On the affine Springer fibers inside the invariant center of the small quantum group

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May 19, 2022

Abstract

Let $\mathfrak{u}_\zeta^\vee$ denote the small quantum group associated with a simple Lie algebra $\mathfrak{g}^\vee$ and a root of unity ζ. Based on the geometric realization of $\mathfrak{u}_\zeta^\vee$ in [8], we use a combinatorial method to derive a formula for the dimension of a subalgebra in the $G^\vee$-invariant part of the center $Z(\mathfrak{u}_\zeta^\vee)^{G^\vee}$ of $\mathfrak{u}_\zeta^\vee$, that conjecturally coincides with the whole $G^\vee$-invariant center. In case $G = SL_n$ we study a refinement of the obtained dimension formula provided by two geometrically defined gradings.

Contents

1 Introduction

Let G be a complex simple simply connected algebraic group, and $\mathfrak{g}$ its Lie algebra. We will fix Cartan and Borel subalgebras $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$ in $\mathfrak{g}$. Denote also by $\mathfrak{t}^\vee \subset \mathfrak{b}^\vee \subset \mathfrak{g}^\vee$ the same data for the Langlands dual Lie algebra. Let ℓ be an odd number that is greater than the Coxeter number h of $\mathfrak{g}$ and coprime to the determinant of the Cartan matrix and to $h+1$.

We denote by $\mathfrak{u}_\zeta^\vee = \mathfrak{u}_\zeta(\mathfrak{g}^\vee)$ the small quantum group associated to the Lie algebra $\mathfrak{g}^\vee$ and a primitive ℓ-th root of unity ζ [33]. Let Λ denote the coweight lattice of G, and W the Weyl group of $\mathfrak{g}$ (and $\mathfrak{g}^\vee$). Then $\mathfrak{u}_\zeta^\vee$ decomposes into a direct sum of blocks enumerated by the orbits of the extended affine Weyl group of $\mathfrak{g}$, $\widetilde{W} = W \ltimes \Lambda$ acting via the ℓ-dilated dot action on Λ, see for example the Introduction to [25]. We denote by $\mathfrak{u}_\zeta^{\vee,\lambda}$ the block corresponding to the $\widetilde{W}$-orbit of $\lambda \in \Lambda$. In particular, $\mathfrak{u}_\zeta^{\vee,0}$ denotes the principal (regular) block of the small quantum group.

In this paper we derive a dimension formula for a subalgebra in the $G^\vee$-invariant part of the center $Z(u_\zeta^\vee)$ of the small quantum group[1]. Conjecturally, this subalgebra coincides with the entire $Z(u_\zeta^\vee)^{G^\vee}$. In case $G = SL_n$ we study a refinement of the obtained dimension formula provided by two gradings. Our treatment is based on the geometric realization of the subalgebra in the $G^\vee$-invariant part of the center of $\mathfrak{u}_\zeta^\vee$ obtained in [8].

Theorem 1.1 (BBASV)**.** *There is an algebra embedding*

$$H^*(\mathrm{Gr}^{\zeta,\gamma})^{\widetilde{W}} \subseteq Z(\mathfrak{u}_\zeta^\vee)^{G^\vee}$$

where the product on the left is the cup product.

Conjecture 1.2 (BBASV)**.** *The embedding above is an isomorphism.*

Here $\mathrm{Gr}^{\zeta,\gamma} = \mathrm{Gr}^\gamma \cap \mathrm{Gr}^\zeta$, where Gr^γ is the affine Springer fiber with $\gamma = st^{\ell-1}$ for a regular element $s \in \mathfrak{t}^{reg}$, and Gr^ζ are the ζ-fixed points for the cyclic group action generated by ζ. The $\widetilde{W}$-action is the one induced by the lattice action on

[1] After this paper was written, we received the complete text of [8], where the same dimension is computed. We use a different argument based on the block decomposition.

the affine Springer fiber, as well as a monodromy action coming from variation of s in a family.

The left-hand side of Theorem 1.1 decomposes naturally in blocks, since $\mathrm{Gr}^{\zeta,\gamma}$ can be written as a finite disjoint union of generalizations of affine Springer fibers, as explained in the next section. This block decomposition respects the one on $Z(\mathfrak{u}_\zeta^\vee)^{G^\vee}$ as explained in *loc. cit.*

We establish an isomorphism of vector spaces between the geometrically defined subspace of the $G^\vee$-invariant part of the center of the regular block $Z(\mathfrak{u}_\zeta^{\vee,0})^{G^\vee}$ of the small quantum group and Gordon's canonical quotient of the space of the diagonal coinvariants, which we will denote $\overline{\mathrm{DR}}_W$ and simply call diagonal coinvariants, hoping this will not cause extra confusion. Assuming Conjecture 1.2, the obtained result agrees with (an ungraded form of) the conjecture formulated in [24].

Further, we extend the result to a similar isomorphism for other blocks of $\mathfrak{u}_\zeta^\vee$, relating the geometrically defined subalgebra in each block of the center $Z(\mathfrak{u}_\zeta^{\vee,\lambda})$ to a respective space of partial diagonal coinvariants $\overline{\mathrm{DR}}_n^{W_\lambda}$.

Combinatorially, this allows us to derive the formula for the dimension of the subalgebra isomorphic to $H^*(\mathrm{Gr}^{\zeta,\gamma})^{\widetilde{W}}$ in the $G^\vee$-invariant part of the center $(Z(\mathfrak{u}_\zeta^\vee))^{G^\vee}$ of the small quantum group in terms of the rational Catalan number associated with the Weyl group of $\mathfrak{g}$.

