GENERALIZED AFFINE SPRINGER THEORY AND HILBERT SCHEMES ON PLANAR CURVES

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Abstract. We show that Hilbert schemes of planar curve singularities and their parabolic variants can be interpreted as certain generalized affine Springer fibers for GL_n, as defined by Goresky-Kottwitz-MacPherson. Using a generalization of affine Springer theory for Braverman-Finkelberg-Nakajima's Coulomb branch algebras, we construct a rational Cherednik algebra action on the homology of the Hilbert schemes, and compute it in examples. Along the way, we generalize to the parahoric setting the recent construction of Hilburn-Kamnitzer-Weekes, which may be of independent interest. In the spherical case, we make our computations explicit through a new general localization formula for Coulomb branches. Via results of Hogancamp-Mellit, we also show the rational Cherednik algebra acts on the HOMFLY homologies of torus knots. This work was inspired in part by a construction in three-dimensional $\mathcal{N} = 4$ gauge theory.

1. Introduction

Let $\widehat{C} = \operatorname{Spec} \frac{\mathbb{C}[[x,t]]}{f}$ be the germ of a complex plane curve singularity. In this paper, we investigate a relationship between the Hilbert scheme of points on $\widehat{C}$ (plus its parabolic flag versions) and certain generalized affine Springer fibers in the sense of [GKM06].

The Hilbert schemes of points on singular curves have been objects of intense study due to their connections to a wide range of topics including knot theory [ORS18, GORS14], representation theory [GORS14, Nak97, Kiv19, OY16, EGL15], and curve counting [PT10, Pan]. Affine Springer fibers, and their various generalizations, have also seen a wide range of study in combinatorics [Hik14], geometry [Ngô04, LS91], number theory [Ngô04, Yun16], and representation theory [OY16, VV09].

1.1. Hilbert schemes and affine Springer fibers.

We now describe our approach in some detail. Using the classical interpretation of torsion-free modules of $R := \frac{\mathbb{C}[[x,t]]}{f}$ as lattices in the total ring of fractions $\operatorname{Frac}(R)$, one can identify classical affine Springer fibers for GL_n with compactified Picard schemes of singular locally planar curves [LN08, MY14], which was a starting point for Ngô's proof of the fundamental lemma.

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When the polynomial f is irreducible, the classical affine Springer fibers for SL_n can further be related to compactified Jacobians of the singularity. The compactified Jacobians have been related to the representation theory of rational Cherednik algebras by means of affine Springer theory and a perverse filtration [OY16] which, thanks to results of Maulik-Yun and Migliorini-Shende [MY14, MS13], we know comes from the Hilbert schemes of points via an Abel-Jacobi map.

We take the relation between affine Springer theory of GL_n and Hilbert schemes further by interpreting (flags of) ideals of $R := \frac{\mathbb{C}[[x,t]]}{f}$ as (flags of) lattices in $\mathrm{Frac}(R)$. These moduli spaces of lattices also have a realization as *generalized* affine Springer fibers in the sense of [GKM06].

Recall that the generalized affine Springer fibers of a reductive group are sub(ind-)schemes of partial affine flag varieties, depending on a representation $N \in \mathrm{Rep}(G)$, a parahoric subgroup $\mathbf{P}$ and a lattice $N_{\mathbf{P}} \subset N_{\mathcal{K}}$. More precisely, they are "fibers" of the map

$$G_{\mathcal{K}} \times_{\mathbf{P}} N_{\mathbf{P}} \to N_{\mathcal{K}}$$

and can be thought of as affine generalizations of Hessenberg varieties. In particular, our first main result is the following. We only state it in the spherical case to keep this introduction more readable.

Theorem 1.1 (Theorem 3.5)**.** *Let $\widehat{C} := \mathrm{Spec}\, R$ be a germ of a plane curve singularity and write $R = \frac{C[[x,t]]}{f}$. If f has x-degree n then there is a generalized $\mathrm{Ad} \oplus V$-affine Springer fiber $M_v \subset \mathrm{Gr}_{GL_n}$ so that there is an isomorphism of (ind-)varieties*

$$\varphi : M_v \to \mathrm{Hilb}^{\bullet}(\widehat{C}).$$

Similarly, the parabolic flag Hilbert schemes from e.g. [GSV20] and the incidence varieties in [ORS18] defined in terms of flag Hilbert schemes of $\widehat{C}$ have natural interpretations as parahoric affine Springer fibers, as we explain in Theorem 3.5.

Remark 1.2. By the Weierstrass preparation theorem, there is no loss of generality in assuming that f has finite degree in x.

Remark 1.3. While it would be tempting to interpret *all* generalized affine Springer fibers of this form as variants of Hilbert schemes of points, a moment's thought shows that this is not possible.

1.2. **Generalized affine Springer theory.** The (co)homologies of the classical affine Springer fibers have an action of the trigonometric double affine Hecke algebra, at least in the "homogeneous cases" [OY16, VV09], similar to the classical Springer action of the graded affine Hecke algebra on the (co)homologies of Springer fibers. Therefore, it is natural to expect there to be a Springer-type action of some algebra on the homologies of the generalized affine Springer fibers as well, for arbitrary (G, N) (see [BFN16b, Remark 3.9.(4)]), and in particular in this case of Hilbert schemes of points.

This turns out to be the case, as recently explored by Hilburn-Kamnitzer-Weekes [HKW20] in the spherical case. The algebras in question turn out to be the ("three-dimensional $\mathcal{N} = 4$") *Coulomb branch algebras*, as mathematically defined in [BFN16b] by a convolution algebra construction modeled on the affine Grassmannian (and in our case, other partial affine flag varieties). We generalize the results of [HKW20] to their natural maximum, allowing in particular for generalized affine Springer fibers in any partial affine flag variety. Presumably, combining our construction with the results of [HKW20] will give us more insight to the nature of Springer representations of various algebras arising as Coulomb branches.

It was shown by Kodera-Nakajima [KN18] that the Coulomb branch algebra, for the datum $G = GL_n$ and $N = \mathrm{Ad} \oplus V$, where Ad is the adjoint representation and V is the vector representation, is isomorphic to the spherical rational Cherednik algebra of $\mathfrak{gl}_n$. In addition, [Web19, BEF16] prove that the Iwahori version of the Coulomb branch in question is naturally isomorphic to the full rational Cherednik algebra. See Theorem 4.12 and subsequent discussion for the precise statements.

Combining the above ingredients, we find an action of the spherical rational Cherednik algebra of $\mathfrak{gl}_n$ on the equivariant Borel-Moore homology of $\mathrm{Hilb}^\bullet(\widehat{C})$ as a type of "generalized affine Springer theory" similar to the orbital variety version in [CG09, Section 6.5.]. The Iwahori generalization of this yields an action of the full rational Cherednik algebra on the parabolic flag Hilbert schemes $\mathrm{PHilb}^{[\bullet,\bullet+(1,\ldots,1)]}(\widehat{C})$. More precisely, we have

Proposition 1.4 (Theorem 4.9)**.** *The rational Cherednik algebra $\mathcal{H}_n$ of $\mathfrak{gl}_n$ acts on*

$$\bigoplus_{m\geq 0} H_*^{L_v}(\mathrm{PHilb}^{[m,m+(1,\ldots,1)]}(\widehat{C}))$$

and the spherical rational Cherednik algebra $e\mathcal{H}_n e$ acts on

$$\bigoplus_{m\geq 0} H_*^{L_v}(\mathrm{Hilb}^m(\widehat{C}))$$

via a natural convolution product.

This fits well with the results of [GORS14, ORS18, EGL15, OY16, GSV20], see e.g. Section 4.3.2, where we compare our calculations with the recent results of Gorsky-Simental-Vazirani.

For the case where the plane curve singularity $\widehat{C} = \widehat{C}_{n,k}$ is quasi-homogeneous and given by $f = x^n - t^k$, we find the above actions with parameter $m = -\frac{k}{n}\hbar$ (to match with most conventions, we must specialize $\hbar \to -1$) on the equivariant Borel-Moore homology with respect to the stabilizer $\mathbb{C}^\times \subset \mathbb{C}^\times_{rot} \times \mathbb{C}^\times_{dil}$ of a specific element $v \in N_\mathcal{O}$, realizing an expectation of [ORS18]. When $\gcd(n,k) = 1$, the Hilbert scheme of points $\mathrm{Hilb}^\bullet(\widehat{C}_{n,k})$ has isolated $\mathbb{C}^\times$-fixed points and we can take the analysis quite far. We compute the action in the basis of fixed points by means of an "abelianization procedure" akin to [BDGH16, BFN16a, DGGH19] in some cases.

Remark 1.5. This abelianization rests on the rather general localization formula in Proposition 4.21. According to the introduction of [HKW20], this might be of independent interest in showing the coincidence of the "BFN Springer action" on homologies of quasimaps' spaces and the $\mathcal{U}_\hbar\mathfrak{gl}_n$-action of Feigin-Finkelberg-Frenkel-Rybnikov on Laumon spaces [FFFR11].

As a concrete application of the previous Proposition, we prove the following Theorem (which is natural to expect to hold true, and was anticipated already in [GORS14]).

Theorem 1.6 (Theorem 5.4)**.** *When* $\gcd(n,k)=1$*, we have*

$$H_*^{\mathbb{C}^\times}(\mathrm{Hilb}^\bullet(\widehat{C}_{n,k})) \simeq eL_{k/n}(triv)$$

as modules for the spherical rational Cherednik algebra of $\mathfrak{gl}_n$*.*

Remark 1.7. For the case of $(2,2\ell+1)$ torus knots we show that this directly, see Appendix B. For the remaining cases the direct analysis becomes cumbersome, so we combine earlier work of [Kiv19, VR18, OY16] to conclude the result. It is however remarkable that our approach is, in principle, amenable to completely explicit computation, when compared with e.g. [OY16]. We also note that Theorem 1.6 is compatible with the earlier results and conjectures of [VV09, OY16, ORS18, GORS14] relating modules for the spherical rational Cherednik algebra and $\mathrm{Hilb}^\bullet(\widehat{C}_{n,k})$.

1.3. **HOMFLY homology of torus knots.** The links of the quasi-homogeneous ($\widehat{C}_{n,k}=\{x^n=t^k\}$) singularities correspond to (positive) (n,k)-torus links, and it has been known for a while that the representations constructed above are closely connected with corresponding "lowest a-degree parts" of the HOMFLY-PT homologies of these links. In particular, our approach combined with recent results of Hogancamp-Mellit [HM19] (and the older philosophies of Gorsky-Oblomkov-Rasmussen-Shende [GORS14, ORS18]) quite directly shows the fact that the rational Cherednik algebra of $\mathfrak{gl}_n$ acts on these link homologies, *par transport de structure.* This is the subject of Section 5.

Remark 1.8. The higher a-degrees also have natural interpretations from the parahoric viewpoint, and the full Iwahori invariant is likely related to the annular invariant introduced in Trinh's thesis [Tri20]. Conditionally on the ORS conjecture [ORS18], our results also imply the rational Cherednik algebra acts on the HOMFLY homology of *any* algebraic link. We do not pursue these directions further.

Remark 1.9 (For the physically minded reader)**.** As is clear from the introduction, we were inspired in part by the physics of three-dimensional $\mathcal{N}=4$ gauge theory [DGH$^+$] and its relationship to a recent construction of the triply graded HOMFLY-PT homology [OR18], whereby the various a-degrees are realized within a certain category of matrix factorizations.

In the upcoming (companion) work [DGH$^+$], the construction of [OR18] is interpreted as a computation in the B-twist of $U(n)$ gauge theory with hypermultiplets transforming in the representation T^*R for $R = \mathrm{Ad} \oplus V$. For the ℓ-th possible a-degree, one computes the supersymmetric Hilbert space of the theory in the presence of a Wilson line in the representation $\bigwedge^\ell V$ subject to a certain boundary condition whose parameters specify the knot in question.

The three-dimensional mirror of this construction is a computation in the A-twist of the same theory. Again, one computes the supersymmetric Hilbert space of the theory but now in the presence of a vortex line and subject to a different boundary condition. The parameters of this boundary condition translate to the eigenvalues of one of the adjoint fields, which braid around one another along the boundary. For algebraic links, this computation can be reformulated algebraically and one finds that the supersymmetric Hilbert space associated to the lowest a-degree component of HOMFLY-PT homology can be computed as the homology of the generalized affine Springer fibers we discuss below.

In the general context of three-dimensional $\mathcal{N} = 4$ theories, the supersymmetric Hilbert spaces associated to boundary conditions and the action of the quantized Coulomb branch on them appeared previously in [BDGH16] and [BDG$^+$16], and we make their geometric action rigorous via the BFN presentation in Section 4. In many cases of interest, we can realize the action of the Coulomb branch using an "abelianization procedure," c.f. [BFN16a, BDG17, Web16].

A generalization of these Hilbert spaces, and the local operators that act upon them, that includes ($\frac{1}{2}$-BPS) vortex line operators appeared briefly in [BDGH16] and was the central aim of [DGGH19]. The results of the current paper have a straightforward generalization to higher a-degrees; namely, there is a generalization of the construction in Section 3 to the incidence varieties of [ORS18]. The homologies of these incidence varieties (supersymmetric Hilbert spaces in the presence of the above boundary conditions and vortex lines) are naturally endowed with actions of convolution algebras (the algebra of local operators bound to the vortex line) generalizing the Coulomb branch construction of BFN. Some features of this generalization will be discussed in [DGH$^+$]. Understanding the module structure of these homologies is a direction for future work.

Remark 1.10. Most of our results, including the computations with fixed-point localization, make sense over other algebraically closed fields, in particular $\overline{\mathbb{F}}_q$ with $\overline{\mathbb{Q}}_\ell$-coefficients in cohomology. But since it makes life easier, and the results of [BFN16b] are also written in the language of algebraic geometry over $\mathbb{C}$, we have decided to work over $\mathbb{C}$ throughout. This also makes the comparison to link homology more transparent.

The paper is organized as follows. In Section 2 we recall the necessary definitions of generalized affine Springer fibers M_v. In Section 3 we identify the generalized affine Springer fiber (for the datum $(GL_n, \mathrm{Ad} \oplus V)$) isomorphic to $\mathrm{Hilb}^\bullet(\widehat{C})$, and generalizations thereof, for $\widehat{C}$ the germ of a plane curve singularity. In Section 4 we define a convolution action of the quantized Coulomb branches of [BFN16b] on the equivariant (Borel-Moore) homology of the generalized affine Springer fibers M_v, specializing in particular to the action of the spherical rational Cherednik algebra on the equivariant homology of the Hilbert schemes. The proof that the convolution really defines an action is relegated to Appendix A. In Section 5 we discuss the quasi-homogeneous singularities $\widehat{C}_{n,k}$ related to (n,k) torus links and show how they relate to rational Cherednik algebra representations. In Appendix B we discuss $(2, 2\ell+1)$ torus knots in detail.

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2. Generalized affine Springer theory

This section is written in more generality than is needed for most of our main results. Let $G/\mathbb{C}$ be reductive, $\mathfrak{g} = \mathrm{Lie}(G)$, and N be an algebraic representation of G. Let $\mathcal{K} = \mathbb{C}((t))$ and $\mathcal{O} = \mathbb{C}[[t]]$. Let $\mathbf{P}$ be a parahoric subgroup of $G(\mathcal{K})$ and $N_{\mathbf{P}} \subset N(\mathcal{K})$ a lattice stable under $\mathbf{P}$. In later sections, we only use standard parahorics $\mathbf{P} \subset G(\mathcal{O})$ coming as preimages of parabolic subgroups in $G(\mathbb{C})$ via the "evaluation at zero" map, but it should be clear where this assumption can be dropped. Let Gr_G be the affine Grassmannian of G, Fl_G the affine flag variety of G and, more generally, $\mathrm{Fl}_{\mathbf{P}}$ the partial affine flag variety associated to $\mathbf{P}$. On the level of $\mathbb{C}$-points, $\mathrm{Fl}_{\mathbf{P}}(\mathbb{C}) = G(\mathcal{K})/\mathbf{P}$.

Definition 2.1. Let $v \in N(\mathcal{K})$. Define the *generalized affine Springer fiber* (GASF) associated to the datum $(v, \mathbf{P}, N_{\mathbf{P}})$ as the reduced ind-scheme whose closed points are

$$M_v^{\mathbf{P},N_{\mathbf{P}}}(\mathbb{C}) := \{g \in G(\mathcal{K}) | g^{-1}.v \in N_{\mathbf{P}}\}/\mathbf{P}.$$

Remark 2.2. Note that the definition of $M_v^{\mathbf{P},N_{\mathbf{P}}}$ also depends on G. Since we will only be working with $G = GL_n$, we mostly omit these from the notation. When $\mathbf{P} = G(\mathcal{O}), N_{\mathbf{P}} = \mathrm{Ad}(\mathcal{O}) \oplus \mathcal{O}^n$, we simply denote $M_v^{(\mathbf{P},N_{\mathbf{P}})}$ by M_v. Similarly, when $\mathbf{P} = \mathbf{I}, N_{\mathbf{P}} = \mathrm{Lie}(\mathbf{I}) \oplus \mathcal{O}^n$ for $\mathbf{I}$ an Iwahori subgroup we use $\widetilde{M}_v$ and, more generally, when $N_{\mathbf{P}} = \mathrm{Lie}(\mathbf{P}) \oplus \mathcal{O}^n$ we use $M_v^{\mathbf{P}}$.

Remark 2.3. The "classical" affine Springer fibers are the case when $N = \mathrm{Ad}$ and $N_{\mathbf{P}}$ is the Lie algebra of $\mathbf{P}$. As explained in [GKM06], the GASF can be thought of as an affine analog of Hessenberg varieties. Note that both our GASF and those of [GKM06] are different from the Kottwitz-Viehmann varieties, which are group versions of affine Springer fibers.

In [Vas05, VV09, OY16], an action of the (degenerate) double affine Hecke algebra of $\mathfrak{sl}_n$ was constructed on the equivariant (K-)homology of certain (usual) affine Springer fibers using the convolution algebra technique (see e.g. [CG09]).

Just as affine Springer fibers are a source of affine Springer representations of affine Weyl groups and Cherednik algebras, generalized affine Springer fibers can be used to construct representations of certain convolution algebras associated to the datum (G, N) as defined in [BFN16b]. These are the "quantized Coulomb branches" of three-dimensional $\mathcal{N} = 4$ field theories, or "BFN algebras." In the classical case $N = \mathrm{Ad}$, the K-theoretic analog of the Coulomb branch algebra is the DAHA, as explained e.g. in [FT19].

In particular, in [HKW20], the convolution algebra technique from above was extended to *any* Coulomb branch algebra. The authors of *loc. cit.* were kind enough to share their preliminary results on the topic with us, and we expand upon these results in Section 4 (which focuses on the $N = \mathrm{Ad} \oplus V$ case) and in Appendix A. We also define the maximal parahoric generalization of the generalized affine Springer theory, using natural variations of the techniques in [BFN16b, HKW20].

