, X) \cong \mathrm{Tor}_n^{\mathbb{Z}[B]}(\mathbb{Z}[C], X)$ *for any left $\mathbb{Z}[B]$-module X.*

Proof:

1. As an abelian group, $\mathbb{Z} \otimes_{\mathbb{Z}[A]} \mathbb{Z}[B]$ is free on the set of right cosets B/A which is C.

2. Let P_* be a $\mathbb{Z}[B]$-projective resolution of X. Then P_* is also a $\mathbb{Z}[A]$-projective resolution of X and $\mathbb{Z} \otimes_{\mathbb{Z}[A]} P_* \cong \mathbb{Z} \otimes_{\mathbb{Z}[A]} \mathbb{Z}[B] \otimes_{\mathbb{Z}[B]} P_* \cong \mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} P_*$

$\square$

Proposition 6.6.8 *Let $\phi : B \to C$ be a group homomorphism with kernel A. Then*

1. *Conjugation induces an action of C on A_{ab}.*

2. *There is an induced short exact sequence $0 \to A_{\mathrm{ab}} \to \mathbb{Z}C \otimes_{\mathbb{Z}[B]} IB \to IC \to 0$ of $Z[C]$-modules.*

Proof:

1. Given $c \in C$ and $[a] \in A_{\mathrm{ab}}$ define $c \cdot [a] := [bab^{-1}] \in A_{\mathrm{ab}}$, where $b \in \phi^{-1}(c) \subset B$. If b' also satisfies $\phi(b') = b$ then $b' = b\bar{a}$ for some $\bar{a} \in A$. Write $a_1 := bab^{-1}$ and $a_2 := b'\bar{a}b'^{-1}$ in A. Then $b'ab'^{-1} = b\bar{a}a\bar{a}^{-1}b^{-1} = a_2 a_1 a_2^{-1}$ which is congruent to a_1 modulo $[A, A]$. Thus $c \cdot [a]$ is well defined and it is easy to see that it is a group action.

2. Applying $\mathbb{Z}C \otimes_{\mathbb{Z}[B]} (\ )$ to the short exact sequence $0 \to IB \to \mathbb{Z}[B] \to \mathbb{Z} \to 0$ yields the exact sequence

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}[B]}(\mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} \mathbb{Z}, \mathbb{Z}) \to \mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} IB \to \mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} \mathbb{Z}[B] \to \mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} \mathbb{Z} \to 0$$

of $\mathbb{Z}[B]$-modules. We have $\mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} \mathbb{Z}[B] \cong \mathbb{Z}[C]$ and since $B \to C$ is onto, $\mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} \mathbb{Z} \cong \mathbb{Z}$. The map $\mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} \mathbb{Z}[B] \to \mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} \mathbb{Z} \to 0$ corresponds to the augmentation $\mathbb{Z}[C] \to \mathbb{Z}$ under these isomorphisms. Since $A_{\mathrm{ab}} \cong \mathrm{Tor}_1^{\mathbb{Z}[A]}(\mathbb{Z}, \mathbb{Z}) \cong \mathrm{Tor}_1^{\mathbb{Z}[B]}(\mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} \mathbb{Z}, \mathbb{Z})$ using Lemma 6.6.7, the result follows.

$\square$

Theorem 6.6.9 *Let $\phi : B \twoheadrightarrow C$ be a surjective group homomorphism with kernel A. Let X be a left $\mathbb{Z}[C]$-module. Then there is an exact sequence*

$$H_2(B, X) \to H_2(C, X) \to A_{\mathrm{ab}} \otimes_{\mathbb{Z}[C]} X \to H_1(B, X) \to H_1(C, X) \to 0$$

Proof: Applying $(\ ) \otimes_{\mathbb{Z}[C]} X$ to the second part of the preceding proposition gives the exact sequence

$$\mathrm{Tor}_1^{\mathbb{Z}[C]}(\mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} IB, X) \to \mathrm{Tor}_1^{\mathbb{Z}[C]}(IC, X) \to A_{\mathrm{ab}} \otimes_{\mathbb{Z}[C]} X \to IB \otimes_{\mathbb{Z}[B]} X \to IC \otimes_{\mathbb{Z}[C]} X \to 0. \tag{6.1}$$

Let $P \twoheadrightarrow X$ be a projective $\mathbb{Z}[C]$ presentation with kernel M.