Theorem 1.3. *Suppose that ℓ is as in the beginning of the Introduction. Then*

$$\dim H^*(\mathrm{Gr}^{\zeta,\gamma})^{\widetilde{W}} = \mathrm{Cat}_W((h+1)\ell - h, h),$$

where Cat_W is the generalized rational Coxeter-Catalan number of W, and h the Coxeter number associated with the root system of $\mathfrak{g}$.

In case $\mathfrak{g} = \mathfrak{sl}_n$ our result for the dimension of the subalgebra in the center coincides with the formula conjectured by Igor Frenkel for the whole $G^\vee$-invariant part (see [25]):

Corollary 1.4. *Let $G = SL_n$, and suppose that $n \not\equiv 0, -1$ mod ℓ. Then*

$$\dim H^*(\mathrm{Gr}^{\zeta,\gamma})^{\widetilde{W}} = c_{(n+1)\ell-n,n} = \frac{1}{(n+1)\ell}\binom{(n+1)\ell}{n},$$

the rational $((n+1)\ell - n, n)$-Catalan number.

The $G^\vee$-invariant part of the blocks of the center, indeed the entire blocks of the center can in fact be equipped with two gradings that arise from the isomorphism (as a bigraded vector space) of the center of each block with certain equivariant cohomologies of coherent sheaves on the Springer resolution [7] (see Theorem 4.13).

In case $G = SL_n$, assuming the main conjecture in [11], we define a bigrading on the side of the affine Springer fibers, which coincides with the sign-twisted bigraded structure on the diagonal coinvariants (see Theorem 4.11). This bigrading comes from the realization of the invariant piece of the cohomology as a quotient of the BM homology of the "positive part" of the affine Springer fiber, up to a linear dual.

We also study another model for the bigrading, coming from the perverse filtration on certain parabolic Hitchin fibers. This formulation is more geometric and allows us, for example, to define an $\mathfrak{sl}_2$ action on the blocks of the cohomology of affine Springer fibers. We expect that our bigrading (either version) carried over to the blocks of the center coincides with the one coming from the equivariant cohomologies of the coherent sheaves on the Springer resolution. This would imply in particular that the constructed $\mathfrak{sl}_2$-action on the affine Springer fiber side coincides with the $\mathfrak{sl}_2$ action "along the diagonals" as in [24, Section 4]. Finally, we exhibit a spectral curve construction which can hopefully be used to relate the second model of the bigrading to the (much better behaved) elliptic homogeneous affine Springer fibers studied e.g. in [12].

Acknowledgements

We would like to thank Erik Carlsson and Peng Shan for helpful explanations of their work. We also thank Roman Bezrukavnikov for sharing the paper [8] with us. N.H. appreciates the support of the grant number 182767 of the Swiss National Science Foundation. A.L. is grateful for the support from Faculté des Sciences de Base at École Polytechnique Fédérale de Lausanne.

2 Affine Springer fibers

Let ζ be a primitive ℓ-th root of unity, $\mathcal{O} := \mathbb{C}[[t]], \mathcal{K} := \mathbb{C}((t))$, and $s \in \mathfrak{t}^{reg}$. Define $\gamma = st^{\ell-1}$ and consider

$$\mathrm{Gr}^{\gamma,\zeta} = \mathrm{Gr}^{\gamma} \cap \mathrm{Gr}^{\zeta}$$

where Gr^{γ} is the usual affine Springer fiber

$$\mathrm{Gr}^{\gamma} := \{gG(\mathcal{O}) | \mathrm{Ad}_{g^{-1}}\gamma \in \mathfrak{g}(\mathcal{O})\}$$

and Gr^{ζ} are the ζ-fixed points for the cyclic group action generated by $\zeta \in \mathbb{G}_m^{rot}$. More generally, denote by

$$\mathcal{F}l_{\mathbf{P}}^{\gamma} = \{g\mathbf{P} | \mathrm{Ad}_{g^{-1}}\gamma \in \mathrm{Lie}(\mathbf{P})\}$$

the affine Springer fiber of γ for $\mathcal{F}l_{\mathbf{P}} = G(\mathcal{K})/\mathbf{P}$ the partial affine flag variety of some parahoric subgroup (or some congruence subgroup thereof) of $G(\mathcal{K})$.

Note first that we may view Gr^{γ} as an affine Springer fiber in the partial affine flag variety for the first congruence subgroup $G^{(1)}(\mathcal{O}) = \ker(G(\mathcal{O}) \twoheadrightarrow G(\mathbb{C}))$. Namely,

$$\mathrm{Gr}^{\gamma} \cong \{[g] \in G(\mathcal{K})/G^{(1)}(\mathcal{O}) | \mathrm{Ad}_{g^{-1}}st^{\ell} \in t\mathfrak{g}(\mathcal{O})\}.$$

Remark 2.1. The explanation for taking $\gamma = st^{\ell-1}$ as opposed to $\gamma = st^{\ell}$ is given by the rewriting of Gr^{γ} in terms of st^{ℓ} and the congruence subgroup $G^{(1)}(\mathcal{O})$ above. Ultimately, we will not be interested in the $\mathcal{F}l_{\mathbf{P}}^{\gamma}$ for standard parahorics, but their close analogs $\mathrm{fl}_{\mathbf{P}}^{\gamma}$, to be defined below. They can be directly defined using st^{ℓ}. This should be compared to [8, Lemma 2.6.], where the terminology "affine Spaltenstein fiber" and notations such as ${}^{0}\mathcal{G}\mathfrak{r}_{\gamma}$ are used.