Remark 2.4. In analogy with [Yun11], we expect there to be a "global" Springer theory defined on certain generalized Hitchin spaces (quasimaps' spaces) at least for N with good invariant-theoretic properties. This direction will be pursued in future work.

3. Hilbert schemes of points on curve singularities

Let $\widehat{C} := \operatorname{Spec} R$ be the germ of a plane curve singularity and write $R = \frac{\mathbb{C}[[x,t]]}{f}$.

Definition 3.1. The *Hilbert scheme of m points on $\widehat{C}$* is defined as the reduced scheme

$$\widehat{C}^{[m]} := \mathrm{Hilb}^m(\widehat{C}) := \{\text{colength } m \text{ ideals in } R\}.$$

Similarly, given a partition $\vec{p} = (p_1, ..., p_d)$ of n, the *$\vec{p}$-flag Hilbert scheme of $m + n$ points on $\widehat{C}$* is defined as the reduced scheme

$$\widehat{C}^{[m,m+\vec{p}]} := \mathrm{Hilb}^{[m,m+\vec{p}]}(\widehat{C}) := \{I_d \subset ... \subset I_0 \subset R | I_i \text{ is a colength } m + \sum_{j=1}^{i} p_j \text{ ideal in } R\}.$$

Remark 3.2. In particular, the reduced scheme

$$\mathrm{Hilb}^{\bullet}(\widehat{C}) := \bigsqcup_{m\geq 0} \mathrm{Hilb}^m(\widehat{C})$$

is naturally the moduli space of finite length subschemes on $\widehat{C}$, whereas

$$\mathrm{Hilb}^{[\bullet,\bullet+\vec{p}]}(\widehat{C}) := \bigsqcup_{m\geq 0} \mathrm{Hilb}^{[m,m+\vec{p}]}(\widehat{C})$$

is naturally the moduli space of flags of such subschemes.

Remark 3.3. Requiring flags of ideals such that $I_d = tI_0$ puts a natural constraint on the allowed partitions p; if f is a polynomial in x of degree n then p must be a partition of n. When $\vec{p} = (1, \dots, 1)$ is the one-column partition of n, the relevant Hilbert scheme is the *parabolic flag Hilbert scheme* $\mathrm{PHilb}^\bullet(\widehat{C})$ (see e.g. [GSV20]), consisting of full flags of ideals of length n, with the condition that $I_n = tI_0$. More generally, if $\vec{p} = (p_1, ..., p_d)$ is any partition of n we can define the *$\vec{p}$-parabolic flag Hilbert scheme* $\mathrm{PHilb}^{[\bullet,\bullet+\vec{p}]}(\widehat{C})$.

Definition 3.4. The *$\vec{p}$-parabolic flag Hilbert scheme* $\mathrm{PHilb}^{[m,m+\vec{p}]}(\widehat{C})$ is defined as the reduced scheme

$$\mathrm{PHilb}^{[m,m+\vec{p}]}(\widehat{C}) := \{I^\bullet \in \mathrm{Hilb}^{[m,m+\vec{p}]}(\widehat{C}) | I_d = tI_0)\}.$$

We now state and prove our first main theorem.

Theorem 3.5. *For any $\widehat{C}$, there is a generalized* $\mathrm{Ad} \oplus V$*-affine Springer fiber $M_v \subset \mathrm{Gr}_G$ so that there is an isomorphism of schemes*

$$\varphi : M_v \to \mathrm{Hilb}^\bullet(\widehat{C}).$$

More generally, there is a generalized $\mathrm{Ad} \oplus V$*-affine Springer fiber*

$$M_v^{\mathbf{P}} \subset \mathrm{Fl}_{\mathbf{P}}$$

so that there is an isomorphism of schemes

$$\varphi_{\mathbf{P}} : M_v^{\mathbf{P}} \to \mathrm{PHilb}^{[\bullet,\bullet+\vec{p}]}(\widehat{C}).$$

Proof. Note that we can interpret $\widehat{C}$ and $\widehat{C}^{[m]}$ as follows. By Weierstrass preparation, we can assume $f(x,t)$ is a degree n polynomial in x. Then we may write as $\mathbb{C}[[t]] = \mathcal{O}$-modules that

$$\frac{\mathbb{C}[[x,t]]}{f} = \langle 1, x, \dots, x^{n-1} \rangle_{\mathcal{O}}, \tag{3.1}$$

where $\langle S \rangle_{\mathcal{O}}$ denotes the free $\mathcal{O}$-module generated by a set S.

Taking the total ring of fractions of R, we see that as $\mathbb{C}((t)) = \mathcal{K}$-vector spaces $\mathrm{Frac}(R) \cong (\mathcal{K}^n)^*$ ($\mathcal{K}$-linear dual of $\mathcal{K}^n$) as follows. If f is square-free so that $\widehat{C}$ is reduced, $\mathrm{Frac}(R) \cong \prod_{i=1}^d F_i$ where d is the number of irreducible factors over $\mathcal{K}$ of f and F_i are finite extensions of $\mathcal{K}$ so that $\sum_i [F_i : \mathcal{K}] = n$.

If f has a repeated factor, we take $R \cong \prod_{i=1}^d \mathcal{O}_i$ where each $\mathcal{O}_i$ is some finite ring extension of $\mathcal{O}$ which is torsion-free over $\mathcal{O}$. Since $\mathcal{O}$ is a domain, $\mathrm{Frac}(\mathcal{O}_i) \cong \mathcal{O}_i \otimes_{\mathcal{O}} \mathcal{K}$. In particular, $\mathrm{Frac}(R) \cong (\mathcal{K}^n)^*$.

There is a natural injection $R \hookrightarrow \mathrm{Frac}(R)$, and we choose an isomorphism $\phi^* : \mathrm{Frac}(R) \cong (\mathcal{K}^n)^*$ identifying R with $(\mathcal{O}^n)^*$ and $1 \in R$ with the vector $e_1^* = (1, 0, \dots, 0)$ in $(\mathcal{K}^n)^*$. We may moreover choose ϕ^* so that in the costandard basis of $(\mathcal{K}^n)^*$, x

has the form

$$\gamma = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{pmatrix}$$

so that $\{e_k^* = e_1^* \gamma^{k-1}\}_{k=1}^n$ is a $\mathcal{O}$-basis of $(\mathcal{O}^n)^*$. Recall that a matrix of the above form is called the *companion matrix* of the polynomial $x^n - a_{n-1}x^{n-1} - \cdots - a_1 x - a_0$. Also, it is worth noting that $\gamma e_{k+1} = e_k + a_k e_n$ for $k = 1, ..., n-1$.

By definition, $\mathcal{O}$-lattices in $(\mathcal{K}^n)^*$ stable under γ are identified with (nonzero) fractional R-ideals. The variety of nonzero ideals of finite codimension in R is then identified with fractional ideals in $\mathrm{Frac}(R)$ contained in R. Indeed, note that the condition of being a lattice implies that tensoring Λ with $\mathcal{K}$ and projecting to each factor of $\mathcal{K}$ is a surjective map, hence the corresponding ideal is of finite codimension. Under ϕ, we get

$$\mathrm{Hilb}^\bullet(\widehat{C}) \cong X := \{\Lambda \subset (\mathcal{O}^n)^* | \Lambda\gamma \subset \Lambda\}.$$

Now for any Λ, there is an element $g \in G(\mathcal{K})$ so that $\Lambda = (\mathcal{O}^n)^* g^{-1}$. It is well defined up to the stabilizer of $(\mathcal{O}^n)^*$, which is $G(\mathcal{O})$. If $\Lambda \subset (\mathcal{O}^n)^*$ and $\Lambda\gamma \subset \Lambda$, we have

(1) $g^{-1} \in G(\mathcal{K}) \cap \mathfrak{gl}_n(\mathcal{O})$, because $(\mathcal{O}^n)^* g^{-1} = \Lambda \subset (\mathcal{O}^n)^*$, and
(2) $g^{-1}\gamma g \in \mathrm{Ad}(\mathcal{O})$, because $(\mathcal{O}^n)^* g^{-1}\gamma g = \Lambda\gamma g \subset \Lambda g = (\mathcal{O}^n)^*$ and the stabilizer of $(\mathcal{O}^n)^*$ is $\mathfrak{gl}_n(\mathcal{O}) = \mathrm{Ad}(\mathcal{O})$.

If e_i denotes the standard basis in $\mathcal{K}^n$, the first point implies that $g^{-1}e_n$ belongs to $\mathcal{O}^n$.

Let $v := (\gamma, e_n)$ and consider the map

$$\Lambda \mapsto [g]$$

from X to the scheme

$$M_v = \{[g] \in \mathrm{Gr}_G \,| g^{-1}\gamma g \in \mathrm{Ad}(\mathcal{O}), g^{-1}e_n \in \mathcal{O}^n\}.$$

We will construct an inverse to this map. Given any $[g] \in M_v$, we have

(1) $g^{-1} \in G(\mathcal{K}) \cap \mathfrak{gl}_n(\mathcal{O})$, because $g^{-1}e_n \in \mathcal{O}^n$, $g^{-1}\gamma g \in \mathrm{Ad}(\mathcal{O})$ and
$$g^{-1}e_k = (g^{-1}\gamma g)g^{-1}e_{k+1} - a_k g^{-1}e_n \in \mathcal{O}^n$$
for $k = 1, ..., n-1$, and
(2) $(\mathcal{O}^n)^* g^{-1}\gamma \subset (\mathcal{O}^n)^* g^{-1}$, because $g^{-1}\gamma g \in \mathrm{Ad}(\mathcal{O})$.

The first point implies that $\Lambda = (\mathcal{O}^n)^* g^{-1} \subset (\mathcal{O}^n)^*$ and the second implies Λ is closed under the action of γ, i.e. $\Lambda \in X$. As these constructions are inverse to each other, we have $X \cong M_v$.

Finally, composing with the isomorphism to $\mathrm{Hilb}^\bullet(\widehat{C})$ we get that

$$\mathrm{Hilb}^\bullet(\widehat{C}) \cong M_v.$$

By Definition 2.1 the space M_v is the generalized $\mathrm{Ad} \oplus V$-affine Springer fiber for $v = (\gamma, e_n)$.

Now choose a partition $\vec{p} = (p_1, ..., p_d)$ of n and let $\mathbf{P}$ be the corresponding parahoric subgroup. From the above we know that a flag of ideals $tI_0 = I_d \subset ... \subset I_0 \subset R$ can be identified with a flag of lattices $t\Lambda_0 = \Lambda_d \subset ... \subset \Lambda_0 \subset (\mathcal{K}^n)^*$, such that each lattice is closed under the action of γ. Identifying such a flag with an element $[g] \in \mathrm{Fl}_{\mathbf{P}}$ implies that $g^{-1}.v \in \mathrm{Lie}(\mathbf{P}) \oplus \mathcal{O}^n$. Just as above, the identification $tI_0 \subset ... \subset I_0 \subset R \leftrightarrow [g]$ yields the desired isomorphism with $M_v^{\mathbf{P}}$. □

Remark 3.6. It is interesting to consider the generalized affine Springer fiber over the same v as above but with $N_{\mathbf{P}} \neq \mathrm{Lie}(\mathbf{P}) \oplus \mathcal{O}^n$. One such variant yields the incidental varieties "$C^{[m \leq m+l]}$" (note the notational difference to this paper) of [ORS18], where we choose the partition $(l, n-l)$ and require the $\mathrm{Ad}(\mathcal{O})$ element to be proportional to t in the first l columns. This choice of $N_{\mathbf{P}}$ ensures that the flag of lattices $t\Lambda_0 \subset \Lambda_1 \subset \Lambda_0$ satisfies $\Lambda_0\gamma \subset \Lambda_1$. In terms of ideals, this latter point implies that $MI_0 \subset I_1 \subset I_0$, where $M = \langle x, t\rangle$ is the maximal ideal of R. See [DGH$^+$] for more details. Note that in the $G = SL_n, N = \mathrm{Ad}$-case similar incidental varieties appear in the work of Cherednik and Philipp [CP18] under the name of flagged Jacobian factors.

Remark 3.7. An equivalent, perhaps preferred, description of $\mathrm{Hilb}^\bullet(\widehat{C})$ is as lattices $\Lambda \subset \mathcal{O}^n$. If we identify $1 \leftrightarrow e_1$, then following the above proof one finds an isomorphism to the generalized $\mathrm{Ad} \oplus V^*$-affine Springer fiber M'_w for the vector $w = (\gamma^T, e_n^*) \in \mathrm{Ad}(\mathcal{O}) \oplus (\mathcal{O}^n)^*$, c.f. [Yun16].

Remark 3.8. Note that the proof doesn't assume $\widehat{C}$ to be reduced. In particular, this suggests us to define the "compactified Picard variety" $\overline{\mathrm{Pic}}(\widehat{C})$ for these non-reduced curves as the classical GL_n-affine Springer fiber, although it is usually not considered in the literature. For example, when γ is the regular nilpotent matrix, the ASF in question gives an infinite-dimensional affine Springer fiber whose homology coincides with that of the affine Grassmannian. Similarly, the GASF in question yields the Hilbert schemes of points on the non-reduced curve $\{x^n = 0\}$, which are now finite-dimensional projective subvarieties of the "negative part" of the affine Grassmannian (as opposed to the *positive part*, i.e. the lattices containing $\mathcal{O}^n$, although somewhat misleadingly these contain exactly torus-fixed lattices with negative exponents in their defining cocharacters).

Remark 3.9. More generally, note that by

$$\mathrm{Hilb}^\bullet(\widehat{C}) \cong \{\Lambda \subset (\mathcal{O}^n)^* | \Lambda\gamma \subset \Lambda\}$$

we may identify $\mathrm{Hilb}^\bullet(\widehat{C})$ as the intersection

$$\mathrm{Sp}_\gamma \cap \mathrm{Gr}^-_{GL_n}$$

where Sp_γ is the "usual" ($N = \mathrm{Ad}$) affine Springer fiber of γ and $\mathrm{Gr}^-_{GL_n}$ is the negative part of the affine Grassmannian

$$\mathrm{Gr}^-_{GL_n} := \{(\mathcal{O}^n)^* \subset \Lambda \subset (\mathcal{K}^n)^*\}$$

not to be confused with the "negative Grassmannian" which is a distantly related object of intense research. See also [Kiv18, Remark 4.24].

Remark 3.10. Using the decomposition of Gr_G by $\pi_1(G) = \mathbb{Z}$ we find that M_v can be expressed as

$$M_v = \bigsqcup_{m \le 0} M_v^m,$$

where M_v^m is the component of M_v inside the degree m part of Gr_G. Indeed, we have $M_v^m = \mathrm{Hilb}^{|m|}(\widehat{C})$. Thus M_v is a (infinite) disjoint union of projective varieties, because the Hilbert schemes are projective [Gro]. There is a similar decomposition of $M_v^{\mathbf{P},N_{\mathbf{P}}}$ obtained from the decomposition of $\mathrm{Fl}_{\mathbf{P}}$ by $\pi_1(G)$, coming via pullback by the projection $\mathrm{Fl}_{\mathbf{P}} \to \mathrm{Gr}_G$.

3.1. **Links and torus actions.** If $f(x,t)$ is a polynomial, we may interpret $\widehat{C}$ as the germ of the curve $C = \{f = 0\} \subset \mathbb{C}^2$. In this case, the intersection of C with a small three-sphere centered at the origin yields a compact one-manifold

$$\mathcal{L} := \mathrm{Link}_0(C) \hookrightarrow S^3.$$

By work of Oblomkov-Rasmussen-Shende and others (see [Mig19] and references therein) it is conjectured that, topologically, the Hilbert schemes of $\widehat{C}$ are controlled by the HOMFLY-PT homology of the corresponding link $\mathcal{L}$.

Consider f of the form $f = x^n - t^k$ for $n, k \ge 0$. The special form of f in this case means that the singularity is *quasi-homogeneous*, so there is a straightforward $\mathbb{C}^\times$ action on $\mathcal{M}_{(n,k)} := M_v$ coming from scaling x and t. As has been noted by various authors, we thus get an extra torus action on the Hilbert schemes. This is more nontrivial on the generalized affine Springer fiber side.

Namely, let $1 \to G \to \widetilde{G} \to G_F \to 1$ be an extension of algebraic groups over $\mathbb{C}$ and let $\widetilde{G}^{\mathcal{O}}_{\mathcal{K}}$ be the preimage in $\widetilde{G}_{\mathcal{K}}$ of $G_{F,\mathcal{O}}$. With our definition of M_v, we always have an action of the stabilizer of v in $\widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^\times_{rot}$ on M_v (see the next section). Let $G = GL_n, G_F = \mathbb{C}^\times_{dil}, \widetilde{G} = GL_n \times \mathbb{C}^\times_{dil}$, where $\mathbb{C}^\times_{dil}$ acts by dilating the Ad-part in $\mathrm{Ad} \oplus V$. This action is considered in [OY16] in the case of usual affine Springer fibers, where $\mathbb{C}^\times_{rot}, \mathbb{C}^\times_{dil}$ are denoted $\mathbb{G}_m^{rot}, \mathbb{G}_m^{dil}$. For $v = (\gamma, e_n)$ corresponding to $f = x^n - t^k$ as in Theorem 3.5, the stabilizer is given as follows. It is worth noting that we use different conventions from the usual (physical) conventions used for $\mathbb{C}^\times_{rot}$ in some of the literature [BDG17, BDGH16, BDG+16, BFN16b] . In particular, we do *not* include the overall scaling of N by weight $\frac{1}{2}$ in addition to scaling t. These conventions are those used by Webster, see e.g. [Web16].

Lemma 3.11. *We have*

$$L_v := Stab_{\widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^\times_{rot}}(v) \cong \mathbb{C}^\times.$$

Proof. Consider acting with $(g, \mu, \lambda) \in \widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^\times_{rot}$ on $v = (\gamma, e_n)$ for v corresponding to $f = x^n - t^k$. Here μ denotes the flavor part of $\widetilde{g} = (g, \mu) \in \widetilde{G}^{\mathcal{O}}_{\mathcal{K}}$. Preserving the determinant of γ imposes the equation

$$\mu^n \lambda^k = 1.$$

Preserving e_n then says that the last column of g is e_n, thus the last column of g^{-1} is also e_n. From this, we find that the last column of $g\gamma g^{-1}$ is the penultimate column of $g\mu$, so we need this column of g to be $\mu^{-1}e_{n-1}$ for g to preserve the last column of $g\gamma g^{-1}$. This process continues column-by-column so we must have

$$g = \mathrm{diag}(\mu^{1-n}, \dots, \mu^{-1}, 1).$$

In particular, the stabilizer is the image of the cocharacter $\mathbb{C}^\times \to \widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^\times_{rot}$ given by

$$\nu \mapsto (\mathrm{diag}(\nu^{(n-1)k}, \dots, \nu^k, 1), \nu^{-k}, \nu^n).$$

□

Remark 3.12. In general, for inhomogeneous γ, it's *always* the case that the stabilizer is trivial by a similar argument. On the other hand, the same proof shows that γ for the curve $\{x^n = 0\}$ has stabilizer $(\mathbb{C}^\times)^2$ given by $(\mathrm{diag}(\mu^{1-n}, \dots, \mu^{-1}, 1), \mu, \lambda)$.