Naturality gives a commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Tor}_1^{\mathbb{Z}[B]}(IB, P) \arrow[r] \arrow[d] & \mathrm{Tor}_1^{\mathbb{Z}[B]}(IB, X) \arrow[r] \arrow[d] & IB \otimes_{\mathbb{Z}[B]} M \arrow[r] \arrow[d, equal] & IB \otimes_{\mathbb{Z}[B]} P \arrow[d, equal] \\
\mathrm{Tor}_1^{\mathbb{Z}[C]}(\mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} IB, P) \arrow[r] & \mathrm{Tor}_1^{\mathbb{Z}[C]}(\mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} IB, X) \arrow[r] & IB \otimes_{\mathbb{Z}[B]} M \arrow[r] & IB \otimes_{\mathbb{Z}[B]} P
\end{tikzcd}$$

in which the top row is the Tor-sequence obtained by applying $IB \otimes_{\mathbb{Z}[B]} (\ )$ to the short exact sequence $0 \to M \to P \to X \to 0$ (thought of as a short exact sequence of $\mathbb{Z}[B]$-modules via the ring homomorphism $\mathbb{Z}[B] \to \mathbb{Z}[C]$) and the bottom is the one obtained by applying $(\mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} IB) \otimes_{\mathbb{Z}[C]} (\ )$ to the same sequence. The map

$$\mathrm{Tor}_1^{\mathbb{Z}[B]}(IB, X) \to \mathrm{Tor}_1^{\mathbb{Z}[C]}(\mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} IB, X)$$

in the diagram is a lift of the canonical map $\mathrm{Tor}_1^{\mathbb{Z}[B]}(IB, X) \to \mathrm{Tor}_1^{\mathbb{Z}[C]}(IB, X)$ snd since $\mathrm{Tor}_1^{\mathbb{Z}[C]}(\mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} IB, P) = 0$ (as P is $\mathbb{Z}[C]$-projective) diagram chasing shows that it is surjective. That is, $\mathrm{Im}\big(\mathrm{Tor}_1^{\mathbb{Z}[C]}(\mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} IB, X) \to \mathrm{Tor}_1^{\mathbb{Z}[C]}(IB, X)\big)$ equals $\mathrm{Im}\big(\mathrm{Tor}_1^{\mathbb{Z}[B]}(IB, X) \to \mathrm{Tor}_1^{\mathbb{Z}[C]}(IB, X)\big)$. Thus making the subsitutions Equation 6.1 $\mathrm{Tor}_1^{\mathbb{Z}[B]}(IB, X) \cong H^2(B, X)$ and $\mathrm{Tor}_1^{\mathbb{Z}[C]}(IC, X) \cong H^2(C, X)$ gives the exact sequence

$$H_2(B, X) \to H_2(C, X) \to A_{\mathrm{ab}} \otimes_{\mathbb{Z}[C]} X \to IB \otimes_{\mathbb{Z}[B]} X \to IC \otimes_{\mathbb{Z}[C]} X \to 0$$

Diagram-chasing in

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
H_1(B;X) \arrow[r] \arrow[d] & H_1(C;X) \arrow[d] & \\
IB \otimes_{\mathbb{Z}[B]} X \arrow[r] \arrow[d] & IC \otimes_{\mathbb{Z}[C]} X \arrow[r] \arrow[d] & 0 \\
B \arrow[r, equal] & B &
\end{tikzcd}$$

shows that $H_1(B;X) \to H_1(C;X)$ is onto and that its kernel is the same as the kernel of $IB \otimes_{\mathbb{Z}[B]} X \to IC \otimes_{\mathbb{Z}[C]} X$. □

We now consider the special case where $X := \mathbb{Z}$ as a trivial $\mathbb{Z}[C]$-module.

Proposition 6.6.10 *Let $\phi : B \to C$ be a group homomorphism with kernel A. Then the group $A_{\mathrm{ab}} \otimes_{\mathbb{Z}[C]} \mathbb{Z}$ is isomorphic to $A/[B,A]$.*

Proof: Recall that the (right) action of $\mathbb{Z}[C]$ on A_{ab} is given by $[a] \cdot c := [b^{-1}ab]$ where $\phi(b) = c$. Thus $A_{\mathrm{ab}} \otimes_{\mathbb{Z}[C]} \mathbb{Z}$ is the same as $A_{\mathrm{ab}}/ \sim$ where $[a] \sim [a] \cdot c = [b^{-1}ab]$. Define $\theta : A_{\mathrm{ab}} \to A/[B,A]$ by $\theta([a]) := [a]$. If $\phi(b) = c$ then $\theta([a \cdot c]) = \theta(b^{-1}ab) = [b^{-1}ab]$ which is congruent to $\theta([a]) = a$ modulo the commutator $b^{-1}aba^{-1}$. Therefore θ induces a well defined map $\bar{\theta} : A_{\mathrm{ab}} \otimes_{\mathbb{Z}[C]} \mathbb{Z} \to A/[B,A]$. Conversely, let $q : A \to A_{\mathrm{ab}} \otimes_{\mathbb{Z}[C]} \mathbb{Z} \cong A/\sim$ be the quotient map. Since $q(aba^{-1}b^{-1}) = q(a)$ there is a well-defined induced map $\bar{q} : A/[B,A] \to A_{\mathrm{ab}} \otimes \mathbb{Z}[C]\mathbb{Z} \cong A/\sim$. It is clear that $\bar{\theta}$ and $\bar{q}$ are inverse isomorphisms. □