By [32, Proposition 4.6.] we have that

$$\mathrm{Gr}^{\zeta} = \bigsqcup_{\lambda \in \Lambda / \widetilde{W}_{\ell}} \mathcal{F}l_{\mathbf{P}_{\lambda}}$$

where $\mathbf{P}_{\lambda}$ is the parahoric group scheme in $G((z^{\ell}))$ associated to λ. Here $\widetilde{W}_{\ell}$ is the extended affine Weyl group acting by the ℓ-dilated action on Λ.

Since the actions of ζ and γ commute,

$$\mathrm{Gr}^{\gamma,\zeta} = \bigsqcup_{\lambda \in \Lambda / \widetilde{W}_{\ell}} \mathrm{fl}^{\gamma}_{\mathbf{P}_{\lambda}} \tag{2.1}$$

where

$$\mathrm{fl}^{\gamma}_{\mathbf{P}_{\lambda}} = \{[g] \in \mathcal{F}l_{\mathbf{P}_{\lambda}} | \mathrm{Ad}_{g^{-1}} st^{\ell} \in \mathrm{Lie}(\mathrm{Rad}(\mathbf{P}_{\lambda}))\}.$$

The $\Lambda \cong T(\mathcal{K})/T(\mathcal{O})$-action commutes with ζ, so we get a Λ-action on $\mathrm{Gr}^{\zeta,\gamma}$ and this preserves the decomposition into $\mathrm{fl}^{\gamma}_{\mathbf{P}_{\lambda}}$. On $H^*(\mathrm{Gr}^{\gamma,\zeta})$, there is also an action of W coming from varying $s \in \mathfrak{t}^{reg}$. These assemble to a (left) action of $\widetilde{W}$ on the cohomology. There is also another commuting (right) action of $\widetilde{W}$ coming from the Springer action. Taking invariants for the left action and antisymmetrizing on the right by the idempotents

$$\mathbf{e}^{-}_{\lambda} = \frac{1}{|W_{\lambda}|} \sum_{w \in W_{\lambda}} (-1)^{l(w)} w$$

for W_{λ} a parabolic subgroup of $\widetilde{W}$, we have

Proposition 2.1. *As (ungraded) W-representations, for the Springer action of W,*

$$H^*(\mathrm{fl}^{\gamma}_{\mathbf{P}_{\lambda}})^{\widetilde{W}} \cong \mathbb{C}[Q/(h+1)Q]\mathbf{e}^{-}_{\lambda}.$$

Here Q is the coroot lattice. In particular,

$$H^*(\mathrm{fl}^{\gamma}_{\mathbf{P}_{\lambda}})^{\widetilde{W}} \cong \overline{\mathrm{DR}}_{W}^{W_{\lambda}},$$

where $\overline{\mathrm{DR}}_{W}$ is defined in the beginning of Section 3.1. Here W_{λ} is the stabilizer of $\lambda \in \Lambda$ inside W.

Proof. By [9, Theorem 1.2.] we have that $H^*(\mathrm{fl}^{\gamma}_{\mathbf{I}})^{\widetilde{W}}$ is isomorphic to

$$\mathbb{C}[Q/(h+1)Q]$$

as a $W-$representation. This is the sgn-twist of $\overline{\mathrm{DR}}_{W}$ [16].

On the other hand, we claim that $H^*(\mathrm{fl}^{\gamma}_{\mathbf{P}_{\lambda}})^{\widetilde{W}} \cong H^*(\mathrm{fl}^{\gamma}_{\mathbf{I}})^{\widetilde{W}}\mathbf{e}^{-}_{\lambda}$.

Note that there is a natural inclusion

$$\mathrm{fl}^{\gamma}_{\mathbf{P}_{\lambda}} \to \mathcal{F}l^{\gamma}_{\mathbf{P}_{\lambda}} := \{[g] \in \mathcal{F}l_{\mathbf{P}_{\lambda}} | \mathrm{Ad}_{g^{-1}} st^{\ell} \in \mathrm{Lie}(\mathbf{P}_{\lambda})\}$$

And that we always (i.e. for any parahoric containing **I** and any regular semisimple γ) have a Cartesian diagram

$$\begin{array}{ccc}\mathcal{F}l^{\gamma}_{\mathbf{I}} & \longrightarrow & \left[\widetilde{\mathfrak{l}}_{\mathbf{P}}/L_{\mathbf{P}}\right] \\ \downarrow & & \downarrow \\ \mathcal{F}l^{\gamma}_{\mathbf{P}} & \longrightarrow & [\mathfrak{l}_{\mathbf{P}}/L_{\mathbf{P}}]\end{array}$$

where the right-hand column is the Grothendieck-Springer resolution for $L_{\mathbf{P}}$, the Levi quotient of $\mathbf{P}$. Taking the fiber at 0 of the bottom map gives exactly $\mathrm{fl}^{\gamma}_{\mathbf{P}}$. The cohomology of this fiber is exactly the W_λ-antisymmetric part of the pull-back of the Springer sheaf (see [17, Lemma 2.2]), so after noting that everything commutes with the $\widetilde{W}$-action, we are done. □

Remark 2.2. It would be interesting to know if there is an extension of this Proposition to the singly or doubly graded cases. For the doubly graded case in type A, see Corollary 4.12.