Remark 3.13. Since Sp_γ has a $T \times \mathbb{C}^\times_{rot}$-action in the non-coprime/multiple component case and $\mathrm{Gr}^-_{GL_n}$ is a stable subset for this action, we also get a large torus action on $\mathrm{Hilb}^\bullet(\widehat{C})$. This has not been considered in the literature and seems harder to describe from the point of view of the Hilbert scheme.

Proposition 3.14. *In the case* $\gcd(n, k) = 1$*, the action of* L_v *on* M_v *has isolated fixed points labeled by cocharacters* A *of the maximal torus* $T \subset GL_n$ *such that*

$$\langle A, \omega_n \rangle \geq 0 \qquad \langle A, \alpha_i \rangle \geq 0 \qquad \sum_{i=1}^{n-1} \langle A, \alpha_i \rangle \leq k, \tag{3.2}$$

where ω_n *is the* n*-th fundamental weight of* GL_n*,* α_i *are the simple roots of* GL_n *and* $\langle , \rangle$ *is the pairing of cocharacters and weights.*

Remark 3.15. If we write $A = (A_1, ..., A_n)$ the above constraint corresponds to

$$0 \leq A_n \leq A_{n-1} \leq \ldots \leq A_1 \leq A_n + k$$

This fixed point corresponds to the ideal generated by $(t^{A_1}, t^{A_2}x, ..., t^{A_{n-1}}x^{n-2}, t^{A_n}x^{n-1})$. In this language, the constraint on A is to ensure that this is indeed an ideal. Namely, the set generated by these over $\mathcal{O}$ is closed under multiplication by x.

Proof. The action of $\nu \in L_v$ on $[g] \in M_v$ is simply $[\nu g]$, where the product of L and $G(\mathcal{K})$ are viewed within $\widetilde{G}(\mathcal{K}) \rtimes \mathbb{C}^\times$. In particular, we have

$$\nu.[g(t)] = [\nu^{((n-1)k,\dots,k,0)} g(\nu^n t)]$$

where $\nu^{(m_1,\dots,m_n)} := \text{diag}(\nu^{m_1},\dots,\nu^{m_n})$. Define the "orbital variety" (see the next section for motivation)

$$\widehat{\mathcal{V}}^v := G_{\mathcal{K}}.v \cap N_{\mathcal{O}}.$$

A point $g(t) \in \widehat{\mathcal{V}}^v$ will not be invariant under L but will require a compensating $G(\mathcal{O})$ transformation.

We now describe $M_v \cong \widehat{\mathcal{V}}^v/G(\mathcal{O})$. By the Iwasawa decomposition of $G(\mathcal{K})$, we can choose to represent elements of Gr_G by a lower-triangular matrix in $G(\mathcal{K})$ of the form $h = t^{-A} + q$, where q is strictly lower triangular. Moreover, we can always use $G(\mathcal{O})$ to make the (non-zero) q_{ij} Laurent polynomials and with no terms of degree larger than $-A_i - 1$, c.f. [LS91]. We interpret A as a cocharacter of $T \subset GL_n$.

Under the action of ν, the diagonal entries of h transform as $t^{-A_i} \mapsto \nu^{(n-i)k-nA_i} t^{-A_i}$ whereas $u_{ij}(t) \mapsto \nu^{(n-i)k} u_{ij}(\nu^n t)$. We can always return the diagonal entries to t^{-A_i} by means of a diagonal $G(\mathcal{O})$ transformation, sending $\nu^{(n-i)k} u_{ij}(\nu^n t) \mapsto \nu^{k(j-i)+nA_j} u_{ij}(\nu^n t)$. Since the non-zero entries of u are (Laurent) polynomial and have degree at most $-A_i - 1$ in row i, it follows that there is no lower-triangular matrix that can send this back to h. For example, when $j = i-1$ we must solve the equation

$$\nu^{nA_{i-1}-k} q_{ii-1}(\nu^n t) + t^{-A_i} p_i(t) = q_{ii-1}(t)$$

for $p_i(t) \in \mathcal{O}$. This requires $t^{A_i}(q_{ii-1}(t) - \nu^{nA_{i-1}-k} q_{ii-1}(\nu^n t))$ to belong to $\mathcal{O}$, hence

$$q_{ii-1}(t) - \nu^{nA_{i-1}-k} q_{ii-1}(\nu^n t) = 0$$

since q_{ii-1} has no terms of degree more than $-A_i - 1$. Finally, since k is coprime to n we conclude that $q_{ii-1}(t) = 0$. With $q_{ii-1} = 0$, it is straightforward to inductively show that $q = 0$.

For $t^A.v$ to belong to $\widehat{\mathcal{V}}^v$, for v corresponding to the (n,k) torus knot, requires

$$\langle A, \omega_n \rangle \geq 0 \qquad \langle A, \alpha_i \rangle \geq 0 \qquad \sum_{i=1}^{n-1} \langle A, \alpha_i \rangle \leq k.$$

□

Remark 3.16. When n and k are *not* coprime it is possible to have

$$q_{ii-1}(t) - \nu^{nA_{i-1}-k} q_{ii-1}(\nu^n t) = 0$$

for $q_{ii-1}(t)$ nonzero. In these circumstances there are still fixed points but they need not be isolated. See also Remark 3.13.

Remark 3.17. The above proof works, up to Weyl group elements, for the Iwahori case of L_v acting on $\widetilde{M}_v$. In particular, when $\gcd(n,k) = 1$ there are isolated fixed points which can be represented by matrices $h = t^{-A}\sigma^{-1}$ for cocharacters A of the maximal torus of $T \subset GL_n$ and Weyl group elements $\sigma \in \mathfrak{S}_n$. For $h^{-1}.v = \sigma t^A.v$ to belong to $\mathrm{Lie}(\mathbf{I}) \oplus \mathcal{O}^n$, the non-negative integers $(A_1, ..., A_n)$ must have forced jumps. In particular, if $i \in \{1, 2, ..., n-1\}$ we have

$$A_i \geq \begin{cases} A_{i+1} + 1 & \sigma(i+1) < \sigma(i) \\ A_{i+1} & \sigma(i+1) > \sigma(i) \end{cases}$$

and for $i = n$ we have

$$A_n + k \geq \begin{cases} A_1 + 1 & \sigma(1) < \sigma(n) \\ A_1 & \sigma(1) > \sigma(n) \end{cases}$$

In comparison to the discussion in [GSV20], the class $|A, \sigma\rangle$ of this fixed point corresponds to their "renormalized" vector $\widetilde{v}_{\sigma(A)}$.

Proposition 3.18. *In the case* $\gcd(n, k) = 1$*, the action of* L_v *on* $\widetilde{M_v}$ *has isolated fixed points labeled by cocharacters* A *of the maximal torus* $T \subset GL_n$ *and* $\sigma \in \mathfrak{S}_n$ *such that*

$$\langle A, \omega_n\rangle \geq 0 \qquad \langle A, \alpha_i\rangle \geq \tau(i) \qquad \sum_{i=1}^{n-1} \langle A, \alpha_i\rangle \leq k - \tau(n). \tag{3.3}$$

where $\tau(i) = 1$ *if* $\sigma(i+1) < \sigma(i)$ *and* $\tau(i) = 0$ *if* $\sigma(i+1) > \sigma(i)$*, with* $\sigma(n+1) := \sigma(1)$*.*

4. Action of the rational Cherednik algebra

In this section, we construct an action of the rational Cherednik algebras on equivariant BM homologies of Hilbert schemes of $\widehat{C}$ and some of its variants.

We first recall the construction of the BFN algebras in general. This is a minor parahoric variant of the construction in [BFN16b]. Suppose $1 \to G \to \widetilde{G} \to G_F \to 1$ is an extension of algebraic groups and let $\widetilde{G}^{\mathcal{O}}_{\mathcal{K}}$ be the preimage in $\widetilde{G}_{\mathcal{K}}$ of $G_{F,\mathcal{O}}$. Let $ev_0 : \widetilde{G}_{\mathcal{O}} \to \widetilde{G}$ be the homomorphism sending $t \mapsto 0$, and $\widetilde{\mathbf{P}} := ev_0^{-1}(G_F ev_0(\mathbf{P}))$.

Note that

$$\mathrm{Fl}_{\mathbf{P}} \cong \widetilde{G}^{\mathcal{O}}_{\mathcal{K}}/\widetilde{\mathbf{P}} \cong (\widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^{\times}_{rot})/(\widetilde{\mathbf{P}} \rtimes \mathbb{C}^{\times}_{rot})$$

and in particular

$$\mathrm{Gr}_G \cong \widetilde{G}^{\mathcal{O}}_{\mathcal{K}}/\widetilde{G}_{\mathcal{O}} \cong (\widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^{\times}_{rot})/(\widetilde{G}_{\mathcal{O}} \rtimes \mathbb{C}^{\times}_{rot}).$$

Let N be an algebraic representation of $\widetilde{G}$.

Definition 4.1. Define the *BFN space* of $(G, N, \mathbf{P}, N_{\mathbf{P}})$ as

$$\mathcal{R}_{G,N,\mathbf{P},N_{\mathbf{P}}} = \{([g], s) \in \mathrm{Fl}_{\mathbf{P}} \times N_{\mathbf{P}} | g^{-1}.s \in N_{\mathbf{P}}\}.$$

Remark 4.2. If $\mathbf{P} = G_{\mathcal{O}}, N_{\mathbf{P}} = N_{\mathcal{O}}$ we omit the subscripts $\mathbf{P}, N_{\mathbf{P}}$. We naturally have

$$\mathcal{R}_{G,N,\mathbf{P},N_{\mathbf{P}}} \subset \mathcal{T}_{G,N,\mathbf{P},N_{\mathbf{P}}} := G_{\mathcal{K}} \times_{\mathbf{P}} N_{\mathbf{P}} \cong \{([g], s) \in \mathrm{Fl}_{\mathbf{P}} \times N_{\mathcal{K}} | g^{-1}.s \in N_{\mathbf{P}}\}$$

The last isomorphism is given by the embedding $[g, s'] \mapsto ([g], g.s')$, see [BFN16b, discussion on p.6]. We use these descriptions interchangeably. When $\mathbf{P} = G_{\mathcal{O}}, N_{\mathbf{P}} = N_{\mathcal{O}}$, $\mathcal{T}_{G,N}$ has the modular interpretation

$$\mathcal{T}_{G,N} \cong \{(P, \sigma, s) | P \text{ is a } G\text{−torsor on the formal disk } D, \sigma : P|_{D^\times} \xrightarrow{\cong} G|_{D^\times}, s \in \Gamma(D, P \times_G N)\}.$$

The locally closed sub-ind-scheme $\mathcal{R}_{G,N}$ consists of those triples where $\sigma(s)$ extends to a section over D. The versions with $\mathbf{P}$ incorporate appropriate parabolic structure; i.e. we impose that P have a $\mathbf{P}$-reduction and require s to be compatible with this reduction.

Theorem 4.3 (Braverman-Finkelberg-Nakajima)**.** *There is a natural convolution product on* $\mathcal{A}_{G,N} := H_*^{G_{\mathcal{O}}}(\mathcal{R}_{G,N})$ *and* $\mathcal{A}^{\hbar}_{G,N} := H_*^{G_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot}}(\mathcal{R}_{G,N})$*, making them associative algebras with unit. Moreover,* $\mathcal{A}^{\hbar}_{G,N}$ *is a filtered quantization of* $\mathcal{A}_{G,N}$*, which is commutative.*

Definition 4.4. We will call either of these algebras the *BFN algebra* or the (quantized) *Coulomb branch.*

Remark 4.5. The BFN algebra $\mathcal{A}_{G,N}$ and its quantization have natural deformations given an extension as above. Namely, the homologies $\widetilde{\mathcal{A}}_{G,N} := H_*^{\widetilde{G}_{\mathcal{O}}}(\mathcal{R}_{G,N})$ and $\widetilde{\mathcal{A}}^{\hbar}_{G,N} := H_*^{\widetilde{G}_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot}}(\mathcal{R}_{G,N})$ have the structures of algebras that deform $\mathcal{A}_{G,N}$ and $\mathcal{A}^{\hbar}_{G,N}$, respectively, with $\widetilde{\mathcal{A}}^{\hbar}_{G,N}$ a filtered quantization of the commutative $\widetilde{\mathcal{A}}_{G,N}$. See [BFN16b, Section 3(viii)] for more details. This physically corresponds to turning on complex mass parameters for the flavor group G_F. In that context, one assumes that G_F is a torus.

4.0.1. *Parahoric versions.* A slight modification of the construction in [BFN16b] gives

Theorem 4.6. *There is a natural convolution product on* $\mathcal{A}_{G,N,\mathbf{P},N_{\mathbf{P}}} := H_*^{\mathbf{P}}(\mathcal{R}_{G,N,\mathbf{P},N_{\mathbf{P}}})$ *and* $\mathcal{A}^{\hbar}_{G,N,\mathbf{P},N_{\mathbf{P}}} := H_*^{\mathbf{P} \rtimes \mathbb{C}^\times_{rot}}(\mathcal{R}_{G,N,\mathbf{P},N_{\mathbf{P}}})$*, making them associative algebras with unit. Moreover,* $\mathcal{A}^{\hbar}_{G,N,\mathbf{P},N_{\mathbf{P}}}$ *is a filtered quantization of* $\mathcal{A}_{G,N,\mathbf{P},N_{\mathbf{P}}}$*.*

Remark 4.7. Similarly, one defines the flavor-deformed versions $\widetilde{\mathcal{A}}_{G,N,\mathbf{P},N_{\mathbf{P}}}$ and so on. Note that unless $\mathbf{P} = G_{\mathcal{O}}$, the algebra $\mathcal{A}_{G,N,\mathbf{P},N_{\mathbf{P}}}$ is in general *not* commutative. For example, $\widetilde{\mathcal{A}}_{G,N,\mathbf{P},N_{\mathcal{O}}}$ is a matrix algebra (of size $\dim_{\mathbb{C}} G_{\mathcal{O}}/\mathbf{P} \times \dim_{\mathbb{C}} G_{\mathcal{O}}/\mathbf{P}$) over $\widetilde{\mathcal{A}}_{G,N}$.

Remark 4.8 (For the physically minded reader)**.** The algebra $\mathcal{A}_{G,N,\mathbf{P},N_{\mathbf{P}}}$ encapsulates the algebra of local operators bound to a ($\frac{1}{2}$-BPS) vortex line operator labeled by the algebraic data $\mathbf{P}, N_{\mathbf{P}}$. As described in [DGGH19], the choice of $\mathbf{P}$ is a breaking of the gauge group in the vicinity of the line operator. The choice $N_{\mathbf{P}}$ is related to a choice of superpotential (compatible with the choice of symmetry breaking) coupling the bulk degrees of freedom to the degrees of freedom on the line operator.

Examples of such line operators have been used to obtain non-commutative resolutions of Coulomb branches [BFN16a] and played a central role in understanding of symplectic duality between Higgs and Coulomb branches [Web16].

4.1. **Convolution action of Coulomb branches on GASF.** Recall that we have defined the BFN space $\mathcal{R}_{\mathbf{P},N_{\mathbf{P}}} := \mathcal{R}_{G,N,\mathbf{P},N_{\mathbf{P}}}$ of a representation N. We will also consider the infinite-rank vector bundle

$$\mathcal{T}_{\mathbf{P},N_{\mathbf{P}}} := \mathcal{T}_{G,N,\mathbf{P},N_{\mathbf{P}}} := G_{\mathcal{K}} \times_{\mathbf{P}} N_{\mathbf{P}} \to \mathrm{Fl}_{\mathbf{P}}\,.$$

Define

$$\mathcal{V}^v_{N_{\mathbf{P}}} := (\widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^\times_{rot}).v \cap N_{\mathbf{P}}.$$

This is analogous to the *orbital varieties* in [CG09], and is also called such by [HKW20] in the case $\mathbf{P} = G_{\mathcal{O}}$. Note that on the level of closed points (which is what we are concerned with, since we only work with the reduced structure), it is clear that $\mathcal{V}^v_{N_{\mathbf{P}}}/(\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}) = M_v^{\mathbf{P},N_{\mathbf{P}}}$.

We now define the convolution action of $\widetilde{\mathcal{A}}^{\hbar}_{\mathbf{P},N_{\mathbf{P}}} := \widetilde{\mathcal{A}}^{\hbar}_{G,N,\mathbf{P},N_{\mathbf{P}}}$, following [BFN16b] and [HKW20] (which consider the case $\mathbf{P} = G_{\mathcal{O}}, N_{\mathbf{P}} = N_{\mathcal{O}}$).

Theorem 4.9. *Suppose the stabilizer L_v of v is contained in $\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}$. Then there is an action of $\widetilde{\mathcal{A}}^{\hbar}_{\mathbf{P},N_{\mathbf{P}}}$ on $H^{L_v}_*(M_v^{\mathbf{P},N_{\mathbf{P}}})$.*

Proof. Note that there is a natural map

$$p : \widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^\times_{rot} \times N_{\mathbf{P}} \to \mathcal{T}_{\mathbf{P},N_{\mathbf{P}}} \times N_{\mathbf{P}} \tag{4.1}$$

given by

$$(g, s) \mapsto ([g, s], s).$$

Let L_v be the stabilizer of v in $\widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^\times_{rot}$. If $X_v^{\mathbf{P},N_{\mathbf{P}}} := \{g \in \widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^\times_{rot} | g^{-1}.v \in N_{\mathbf{P}}\}$, there are two natural projections to $M_v^{\mathbf{P},N_{\mathbf{P}}}$ and $\mathcal{V}^v_{\mathbf{P},N_{\mathbf{P}}}$, which are $\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}$ and L_v-torsors, respectively. Taking the equivariant cohomology of the dualizing sheaves, we get

$$H^{L_v}_*(M_v^{\mathbf{P},N_{\mathbf{P}}}) \simeq H^{\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}}_*(\mathcal{V}^v_{\mathbf{P},N_{\mathbf{P}}}),$$

where the left-hand side makes sense because L_v is compact.

Consider the groupoid

$$\mathcal{P}_{\mathbf{P},N_{\mathbf{P}}} := \{(g, v) \in \widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^\times_{rot} \times N_{\mathbf{P}} | g^{-1}.v \in N_{\mathbf{P}}\} \xrightarrow{\pi_1 : (g,v) \mapsto v} N_{\mathbf{P}}.$$

Note that there is another projection map π_2 to $N_{\mathbf{P}}$ given by $(g, v) \mapsto g^{-1}.v$. Then for $\mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}} := \omega_{\mathcal{V}^v_{\mathbf{P},N_{\mathbf{P}}}}[-2 \dim \widetilde{\mathbf{P}}]$ (which is an object in the $\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}$-equivariant derived category of $N_{\mathbf{P}}$) we have a natural isomorphism

$$\pi_1^* \mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}} \cong \pi_2^* \mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}}.$$

By definition we have $p^{-1}(\mathcal{R}_{\mathbf{P},N_{\mathbf{P}}} \times N_{\mathbf{P}}) = \mathcal{P}_{\mathbf{P},N_{\mathbf{P}}}$, and that $m \circ q = \pi_2$, $\pi \circ j = \pi_1$, where $\pi : \widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^\times_{rot} \times N_{\mathbf{P}} \to N_{\mathbf{P}}$ is the projection.