Corollary 6.6.11 *(Hopf)*

Let $F \twoheadrightarrow G$ be a free presentation of a group G and let R be its kernel. Then $H_2(G) \cong R \cap \big([F,F]/[F,R]\big)$.

Proof: Applying the preceding theorem with $X := \mathbb{Z}$ gives

$$H_1(G) \cong \ker(R/[F,R] \to F/[F,F]) = R \cap \big([F,F]/[F,R]\big)$$

□

The cohomology version of Theorem 6.6.9 is

Theorem 6.6.12 *Let $\phi : B \to C$ be a group homomorphism with kernel A. Let X be a left $\mathbb{Z}[C]$-module. Then there is an exact sequence*

$$0 \to H^1(C;X) \to H^1(B;X) \to \mathrm{Hom}_{\mathbb{Z}[C]}(A_{\mathrm{ab}}, X) \to H^2(C;X) \to H^2(B;X)$$

Proof: Applying $\mathrm{Hom}_{\mathbb{Z}[C]}(\ , X)$ to the second part of Proposition 6.6.8 gives the exact sequence

$$\begin{aligned}0 \to \mathrm{Hom}_{\mathbb{Z}[C]}(IC, X) \to \mathrm{Hom}_{\mathbb{Z}[B]}(IB, X) \to \mathrm{Hom}_{\mathbb{Z}[C]}(A_{\mathrm{ab}}, X) \to \mathrm{Ext}_{\mathbb{Z}[C]}(IC, X)& \\ \to \mathrm{Ext}_{\mathbb{Z}[C]}(\mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} IB, X)&\end{aligned}$$

Let $P \twoheadrightarrow X$ be a presentation with kernel M.
Naturality gives the commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}[B]}(IB, P) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathbb{Z}[B]}(IB, M) \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_{\mathbb{Z}[B]}(IB, X) \arrow[r] \arrow[d] & \mathrm{Ext}^1_{\mathbb{Z}[B]}(IB, P) \arrow[d] \\
\mathrm{Hom}_{\mathbb{Z}[B]}(IB, P) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}[B]}(IB, M) \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}[C]}(\mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} IB, X) \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}[C]}(\mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} IB, P)
\end{tikzcd}$$

in which $\mathrm{Ext}^1_{\mathbb{Z}[B]}(IB, P) = 0$. The map $\mathrm{Ext}^1_{\mathbb{Z}[B]}(IB, X) \to \mathrm{Ext}^1_{\mathbb{Z}[C]}(\mathbb{Z}[C] \otimes_{\mathbb{Z}[B]} IB, X)$ in the diagram extends to the canonical map $\mathrm{Ext}^1_{\mathbb{Z}[B]}(IB, X) \to \mathrm{Ext}^1_{\mathbb{Z}[C]}(IB, X)$ and diagram-chasing shows that is injective. This gives the exact sequence

$$0 \to \mathrm{Hom}_{\mathbb{Z}[C]}(IC, X) \to \mathrm{Hom}_{\mathbb{Z}[B]}(IB, X) \to \mathrm{Hom}_{\mathbb{Z}[C]}(A_{\mathrm{ab}}, X) \to H^2(C, X) \to H^2(B;X)$$

in which we have made the substitutions $\mathrm{Ext}^1_{\mathbb{Z}[B]}(IB, X) \cong H^2(B, X)$ and $\mathrm{Ext}^1_{\mathbb{Z}[C]}(IC, X) \cong H^2(C, X)$.

Diagram-chasing in

$$\begin{tikzcd}
 & B \arrow[r, equal] \arrow[d] & B \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}(IC, X) \arrow[r] \arrow[d] & \mathrm{Hom}(IB, X) \arrow[d] \\
 & H^1(C; X) \arrow[r] \arrow[d] & H^1(B; X) \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

shows that $H^1(C; X) \to H^1(B; X)$ is injective and that its cokernel is the same as the cokernel of $\mathrm{Hom}(IC, X) \to \mathrm{Hom}(IB, X)$ giving the desired sequence. □

If X and Y are groups, a sequence of group homomorphism $Y \to Z \twoheadrightarrow X$ in which $Z \to X$ is surjective with kernel $Y \to Z$ is called an "extension" of X by Y. We will write $Y \rightarrowtail Z \twoheadrightarrow X$ for an extension.