The relation with the center of the small quantum group is as follows.

Proposition 2.2. *We have*

$$H^*(\mathrm{Gr}^{\zeta,\gamma})^{\widetilde{W}} \cong \bigoplus_{\lambda\in\Lambda/\widetilde{W}_\ell} \overline{\mathrm{DR}}_W^{W_\lambda}$$

where $\overline{\mathrm{DR}}_W^{W_\lambda}$ *is as in Eq.* (3.1). *Assuming Conjecture 1.2, this also gives the block decomposition for* $Z(\mathfrak{u}_\zeta^\vee)^{G^\vee}$.

To compute the dimension of $H^*(\mathrm{Gr}^{\zeta,\gamma})^{\widetilde{W}}$, it will be enough to understand the structure of the block decomposition and the dimensions of the $\overline{\mathrm{DR}}_W^{W_\lambda}$. This will be done in Sections 3.2 and 3.3.

In analogy with the proof of Proposition 2.1, we will need the following simple fact. Let

$$\mathbf{e} = \frac{1}{|W|}\sum_{w\in W} w,\ \mathbf{e}^- = \frac{1}{|W|}\sum_{w\in W}(-1)^{\ell(w)}w$$

be the symmetrizing and antisymmetrizing idempotents for W.

Lemma 2.3. *Let* $\gamma \in \mathfrak{g}(\mathcal{K})$ *be* any *regular semisimple element. As singly graded* W*-representations, we have that*

$$H^*(\mathcal{F}l^{t\gamma}_{\mathbf{I}})\mathbf{e}^- \cong H^*(\mathrm{Gr}^\gamma)[-2\dim G/B]$$

Proof. This is [17, Lemma 2.2.]. □

The following corollary is probably well-known but we couldn't find a proof in the literature and give a Springer-theoretic proof here, which may be of independent interest.

Corollary 2.4. *Let* $m > h$ *be coprime to* h*. Then*

$$\mathbb{C}[Q/mQ]\mathbf{e}^- \cong \mathbb{C}[Q/(m-h)Q]\mathbf{e}$$

Proof. For any m coprime to h, it is known by [34] that $\mathbb{C}[Q/mQ]$ can be realized as a $\widetilde{W}$-representation using the cohomology of the affine Springer fiber corresponding to a certain "elliptic homogeneous element $\gamma_{m/h}$ of slope m/h". More precisely, let Φ_r denote the set of roots for $\mathfrak{g}$ of height r. Write $m = ah + b$ where $0 \leq b < h$ and define

$$\gamma_{m/h} = t^a(t \sum_{\phi \in \Phi_{h-b}} e_\phi + \sum_{\phi \in \Phi_{-b}} e_\phi) \tag{2.2}$$

On the other hand, if $m > h$, the element is divisible by t. Dividing by t, we get an elliptic homogeneous element in the same family, but of slope $m/h - 1 = (m-h)/h$. Now apply Lemma 2.3.

□

Remark 2.3. This corresponds to the relation between the number of all orbits vs. the regular orbits of W in Q/mQ, which are by [20, Theorem 7.4.4.] given by Eq. (3.3) and Eq. (3.4).

3 Combinatorics of the blocks

3.1 Singular blocks from the principal block

For the Lie algebra $\mathfrak{g}$, fix a Cartan and Borel subalgebras $\mathfrak{t} \subset \mathfrak{b} \subset \mathfrak{g}$. Then $\mathfrak{t}$ carries an irreducible representation of the Weyl group W. Let $\overline{\mathrm{DR}}_W$ denote Gordon's canonical quotient of the diagonal coinvariants for W. The latter is by definition the quotient ring of $\mathbb{C}[\mathfrak{t} \times \mathfrak{t}^*]$ over the natural doubly homogeneous ideal containing the invariants without the constant term with respect to the diagonal action of W, and in [16] the further quotient $\overline{\mathrm{DR}}_W$ and its structure as a W-module is studied. In particular, the dimension of $\overline{\mathrm{DR}}_W$ is $(h+1)^r$, where $r = \mathrm{rank}(\mathfrak{g})$, and h its Coxeter number [16, Theorem 1.4.] and as a $W-$representation, $\overline{\mathrm{DR}}_W \cong \mathrm{sgn} \otimes \mathbb{C}[Q/(h+1)Q]$, where sgn is the sign representation of W.

Let $\lambda \in \Lambda$, and W_λ be the stabilizer of λ, and consider

$$\overline{\mathrm{DR}}_W^\lambda := \overline{\mathrm{DR}}_W^{W_\lambda} = \mathrm{Hom}_{W_\lambda}(\mathrm{triv}, \overline{\mathrm{DR}}_W). \tag{3.1}$$

By Frobenius reciprocity, the latter is the same as $\mathrm{Hom}_W(\mathrm{Ind}_{W_\lambda}^W(\mathrm{triv}), \overline{\mathrm{DR}}_W)$.