Consider then the following diagram:

$$\begin{array}{ccccccc} \mathcal{R}_{\mathbf{P},N_{\mathbf{P}}} \times N_{\mathbf{P}} & \xleftarrow{p} & \mathcal{P}_{\mathbf{P},N_{\mathbf{P}}} & \xrightarrow{q} & q(\mathcal{P}_{\mathbf{P},N_{\mathbf{P}}}) & \xrightarrow{m} & N_{\mathbf{P}} \\ \Big\downarrow i & & \Big\downarrow j & & & & \\ \mathcal{T}_{\mathbf{P},N_{\mathbf{P}}} \times N_{\mathbf{P}} & \xleftarrow{p} & \widetilde{G}_{\mathcal{K}}^{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot} \times N_{\mathbf{P}} & & & & \end{array}$$

Here p is as above, q is quotient by the $\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}$-action $h.(g,s) = (gh^{-1}, h.s)$ and m is the multiplication map $[g,s] \mapsto g.s$. The composition $m \circ q$ is the above map $\pi_2 : \mathcal{P}_{\mathbf{P},N_{\mathbf{P}}} \to N_{\mathbf{P}}$.

Using the "restriction with support" map of Section A.1 (see [BFN16b, Section 3(ii)]) applied to the leftmost Cartesian square, and the map

$$p^*\omega_{\mathcal{T}_{\mathbf{P},N_{\mathbf{P}}}}[-2\dim N_{\mathbf{P}}] \boxtimes \mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}} \cong \omega_{\widetilde{G}_{\mathcal{K}}^{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot}}[-2\dim \widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}] \boxtimes \mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}}$$

we get a map (omitting the shifts for sake of readability)

$$\begin{aligned} p^* : & H^{-*}_{\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot} \times \widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}}(\mathcal{R}_{\mathbf{P},N_{\mathbf{P}}} \times N_{\mathbf{P}}, \omega_{\mathcal{R}_{\mathbf{P},N_{\mathbf{P}}}} \boxtimes \mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}}) \\ & = H_*^{\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}}(\mathcal{R}_{\mathbf{P},N_{\mathbf{P}}}) \otimes H_*^{\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}}(\mathcal{V}^v_{N_{\mathbf{P}}}) \to H^*_{\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}}(\mathcal{P}_{\mathbf{P},N_{\mathbf{P}}}, \pi_1^! \mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}}). \end{aligned} \tag{4.2}$$

Since $\mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}}$ is a $\mathbf{P}$-equivariant complex, we have $\pi_1^! \mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}} \cong \pi_2^! \mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}}$ and since $\pi_2 = m \circ q$, we get

$$H^*_{\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}}(\mathcal{P}_{\mathbf{P},N_{\mathbf{P}}}, \pi_1^! \mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}}) = H^*_{\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}}(q(\mathcal{P}_{N_{\mathbf{P}}}), m^! \mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}})$$

Finally, m is proper, so that using the adjunction $m_! m^! \to \mathrm{id}$ we get a map

$$(m \circ q)_* : H_*^{\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}}(q(p^{-1}(\mathcal{P}_{\mathbf{P},N_{\mathbf{P}}}), m^! \mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}}) \to H_*^{\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}}(\mathcal{V}^v_{\mathbf{P},N_{\mathbf{P}}}).$$

In particular, composing gives us an "intersection pairing"

$$\star := (m \circ q)_* p^* : H_*^{\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}}(\mathcal{R}_{\mathbf{P},N_{\mathbf{P}}}) \times H_*^{L_v}(M_v^{\mathbf{P},N_{\mathbf{P}}}) \to H_*^{L_v}(M_v^{\mathbf{P},N_{\mathbf{P}}}).$$

This is clearly bilinear over $\mathbb{Q}$. We prove the associativity in Lemma A.3 and the fact that the identity acts by 1 in Lemma A.4. □

4.1.1. *The case of Hilbert schemes.* Specializing the construction of the Theorem to $N = \mathrm{Ad} \oplus V$ and $\mathbf{P} = G_{\mathcal{O}}, N_{\mathbf{P}} = N_{\mathcal{O}}$, the L_v-equivariant homology of GASF admits an action of the spherical rational Cherednik algebra of $\mathfrak{gl}_n$. Similarly, for $\mathbf{P} = \mathbf{I}, N_{\mathbf{P}} = \mathrm{Lie}(\mathbf{I}) \oplus \mathcal{O}^n$ we get an action of the (full) RCA of $\mathfrak{gl}_n$, as we now describe.

Definition 4.10. The rational Cherednik algebra of $\mathfrak{gl}_n$ is the quotient algebra

$$\mathcal{H}_n = \frac{\mathbb{C}[\hbar, m]\langle x_1, \ldots, x_n, y_1, \ldots, y_n \rangle \rtimes \mathbb{C}\mathfrak{S}_n}{\sim}$$

where $\sim$ consists of the relations $[x_i, x_j] = [y_i, y_j] = 0$ for all i, j, and

$$[y_i, x_j] = \begin{cases} -\hbar + m \sum_{k \neq i} (i\,k) & \text{if } i = j, \\ -m(i\,j) & \text{if } i \neq j. \end{cases}$$

The *spherical subalgebra* is defined as $e\mathcal{H}_n e$ where $e = |\mathfrak{S}_n|^{-1}\sum_{w\in\mathfrak{S}_n} w$. We often refer to the spherical subalgebra simply as the spherical rational Cherednik algebra of $\mathfrak{gl}_n$.

Remark 4.11. To match with the conventions in most other sources, we should specialize $\hbar \to -1$. Indeed, it is the specialized algebra which will act on the equivariant homology as in Theorem 5.4 and for example [OY16].

We record the following theorems of Kodera-Nakajima and Braverman-Etingof-Finkelberg [KN18, BEF16] (see also [Web19, LW19]).

Theorem 4.12 (Kodera-Nakajima)**.** *For $G = GL_n, N = \mathrm{Ad} \oplus V$, the quantized BFN algebra $\widetilde{\mathcal{A}}^{\hbar}_{G,N}$ is isomorphic to the* spherical rational Cherednik algebra *of $\mathfrak{gl}_n$.*

Theorem 4.13 (Braverman-Etingof-Finkelberg)**.** *For $G = GL_n, N = \mathrm{Ad} \oplus V, \mathbf{P} = \mathbf{I}, N_{\mathbf{P}} = \mathrm{Lie}(\mathbf{I}) \oplus \mathcal{O}^n$, the quantized BFN algebra $\widetilde{\mathcal{A}}^{\hbar}_{G,N,\mathbf{P},N_{\mathbf{P}}}$ is isomorphic to the* rational Cherednik algebra *of $\mathfrak{gl}_n$.*

Remark 4.14. The extended group $\widetilde{G}$ in the above theorems is simply $G \times G_F$ where $G_F = \mathbb{C}^{\times}_{dil}$ acts by scaling Ad with weight 1 and V with weight 0.

In the situation of Theorem 3.5 we get

Corollary 4.15. *The spherical rational Cherednik algebra $e\mathcal{H}_n e$ of $\mathfrak{gl}_n$ acts on $H^{L_v}_*(\mathrm{Hilb}^\bullet(\widehat{C}))$ where L_v is the stabilizer in $\widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^{\times}_{rot}$ of $v \in \mathrm{Ad}(\mathcal{K}) \oplus \mathcal{K}^n$ associated to $\widehat{C}$ as in Theorem 3.5.*

Corollary 4.16. *The rational Cherednik algebra $\mathcal{H}_n$ of $\mathfrak{gl}_n$ acts on $H^{L_v}_*(\mathrm{PHilb}^\bullet(\widehat{C}))$.*

Remark 4.17. The action in Corollary 4.16 coincides by [Web19, Section 7] with that studied in [GSV20]. Both papers use a different set of generators than us, and we compare their construction to ours in Section 4.3.2.

4.2. **Comparison of the convolution action to an action by correspondences.** For many of our results, in particular Theorem 5.4, we will need to compare the convolution action from Theorem 4.9 to another action by correspondences. We will do this again in greater generality than needed for the rest of the paper. In particular, we make rigorous expectations from [BDG17] and [BDGH16].

Definition 4.18. Define the *raviolo space/Hecke stack* for v which has $\mathbb{C}$-points given by

$$\mathcal{R}^v_{\mathbf{P},N_{\mathbf{P}}}(\mathbb{C}) = \{(s_2, g, s_1) \in \mathcal{V}^v_{N_{\mathbf{P}}} \times \widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^{\times}_{rot} \times \mathcal{V}^v_{N_{\mathbf{P}}} | g.s_1 = s_2\}/\widetilde{\mathbf{P}} \rtimes \mathbb{C}^{\times}_{rot}.$$

Here the $\widetilde{\mathbf{P}} \rtimes \mathbb{C}^{\times}_{rot}$-action is on s_1 and the right of g.

Definition 4.19. Define also

$$\mathcal{T}^v_{\mathbf{P},N_{\mathbf{P}}}(\mathbb{C}) = \{(s_2, g, s_1) \in \mathcal{W}^v \times \widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^{\times}_{rot} \times \mathcal{V}^v_{N_{\mathbf{P}}} | g.s_1 = s_2\}/\widetilde{\mathbf{P}} \rtimes \mathbb{C}^{\times}_{rot},$$

where $\mathcal{W}^v := (\widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^{\times}_{rot}).v \subset N(\mathcal{K})$.

Next, note that $\mathcal{R}^v_{\mathbf{P},N_\mathbf{P}}$ is a locally closed sub-ind-variety of $\mathcal{R}_{\mathbf{P},N_\mathbf{P}}$ via $[s_2, g, s_1] \mapsto [g, s_1]$ and therefore inherits a stratification $\mathcal{R}^v_{\mathbf{P},N_\mathbf{P},\leq w} := \mathcal{R}^v_{\mathbf{P},N_\mathbf{P}} \cap \mathcal{R}_{\mathbf{P},N_\mathbf{P},\leq w}$ (ditto for $\mathcal{T}^v_{\mathbf{P},N_\mathbf{P}}, \mathcal{T}_{\mathbf{P},N_\mathbf{P}}$.) Here $w \in W^{aff}/W_\mathbf{P}$ is a coset for the extended affine Weyl group of G. We also have maps

$$\begin{array}{ccc} & \mathcal{R}^v_{\mathbf{P},N_\mathbf{P}} & \\ {}^{\varphi_1}\swarrow & & \searrow^{\varphi_2} \\ M_v^{\mathbf{P},N_\mathbf{P}} & & N_\mathbf{P} \end{array}$$

where φ_1 is the $\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}$- equivariant projection map

$$\varphi_1 : [s_2, g, s_1] \mapsto [s_1],$$

whose restriction to $\mathcal{R}^v_{\mathbf{P},N_\mathbf{P},\leq w}$ is smooth, and φ_2 is another proper equivariant projection given by

$$\varphi_2 : [s_2, g, s_1] \mapsto s_2,$$

whose image is naturally identified with $\mathcal{V}^v_{N_\mathbf{P}}$.

The map in Equation (4.1) restricts to

$$p : p^{-1}(\mathcal{R}_{\mathbf{P},N_\mathbf{P}} \times \mathcal{V}^v_{N_\mathbf{P}}) \;\; \to \mathcal{R}^v_{\mathbf{P},N_\mathbf{P}} \times \mathcal{V}^v_{N_\mathbf{P}} \tag{4.3}$$

$$p : \widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^\times_{rot} \times \mathcal{V}^v_{N_\mathbf{P}} \quad \to \mathcal{T}^v_{\mathbf{P},N_\mathbf{P}} \times \mathcal{W}^v \tag{4.4}$$

and $q(p^{-1}(\mathcal{R}_\mathbf{P} \times \mathcal{V}^v_\mathbf{P})) \cong \mathcal{R}^v_\mathbf{P}$ by the right quotient. Note that when the stabilizer of v is trivial, we have $p^{-1}(\mathcal{R} \times \mathcal{V}^v) \cong \mathcal{V}^v \times \mathcal{V}^v$ by $(g, s) \mapsto (s, g.s)$. Our goal is to interpret the "push-pull" maps in equivariant cohomology of $\mathcal{V}^v$ giving rise to the action.

Note that $q_* p^*$, where p^* is defined in Theorem 4.9, defines a map

$$H_*^{\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times}(\mathcal{R}^v_{\mathbf{P},N_\mathbf{P}} \times \mathcal{V}^v_{N_\mathbf{P}}) \to H^*_{\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}}(\mathcal{P}_{N_\mathbf{P}}, \pi_1^! \mathcal{F}^v_{\mathbf{P},N_\mathbf{P}}) = H^*_{\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}}(\mathcal{R}^v_{\mathbf{P},N_\mathbf{P}}).$$

Given a class $[\mathcal{R}_{\mathbf{P},N_\mathbf{P},\leq w}] \in \mathcal{A}^\hbar_{\mathbf{P},N_\mathbf{P}}$ and $\alpha \in H_*^{L_v}(M_v^{\mathbf{P},N_\mathbf{P}}) \cong H_*^{\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times}(\mathcal{V}^v_{N_\mathbf{P}})$ we have that $q_* p^*([\mathcal{R}_{\mathbf{P},N_\mathbf{P},\leq w}] \otimes \alpha)$ is identified with the restriction of the map $q_* p^*$ to $\mathcal{R}^v_{\mathbf{P},N_\mathbf{P},\leq w} \times \mathcal{V}^v_{N_\mathbf{P}}$. In particular, by smoothness of the maps in Eq. (4.3) and the natural inclusion $\mathcal{R}_{\mathbf{P},N_\mathbf{P}} \to \mathcal{T}_{\mathbf{P},N_\mathbf{P}}$ we may use the "classical" refined pullback map as in [Ful13] to compute $q_* p^*([\mathcal{R}_{\mathbf{P},N_\mathbf{P},\leq w}] \otimes \alpha)$ given good enough understanding of $\mathcal{R}^v_{\mathbf{P},N_\mathbf{P}}$ and how it sits in $\mathcal{T}^v_{\mathbf{P},N_\mathbf{P}}$. Moreover, $m_* : H_*^{\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}}(\mathcal{R}^v_{\mathbf{P},N_\mathbf{P}}) \to H_*^{L_v}(M_v^{\mathbf{P},N_\mathbf{P}})$ as given as in Theorem 4.9 is identified with $\varphi_{2,*}$. In Section 4.3 we will see that it is possible to compute $(m \circ q)_* p^*$ using this interpretation in the abelian setting, which enormously simplifies computations.

4.2.1. *The case of Hilbert schemes.* Suppose now $\lambda = (1, \ldots, 0)$ and we are in the setting of Theorem 3.5. Then $H_*^{\widetilde{G}_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathcal{R}^v_{\leq \lambda}) \cong H_*^{L_v}(\mathrm{Hilb}^{\bullet,\bullet+1}(C))$ where $\mathrm{Hilb}^{\bullet,\bullet+1}(C)$ is the flag Hilbert scheme and (after forgetting equivariance) the map $\varphi_1^* = q_* p^*$ can be identified with the refined pullback map also denoted "p" in [Kiv19, Theorem

1] restricted to the punctual Hilbert scheme (the versal deformations needed in *loc. cit.* work locally, whereas global curves are needed for the rest of the arguments). Similarly, if $\lambda = (-1, \dots, 0)$, we recover the map "q^*" of *loc. cit.*

Let us now explain why this happens. The affine Grassmannian of GL_n is the increasing union of the projective varieties

$$\mathrm{Gr}^d_{\mathrm{GL}_n} := \{\Lambda \subset \mathcal{K}^n | t^d \mathcal{O}^n \subset \Lambda \subset t^{-d}\mathcal{O}^n\}.$$

It is clear that M_v as in Theorem 3.5 corresponding to the germ of a curve $\widehat{C}$ has $M_v^m := \bigsqcup_{i=0}^m \mathrm{Hilb}^i(\widehat{C}) \subset M_v$ contained in $\mathrm{Gr}^d_{\mathrm{GL}_n}$ for all m and some d depending on m.

Let moreover $N_d := N(\mathcal{O})/t^d N(\mathcal{O})$ and $\mathcal{V}^v_d$ be the image in the quotient. Let also $\mathcal{R}^d := \{[g, s] \in \mathrm{Gr}^d_G \times^{G(\mathcal{O})/t^d} N_d | g^{-1}.s \in N_d\}$. Then $\mathcal{R}$ is the colimit of $\mathcal{R}^d$ for the inclusions coming from $\mathrm{Gr}^d \hookrightarrow \mathrm{Gr}^{d+1}_G$, in particular the equivariant Borel-Moore homology is the corresponding colimit.

Choose $d \gg 0$ and some open neighborhood U of $v \in N_d$. Then choosing some transversal slice S to $\mathcal{V}^v_d$, we locally have $\mathcal{V}^v_d \times S \cong U$. In particular, if we let $\varphi : \mathcal{R}^d \to N_d$ be the projection, and

$$\Sigma := \varphi^{-1}(\mathcal{V}^v_d),\ \Sigma_U := \varphi^{-1}(U \cap \mathcal{V}^v_d)$$

then

$$\Sigma_U \cong (\mathcal{V}^v \cap U) \times M^d_v.$$

Consider the inclusion $\mathcal{V}^v_d \cap U \hookrightarrow U$. The map $\varphi^{-1}(U) \to U$ is smooth, so we get a refined pullback map [Ful13]

$$\varphi^* : H_*^{\widetilde{G}(\mathcal{O})/t^d \rtimes \mathbb{C}^\times}(\mathcal{V}^v_d \cap U) \to H_*^{\widetilde{G}(\mathcal{O})/t^d \rtimes \mathbb{C}^\times}(\Sigma_U).$$

We will in fact abuse notation and denote by φ^* the composition of this map and the pushforward

$$H_*^{\widetilde{G}(\mathcal{O})/t^d \rtimes \mathbb{C}^\times}(\Sigma_U) \to H_*^{\widetilde{G}(\mathcal{O})/t^d \rtimes \mathbb{C}^\times}(\Sigma).$$

Possibly further increasing d and throwing away some high codimension subset of U not containing v, note that by Theorem 3.5 it is possible to identify $\varphi^{-1}(U) \to U$ with the family of Hilbert schemes of $0, 1, \dots, d$ points (i.e. the union thereof) since having a cyclic vector is an open condition.

Since N_d is the space of *all* matrices and vectors in $\mathcal{O}/t^d$, the associated family of (germs of) planar curves is versal for large enough U. By results of Shende and others (see e.g. Sections 2 and 3 of [Kiv19] for discussion and references), the associated total space is smooth.