Let G be a group and let A be abelian. Then, any extension of $A \rightarrowtail E \overset{\beta}{\twoheadrightarrow} G$ gives rise to an action of G by A by $g \cdot a := xax^{-1}$ where $x \in E$ satisfies $\beta(x) = g$. Conversely, given an action of G on an abelian group A, we wish to determine the possible extensions of G by A which give rise to this action.

Write $(A \rightarrowtail E \twoheadrightarrow G) \sim (A \rightarrowtail E' \twoheadrightarrow G)$ if there exist an isomorphism $\phi : E \to E'$ such that

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, equal] & E \arrow[r, two heads] \arrow[d, "\phi"] & G \arrow[d, equal] \\
A \arrow[r, tail] & E' \arrow[r, two heads] & G
\end{tikzcd}$$

Observe that if $(A \rightarrowtail E \twoheadrightarrow G) \sim (A \rightarrowtail E' \twoheadrightarrow G)$ then the action of G on A coming from $A \rightarrowtail E \twoheadrightarrow G$ agrees with that coming from $A \rightarrowtail E' \twoheadrightarrow G$. Let $\mathrm{M}(G, A)$ denote the set of equivalences classes of extension of G by A under this relation which yield to the given action of G on A. We will construct a bijection from $\mathrm{M}(G, A)$ to $H^2(G, A)$.

Let $A \rightarrowtail E \twoheadrightarrow G$ be an extension of G by A which yields the given action of G on A. By Theorem 6.6.12 in the special case $X := A$, there is an exact sequence

$$0 \to H^1(G; A) \to H^1(E; A) \to \mathrm{Hom}_{\mathbb{Z}[G]}(A, A) \xrightarrow{\delta} H^2(G; A) \to H^2(E; A).$$

If $(A \rightarrowtail E \twoheadrightarrow G) \sim (A \rightarrowtail E' \twoheadrightarrow G)$ then naturality gives a commuting diagram showing that the element $\delta(1_A)$ is independent of the choice of representative for the equivalence class containing $(A \rightarrowtail E \twoheadrightarrow G)$. That $\delta(1_A)$ induces a well defined $\theta : \mathrm{M} \to H^2(G; A)$.

Theorem 6.6.13 *$\theta : \mathrm{M} \to H^2(G; A)$ is a bijection.*

Proof: We first show that θ is onto. Let $q : F \twoheadrightarrow G$ be any free presentation of G with kernel $j : R \to F$. Theorem 6.6.12 gives an exact sequence

$$H^1(F, A) \to \mathrm{Hom}_{\mathbb{Z}[G]}(R_{\mathrm{ab}}, A) \to H^2(G; A) \longrightarrow 0.$$

Given an element of $H^2(G; A)$, choose a pre-image in $\bar{\alpha} \in \mathrm{Hom}_{\mathbb{Z}[G]}(R_{\mathrm{ab}}, A)$ and let α be the composite $R \twoheadrightarrow R_{\mathrm{ab}} \xrightarrow{\bar{\alpha}} A$.

Composing the auction map $G \to \mathrm{Aut}(A)$ with q gives an action of F on A. We write a^f for $f \cdot a$, noting that $(a^x)^y = a^{yx}$. Let V be the semidirect product $V := A \rtimes F$.

Set $U := \big\{\big(\alpha(r), r^{-1}\big) \mid r \in R\big\} \subset V$.

Claim: U is a normal subgroup of V.

Proof of Claim:

The restriction to R of the action on A is defined as the composition $R \hookrightarrow F \twoheadrightarrow G \to \mathrm{Aut}(A)$ which is trivial. Therefore, writing '+' for the group operation in the abelian group A, we have $\big(\alpha(r_1), r_1^{-1}\big) + \big(\alpha(r_2), r_2^{-1}\big) = \big(\alpha(r_1) + \alpha(r_2), r_1^{-1} r_2^{-1}\big) = \big(\alpha(r_2 r_1), (r_2 r_1)^{-1}\big) \in U$ and $\big(\alpha(r), r^{-1}\big)^{-1} = \big(\alpha(r^{-1}), r\big) \in U$ and so U is a subgroup of V.