The bigraded dimension of this space is given by the "Hall inner product" of Frobenius characters:

$$\dim_{q,t}(\overline{\mathrm{DR}}_W^\lambda) = \langle \mathrm{Frob}_{q,t}(\mathrm{Ind}_{W_\lambda}^W(\mathrm{triv})), \mathrm{Frob}_{q,t}(\overline{\mathrm{DR}}_W) \rangle. \tag{3.2}$$

Here the Frobenius character $\mathrm{Frob}_{q,t} : \mathrm{Rep}_{\mathbb{Z}^2-\text{graded}}(W) \to K_0(\mathrm{Rep}_{\mathbb{Z}^2-\text{graded}}(W))$ takes a doubly graded representation to its class in the Grothendieck group, where a representation in bigrading (i,j) is weighted by $q^i t^j$. When $G = SL_n$, so that $W = S_n$, we use the notation $\mathrm{Frob}_{q,t}$ to also denote the composition of the above map with $K_0(\mathrm{Rep}_{\mathbb{Z}^2-\text{graded}}(W)) \to \mathrm{Sym}_{q,t}$ sending the Specht module labeled by λ to the Schur function s_λ, see the next Section.

3.1.1 Type A

Let us now upgrade the type A result to include the natural bigrading. In case $\mathfrak{g} = \mathfrak{sl}_n$ we have $W = S_n$ and we write

$$\overline{\mathrm{DR}}_{S_n} = \overline{\mathrm{DR}}_n = \mathrm{DR}_n = \frac{\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]}{\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]_+^W}.$$

Let $\mathrm{Sym}_{q,t}[X]$ be the ring of symmetric functions over $\mathbb{Q}(q, t)$ in the alphabet $X = \{x_1, x_2, \ldots\}$ and let ∇ be the nabla operator of [6], diagonal in the basis of modified Macdonald polynomials. Let $\{e_\lambda\}, \{p_\lambda\}, \{h_\lambda\}, \{m_\lambda\}, \{s_\lambda\}$ be the bases of elementary, power sum, complete homogeneous, monomial, and Schur symmetric functions, and $\omega = \omega_X$ the usual involution on symmetric functions.

In this case we have the following more explicit statement about the bigraded dimension of $\overline{\mathrm{DR}}_n^{W_\lambda}$.

Proposition 3.1. *Let $\lambda \in \Lambda$ and $W_\lambda \subset S_n$ be the stabilizer of λ. Then*

$$\dim_{q,t}(\overline{\mathrm{DR}}_n^\lambda) = \langle h_\lambda, \nabla e_n \rangle,$$

where h_λ is the homogeneous symmetric function associated with λ, e_n the n-th elementary symmetric function, and ∇ the Garsia-Haiman nabla operator acting on symmetric functions.

Proof. It is clear that $\overline{\mathrm{DR}}_n$ is bigraded by x, y-degree and that this decomposition respects the W-action. The q, t-Frobenius character is given by Haiman [19] as the symmetric function ∇e_n where ∇ is the Garsia-Haiman ∇-operator on symmetric functions and e_n is the n-th elementary symmetric function.

It is also well known that the (trivially graded) Frobenius character of $\mathrm{Ind}_{W_\lambda}^W(triv)$ is given by h_λ, the homogeneous symmetric function attached to λ. Therefore, we compute using (3.2):

$$\dim_{q,t}(\overline{\mathrm{DR}}_n^\lambda) = \langle \mathrm{Frob}_{q,t}(\mathrm{Ind}_{W_\lambda}^W(\mathrm{triv})), \mathrm{Frob}_{q,t}(\overline{\mathrm{DR}}_n) \rangle = \langle h_\lambda, \nabla e_n \rangle.$$

$\square$

Remark 3.1. We will later use the interpretation of the Frobenius characters as symmetric functions when $G = SL_n$. If we are for example interested in the bigraded multiplicity of λ, it is given by

$$\langle \nabla e_n, s_\lambda \rangle.$$

Proposition 3.2. *When $W = S_n$ and λ is subregular, we have*

$$\dim(\overline{\mathrm{DR}}_n^\lambda) = \langle h_\lambda, \nabla e_n \rangle = \sum_{k=0}^{n-1} \sum_{i+j=k} q^i t^j,$$

in particular the bigraded dimensions of $\overline{\mathrm{DR}}_n$ *are*

$$\begin{array}{|c|c|c|c|c|}
\hline
1 & 1 & \cdots & 1 & 1 \\
\hline
1 & 1 & \cdots & 1 \\
\cline{1-4}
\vdots & \vdots & \vdots \\
\cline{1-3}
1 & 1 \\
\cline{1-2}
1 \\
\cline{1-1}
\end{array}$$

Proof. The Shuffle Theorem of Carlsson-Mellit [11] states that

$$\nabla e_n = \sum_{\pi \in PF_n} q^{\mathrm{area}(\pi)} t^{\mathrm{dinv}(\pi)} x_\pi$$

where $\pi \in PF_n$ is a parking function on n letters, and area and dinv are certain combinatorial statistics (see [11] for the definition). The monomial x_π is a monomial in the alphabet $\{x_1, \ldots, x_n, \ldots\}$ associated to π. Collecting all the monomials in the S_n-orbit of a fixed π and using the orthogonality of the bases $\{m_\lambda\}, \{h_\mu\}$ for the Hall inner product, we see that $\langle h_\lambda, \nabla e_n \rangle$ is a weighted count of Dyck paths whose associated monomial is λ. For $\lambda = (n-1, 1)$, these are Dyck paths differing from the one with minimal area by allowing an extra horizontal step (compare [15], where a similar result is proved using Schröder paths). Fixing the length of this step, we get $n -$ length Dyck paths, each of which has the same area. It is easy to see that they all have a different dinv statistic. In total, we get $\binom{n+1}{2}$ Dyck paths, each with distinct statistics. This completes the proof. □