Further restricting φ to $\varphi^{-1}(U) \cap \mathcal{R}^{\leq \lambda}_d$ for the cocharacter $\lambda = (1, \dots, 0)$ identifies the refined intersection map p^* for the inclusion $v \hookrightarrow U$ in [Kiv19, Definition 3.4] with $\varphi^*_{\leq \lambda}$. The other case is similar.

In particular, this gives an interpretation of one of the Weyl algebras appearing in [Kiv19, VR18]. The other one has to do with the Hilbert schemes of global curves and cannot be defined in our setting. Indeed, the other Weyl algebra depends on the number of components of the curve, whereas our Cherednik algebra depends on the *degree* of the curve.

Remark 4.20. It is remarkable to note that the convolution action works on the level of punctual Hilbert schemes directly. In [Kiv19] and [VR18], one of the main points is to define convolution maps for the Hilbert schemes of (locally planar) singular curves using refined intersection products, which are constructed by deforming the singularities as we saw above. The role of the deformation in our context is played by considering the infinite-dimensional ind-variety $\mathcal{V}^v$ in place of M_v. Note also that the "restriction with supports" map is a refined intersection product in the case of a regular embedding, while here we use a rather special form of the map p, which is very far from anything like a regular embedding, but rather a principal bundle.

4.3. **Localization to fixed points.** Let us analyze the construction of Theorem 4.9 first in the case $G = T$ is a torus. In this case, $\mathcal{R}_T$ is a collection of (infinite rank) vector bundles over a discrete set $\mathrm{Gr}_T \cong X^*(T)$, of finite codimension in $\mathcal{T}$. Its complex points are

$$\mathcal{R}_T(\mathbb{C}) = \{(g, s) \in \widetilde{T}_{\mathcal{K}}^{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot} \times N(\mathcal{O}) : g^{-1}.s \in N_{\mathcal{O}}\}/\widetilde{T}_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot},$$

and the map $\pi_T : \mathcal{R}_T \to \mathrm{Gr}_T$ given by forgetting s. The map

$$\widetilde{T}_{\mathcal{K}}^{\mathcal{O}} \times N_{\mathcal{O}} \to \mathcal{T}_T \times N_{\mathcal{O}}$$

is simply many copies of the quotient map

$$\mathbb{C}((t))^\times \to \mathbb{C}((t))^\times/\mathbb{C}[[t]]^\times.$$

Fix now G reductive and T a maximal torus in it. We may think of $\mathcal{R}_T$ as an "abelianized" BFN space for G, as it also admits an inclusion map $\iota : \mathcal{R}_T \hookrightarrow \mathcal{R}$ via inclusion of $\mathrm{Gr}_T \hookrightarrow \mathrm{Gr}_G$. The space $\mathcal{R}_T$ has a natural convolution product and it admits a natural action of the Weyl group W. By Lemma 5.10 of [BFN16b] there is an algebra homomorphism $(\iota^W_{\mathcal{R}})_* : (\mathcal{A}^\hbar_T)^W \to \mathcal{A}^\hbar$ coming from the inclusion $\iota_{\mathcal{R}} : \mathcal{R}_T \hookrightarrow \mathcal{R}$. We call $\mathcal{A}^\hbar_T$ the "abelianized" BFN algebra. This construction generalizes to the flavor deformed algebras $(\widetilde{\mathcal{A}}^\hbar_T)^W \to \widetilde{\mathcal{A}}^\hbar$, where $\widetilde{\mathcal{A}}^\hbar_T := H_*^{\widetilde{T}_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot}}(\mathcal{R}_T)$.

Consider $\mathcal{R}^v_T = \mathcal{R}_T \cap \mathcal{R}^v$. By definition of the generalized affine Springer fiber for v, where we consider N as a representation of $T \subset G$, we see that $\mathcal{R}^v_T$ is the Hecke stack associated to the datum (T, N, v). Using the convolution action of Theorem 4.9 for (T, N), we get an action of $\widetilde{\mathcal{A}}^\hbar_T$ on $H_*^{L_{T,v}}(M_{v,T})$ where $L_{T,v}$ is the stabilizer of v in T.

We can now try to compare the two actions.

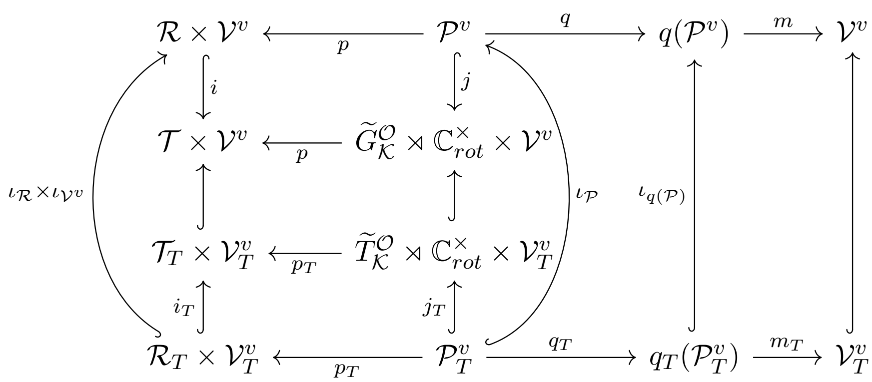
Proposition 4.21. *Suppose* Gr_G *has isolated fixed points under the stabilizer* $L_v \subset \widetilde{G}_{\mathcal{K}}^{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot}$ *of* v *and that* L_v *is contained in* $\widetilde{T}_{\mathcal{O}} \times \mathbb{C}^\times_{rot}$. *Then*

(1) $M_{v,T} = M_v^{L_v}$
(2) $(\iota_{M_v})_* : H_*^{L_v}(M_v^{L_v}) \to H_*^{L_v}(M_v)$ *becomes an isomorphism after inverting countably many characters of* L.

(3) $(\iota_{M_v})_*$ *intertwines the actions of* $(\widetilde{\mathcal{A}}^{\hbar}_{T,N})^W$ *and* $\widetilde{\mathcal{A}}^{\hbar}_{G,N}$.

Proof. The first assertion follows from the fact that the L_v-fixed points are contained in the $L_v = T$-fixed points on the affine Grassmannian, which for T is topologically a discrete set of points coinciding with Gr_T. The second assertion is the Atiyah-Bott localization theorem.

Consider the following diagram:



Here i, j, p, q, m are as before, and the versions with subscript T are the corresponding maps for $T \subset G$. The inclusions $\iota_?$ come from the maps $T \hookrightarrow G, \mathrm{Gr}_T \hookrightarrow \mathrm{Gr}_G$ and variations. The space $\mathcal{P}^v$ is defined as $\mathcal{P}^v := p^{-1}(\mathcal{R} \times \mathcal{V}^v)$ and $\mathcal{P}^v_T$ by replacing G with T.

Note that the upper and lower squares on the left tower of squares are clearly Cartesian. We claim that the middle one is so too. By definition the fiber product

$$(\widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^\times_{rot} \times \mathcal{V}^v) \times_{\mathcal{T} \times \mathcal{V}^v} (\mathcal{T}_T \times \mathcal{V}^v_T)$$

consists of $(g, v'', [t, v'])$ so that $[g, v''] = [t, v']$ and $v' = v''$. In particular, there is some $g' \in L_v$ such that $gg' = t$. But since L_v is contained in $\widetilde{T}_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot}$, we must have $g \in \widetilde{T}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^\times_{rot}$. So every square in the tower is Cartesian. Note that this is not true without our assumptions (take for example $N = 0, v = 0$).

Let $\mathcal{F} = \omega_{\mathcal{V}^v}[-2\dim \widetilde{G}_{\mathcal{O}}]$ and $\mathcal{F}_T = \omega_{\mathcal{V}^v_T}[-2\dim \widetilde{T}_{\mathcal{O}}]$. Let $\iota_{\mathcal{V}^v} : \mathcal{V}^v_T \hookrightarrow \mathcal{V}^v$. Then $\iota^*_{\mathcal{V}^v_T}\mathcal{F} = \mathcal{F}_T$. Let then

$$r \otimes \alpha \in H^{\widetilde{G}_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot}}_*(\mathcal{R}) \otimes H^{L_v}_*(M_v) \cong H^{-*}_{\widetilde{G}_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot} \times \widetilde{G}_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot}}(\mathcal{R} \times \mathcal{V}^v, \omega_{\mathcal{R}} \boxtimes \mathcal{F}).$$

By Lemma 5.10. of [BFN16b], the pushforward map

$$(\iota_{\mathcal{R}})^W_* : (\widetilde{\mathcal{A}}^{\hbar}_T)^W \to \widetilde{\mathcal{A}}^{\hbar}$$

given by taking the W-invariants of the $\widetilde{T}_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot}$-equivariant pushforward becomes an isomorphism after localizing at countably many characters of $\widetilde{T} \times \mathbb{C}^\times_{rot}$. By parts (1) and (2),

$$(\iota_{\mathcal{V}^v})_* : H^{\widetilde{T}_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot}}_*(\mathcal{V}^v_T) \cong H^{L_v}_*(M_v^{L_v}) \to H^{L_v}_*(M_v) \cong H^{\widetilde{G}_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot}}_*(\mathcal{V}^v)$$

also becomes an isomorphism after localizing at countably many characters of L_v.

If we define moreover

$$\iota_* := (\iota_{\mathcal{R}})_*^W \otimes (\iota_{\mathcal{V}^v})_*$$

and work in this localization, the intertwining property we need to show becomes

$$\iota_*(m_T \circ q_T)_* p_T^*((\iota_*)^{-1}(r \otimes \alpha)) = (m \circ q)_* p^*(r \otimes \alpha).$$

Define

$$A := \omega_{\mathcal{T}}[-2\dim N_{\mathcal{O}}] \boxtimes \mathcal{F}, A_T := \omega_{\mathcal{T}_T}[-2\dim N_{\mathcal{O}}] \boxtimes \mathcal{F}_T$$

and

$$B := \omega_{\widetilde{G}_{\mathcal{K}}^{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot}}[-2\dim \widetilde{G}_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot}] \boxtimes \mathcal{F}, B_T := \omega_{\widetilde{T}_{\mathcal{K}}^{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot}}[-2\dim \widetilde{T}_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot}] \boxtimes \mathcal{F}_T$$

The restriction with support map p^* from Theorem 4.9 and Definition A.1 is (the induced map in hypercohomology of) the composition

$$i^! A \to i^! p_* p^* A = p_* j^! p^* A \to p_* j^! B.$$

Similarly we have

$$i_T^! A_T = (\iota_{\mathcal{R}} \times \iota_{\mathcal{V}^v})^! i^! A \to i_T^! p_{T*} p_T^* A_T \to p_{T*} j_T^! B_T$$

Using proper base change, we rewrite this as

$$(\iota_{\mathcal{R}} \times \iota_{\mathcal{V}^v})^! i^! A \to i_T^! p_{T*} p_T^* A_T = (\iota_{\mathcal{R}} \times \iota_{\mathcal{V}^v})^! i^! p_* p^* A \to p_{T*} j_T^! B_T = (\iota_{\mathcal{R}} \times \iota_{\mathcal{V}^v})^! p_* j^! B.$$

Passing to $\widetilde{T}_{\mathcal{O}} \rtimes \mathbb{C}^\times$-equivariant hypercohomology, we get that the square

$$\begin{array}{ccc} H^{-*}_{\widetilde{T}_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot} \times \widetilde{G}_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot}}(\mathcal{R} \times \mathcal{V}^v, i^! A) & \xrightarrow{p^*} & H^{-*}_{\widetilde{T}_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot} \times \widetilde{G}_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot}}(\mathcal{P}^v, \pi_1^! \mathcal{F}) \\ \uparrow{\scriptstyle (\iota_{\mathcal{R}})_* \otimes (\iota_{\mathcal{V}^v})_*} & & \uparrow{\scriptstyle (\iota_{\mathcal{P}})_*} \\ H^{-*}_{\widetilde{T}_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot} \times \widetilde{T}_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot}}(\mathcal{R}_T^v \times \mathcal{V}_T^v, i^! A_T) & \xrightarrow{p_T^*} & H^{-*}_{\widetilde{T}_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot} \times \widetilde{T}_{\mathcal{O}} \rtimes \mathbb{C}^\times_{rot}}(\mathcal{P}_T^v, \pi_{1,T}^! \mathcal{F}_T) \end{array}$$

commutes. Now taking W-invariants on the $\mathcal{R}$-factor everywhere and passing to the localization where the left column becomes an isomorphism, we get

$$p_T^*((\iota_*)^{-1}(r \otimes \alpha)) = (\iota_{\mathcal{P}*})^{-1} p^*(r \otimes \alpha).$$

Since the right large square is also Cartesian and $\iota_{\mathcal{P}}$ is a closed embedding, using proper base change once more we get

$$\iota_*(m_T \circ q_T)_*(\iota_{\mathcal{P}})_*^{-1} p^*(r \otimes \alpha) = (m \circ q)_* p^*(r \otimes \alpha).$$

□

Remark 4.22. Parts (1) and (2) of the above Proposition were also obtained in [HKW20, Theorem 5.13].

Remark 4.23. While it is natural to anticipate similar localization results for the parahoric cases, we do not know how these work due to a lack of an obvious replacement for the map $\mathrm{Gr}_T \hookrightarrow \mathrm{Gr}_G$ respecting the convolution structure in the case of other partial affine flag varieties.

4.3.1. *Computations in the spherical case.* Let Gr_G^λ be the $G_{\mathcal{O}}$-orbit of $t^\lambda \in \mathrm{Gr}_G$ and set $\mathcal{R}^{\leq\lambda} = \mathcal{R} \cap \pi^{-1}(\overline{\mathrm{Gr}_G^\lambda})$, where $\pi : \mathcal{R} \to \mathrm{Gr}_G$ is the projection forgetting $N_{\mathcal{O}}$. In what follows we will determine the action of various classes in $\widetilde{\mathcal{A}}^\hbar$ by means of two-fold fixed-point localization. Recall that there are commutative subalgebras $H^*_{\widetilde{G}\times\mathbb{C}^\times_{rot}}(pt) \hookrightarrow \widetilde{\mathcal{A}}^\hbar$ and $H^*_{\widetilde{T}\times\mathbb{C}^\times_{rot}}(pt) \hookrightarrow \widetilde{\mathcal{A}}^\hbar_T$. Denote the equivariant parameters the maximal $\widetilde{T} \times \mathbb{C}^\times_{rot}$ collectively by φ (for T), m (for G_F) and $\hbar$ (for $\mathbb{C}^\times_{rot}$).

Let $[t^\lambda]$ denote the fundamental class of $\mathcal{R}_T \cap p_T^{-1}(\overline{\mathrm{Gr}_T^\lambda})$, often called an "abelianized monopole" [BDG17, BDGH16]. For λ dominant with Gr_G^λ closed we can then write the following localization formula, c.f. [BFN16b] Proposition 6.6:

$$[\mathcal{R}^{\leq\lambda}] = \iota_*\Bigg(\sum_{w\in W/W_\lambda} \frac{[t^{w.\lambda}]}{e(T_{w.\lambda}\,\mathrm{Gr}_G^{\leq\lambda})}\Bigg), \tag{4.5}$$

where W_λ is the stabilizer of λ in the Weyl group W.[1] The unit of the algebra $\widetilde{\mathcal{A}}^\hbar$ is $1 := [\mathcal{R}^{\leq 0}]$. Other generators of $\widetilde{\mathcal{A}}^\hbar$ can be constructed by including a W_λ-invariant function $f(\varphi, m, \hbar)$ to the numerator of this expression:

$$[\mathcal{R}^{\leq\lambda}][f] = \iota_*\Bigg(\sum_{w\in W/W_\lambda} \frac{(w.f)[t^{w.\lambda}]}{e(T_{w.\lambda}\,\mathrm{Gr}_G^{\leq\lambda})}\Bigg) \tag{4.6}$$

These are called "dressed" monopole operators, which are known to generate $\widetilde{\mathcal{A}}^\hbar$ [BFN16a, Wee19].

Remark 4.24. More precisely, it was shown in [Wee19] that the $[\mathcal{R}_{\leq\lambda}][f]$ with *minuscule* λ and a slightly smaller collection of f's generate $\widetilde{\mathcal{A}}^\hbar_{G,N}$ for any quiver gauge theory; the quiver in this case is a Jordan quiver with a framing node of rank 1.

Remark 4.25. The terminology "dressed monopole" has its origins in the physics literature, in our context they appear for example in [CHZ14]. These operators also appear as the dimensional reduction of the four-dimensional mixed Wilson-'t Hooft operators of [Kap06].

Assume the hypothesis of Proposition 4.21 and, moreover, that the map $L_v \to G_F \times \mathbb{C}^\times_{rot}$ is injective. Thus, the action of $H^*_{\widetilde{T}\times\mathbb{C}^\times}(pt)$ factors through the action of $H^*_{L_v}(pt)$ [HKW20].

A representative in $T_{\mathcal{K}}$ of a fixed point $p \in M_v$ will generically not be exactly fixed by L_v, instead requiring a compensating $T \subset T_{\mathcal{O}}$ transformation. The requirement that $L_v \to G_F \times \mathbb{C}^\times_{rot}$ is injective implies that there is a unique such compensating transformation, hence the action of $H^*_T(pt) \subset H^*_{\widetilde{T}\times\mathbb{C}^\times_{rot}}(pt)$ on the fixed point class $|p\rangle$ is uniquely determined by the action of $H^*_{T_F\times\mathbb{C}^\times}(pt)$ on $|p\rangle$. We write $\varphi\,|p\rangle = \varphi(p)\,|p\rangle$.

[1] Since the φ do not commute with $[t^\lambda]$, we take the convention that the denominator is to the right of the numerator in writing this formula.

The action of $H^*_{T_F\times\mathbb{C}^\times}(pt)$ is then determined by the injection $L_v \to G_F \times \mathbb{C}^\times_{rot}$, which imposes rank $G_F + 1 -$ rank L_v linear relations on the $m\,|p\rangle\,, \hbar\,|p\rangle$. This is the source of the specialization discussed earlier.

Remark 4.26. The bra-ket notation used to denote the fixed point classes $|p\rangle$ is used due to the realization of these classes as vectors in the supersymmetric Hilbert space in the gauge theory setup. It is important to note that this isn't an honest Hilbert space as the twisted theory need not be unitary. Nonetheless, there is a natural symmetric, non-degenerate pairing of classes, c.f. [BDG+16, Section 3.3].

Lemma 4.27. *Assume that M_v has isolated fixed points under the action of $L_v \subset \widetilde{T} \times \mathbb{C}^\times_{rot}$ and that the map $L_v \to G_F \times \mathbb{C}^\times_{rot}$ is injective.*

For λ a minuscule cocharacter and $f(\varphi, m, \hbar)$ a W_λ-invariant function we have

$$[\mathcal{R}_{\leq\lambda}][f]\,|p\rangle = \sum_{w\in W/W_\lambda} \frac{\big(w.f(\varphi(t^{w.\lambda}p), m, \hbar)\big)e(E_{p,w.\lambda})}{e(T_{w.\lambda}\,\mathrm{Gr}_G^{\leq\lambda})}\,|t^{w.\lambda}p\rangle\,, \tag{4.7}$$

where $E_{p,\nu}$ is an excess intersection factor. The denominator in this formula should be understood as replacing φ in the polynomials $e(T_{w.\lambda}\,\mathrm{Gr}_G^{\leq\lambda})$ with $\varphi(p)$.