Let $(a, f) \in V$ and $\big(\alpha(r), r^{-1}\big) \in U$. Write $r' := r^f \in R$. Then

$$\begin{aligned}
(a, f)\big(\alpha(r), r^{-1}\big)(a, f)^{-1} &= \big(a + \alpha(r)^f, f r^{-1}\big)\big((-a)^{f^{-1}}, f^{-1}\big) \\
&= \Big(a + \alpha(r)^f + \big((-a)^{f^{-1}}\big)^{f r^{-1}}, f r^{-1} f^{-1}\Big) \\
&= \Big(a + \alpha(r)^f + (-a)^{(r')^{-1}}, (r')^{-1}\Big) \\
&= \big(a + \alpha(r)^f - a, (r')^{-1}\big) \\
&= \big(\alpha(r'), (r')^{-1}\big)
\end{aligned}$$

which belongs to U. Thus U is a normal subgroup of V completing the proof of the claim.

Set $E := V/U$. We have an inclusion $A \hookrightarrow E$ given by $a \mapsto [(a,1)]$. The map $p : V \to G$ given by $(a,f) \mapsto q(f)$ restricts trivially to E and so induces $\bar{p} : E \to G$ whose kernel is A. Thus we have produced an extension $A \rightarrowtail E \twoheadrightarrow G$ of G by A. Define $\phi : F \to E$ by $\phi(f) = [(0,f)]$. The restriction of ϕ to kernels is $\alpha : R \to A$ creating a diagram

$$\begin{tikzcd}
R \arrow[r, tail] \arrow[d, "\alpha"] & F \arrow[r, two heads, "q"] \arrow[d, "\phi"] & G \arrow[d, equal] \\
A \arrow[r, tail] & E \arrow[r, two heads, "\bar{p}"] & G
\end{tikzcd}$$

Naturality of Theorem 6.6.12 then gives a diagram

$$\begin{tikzcd}
H^1(E,A) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbb{Z}[G]}(A,A) \arrow[r] \arrow[d, "\bar{\alpha}^*"] & H^2(G;A) \arrow[r] \arrow[d, equal] & H^2(E,A) \arrow[d] \\
H^1(F,A) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}[G]}(R_{\mathrm{ab}},A) \arrow[r] & H^2(G;A) \arrow[r] & 0
\end{tikzcd}$$

which, by definition of $\bar{\alpha}$, shows that our chosen element of $H^2(G,A)$ lies in the image of θ.

We next show that θ is injective. Suppose that $(A \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} G)$ and $(A \overset{i'}{\rightarrowtail} E' \overset{p'}{\twoheadrightarrow} G)$ are extensions whose equivalence classes in M have the same image under θ. Let F be a free group on $E \amalg E'$. Let $f : F \to E$ and $f' : F \to E'$ be the canonical maps, and let $q : F \to G$ be the map whose restrictions to E and E' are p and p' respectively. Let R be the kernel of q. Then we have a diagram

$$\begin{tikzcd}
R \arrow[r, tail, "j"] \arrow[d, dotted, "\bar{f}"] & F \arrow[r, two heads, "q"] \arrow[d, "f"] & G \arrow[d, equal] \\
A \arrow[r, tail, "i"] & E \arrow[r, two heads, "\bar{p}"] & G
\end{tikzcd}$$

and a similar diagram for f'. The induced maps $\bar{f}$ and $\bar{f}'$ on kernels are surjective.

By naturality we have diagram

$$\begin{tikzcd}
H^1(E;A) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbb{Z}[G]}(A,A) \arrow[r, "\delta"] \arrow[d, "\bar{f}^*"] & H^2(G;A) \arrow[r] \arrow[d, equal] & H^2(E,A) \arrow[d] \\
H^1(F;A) \arrow[r, "j^*"] & \mathrm{Hom}_{\mathbb{Z}[G]}(R_{\mathrm{ab}},A) \arrow[r, "\hat{\delta}"] & H^2(G;A) \arrow[r] & 0
\end{tikzcd}$$

and a similar diagram for the other extension. The images of the extension under θ is defined as $\delta(1)$, so the commutativity of the diagram says that $\hat{\delta}\big(\bar{f}^*(1)\big) = \hat{\delta}\big(\bar{f}'^*(1)\big)$. Thus there exists $d \in H^1(F;A) = \mathrm{Der}(F,A)$ such that $j^*(d) = \bar{f}^*(1) - \bar{f}'^*(1)$. That is, we have a crossed homomorphism $d : F \to A$ such that $dj = \bar{f} - \bar{f}'$. Define $f'' : F \to E'$ by $f''(x) := \big(i'd(x)\big)\big(f'(x)\big)$. Note that the equation $dj = \bar{f} - \bar{f}'$ implies that the restriction of f'' to R equals the restriction of f to R.