Corollary 3.3. *Let $\mathfrak{g}^\vee = \mathfrak{sl}_n$ and let $Z(\mathfrak{u}_\zeta^{\vee,\lambda})$ denote the block of the center of the small quantum group $\mathfrak{u}_\zeta(\mathfrak{g}^\vee)$ with λ a subregular weight. Let $P_\lambda \subset G = SL_n$ be the parabolic subgroup associated to λ and $\widetilde{N_\lambda} \simeq T^*(G/P_\lambda)$ the Springer resolution. The additional grading of the coherent sheaf of poly-vectorfields $\wedge^j T\widetilde{N_\lambda}$ is given by the induced action of $\mathbb{C}^*$ along the fibers of the Springer resolution. Then there is an isomorphism of bigraded vector spaces*

$$Z(\mathfrak{u}_\zeta^{\vee,\lambda})^{i,j} \simeq H^i(\widetilde{N_\lambda}, \wedge^j T\widetilde{N_\lambda})^{-i-j} \simeq \left(\overline{\mathrm{DR}}_n^\lambda\right)^{\binom{n+1}{2} - \frac{i+j}{2}, \frac{j-i}{2}}.$$

Proof. The first isomorphism is a particular case of theorem 7 in [7]. The bigraded dimensions of the equivariant coherent sheaf cohomologies in case when λ is subregular and $G/P_\lambda \simeq \mathbb{P}^k$ are computed in Theorem 3.3 of [25]. They match exactly the bigraded dimensions of $\overline{\mathrm{DR}}_n^\lambda$ obtained in Proposition 3.2. □

This shows in particular that for G of type A_n and a singular block $Z(\mathfrak{u}_\zeta^{\vee,\lambda})$ such that G/P_λ is a projective space, the cohomology of the corresponding affine Springer fiber is isomorphic to the whole singular block of the center. This also confirms Conjecture 4.9(3) in [24] at the level of the bigraded vector spaces. Note that in this case the whole block of the center of the small quantum group is $G^\vee$-invariant.

3.2 Enumeration of blocks

Let $\mathfrak{a}_\ell$ be the ℓ-dilated fundamental alcove for G. We would like to compute the number of blocks $u_\zeta^{\vee,\lambda}$ for the small quantum group for λ with a given stabilizer $W_\lambda \in W$. Therefore, we have to enumerate the singular coroot weights in $\mathfrak{a}_\ell \cap X^+$, or equivalently in $Q/\ell Q$, with a given type of stabilizer in affine Weyl group. By [20], the total number of orbits is

$$\frac{1}{|W|} \prod (\ell + d_i) \tag{3.3}$$

and the number of regular orbits is

$$\frac{1}{|W|}\prod(\ell - d_i). \tag{3.4}$$

The first quantity merits a name and plays a significant role in so called Coxeter-Catalan combinatorics, see for example [35].

Definition 3.4. *Let m be coprime to h. The m/h-*Coxeter-Catalan number *of W is*

$$\mathrm{Cat}_W(m,h) = \frac{1}{|W|}\prod(m + d_i)$$

Let us first sketch how this plays out for $G = SL_n$. In this case for $\ell = n+1$, we have the classical Catalan number $\mathrm{Cat}_{S_n}(n+1,n) = \frac{1}{n+1}\binom{2n}{n}$. On the other hand, Eq. (3.4) gives 1, i.e. there is a single regular orbit in $Q/(n+1)Q$.

For more general m, since Q/mQ for $(m,n)=1$ is in S_n-equivariant bijection with rational parking functions of slope m/n, we only need to understand the orbits on the latter. One can think of rational parking functions as Dyck paths with labels $\{1,\dots,n\}$ on the vertical runs, where the labels have to be increasing in each run. The S_n-action permutes the labels on the parking functions, and therefore the stabilizer is given by the structure of the vertical runs.

In [3, Proposition 2] the following is proved, giving the general number of orbits of a given type:

Proposition 3.5. *Let $m_i, i = 1,\dots,n$ be the number of vertical runs of length i and let m_0 be such that $\sum_{j=0}^{n} m_j = m$. Then the number of Dyck paths with vertical run structure $m_1,\dots,m_n$ is given by the multinomial coefficient*

$$\frac{(m-1)!}{m_0!\cdots m_n!}$$

Compare also [20, Conjecture 2.4.2] in the case $m = n+1$.

These numbers are known as (rational) *Kreweras numbers*. The goal of this section is to understand the general "rational Coxeter" analogs of Kreweras numbers, as first defined in [34]. These are defined as follows. Consider $Q/\ell Q$ with its natural action of the affine Weyl group $\widetilde{W}$.

The stabilizer in the finite Weyl group $W \subset \widetilde{W}$ of $q \in Q/\ell Q$ is by [34, Proposition 4.1.] a parabolic subgroup of W. Let $\{W_\lambda\}$ be a set of representatives of parabolic subgroups of W. Now, $\mathbb{C}[Q/\ell Q]$ is by definition a permutation representation of W, so by orbit-stabilizer splits as

$$\mathbb{C}[Q/\ell Q] = \bigoplus_\lambda d_{\lambda,\ell}\mathrm{Ind}_{W_\lambda}^W(1)$$

Definition 3.6. *The ℓ/h-*rational Kreweras number of type λ *for G is by definition the coefficient $d_{\lambda,\ell}$ in the above decomposition. In other words, it is the number of W-orbits in $Q/\ell Q$ with a given stabilizer.*

Remark 3.2. Without our assumptions on ℓ, which for example imply ℓ is "very good" in the sense of [34], the W_λ appearing above are in general only so called *quasi-parabolic* subgroups of W. We will however not need them.