Proof. By the previous Proposition we only need to compute this inside $H_*^{L_v}(M_v^{L_v})\otimes \mathbb{C}(\mathfrak{l})$. Let $|p\rangle$ be (the inclusion of) the fundamental class of a fixed point in $M_v \subset N(\mathcal{O})/G(\mathcal{O})$. The subalgebra $H^*_T(pt) = \mathbb{C}[\mathfrak{t}] \subset \widetilde{\mathcal{A}}^\hbar_T$ acts as $\varphi_a\,|p\rangle = \varphi_a(p)\,|p\rangle$. Since $\pi_T^{-1}(\mathrm{Gr}_T^\lambda)$ is a vector bundle over a point, using the excess intersection formula for the refined pullback p^* (see Fulton [Ful13, Section 6.3]) we have

$$[t^\lambda]\,|p\rangle = (m\circ q)_* p^*([t^\lambda]\otimes[p]) = e(E_{p,\lambda})\,|t^\lambda p\rangle\,.$$

As a vector space over $\mathbb{C}$, $E_{p,\lambda}$ can be expressed as

$$E_{p,\lambda} \simeq N(\mathcal{O})/(N(\mathcal{O}) \cap t^{-\lambda}N(\mathcal{O})).$$

The equivariant structure of this vector space is determined by λ and p; $E_{p,\lambda}$ should be thought of as a quotient of tangent spaces at $(t^\lambda p, t^\lambda, p) \in \mathcal{T}_T$. A straightforward computation shows that

$$e(E_{p,\lambda}) = \prod_{\widetilde{\mu}\text{ s.t. }\langle\mu,\lambda\rangle<0} [\langle\widetilde{\mu}, \varphi(p) + m\rangle]^{\langle\mu,\lambda\rangle},$$

where the product runs over the $\widetilde{G}$ weights $\widetilde{\mu}$ of N, with μ its restriction to G. We also use the notation that

$$[x]^r = \begin{cases} \prod_{j=0}^{r-1}(x + j\hbar) & r > 0 \\ 1 & r = 0 \\ \prod_{j=1}^{|r|}(x - j\hbar) & r < 0 \end{cases}.$$

It is worth noting that if t^λ maps p outside $N(\mathcal{O})$ then $E_{p,\lambda}$ will necessarily have a vector that transforms trivially under $\mathbb{C}^\times$, i.e. $e(E_{p,\lambda}) = 0$. By Eq. (4.6) the result follows. □

Remark 4.28. The above localization computations and the "abelianization procedure" appear in [BFN16a] as an embedding of the algebra $\widetilde{\mathcal{A}}^\hbar$ to an algebra of differential(-difference) operators on the maximal torus $T \subset G$.

4.3.2. *Comparison to results of Gorsky-Simental-Vazirani.* In the recent preprint [GSV20], when $\widehat{C} = \{x^n = t^k\}, \gcd(n,k) = 1$, another action of the rational Cherednik algebra of $\mathfrak{gl}_n$ (see Definition 4.10) is defined on the localized equivariant (Borel-Moore) homology of the parabolic flag Hilbert schemes $\mathrm{PHilb}_{m,m+n}(\widehat{C})$. (In the above notation, this would correspond to $\mathrm{PHilb}^{m,m+(1,1,\ldots,1)}(\widehat{C})$.) By Remark 3.3 and Theorem 3.5,

$$\mathrm{PHilb}^\bullet(\widehat{C}) := \bigsqcup_{m\geq 0} \mathrm{PHilb}_{m,m+n}(\widehat{C}) \cong \widetilde{M}_v := M_v^{\mathbf{I},\mathrm{Lie}(\mathbf{I})\oplus\mathcal{O}^n}$$

where $\mathbf{I}$ is the standard Iwahori of $G_\mathcal{K}$ and v is associated to $\widehat{C}$ as in Theorem 3.5.

We show that the actions defined in Theorem 4.9 and [GSV20, Theorem 7.14] coincide.

Theorem 4.29. *The action in* [GSV20, Theorem 7.14] *on the module*

$$H_*^{\mathbb{C}^\times}\Big(\bigsqcup_{m\geq 0} PHilb_{m,m+n}(\widehat{C})\Big)[\hbar^{-1}]$$

agrees with the action defined by Theorem 4.9 on $H_*^{\mathbb{C}^\times}(\widetilde{M}_v)[\hbar^{-1}]$.

Proof. After inverting $\hbar$, the Atiyah-Bott localization formula implies the fixed point classes are a *basis* for the equivariant BM homology. As proven in [Web19, LW19, GSV20], the rational Cherednik algebra $\mathcal{H}_n$ is generated by the Dunkl-Opdam subalgebra, the finite symmetric group $\mathfrak{S}_n$, as well as two elements τ, λ, which can be identified with $\pi, \pi^{-1} \in \mathfrak{S}_n^{aff}$ under Suzuki's embedding of $\mathcal{H}_n$ to the trigonometric Cherednik algebra (see [KN18, GSV20]). We only need to identify these generators on both sides - the relations they satisfy are proved in [Web19, LW19, GSV20].

The Springer action is induced by the following diagram

$$\begin{array}{ccc} \mathcal{R}_{N_\mathbf{I},\mathbf{I}} & \xrightarrow{\varphi'} & [\widetilde{\mathfrak{g}}/G] = [\mathfrak{b}/B] \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi'} \\ \mathcal{R}_{N_\mathcal{O},G_\mathcal{O}} & \xrightarrow{\varphi} & [\mathfrak{g}/G] \end{array} \tag{4.8}$$

and the action of the simple reflections $s_1, \ldots, s_{n-1}$ comes from convolution with $[\mathcal{R}^{\leq s_i}_{N_\mathbf{I},\mathbf{I}}]$, which come about via pullback from classical correspondences on the Steinberg variety. The Springer action of [GSV20] is the usual one coming from projections to smaller affine flag varieties which is also defined via pullback from the right. The coincidence of the two is a classical result.

The equivariant cohomology classes $u_i \in H^*_T(pt)$ are identified with cap product by the Chern classes $c(\mathcal{L}_i)$ of the natural line bundles on the affine flag variety. The identification of these two is e.g. [OY16, Lemma 5.1.6].

Finally, we need the $\pi = \tau$ and $\pi^{-1} = \lambda$ operators. In [LW19, Theorem 5.2] and [Web19, Lemma 4.2], τ is identified with convolution by the correspondence in Section 4.2 corresponding to the space

$$X_\tau := \{(V_\bullet, V'_\bullet) | V_i = V'_{i+1}\} = \{(g\mathbf{I}, g'\mathbf{I}) | g\mathbf{t} = g'\}$$

and similarly σ is identified with convolution by the correspondence

$$X_\sigma := \{(V_\bullet, V'_\bullet) | V_i = V'_{i-1}\} = \{(g\mathbf{I}, g'\mathbf{I}) | g'\mathbf{t} = g\},$$

where $\mathbf{t}$ is the matrix sending $e_i \mapsto e_{i+1}, i = 1, \ldots, n-1$ and $e_n \mapsto te_1$ in the standard basis of $\mathcal{K}^n$. It is immediate that these coincide with the maps T, Λ on $H^{\mathbb{C}^\times}_*(\bigsqcup_{m\geq 0} \mathrm{PHilb}_{m,m+n}(\widehat{C}))[\hbar^{-1}]$ in [GSV20, Theorem 7.14]. □

5. Torus Links and the spherical RCA

In this section we speculate on the relation with a conjecture of Oblomkov-Rasmussen-Shende [ORS18] concerning the relation between the homology of the Hilbert schemes of points on plane curve singularities and minimal a-degree HOMFLY homology of the associated link. These cases of the ORS conjecture follow from the results of [ORS18] and Hogancamp-Mellit's computation of the HOMFLY homologies of torus knots [HM19].

It is still unclear whether the rational Cherednik algebra acts naturally on the triply graded homologies of algebraic links. Of course, assuming the ORS conjecture's validity (which we have in the toric cases), one has such an action *par transport de structure*. It would be interesting to know what this action means in terms of knot homology. For some speculations one can consult [GORS14].

Fix $G = GL_n, N = \mathrm{Ad} \oplus V, G_F = \mathbb{C}^\times_{dil}$ and set $\mathcal{R} = \mathcal{R}_{G,N}, \mathcal{A}^\hbar = \mathcal{A}^\hbar_{G,N}$. We focus on the case of $v \in N(\mathcal{O})$ corresponding to positive (n, k) torus knots, which can be realized by the plane curve singularities $\widehat{C}_{n,k}$ associated to $f = x^n - t^k$. Based on the relation between Hilbert schemes of points on $\widehat{C}_{n,k}$ and GASF in Theorem 3.5, we see that

$$\mathcal{M}_{(n,k)} := \mathrm{Hilb}^\bullet(\widehat{C}_{n,k}) = M_v$$

for, e.g. $v = (\gamma, e_n)$ with

$$\gamma = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ t^k & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

5.1. **Computation of the convolution action.** The assumptions of Lemma 4.27 hold for (n,k) torus knots due to Lemma 3.11. The L_v-fixed points are labeled by cocharacters as described in Proposition 3.14. We will label the fixed point classes inside $\mathbb{C}[\hbar^{\pm 1}]H_*^{\mathbb{C}^\times}(\mathcal{M}_{(n,k)})$ by $|A\rangle$. The map $L_v \to G_F \times \mathbb{C}^\times_{rot} \cong \mathbb{C}^\times_{dil} \times \mathbb{C}^\times_{rot}$ is realized by $\nu \mapsto (\nu^{-k}, \nu^n)$. This implies the relation $(nm + k\hbar)|A\rangle = 0$ for all A. We explicitly solve this by replacing $m|A\rangle = -\frac{k}{n}\hbar|A\rangle$. Let φ_a, $a = 1, ..., n$ be the components of φ in the standard basis.

Lemma 5.1. *The action of* $\mathbb{C}[\mathfrak{t}]$ *is given by*

$$\varphi_a |A\rangle = \left((n-a)\tfrac{k}{n} - A_a\right)\hbar |A\rangle$$

and the action of $[t^\lambda]$ *is given by*

$$[t^\lambda]|A\rangle = \left(\prod_{\lambda_a<0}\prod_{\alpha=0}^{|\lambda_a|-1}((n-a)\frac{k}{n} - A_a + \alpha)\hbar\right)\left(\prod_{\lambda_a>\lambda_b}\prod_{\beta=0}^{\lambda_a-\lambda_b-1}((b-a+1)\frac{k}{n} - A_a + A_b + \beta)\hbar\right)|A+\lambda\rangle\,.$$

Proof. This is a direct application of Lemma 4.27. □

Using these ingredients and equation (4.7) one can obtain an expression for the action of any $[\mathcal{R}_{\le\lambda}][f]$. Therefore, for λ a minuscule cocharacter we have
(5.1)

$$[\mathcal{R}_{\le\lambda}][1]|A\rangle = \sum_{\lambda' \in W\cdot\lambda} \frac{\left(\prod_{\lambda'_a<0}\prod_{\alpha=1}^{|\lambda'_a|}(\varphi_a - \alpha\hbar)\right)\left(\prod_{\lambda'_a>\lambda'_b}\prod_{\beta=1}^{\lambda'_a-\lambda'_b}(\varphi_b - \varphi_a + m - \beta\hbar)\right)}{\left(\prod_{\lambda'_a>\lambda'_b}\prod_{\gamma=1}^{\lambda'_a-\lambda'_b}(\varphi_b - \varphi_a - \gamma\hbar)\right)}|A+\lambda'\rangle\,.$$

There is a similar expression for the action of $[\mathcal{R}_{\le\lambda}][f]$ for $f(\varphi, m, \hbar)$ a W_λ-invariant function, though we will not need it in the following.

Proposition 5.2. *Comparing to* [BFN16a, A(iii)], *we have an identification (up to numerical factors)* $E_r[f] = [\mathcal{R}_{\le\lambda_r}][f]$ *and* $F_r[f] = [\mathcal{R}_{\le -\lambda_r}][\widetilde{f}]$ *where* $\lambda_r = (1,1,...,1,0,0...,0)$ *with* r *1's and* $\widetilde{f}(\varphi) = f(\varphi - \hbar)$.

Using this presentation of the algebra, the following result is straightforward.

Lemma 5.3. *For coprime* (n,k), $H_*^{\mathbb{C}^\times}(\mathcal{M}_{(n,k)})$ *is irreducible as the module for the spherical rational Cherednik algebra at parameter* $m = -\frac{k}{n}\hbar$.

Proof. We show that this module is irreducible by identifying the unique singular vector, namely $|0\rangle$. Recall that being a singular vector for the spherical rational Cherednik algebra corresponds to being in the kernel of all $F_r[f] = [\mathcal{R}_{\le -\lambda_r}][f]$. First consider the kernel of $F_n[f]$, or the classes corresponding to the cocharacter

$\lambda = (-1,-1,...,-1)$. The choice of f is that of a W invariant polynomial $f(\varphi, m, \hbar)$. From the action given in (5.1), we find that

$$F_n[1]\,|A\rangle = \prod_{b=1}^{n}\big(\varphi_b - \hbar\big)\hbar\,|A-(1,1,...,1)\rangle = \prod_{b=1}^{n}\big((n-b)\frac{k}{n} - A_b\big)\hbar\,|A-(1,1,...,1)\rangle\,.$$

Since $\gcd(n,k)=1$, the factor $((n-b)\frac{k}{n} - A_b)$ can only vanish for $b=n$ and $A_n = 0$. It follows that the kernel of $F_n[1]$ is exactly those classes $|A\rangle$ with $A_n = 0$. Moreover, such classes are in the kernel of $F_n[f]$ for all f.

Now consider the action of $F_{n-1}[f]$ on sums of fixed point classes with $A_n = 0$. Using Eq. (4.7) for we have, after a dramatic simplification following from $A_n = 0$,

$$F_{n-1}[1]\,|A_1,...,A_{n-1},0\rangle = \left(\prod_{b=1}^{n-1}\big((n-1-b)\frac{k}{n} - A_b\big)\hbar\right)|A_1-1,...,A_{n-2}-1,0\rangle\,.$$

Again, since $\gcd(n,k)=1$, the factor $((n-1-b)\frac{k}{n} - A_b)$ can only vanish for $b=n-1$ and $A_{n-1}=0$. Therefore $|A_1,...,A_{n-1},0\rangle$ is in the kernel of $F_{n-1}[1]$ if and only if $A_{n-1}=0$. Thus $\ker F_n[1]\cap\ker F_{n-1}[1]$ only contains classes with $A_n = A_{n-1} = 0$. Moreover, these classes belong to the kernel of $F_{n-1}[f]$ for all f. Continuing this process shows that

$$\ker F_n[1]\cap\ker F_{n-1}[1]\cap ... \cap F_1[1] = \mathrm{span}\{|0\rangle\}$$

and that it also belongs to the kernel of all $F_r[f]$. □

Now we state and prove the main theorem of this section.

Theorem 5.4. *For coprime (n,k), $H_*^{\mathbb{C}^\times}(\mathcal{M}_{(n,k)})$ can be identified with the irreducible representation $eL_{k/n}(triv)$ of the spherical rational Cherednik algebra of $\mathfrak{gl}_n$ at parameter $m = -\frac{k}{n}\hbar$. That is, setting the equivariant parameter $\hbar$ in $H_*^{\mathbb{C}^\times}(\mathcal{M}_{(n,k)})$ to -1, the quotient algebra $e\mathcal{H}_n e/(m-\frac{k}{n})$ acts.*

Proof. From [KN18], or a direct computation using (5.1), it follows that for all n the operators $X = [\mathcal{R}_{\le(1,0,...,0)}] = E_1[1]$ and $Y = [\mathcal{R}_{\le(-1,0,...,0)}] = F_1[1]$ generate an appropriately scaled copy of the Heisenberg algebra: $[X,Y] = n\hbar$. Since we have shown that $H_*^{\mathbb{C}^\times}(\mathcal{M}_{(n,k)})$ is irreducible as a module for the spherical rational Cherednik algebra of $\mathfrak{gl}_n$ at parameter $m = -\frac{k}{n}\hbar$ it follows that it must decompose as a product $\mathbb{C}[X]\otimes M$, where M is some irreducible module for the spherical rational Cherednik algebra of $\mathfrak{sl}_n$. Finally, noting that the spherical rational Cherednik algebra of $\mathfrak{sl}_n$ at parameter $m = -\frac{k}{n}\hbar$ has a unique finite dimensional, irreducible module, it suffices to show that $\ker Y \simeq M$ is finite dimensional.

Consider the graded Euler character of this homology, which can easily be computed from counting fixed points. Recall that the fixed points in $\mathcal{M}_{(n,k)}$ are labeled by cocharacters A as in Prop. 3.14, denote the set of such A by $\mathfrak{A}_{(n,k)}$. The degree in the Hilbert scheme is given by

$$d(A) = \sum_{a=1}^{n} A_a$$

and one finds

$$\chi(\mathcal{M}_{(n,k)}) = \sum_{A \in \mathfrak{A}_{(n,k)}} q^{d(A)} = \frac{1}{1-q^n} \begin{bmatrix} n-1+k \\ n-1 \end{bmatrix}_q .$$

Noting that X changes q-degree by 1, we can determine the dimension of M by multiplying the above by $1-q$, counting the $\mathbb{C}[X]$ factor, and setting $q = 1$. One finds

$$\dim_{\mathbb{C}} M = \frac{1}{n}\binom{n+k-1}{n-1} = \dim_{\mathbb{C}} H^*(\overline{\mathcal{J}}_{n,k}),$$

where $\overline{\mathcal{J}}_{n,k}$ is the compactified Jacobian of the curve $\widehat{C}_{n,k}$. □

Remark 5.5. It is worth noting that $\widetilde{\mathcal{A}}^{\hbar}$ is bi-filtered by the degree in Gr_G, called "monopole number" in the physics literature, and by the action induced by scaling $\mathbb{C}[\mathfrak{t}, \hbar]^W$ with weight 2, called "R-charge" in the physics literature. In particular, we assign the degree $(\pm r, r + 2\deg f)$ to $[\mathcal{R}_{\leq \pm\lambda_r}][f]$. The spherical rational Cherednik algebra of $\mathfrak{gl}_n$ is also bi-filtered by total polynomial degree and by difference in degree of x's and y's. That the respective filtrations agree follows from [KN18].

Appendix A

A.1. **Restriction with supports.** In this section, we define the restriction with support homomorphisms used in the definition of p^* in Theorem 4.9. We follow [BFN16b].