We wish to check that f'' is a homomorphism. Writing a^g for the action of G on A and for its induced action on F (via q), for $x, y \in F$ we have

$$f''(xy) = i'd(xy)\, f'(xy) = i'(dx + (dy)^x)\, f'(x)\, f'(y) = i'(dx)\, i'\big((dy)^x\big)\, f'(x)\, f'(y)$$

The action of G on A (which, by definition of "extension" is the same for both extensions) is defined by conjugation with preimages under p'. Thus, using $\bar{p'}f = q$ gives

$$i'\big((dy)^x\big) = f'(x)\, i'dy\, \big(f'(x)\big)^{-1}.$$

Therefore

$$f''(xy) = i'(dx)\, f'(x)\, i'dy\, \big(f'(x)\big)^{-1} f'(x)\, f'(y) = i'(dx)\, f'(x)\, i'dy\, f'(y) = f''(x)f''(y)$$

The definition of f'' implies that the $f''(x)$ differs from $f'(x)$ by multiplication by an element of $A = \ker \bar{p}$ so the map $G \to G$ induced on cokernels by f'' is the same as that induced by f', namely the identity. Since 1_G is surjective and the restriction of f'' to the kernels is $\bar{f}'$ which is surjective, it follows by diagram-chasing (non-abelian Snake Lemma) that f'' is surjective.

For $e \in E$, choose $x \in F$ such that $f(x) = e$ and set $\Phi(e) := f''(x) \in E'$. If y also satisfies $f(y) = e$ then $q(x^{-1}y) = pf(x^{-1}y) = 1 \in G$ and so there exists $r \in R$ such that $j(r) = x^{-1}y$. Then $i\bar{f}(r) = fj(r) = e^{-1}e = 1_E$ which implies that $\bar{f}(r) = 1_A$, since i is an injection. Since, as above $f''j = i\bar{f} = fj$, we get $f''j(r) = 1_{E'}$ which implies that $f''(x) = f''(y)$, and so $\Phi : E \to E'$ is well defined. Given $a \in A$, choose $s \in R$ such that $\bar{f}(s) = a$. Then computing $\Phi(ia)$ using the lift js of ia shows that the restriction of Φ to A is the identity. Also, as above, the map on cokernels induced by f'' is 1_G and so the map of cokernels induced by Φ is 1_G. Thus Φ shows that $(A \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} G) \sim (A \overset{i'}{\rightarrowtail} E' \overset{p'}{\twoheadrightarrow} G)$. Hence θ is injective. $\square$

6.7 Ext and extensions

In this section we discuss a relationship between Ext^n and extensions (explaining the name) analogous to the interpretation of $H^2(G;A)$ in the preceding section. This material is not specifically about group cohomology but is presented in this chapter because of the analogy with the previous section.

An exact sequence of the form $0 \to B \to E_n \to E_{n-1} \to \ldots \to E_1 \to A \to 0$ is called an n-extension of A by B. Let $\sim$ be the equivalence relation generated by $(0 \to B \to E_n \to E_{n-1} \to \ldots \to E_1 \to A \to 0 \sim (0 \to B \to E'_n \to E'_{n-1} \to \ldots \to E_1 \to A \to 0)$ if there exists a diagram

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & E_n \arrow[r] \arrow[d] & E_{n-1} \arrow[r] \arrow[d] & \ldots \arrow[r] & E_1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & E'_n \arrow[r] & E'_{n-1} \arrow[r] & \ldots \arrow[r] & E'_1 \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

(Unlike the case $n = 1$ where the equalities at the ends force the middle arrow to be an isomorphism, in general this relation would not be symmetric if we had not extended it by taking the equivalence relation it generated.) Let $E^n(A,B)$ be the set of equivlances classes of extensions of A by B under this equivalence relation.

A map $\alpha : A' \to A$ induces a map $E^n(A',B) \to E^n(A,B)$ as follows. Let E'_1 be the pullback of $A' \to A$ and $E_1 \to A$. Then $\ker(E'_1 \to A') \cong \ker(E \to A)$ and so we have a map $E_2 \to \ker(E'_1 \to A') \rightarrowtail E'_1$ which makes $E_2 \to E'_1 \to A$ and $E_3 \to E_2 \to E'_1$ exact. Define

$$\alpha^*\big([0 \to B \to E_n \to E_{n-1} \to \ldots \to E_1 \to A]\big) := [0 \to B \to E_n \to E_{n-1} \to \ldots \to E'_1 \to A']$$

Check that α^* is well defined (exercise) and $E(A,B)$ is a contravariant functor in the first variable. Similarly, given $\beta : B \to B'$, use pushout to define E'_n to get a map

$$\beta_*\big([0 \to B \to E_n \to E_{n-1} \to \ldots \to E_1 \to A]\big) ::= [0 \to B \to E'_n \to E_{n-1} \to \ldots \to E_1 \to A]\big)$$

turning $E(A,B)$ into a covariant functor in the second variable.