Explicit formulas for the Kreweras numbers for classical groups can be found in [34]. There is also a general formula in terms of hyperplane arrangements [34, Proposition 5.1.]. We will need the following proposition, which follows directly from the definitions.

Proposition 3.7. *The sum of the Kreweras numbers over the representatives $\{W_\lambda\}$ is the ℓ/h-Coxeter-Catalan number for W:*

$$\sum_\lambda d_{\lambda,\ell} = \mathrm{Cat}_W(\ell, h)$$

Example 3.3. For W of dihedral type, there are three distinct types of orbits, which one can compute by hand or using Eqs. (3.3)–(3.4), generalizing the Alfano-Reiner results from [20, Section 7.5]. For example for G of type B_2, we have 1 maximally singular orbit (the origin), $\frac{(\ell-1)(\ell-3)}{8}$ regular orbits, and $\ell - 1$ subregular orbits.

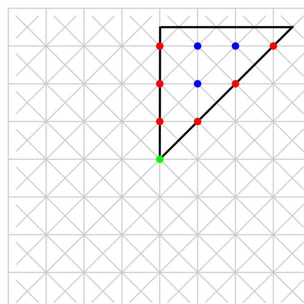

Figure 3.1: The dilated fundamental alcove for B_2 and dominant weights in it for $\ell = 7$. Different colors correspond to different stabilizer types.

For G of type G_2, we have the origin, $\frac{(\ell-1)(\ell-5)}{12}$ regular orbits and $\ell - 1$ subregular orbits.

3.3 Putting it together

Fix G and as in the previous section, let $d_{\lambda,\ell}$ be the number of orbits of type λ in $Q/\ell Q$. We are interested in the sum

$$d_{h,\ell} := \sum_\lambda d_{\lambda,\ell} \dim \overline{\mathrm{DR}}_W^{W_\lambda}\ . \tag{3.5}$$

Theorem 3.8. *Assume ℓ does not divide h or $h+1$ for W. Then We have*

$$d_{h,\ell} = \mathrm{Cat}_W(\ell(h+1) - h, h).$$

The rest of this section will be devoted to a proof of this theorem. Before doing so, we note how this implies Theorem 1.3 from the Introduction.

Corollary 3.9. *With ℓ as above, we have that the dimension of the subalgebra described in Theorem 1.1 of the $G^\vee$-invariant part of the center of the small quantum group at a primitive ℓ-th root of unity ζ is given by*

$$\dim H^*(\mathrm{Gr}^{\zeta,\gamma})^{\widetilde{W}} = \mathrm{Cat}_W((h+1)\ell - h, h).$$

Proof. Combining Proposition 2.1 and Eq. (2.1), we have that

$$\dim H^*(\mathrm{Gr}^{\zeta,\gamma})^{\widetilde{W}}$$

is given by summing $d_{\lambda,\ell} \dim \overline{\mathrm{DR}}_W^{W_\lambda}$ over λ. This is the definition of $d_{h,\ell}$ from above. □

Of Theorem 3.8. We may interpret the summation over λ on the RHS of Eq. (3.5) as follows. Each orbit of type λ contributes a $\mathrm{Ind}_{W_\lambda}^{W} 1$ to the representation $Q/\ell Q$. On the other hand, $\dim \overline{\mathrm{DR}}_W^{W_\lambda}$ is by Frobenius reciprocity

$$\dim \mathrm{Hom}_W(\mathrm{Ind}_{W_\lambda}^{W}(1), \overline{\mathrm{DR}}_W),$$

so we can write $d_{h,\ell} = \dim \mathrm{Hom}_W(\mathbb{C}[Q/\ell Q], \overline{\mathrm{DR}}_W)$. Now note that $\overline{\mathrm{DR}}_W \cong sgn \otimes \mathbb{C}[Q/(h+1)Q]$ as W-representations, where sgn is the sign character of W where all simple reflections act by -1.

Therefore, we are computing the dimension of the anti-invariants:

$$\mathrm{Hom}_W(\mathbb{C}[Q/\ell Q], \overline{\mathrm{DR}}_W) = (C[Q/\ell Q] \otimes \mathbb{C}[Q/(h+1)Q])\mathbf{e}^-.$$

By our assumptions on ℓ, the Chinese remainder theorem implies $\mathbb{C}[Q/\ell Q] \otimes \mathbb{C}[Q/(h+1)Q] \cong \mathbb{C}[Q/\ell(h+1)Q]$ as W-representations.

To conclude the proof, use Lemma 2.3 to note that

$$\mathbb{C}[Q/\ell(h+1)Q]\mathbf{e}^- \cong \mathbb{C}[Q/(\ell(h+1)-h)Q]^W.$$

The dimension of this space is $\mathrm{Cat}_W((h+1)\ell - h, h)$ by definition. □

Remark 3.4. Corollary 3.9 gives the dimension of the cohomology of the affine Springer fibers. By the explicit computation on the coherent side, we know that these dimensions match the dimensions of the $G^\vee$-invariant part of the center of the small quantum group for types $A_1, A_2, A_3, A_4, B_2, G_2$ for all blocks (see [21, Sections 4 and 5]). Therefore, Conjecture 1.2 is confirmed in all these cases.