Definition A.1. Suppose we have a Cartesian diagram of ind-varieties

$$\begin{array}{ccc} Y & \xleftarrow[g]{} & Z \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} \\ W & \xleftarrow[f]{} & X \end{array}$$

and let A, B be (possibly unbounded) complexes of constructible sheaves on W, X. Then suppose we are given $\varphi \in \mathrm{Hom}(A, f_*B) \cong \mathrm{Hom}(f^*A, B)$. Define the morphism of complexes

$$j^!A \to j^!f_*f^*A \cong g_*i^!f^*A \to g_*i^!B$$

as the composition of the adjunction map and φ. This induces a map on hypercohomology:

$$H^*(Y, j^!A) \to H^*(Z, i^!B).$$

We will call this map "restriction with supports".

Remark A.2. Suppose we have a Cartesian diagram of varieties $\begin{array}{ccc} Z & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & W \end{array}$ If the first arrow is a regular embedding, let N be the pullback to Z of the normal bundle $N_{X/W}$. There is a specialization map

$$\sigma : H_*(Y) \to H_*(N),\ [V] \mapsto [C_{(C\cap Z)/V}].$$

The usual refined intersection map/pullback with support is defined as the composition $H_*(Y) \to H_*(N) \to H_*(Z)$.

A.2. **Finite-dimensional approximation.** In many parts of this paper, we consider equivariant complexes on infinite-dimensional ind-varieties, in particular $\mathcal{R}, \mathcal{T}$ and $N_{\mathcal{O}}$ and their substacks. We refer the reader to [BFN16b, Section 2] for more precise definitions in the first two cases, and in the latter case define

$$D^b_{\tilde{G}_{\mathcal{O}}}(N_{\mathcal{O}})$$

to be the direct limit over the finite-dimensional approximations to $N_{\mathcal{O}}$ given by $N_{\mathcal{O}}/t^iN_{\mathcal{O}}$. The degree shifts such as $[-2\dim N_{\mathcal{O}}]$ we use, are also to be understood as in [BFN16b, Section 2].

A.3. **Associativity.** In this section, we prove that the convolution product defined in Theorem 4.9 is associative. We follow the proof of associativity of the convolution product of $\widetilde{\mathcal{A}}^{\hbar}_{G,N,\mathbf{P},N_\mathbf{P}} := \widetilde{\mathcal{A}}^{\hbar}_{\mathbf{P},N_\mathbf{P}}$ in [BFN16b, Section 3] and the rough outline in the preprint [HKW20] .

Lemma A.3. *The convolution product defined in Theorem 4.9 is associative.*

Proof. We consider the following commutative diagram, which is a 'product' of the upper row of (4.1) and the appropriate version of [BFN16b, (3.2)]:
(A.1)

$$\begin{array}{ccccccc}
\mathcal{R}_{\mathbf{P},N_\mathbf{P}} \times N_\mathbf{P} & \xleftarrow{p} & \mathcal{P}_{N_\mathbf{P}} & \xrightarrow{q} & q(\mathcal{P}_{N_\mathbf{P}}) & \xrightarrow{\widetilde{m}} & N_\mathbf{P} \\
\uparrow{\scriptstyle m\times \mathrm{id}_{\mathcal{R}_{\mathbf{P},N_\mathbf{P}}}} & & \uparrow & & \uparrow & & \uparrow{\scriptstyle m} \\
q(p^{-1}(\mathcal{R}_{\mathbf{P},N_\mathbf{P}} \times \mathcal{R}_{\mathbf{P},N_\mathbf{P}})) \times N_\mathbf{P} & \longleftarrow & \boxed{3} & \longrightarrow & \boxed{4} & \longrightarrow & q(\mathcal{P}_{N_\mathbf{P}}) \\
\uparrow{\scriptstyle q\times \mathrm{id}_{N_\mathbf{P}}} & & \uparrow & & \uparrow & & \uparrow{\scriptstyle q} \\
p^{-1}(\mathcal{R}_{\mathbf{P},N_\mathbf{P}} \times \mathcal{R}_{\mathbf{P},N_\mathbf{P}}) \times N_\mathbf{P} & \longleftarrow & \boxed{1} & \longrightarrow & \boxed{2} & \longrightarrow & \mathcal{P}_\mathbf{P} \\
\downarrow{\scriptstyle p_{\mathcal{R}_{\mathbf{P},N_\mathbf{P}}}\times \mathrm{id}_{N_\mathbf{P}}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle p} \\
\mathcal{R}_{\mathbf{P},N_\mathbf{P}} \times \mathcal{R}_{\mathbf{P},N_\mathbf{P}} \times N_\mathbf{P} & \xleftarrow[\mathrm{id}_{\mathcal{R}_{\mathbf{P},N_\mathbf{P}}}\times p]{} & \mathcal{R}_{\mathbf{P},N_\mathbf{P}} \times \mathcal{P}_{N_\mathbf{P}} & \xrightarrow[\mathrm{id}_{\mathcal{R}_{\mathbf{P},N_\mathbf{P}}}\times q]{} & \mathcal{R}_{\mathbf{P},N_\mathbf{P}} \times q(\mathcal{P}_{N_\mathbf{P}}) & \xrightarrow[\mathrm{id}_{\mathcal{R}_{\mathbf{P},N_\mathbf{P}}}\times m]{} & \mathcal{R}_{\mathbf{P},N_\mathbf{P}} \times N_\mathbf{P},
\end{array}$$

where we have defined

$$\boxed{1} = \{(g_1, g_2, v') \in \widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^\times_{rot} \times \widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^\times_{rot} \times N_\mathbf{P} \mid g_2 v', g_1 g_2 v' \in \mathbf{N_P}\},$$

and $\boxed{2}$, $\boxed{3}$, $\boxed{4}$ are quotients of $\boxed{1}$ by $1 \times \widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}$, $\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot} \times 1$, $\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot} \times \widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}$ respectively. Here $\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot} \times \widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}$ acts on $\boxed{1}$ by

$$(h_1, h_2) \cdot (g_1, g_2, v') = (g_1 h_1^{-1}, h_1 g_2 h_2^{-1}, h_2 v') \quad \text{for } (h_1, h_2) \in \widetilde{\mathbf{P}} \times \widetilde{\mathbf{P}}.$$

The horizontal and vertical arrows from $\boxed{1}$, $\boxed{4}$ are given by

$$\text{(A.2)} \qquad \begin{array}{ccc}
(g_1, [g_2, v'], v') \xleftarrow{p_1} (g_1, g_2, v') \in \boxed{1} & & [g_1 g_2, v'] \\
\downarrow{\scriptstyle p_2} & & \uparrow \\
([g_1, g_1 g_2 v'], g_2, v'), & & \boxed{4} \ni [g_1, [g_2, v']] \mapsto [g_1, g_2 v'].
\end{array}$$

Arrows from $\boxed{2}$, $\boxed{3}$ are given by the obvious modification of above ones, as $\boxed{1} \to \boxed{3}$, etc. are fiber bundles. Also, $p_{\mathcal{R}_{\mathbf{P},N_\mathbf{P}}}$ is as defined in [BFN16b], i.e.

$$(g_1, [g_2, s]) \mapsto ([g_1, g_2 s], [g_2, s]).$$

Let $\alpha \in H^{L_v}(M_v^{\mathbf{P},N_\mathbf{P}})$ and $c_1, c_2 \in \widetilde{\mathcal{A}}^{\hbar}_{\mathbf{P},N_\mathbf{P}}$. The convolution product $c_2 \star \alpha$ is given by applying the construction in Theorem 4.9 (i.e. induced homomorphisms in BM homology) to the bottom row from left to right, and $c_1 \star (c_2 \star \alpha)$ is then obtained by going up in the rightmost column. Similarly $(c_1 \star c_2) \star \alpha$ is given by going up the leftmost column using the construction in [BFN16b] and then from left to right along the top row.

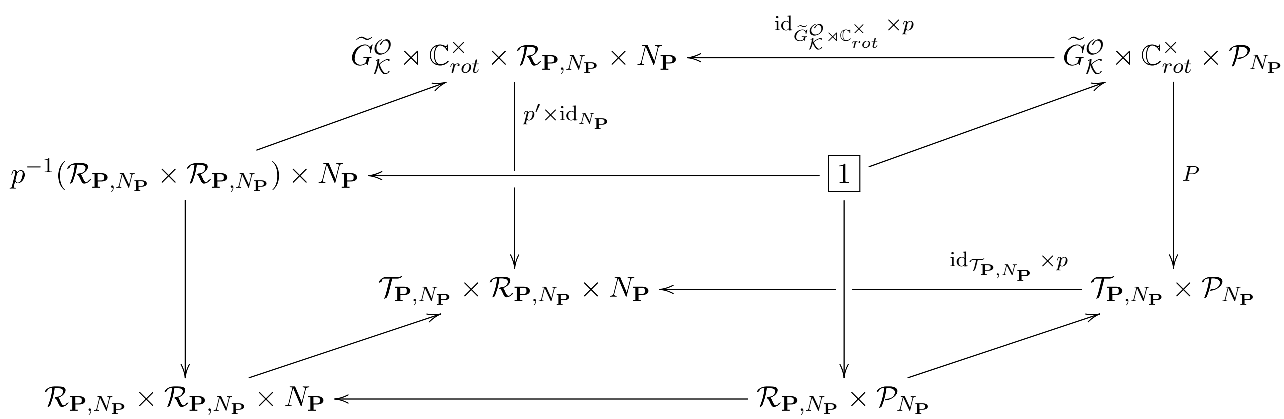
Therefore the associativity of the convolution product is the statement that the induced morphisms

$$-\star(-\star-),(-\star-)\star-: \widetilde{\mathcal{A}}^{\hbar}_{\mathbf{P},N_{\mathbf{P}}}\otimes\widetilde{\mathcal{A}}^{\hbar}_{\mathbf{P},N_{\mathbf{P}}}\otimes H^{L_v}_*(M_v^{\mathbf{P},N_{\mathbf{P}}})\to H^{L_v}_*(M_v^{\mathbf{P},N_{\mathbf{P}}})$$

are equal. This would follow commutativity of the associated "large square" in BM homology. (It might be helpful for the reader to recall the usual diagram for associativity of an algebra action).

We will in fact prove that each square is commutative after applying BM homology.

Let us first look at the bottom left square. We can extend the square to a cube as



Arrows from spaces in the front square to those in the rear square are closed embeddings. Arrows in the rear square are as indicated, where we have defined $P\colon \widetilde{G}^{\mathcal{O}}_{\mathcal{K}}\rtimes\mathbb{C}^\times_{rot}\times\mathcal{P}_{N_{\mathbf{P}}}\to\mathcal{T}_{\mathbf{P},N_{\mathbf{P}}}\times\mathcal{P}_{N_{\mathbf{P}}}$ by $(g_1,g_2,v')\mapsto([g_1,g_1g_2v'],g_2,v')$, just as the downward arrow from $\boxed{1}$ above.

The top, right, left and bottom faces of the cube are Cartesian and we have the isomorphisms

$$P^*(\omega_{\mathcal{T}_{\mathbf{P},N_{\mathbf{P}}}}\boxtimes\pi_1^!\mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}})\cong\omega_{\widetilde{G}^{\mathcal{O}}_{\mathcal{K}}\rtimes\mathbb{C}^\times_{rot}}\boxtimes\pi_1^!\mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}}$$

$$(p'\times\mathrm{id}_{N_{\mathbf{P}}})^*\omega_{\mathcal{T}_{\mathbf{P},N_{\mathbf{P}}}}\boxtimes\omega_{\mathcal{R}_{\mathbf{P},N_{\mathbf{P}}}}\boxtimes\mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}}\cong\omega_{\widetilde{G}^{\mathcal{O}}_{\mathcal{K}}\rtimes\mathbb{C}^\times_{rot}}\boxtimes\omega_{\mathcal{R}_{\mathbf{P},N_{\mathbf{P}}}}\boxtimes\mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}}.$$

This gives us two pullbacks with supports

$$H^*_{\widetilde{\mathbf{P}}\rtimes\mathbb{C}^\times_{rot}\times\widetilde{\mathbf{P}}\rtimes\mathbb{C}^\times_{rot}}(\mathcal{R}_{\mathbf{P},N_{\mathbf{P}}}\times\mathcal{P}_{N_{\mathbf{P}}},\omega_{\mathcal{R}_{\mathbf{P},N_{\mathbf{P}}}}\boxtimes\pi_1^!\mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}})\to H^*_{\widetilde{\mathbf{P}}\rtimes\mathbb{C}^\times_{rot}\times\widetilde{\mathbf{P}}\rtimes\mathbb{C}^\times_{rot}}(\boxed{1},\omega_{\widetilde{G}^{\mathcal{O}}_{\mathcal{K}}\rtimes\mathbb{C}^\times_{rot}}\boxtimes\pi_1^!\mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}})$$

and

$$H^*_{\widetilde{\mathbf{P}}\rtimes\mathbb{C}^\times_{rot}\times\widetilde{\mathbf{P}}\rtimes\mathbb{C}^\times_{rot}}(p^{-1}_{\mathcal{R}_{\mathbf{P},N_{\mathbf{P}}}}(\mathcal{R}_{\mathbf{P},N_{\mathbf{P}}}\times\mathcal{R}_{\mathbf{P},N_{\mathbf{P}}})\times N_{\mathbf{P}},\omega_{\mathcal{R}_{\mathbf{P},N_{\mathbf{P}}}}\boxtimes\omega_{\mathcal{R}_{\mathbf{P},N_{\mathbf{P}}}}\boxtimes\mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}})\to$$

$$H^*_{\widetilde{\mathbf{P}}\rtimes\mathbb{C}^\times_{rot}\times\widetilde{\mathbf{P}}\rtimes\mathbb{C}^\times_{rot}}(\boxed{1},\omega_{\widetilde{G}^{\mathcal{O}}_{\mathcal{K}}}\boxtimes\pi_1^!\mathcal{F}^v_{\mathbf{P},N_{\mathbf{P}}}).$$

We claim that these are the same homomorphism. Consider $\omega_{\mathcal{T}_\mathbf{P}} \boxtimes \omega_{\mathcal{R}_\mathbf{P}} \boxtimes \mathcal{F}^v_{\mathbf{P},N_\mathbf{P}}$ on $\mathcal{T}_{\mathbf{P},N_\mathbf{P}} \times \mathcal{R}_{\mathbf{P},N_\mathbf{P}} \times N_\mathbf{P}$, and consider the pull-backs of $\omega_{\mathcal{T}_{\mathbf{P},N_\mathbf{P}}}$ and $\omega_{\mathcal{R}_{\mathbf{P},N_\mathbf{P}}} \boxtimes \mathcal{F}^v_{\mathbf{P},N_\mathbf{P}}$ separately. Let us first consider $\omega_{\mathcal{R}_{\mathbf{P},N_\mathbf{P}}} \boxtimes \mathcal{F}^v_{\mathbf{P},N_\mathbf{P}}$.

$$\begin{array}{ccc}
P^*(\mathrm{id}_\mathcal{T} \times p)^*(\omega_{\mathcal{T}_{\mathbf{P},N_\mathbf{P}}} \boxtimes \omega_{\mathcal{R}_{\mathbf{P},N_\mathbf{P}}} \boxtimes \mathcal{F}^v_{\mathbf{P},N_\mathbf{P}}) & \longrightarrow & \omega_{\widetilde{G}^\mathcal{O}_\mathcal{K} \rtimes \mathbb{C}^\times_{rot}} \boxtimes \pi_1^! \mathcal{F}^v_{\mathbf{P},N_\mathbf{P}}[2\dim N_\mathbf{P} - 2\dim \widetilde{\mathbf{P}}] \\
\Vert & & \Vert \\
(p' \times \mathrm{id}_{N_\mathbf{P}})^*(\mathrm{id}_{\widetilde{G}^\mathcal{O}_\mathcal{K} \rtimes \mathbb{C}^\times_{rot}} \times p)^*(\omega_{\mathcal{T}_{\mathbf{P},N_\mathbf{P}}} \boxtimes \omega_{\mathcal{R}_{\mathbf{P},N_\mathbf{P}}} \boxtimes \mathcal{F}^v_{\mathbf{P},N_\mathbf{P}}) & \longrightarrow & \omega_{\widetilde{G}^\mathcal{O}_\mathcal{K} \rtimes \mathbb{C}^\times_{rot}} \boxtimes \pi_1^! \mathcal{F}^v_{\mathbf{P},N_\mathbf{P}}[2\dim N_\mathbf{P} - 2\dim \widetilde{\mathbf{P}}]
\end{array}$$

by following left, top arrows and bottom, right arrows in the rear square. They are the same, as both are essentially given by the homomorphism

$$p^*\omega_{\mathcal{R}_{\mathbf{P},N_\mathbf{P}}} \boxtimes \mathcal{F}^v_{\mathbf{P},N_\mathbf{P}} \to \pi_1^! \mathcal{F}^v_{\mathbf{P},N_\mathbf{P}}.$$

Next consider $\omega_{\mathcal{T}_{\mathbf{P},N_\mathbf{P}}}$. The $\mathcal{T}_{\mathbf{P},N_\mathbf{P}}$-component of $(\mathrm{id}_{\mathcal{T}_{\mathbf{P},N_\mathbf{P}}} \times p) \circ P = (\mathrm{id}_{\widetilde{G}^\mathcal{O}_\mathcal{K} \rtimes \mathbb{C}^\times_{rot}} \times p) \circ (p' \times \mathrm{id}_{\mathcal{R}_{\mathbf{P},N_\mathbf{P}}})$ (which is $(g_1, g_2, s) \mapsto [g_1, g_1 g_2.s]$) factors as

$$\widetilde{G}^\mathcal{O}_\mathcal{K} \rtimes \mathbb{C}^\times_{rot} \times \mathcal{P}_{N_\mathbf{P}} \xrightarrow{\mathrm{id}_{\widetilde{G}^\mathcal{O}_\mathcal{K} \rtimes \mathbb{C}^\times_{rot}} \times \Pi'} \widetilde{G}^\mathcal{O}_\mathcal{K} \rtimes \mathbb{C}^\times_{rot} \times N_\mathbf{P} \xrightarrow{p'_{\mathcal{T}_{\mathbf{P},N_\mathbf{P}}}} \mathcal{T}_{\mathbf{P},N_\mathbf{P}},$$

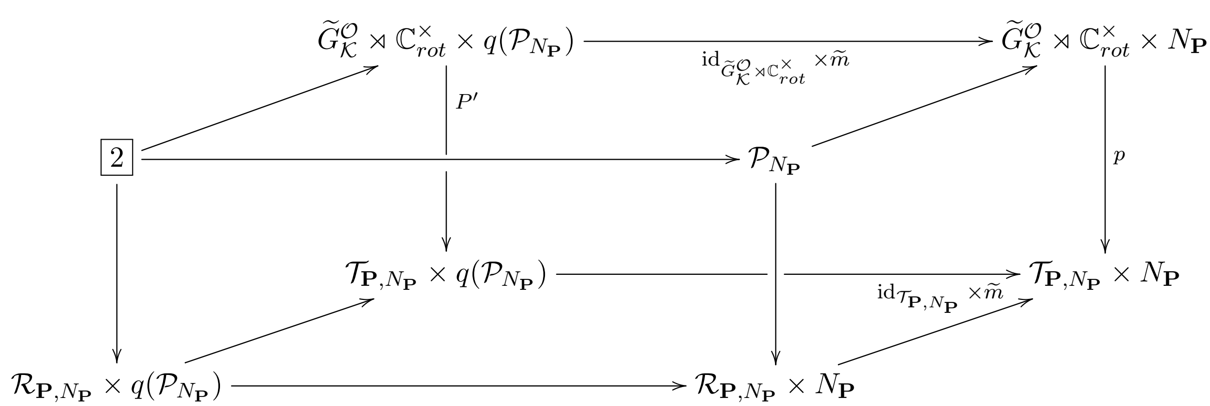
where $\Pi' \colon \mathcal{P}_{N_\mathbf{P}} \to N_\mathbf{P}$ is $(g_2, s) \mapsto g_2.s$. So we have

$$((\mathrm{id}_{\mathcal{T}_{\mathbf{P},N_\mathbf{P}}} \times p) \circ P)^*(\omega_{\mathcal{T}_{\mathbf{P},N_\mathbf{P}}} \boxtimes \omega_{\mathcal{R}_{\mathbf{P},N_\mathbf{P}}} \boxtimes \mathcal{F}^v_{\mathbf{P},N_\mathbf{P}}) \cong \omega_{\widetilde{G}^\mathcal{O}_\mathcal{K} \rtimes \mathbb{C}^\times_{rot}} \boxtimes \pi_1^! \mathcal{F}^v_{\mathbf{P},N_\mathbf{P}}[2\dim N_\mathbf{P} - 2\dim \widetilde{\mathbf{P}}].$$

The two restriction with supports homomorphisms from above constructed by going along left, top arrows and bottom, right arrows in the rear square are thus identical. This completes the proof of the commutativity of the bottom left square.