Theorem 6.7.1 $E^n(A,B) \cong \mathrm{Ext}^n(A,B)$

Proof: Define $\Theta : E^n(A,b) \to \mathrm{Ext}^n(A,B)$ as follows.

Set $K_n := \ker(P_{n-1} \to P_{n-2} \rightarrowtail P_{n-1}$. Then Proposition 3.1.8 yields an exact sequence $\mathrm{Hom}(P_{n-1},B) \to \mathrm{Hom}_(K_n,B) \xrightarrow{\tau} \mathrm{Ext}^n(A,B) \to 0$

Let P_* be a projective resolution of A. Let $[0 \to B \to E_n \to E_{n-1} \to \dots \to E_1 \to A]$ belong to $E_n(A, B)$. Set $D_* =\to 0 \to \dots 0 \longrightarrow B \to E_n \to E_{n-1} \to \dots \to E_1$. Then D_* is acyclic with $H_0(D_*) = A$. By Theorem 3.1.3 there exists a chain map $\phi : P_* \to D_*$ inducing the identity on $H_0(P_*) = H_0(D_*) = A$. That is, we have a commutative diagram

$$\begin{tikzcd}
P_{n+1} \arrow[r] & P_n \arrow[r] \arrow[d, "\phi_n"] & P_{n-1} \arrow[r] \arrow[d, "\phi_{n-1}"] & P_{n-2} \arrow[r] \arrow[d, "\phi_{n-1}"] & \dots \arrow[r] & P_0 \arrow[r] \arrow[d, "\phi_0"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] & E_n \arrow[r] & E_{n-1} \arrow[r] & \dots \arrow[r] & E_1 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

By exactness, we have an induced map $\eta : P_n \twoheadrightarrow K_n$ and ϕ_n factors as $\phi_n = \phi\eta$ wghere $\phi : K_n \to B$.

Define $\Theta([0 \to B \to E_n \to E_{n-1} \to \dots \to E_1 \to A]) := \tau(\phi)$. Check that Θ is well defined. (Exercise)

Conversely given an element $x \in \operatorname{Ext}^n(B, A)$ pick $\phi : K_n \to B$ such that $x = \tau(\phi)$. Let E be the pushout of ϕ with $i : K_n \rightarrowtail P_{n-1}$. The induced map $E \twoheadrightarrow K_{n-1} : \ker(P_{n-2} \to P_{n-3}$ makes $0 \to B \to E \to P_{n-2} \to P_{n-3}$ exact, where $E \to P_{n-2}$ is the compositive $E \twoheadrightarrow K_{n-1} \rightarrowtail P_{n-2}$. Define

$$\Psi(x) := [B \to E \to P_{n-2} \to P_{n-3} \to \dots \to P_0 \to A]$$

Check that $\Psi : \operatorname{Ext}^n(A, B) \to E^n(A, B)$ is a well defined inverse to Θ. □

6.8 (Co)Homology of cyclic groups

Let $G = \mathbb{Z}/(n\mathbb{Z})$. Then $\mathbb{Z}[G] \cong \mathbb{Z}[t]/(t^n - 1)$ where t is a generator of G. Set $N := 1 + t + t^2 + \dots + t^{n-1}$. Then $tN = 0$ in $\mathbb{Z}[G]$.

$$\dots \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{1-t} \mathbb{Z}[G] \to \dots \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{1-t} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{1-t} \mathbb{Z}[G] \tag{6.2}$$

forms a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}[G]$ module. Applying $\mathbb{Z} \otimes_{\mathbb{Z}[G]} (\ )$ to the sequence 6.2 gives

$$\dots \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to \dots \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

Taking homology gives

$$H_q\big(\mathbb{Z}/(n\mathbb{Z})\big) = \begin{cases} \mathbb{Z} & \text{if } q = 0 \\ \mathbb{Z}/(n\mathbb{Z}) & \text{if } q \text{ is odd} \\ 0 & \text{if } q \text{ is even and } q > 0 \end{cases}$$

Similarly, applying Applying $\mathrm{Hom}_{\mathbb{Z}[G]}(\ ,\mathbb{Z})$ to the sequence 6.2 gives

$$Z \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to \dots \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to \dots .$$

and so taking cohomology gives

$$H^q\big(\mathbb{Z}/(n\mathbb{Z})\big) = \begin{cases} \mathbb{Z} & \text{if } q = 0 \\ 0 & \text{if } q \text{ is odd} \\ \mathbb{Z}/(n\mathbb{Z}) & \text{if } q \text{ is even and } q > 0 \end{cases}$$

Alternatively, the cohomology result could be derived from the homology result by means of the Universal Coefficient Theorem.