Since $\widetilde{q} \colon \mathcal{P}_{N_\mathbf{P}} \to q(\mathcal{P}_{N_\mathbf{P}})$ is a fiber bundle with fibers $\widetilde{\mathbf{P}} \rtimes \mathbb{C}^\times_{rot}$, commutativity for squares involving q is obvious.

Let us finally consider the right bottom square. We extend it to a cube:



Arrows from the front to rear are closed embeddings. The map $P' \colon \widetilde{G}^{\mathcal{O}}_{\mathcal{K}} \rtimes \mathbb{C}^{\times}_{rot} \times q(\mathcal{P}_{N_{\mathbf{P}}}) \to \mathcal{T}_{\mathbf{P},N_{\mathbf{P}}} \times q(\mathcal{P}_{N_{\mathbf{P}}})$ is given by

$$(g_1, [g_2, s]) \mapsto ([g_1, g_1 g_2.s], [g_2, s]).$$

The left and right faces of the cube are cartesian, and the commutativity of the rear square in the cube is enough to conclude that the corresponding proper pushforwards give the same map.

Finally, the commutativity of the induced maps in the right top square is clear, as it involves only pushforward homomorphisms. In particular, the whole large square is commutative. $\square$

Lemma A.4. *The class of* $[1] \in H_*^{\widetilde{\mathbf{P}} \rtimes \mathbb{C}^{\times}}(\mathcal{R}_{\mathbf{P},N_{\mathbf{P}}})$ *acts by the identity on* $H_*^{L_v}(M_v^{\mathbf{P},N_{\mathbf{P}}})$.

Proof. Consider the following diagram.

$$\begin{array}{ccccc}
N_{\mathbf{P}} \times N_{\mathbf{P}} & \longleftarrow & \widetilde{\mathbf{P}} \rtimes \mathbb{C}^{\times}_{rot} \times N_{\mathbf{P}} & \longrightarrow & N_{\mathbf{P}} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{R}_{\mathbf{P},N_{\mathbf{P}}} \times N_{\mathbf{P}} & \longleftarrow & \mathcal{P}_{N_{\mathbf{P}}} & \longrightarrow & q(\mathcal{P}_{N_{\mathbf{P}}})
\end{array}$$

The vertical maps are the natural inclusions (where we include $N_{\mathbf{P}} \hookrightarrow \mathcal{R}_{\mathbf{P},N_{\mathbf{P}}}$ as the fiber over $\mathrm{Fl}_{\mathbf{P}}^{\leq 1}$). Since $[1] \otimes c$ is the pushforward of $1 \otimes c$ along the left inclusion, by proper base change, $q_* p^*([1] \otimes c)$ is given by the pushforward along right vertical embedding

$$N_{\mathbf{P}} \to q(\mathcal{P}_{N_{\mathbf{P}}}).$$

Composing with $m : q(\mathcal{P}_{N_{\mathbf{P}}}) \to N_{\mathbf{P}}$, this embedding becomes the identity map on $N_{\mathbf{P}}$, so we must have $m_* q_* p^*([1] \otimes c) = c$. $\square$

Appendix B

B.1. **Modules for** $(2, 2\ell + 1)$ **Torus Knots.** In this section we discuss the module structure of $H_*^{L_v}(\mathcal{M}_{(2,2\ell+1)})$ and $H_*^{L_v}(\widetilde{\mathcal{M}}_{(2,2\ell+1)})$.

Recall that the rational Cherednik algebra of $\mathfrak{gl}_2$ is the quotient algebra

$$\mathcal{H}_n = \frac{\mathbb{C}[\hbar, m]\langle x_1, x_2, y_1, y_2 \rangle \rtimes \mathbb{C}\mathfrak{S}_2}{\sim}$$

where $\sim$ consists of the relations $[x_i, x_j] = [y_i, y_j] = 0$ for all i, j, and

$$[y_i, x_j] = \begin{cases} -\hbar + m(12) & \text{if } i = j, \\ -m(12) & \text{if } i \neq j. \end{cases}$$

The sphericizing element is given by $e = \frac{1}{2}(1 + (12))$.

The spherical subalgebra has generators given by an $\mathfrak{sl}_2$ triple

$$E = -\tfrac{1}{2} e(x_1^2 + x_2^2)e \qquad F = \tfrac{1}{2} e(y_1^2 + y_2^2)e \qquad H = \tfrac{1}{2} e(x_1 y_1 + y_1 x_1 + x_2 y_2 + y_2 x_2)e$$

and a Heisenberg pair

$$X = e(x_1 + x_2)e \qquad Y = e(y_1 + y_2)e$$

transforming in the defining representation of that $\mathfrak{sl}_2$. In particular, the non-zero commutation relations between these generators are those defining $\mathfrak{sl}_2$ and the Heisenberg algebra

$$[E, F] = \hbar H \qquad [H, E] = 2\hbar E \qquad [H, F] = -2\hbar F \qquad [X, Y] = 2\hbar,$$

and those describing the way X, Y transform under $\mathfrak{sl}_2$

$$[E, X] = [F, Y] = 0 \qquad [H, X] = [E, Y] = \hbar X \qquad [H, Y] = -[F, X] = -\hbar Y.$$

Denote $W^+ = \frac{1}{2}X^2, W^0 = -\frac{1}{2}(XY + YX), W^- = -\frac{1}{2}Y^2$, so that the $W^\pm, W^0$ transform in the adjoint representation of the above $\mathfrak{sl}_2$. There is one additional relation amongst these operators:

$$C_2 = 2(EW^- + FW^+) + HW^0 + m(m - \hbar),$$

where $C_2 = 2(EF + FE) + H^2$ is the quadratic Casimir of the $\mathfrak{sl}_2$ triple and m is a complex parameter.

Theorem B.1. *The spherical subalgebra, realized as the quantized BFN algebra* $\widetilde{\mathcal{A}}^{\hbar}_{G,N}$ *for* $G = GL_2$, $N = \mathrm{Ad} \oplus \mathbb{C}^2$, *acts via convolution on* $H^{\mathbb{C}^\times}_*(\mathcal{M}_{(2,2\ell+1)})$ *for* $m = -\frac{2\ell+1}{2}\hbar$. *As a module for the spherical rational Cherednik algebra of* $\mathfrak{gl}_2$, *we have*

$$H^{\mathbb{C}^\times}_*(\mathcal{M}_{(2,2\ell+1)}) \simeq eL_{(2\ell+1)/2}(triv),$$

where e *is the* $\mathfrak{S}_2$ *symmetrizer in rational Cherednik algebra of* $\mathfrak{gl}_2$ *and* $L_{(2\ell+1)/2}(triv)$ *is the simple rational Cherednik algebra (at parameter* $m = -\frac{2\ell+1}{2}\hbar$*) module induced from the trivial representation of* $\mathfrak{S}_2$.

To simplify the expressions below, we will simply write k instead of $2\ell + 1$. The below does *not* apply when k is even.

Proof. First consider the monopole operator $X := [\mathcal{R}_{(1,0)}]$. This arises from the orbit $\mathrm{Gr}^{(1,0)}_{GL_2}$, which form a copy of $\mathbb{P}^1$ parameterized by two affine charts given by

$$\begin{pmatrix} t & 0 \\ a_1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & a_2 \\ 0 & t \end{pmatrix}$$

with transition function $a_2 = \frac{1}{a_1}$. There are $G(\mathcal{O})$ torus fixed points at the origins of these affine charts, and the coordinate a_1 (resp. a_2) transforms with weight $\varphi_2 - \varphi_1$ (resp. $\varphi_1 - \varphi_2$). Applying Eq. (5.1) yields

$$X |A_1, A_2\rangle = \frac{A_1 - A_2 - k}{A_1 - A_2 - \frac{k}{2}} |A_1 + 1, A_2\rangle + \frac{A_1 - A_2}{A_1 - A_2 - \frac{k}{2}} |A_1, A_2 + 1\rangle .$$

Similarly, there is the monopole operator $Y := [\mathcal{R}_{(0,-1)}]$ coming from the orbit $\mathrm{Gr}^{(0,-1)}_{GL_2}$, which forms a copy of $\mathbb{P}^1$ parameterized by two affine charts

$$\begin{pmatrix} 1 & 0 \\ a_1 t^{-1} & t^{-1} \end{pmatrix} \qquad \begin{pmatrix} t^{-1} & a_2 t^{-1} \\ 0 & 1 \end{pmatrix}$$

with transition function $a_2 = \frac{1}{a_1}$. The coordinate a_1 again transforms with weight $\varphi_1 - \varphi_2$. We find that

$$Y\,|A_1, A_2\rangle = \frac{(A_1 - A_2)(\frac{k}{2} - A_1)\hbar}{A_1 - A_2 - \frac{k}{2}}\,|A_1 - 1, A_2\rangle + \frac{A_2(k - A_1 + A_2)\hbar}{A_1 - A_2 - \frac{k}{2}}\,|A_1, A_2 - 1\rangle\,.$$

There are two other monopole operators we will be interested in, namely $E = [\mathcal{R}_{(1,1)}]$ and $F = -[\mathcal{R}_{(-1,-1)}]$. They come from $\mathrm{Gr}^{(1,1)}_{GL_2}$ and $\mathrm{Gr}^{(-1,-1)}_{GL_2}$ respectively, both of which are single points. Applying Eq. (5.1) gives

$$E\,|A_1, A_2\rangle = |A_1 + 1, A_2 + 1\rangle \quad F\,|A_1, A_2\rangle = (\tfrac{k}{2} - A_1)A_2\hbar^2\,|A_1 - 1, A_2 - 1\rangle$$

from which it is straightforward to compute that $H = \hbar - \varphi_1 - \varphi_2$ acts as

$$H\,|A_1, A_2\rangle = (A_1 + A_2 + 1 - \tfrac{k}{2})\hbar\,|A_1, A_2\rangle$$

and makes (E, F, H) an $\mathfrak{sl}_2$ triple. The quadratic Casimir $C_2 = 2(EF + FE) + H^2$ acts as

$$C_2\,|A_1, A_2\rangle = \Big((A_1 - A_2 - \tfrac{k}{2})^2 - 1\Big)\hbar^2\,|A_1, A_2\rangle\,.$$

It is straightforward to check that the desired relations are indeed satisfied with $m = -\frac{k}{2}\hbar$.

From the action of $\mathfrak{sl}_2$, we see that the classes $|A_1, 0\rangle$ are lowest weight vectors with weights $\nu = (A_1 + 1 - \frac{k}{2})\hbar$. Therefore, the homology of this GASF can be expressed as an $\mathfrak{sl}_2$ module as

$$H^{\mathbb{C}^\times}_*(\mathcal{M}_{(2,k)}) = \bigoplus_{A_1=0}^{k} \Lambda_{\left(A_1 + 1 - \frac{k}{2}\right)\hbar},$$

where Λ_ν is the $\mathfrak{sl}_2$ Verma module generated by a lowest weight vector of weight ν. It is also worth noting that $|0, 0\rangle$ is a vacuum vector for the Heisenberg algebra generated by X, Y; hence it is the unique spherical rational Cherednik algebra singular vector. We can therefore identify this with the SCA module:

$$H^{\mathbb{C}^\times}_*(\mathcal{M}_{(2,k)}) \simeq eL_{k/2}(\mathrm{triv}),$$

where e is the $\mathfrak{S}_2$ symmetrizer in the rational Cherednik algebra and $L_{k/2}(\mathrm{triv})$ is the simple rational Cherednik algebra module induced from the trivial representation of $\mathfrak{S}_2$. □

Remark B.2. It is worth noting that there is another presentation of the GL_2 spherical rational Cherednik algebra given by a (different) $\mathfrak{sl}_2$-triple $(\widetilde{E}, \widetilde{F}, \widetilde{H})$ and the Heisenberg pair X, Y.[2] In this presentation X, Y transform trivially under $\mathfrak{sl}_2$ and the quadratic Casimir of the $\mathfrak{sl}_2$-triple is given by

$$\widetilde{C}_2 = (m - \tfrac{3}{2}\varepsilon)(m + \tfrac{1}{2}\varepsilon)$$

with no other constraints. In this presentation we find that the homology of our GASF is given by

$$H_*^{\mathbb{C}^\times}(\mathcal{M}_{(2,2\ell+1)}) \simeq \mathbb{C}[X] \otimes \mathrm{Sym}^\ell \square,$$

where $\mathrm{Sym}^\ell \square$ is the $\ell + 1$ dimensional representation of $\mathfrak{sl}_2$. We can identify $\mathrm{Sym}^\ell \square$ as the cohomology of $\mathbb{P}^\ell$, the compactified Jacobian for the $(2, 2\ell + 1)$ torus knots. This feature was predicted in [ORS18].

We now move to the action of the rational Cherednik algebra on the homology of parabolic Hilbert schemes. In particular, we spell out the comparison in Theorem 4.29 between the action given by Theorem 4.9 and [GSV20].

Theorem B.3. *The action of the rational Cherednik algebra on the homology of* $\mathrm{PHilb}^\bullet(\widehat{C})$ *given in Theorem 4.9 agrees with the action of* [GSV20, Theorem 7.14]*.*

Proof. As discussed at the end of Section 3, we describe the action on classes $|A, \sigma\rangle$ associated to the fixed points $\sigma t^A p$ and match the action of the rational Cherednik algebra given in [GSV20] by identifying these fixed points with their "renormalized basis." We start by identifying

$$|A_1, A_2, ()\rangle = \widetilde{v}_{(A_1,A_2)} \qquad |A_1, A_2, (12)\rangle = \widetilde{v}_{(A_2,A_1)}.$$

The action of the equivariant parameters φ_a on the class $|A, \sigma\rangle$ can be easily seen to be

$$\varphi_1 |A, ()\rangle = (\tfrac{k}{2} - A_1)\hbar |A, ()\rangle \qquad \varphi_1 |A, (12)\rangle = -A_2\hbar |A, (12)\rangle$$

and

$$\varphi_2 |A, ()\rangle = -A_2\hbar |A, ()\rangle \qquad \varphi_2 |A, (12)\rangle = (\tfrac{k}{2} - A_1)\hbar |A, (12)\rangle,$$

which translates to (for $A_2 \le A_1$)

$$\varphi_1 \widetilde{v}_{(A_2,A_1)} = (\tfrac{k}{2} - A_1)\hbar \widetilde{v}_{(A_1,A_2)} \qquad \varphi_2 \widetilde{v}_{(A_1,A_2)} = -A_2\hbar \widetilde{v}_{(A_2,A_1)}$$

and (for $A_1 > A_2$)

$$\varphi_1 v_{(A_2,A_1)} = -A_2\hbar v_{(A_2,A_1)} \qquad \varphi_2 v_{(A_2,A_1)} = (\tfrac{k}{2} - A_1)\hbar v_{(A_2,A_1)}$$

we can thus identify $u_1 = \varphi_1$ and $u_2 = \varphi_2$ in [GSV20, Theorem 7.14].

[2] The change of variables is given by $\widetilde{E} = E - \frac{1}{4}X^2, \widetilde{F} = F + \frac{1}{4}Y^2, \widetilde{H} = H + \frac{1}{4}(XY + YX)$.

The action of the transposition s on $|A,\sigma\rangle$ is given by

$$\begin{aligned} s\,|A_1,A_2,()\rangle &= \frac{k}{2(A_2-A_1)-k}\,|A_1,A_2,()\rangle + \frac{2(A_2-A_1)}{2(A_2-A_1)-k}\,|A_1,A_2,(12)\rangle \\ s\,|A_1,A_2,(12)\rangle &= \frac{2(A_2-A_1+k)}{2(A_2-A_1)-k}\,|A_1,A_2,()\rangle - \frac{k}{2(A_2-A_1)-k}\,|A_1,A_2,(12)\rangle \end{aligned}.$$

from which it follows that $1-s$ acts as

$$\begin{aligned} (1-s)\,|A_1,A_2,()\rangle &= \frac{2(A_1-A_2)}{2(A_2-A_1)-k}(|A_1,A_2,()\rangle - |A_1,A_2,(12)\rangle) \\ (1-s)\,|A_1,A_2,(12)\rangle &= \frac{2(A_1-A_2-k)}{2(A_2-A_1)-k}(|A_1,A_2,(12)\rangle - |A_1,A_2,()\rangle) \end{aligned}$$

In agreement with the action of $1-s$ in [GSV20, Theorem 7.14].

Finally, the actions of T and Λ do not require a fancy localization formula as they correspond to point classes in the affine flag variety. In particular, we find that the excess intersection factors are trivial for T:

$$T\,|A_1,A_2,()\rangle = |A_1+1,A_2,(12)\rangle \qquad T\,|A_1,A_2,(12)\rangle = |A_1,A_2+1,()\rangle$$

and they are $-A_2\hbar$ (resp. $(\frac{k}{2}-A_1)\hbar$) for Λ on $|A_1,A_2,()\rangle$ (resp. $|A_1,A_2,(12)\rangle$):

$$\Lambda\,|A_1,A_2,()\rangle = (\tfrac{k}{2}-A_2)\hbar\,|A_1,A_2-1,(12)\rangle \qquad \Lambda\,|A_1,A_2,(12)\rangle = (\tfrac{k}{2}-A_1)\hbar\,|A_1,A_2-1,()\rangle$$

in agreement with the action of T,Λ on $\widetilde{v}_{\mathbf{a}}$ from [GSV20, Theorem 7.14]. □

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