In the special case $n = 2$, geometrically this result corresponds to the fact that the Eilenberg-Mac Lane space $K(\mathbb{Z}/(2\mathbb{Z}), 1)$ is the projective space $\mathbb{R}P^\infty$.

Next consider the case where G is the infinite cyclic group $\mathbb{Z}$. Then $\mathbb{Z}[G] = Z[t]$

$$0 \to \dots \to 0 \to \mathbb{Z}[G] \xrightarrow{1-t} \mathbb{Z}[G] \tag{6.3}$$

forms a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}[G]$ module. Applying $\mathbb{Z} \otimes_{\mathbb{Z}[G]} (\ )$ to the sequence 6.2 gives

$$0 \to \dots \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

and then taking homology gives

$$H_q(\mathbb{Z}) \cong \mathbb{Z} \begin{cases} \mathbb{Z} & \text{if } q = 0 \text{ of } q = 1 \\ 0 & \text{otherwise} \end{cases}$$

Similarly, or by means of the Universal Coefficient Theorem, we compute

$$H^q(\mathbb{Z}) \cong \mathbb{Z} \begin{cases} \mathbb{Z} & \text{if } q = 0 \text{ of } q = 1 \\ 0 & \text{otherwise} \end{cases}$$

Geometrically this corresponds to the statement that the Eilenberg-Mac Lane space $K(\mathbb{Z}, 1)$ is S^1.

Bibliography

[1] J. F. Adams. "Stable Homotopy and Generalized Homology", Chicago University Press.

[2] M. Atiyah, F. Hirzebruch. "Vector bundles and homogeneous spaces", *Proc. Symp. Pure Math.* **3** (1969), 7–38.

[3] J. M. Boardman. "Conditionally convergent spectral sequences", *Contemporary Math.* **239** (1999), 49–84.

[4] A. K. Bousfield, D. M. Kan. "Homotopy Limits, Completions and Localization", Lecture Notes in Math **304**, Springer-Verlag (1972).

[5] H. Cartan, S. Eilenberg. "Homological Algebra", Princeton University Press (1956).

[6] Eilenberg, J. Moore. "Homology and fibrations I. Coalgebras cotensor product and its derived functors", *Comment. Math. Helv.***40** (1962), 199–236.

[7] A. Grothendieck. "Sur quelques points d'algèbre homologique" *Tôhoku Math. J.***9** (1957), 119–221.

[8] P. J. Hilton, U. Stammbach. "A Course in Homological Algebra", Graduate Texts in Mathematics **4**, Springer-Verlag (1970).

[9] J.-L. Koszul. "Sur l'opérateurs de derivation dans un anneau" *C.R. Acad. Sci. Paris* **225** (1947), 217–219.

[10] J. Leray. "L'anneau d'une representation; Propriétés d'homologie de la projections d'un espace fibré sur sa base; Sur l'anneau d'homologie de l'espace homogène, quotient d'un groupe clos par un sous-groupe abélien, connexe, maximum", *C.R. Acad. Sci.* **222** (1946), 1366–1368; 1419–1422; **223** (1946) 412–415.

[11] W. Massey. "Exact couples in algebraic topology I, II, III, IV, V", *Ann. of Math.* **56** (1952), 363–396; **57** (1953), 248–286.

[12] J. McCleary. "User's Guide to Spectral Sequences", Cambridge Studies in Advanced Mathematics **58**, Cambridge University Press (2001).

[13] J. Milnor. "Constructions of universal bundles, II", *Ann. of Math* **63** (1956), 430–436.

[14] J. Milnor. "On axiomatic homology theory", *Pac. J. of Math* **12** (1962), 337–341.

[15] P. Selick. "Introduction to Homotopy Theory", Fields Institute Monographs **9**, American Mathematical Society (1997).

[16] J.-P. Serre. "Homologie singulière des espaces fibrés", *Ann. of Math* **54** (1951), 425–505.

[17] L. Smith. "Lectures on the Eilenberg-Moore Spectral Sequence", Lecture Notes in Math **134**, Springer-Verlag (1969).

[18] E. C. Zeeman. "A proof of the comparison theorem for spectral sequences", *Proc. Camb. Phil. Soc.* **54** (1958), 57–62.