Example 3.5. We illustrate the dimension formula by a continuation of Example 3.3. In the case of B_2 one computes

$$d_{h,\ell} = 25 \cdot \frac{(\ell-1)(\ell-3)}{8} + 10 \cdot (\ell - 1) + 1$$

and in the case of G_2 one has

$$49 \cdot \frac{(\ell-1)(\ell-5)}{12} + 21 \cdot (\ell - 1) + 1$$

and these can be checked to match Hochschild cohomology computations as in [21].

3.3.1 Type A

Let $d_{\lambda,\ell}$ be the number of orbits of type λ in $Q/\ell Q$. We will give a slightly different proof of Theorem 3.8 for G of type A in this section, which we hope will be illuminating to the reader. Note that

$$d_{n,\ell} := \sum_{\lambda \in P(n)} d_{\lambda,\ell} \langle h_\lambda, \nabla e_n \rangle |_{q=t=1}$$

in this case, where we use the Hall inner product on symmetric functions and h_λ is the complete homogeneous symmetric function.

By Proposition 3.5, we have that

$$d_{\lambda,\ell} = \frac{1}{\ell}\binom{\ell}{m_0(\lambda),\ldots,m_n(\lambda)},$$

where $m_i(\lambda)$ is the number of parts of size i in λ for $i > 0$ and defined as above for $i = 0$.

The final answer we are looking for is the $((n+1)\ell - n, n)$-Catalan number. This is the same as the total number of orbits in $Q/(n(\ell+1)-n)Q$.

Theorem 3.10. *Suppose ℓ is as in the introduction, i.e. odd and $n \not\equiv 0, -1$ mod ℓ. We have*

$$d_{n,\ell} = c_{(n+1)\ell-n,n} = \frac{1}{(n+1)\ell}\binom{(n+1)\ell}{n},$$

the rational $((n+1)\ell - n, n)$-Catalan number.

Proof. Note that $d_{\lambda,\ell} = \langle P_{\ell,n}\cdot 1, m_\lambda\rangle|_{q=t=1}$ where m_λ are the monomial symmetric functions and $P_{m,n}$ for $m, n \geq 0$ are the usual elliptic Hall algebra operators as in [30].

We may combine the summation over λ on the RHS and use linearity of the scalar product to get

$$\langle \sum_\lambda h_\lambda \langle P_{\ell,n}\cdot 1, m_\lambda\rangle, \nabla e_n\rangle|_{q=t=1} = \langle \omega P_{\ell,n}\cdot 1, \nabla e_n\rangle|_{q=t=1} \tag{3.6}$$

Here ω is the usual involution on symmetric functions. On the other hand, on the LHS, we may write $d_{n,\ell} = \langle P_{(n+1)\ell-n,n}\cdot 1, e_n\rangle|_{q=t=1}$. We may interpret this latter pairing as taking the dimension of the invariants in the $((n+1)\ell - n, n)$-parking function module. On the other hand, this is the same as the anti-invariant part of the $((n+1)\ell, n)-$parking function module by Lemma 2.3.

In Eq. (3.6) we may further interpret the pairing as the dimension of the invariants of the tensor product of the sign-twisted (ℓ, n)-parking function module and the $(n+1, n)$-parking function modules.

Now, each of these m, n-parking function modules looks like $\mathbb{C}[Q/mQ]$ where Q is the root lattice of S_n. If $n \not\equiv 0, -1$ mod ℓ, the Chinese remainder theorem implies that $\mathbb{C}[Q/\ell Q \times Q/(n+1)Q] \cong \mathbb{C}[Q/(\ell(n+1))Q]$ as S_n-representations, giving the conclusion.

$\square$

Remark 3.6. We have checked the identity from Theorem 3.10 with a computer for all (not just coprime to $n, n+1$) odd $5 < \ell < 30$ and $1 < n < 10$ and expect it to be true in general.

Remark 3.7. By the rational shuffle conjecture, resulting from e.g. [29], we know the bigraded scalar products $\langle P_{m/n}\cdot 1, h_\lambda\rangle$ for the m/n-case as well. It would be interesting to understand a (bi)graded version of Theorem 3.10 as well.

3.3.2 The Harish-Chandra center, the Higman ideal and the Verlinde quotient

In the proof of Theorem 3.8 we have obtained a (non-multiplicative) combinatorial model of a subalgebra $\overline{Z} \equiv H^*(\mathrm{Gr}^{\zeta,\gamma})^{\widetilde{W}}$ of $Z(\mathfrak{u}_\zeta^\vee)^{G^\vee}$, and hopefully, under Conjecture 1.2, of the whole $G^\vee$-invariant part of the center. It is given by

$$\overline{Z} \cong \mathbb{C}[Q/\ell(h+1)Q]\mathbf{e}^- \cong [\mathbb{C}[Q/(h+1)Q] \otimes \mathbb{C}[Q/\ell Q]]^{W_-} \tag{3.7}$$

Here we denote by $[.]^{W_-}$ taking the isotypical component of the sign representation. We would like to point out remarkable correspondences between certain natural subspaces in $[\mathbb{C}[Q/(h+1)Q] \otimes \mathbb{C}[Q/\ell Q]]^{W_-}$ and the well known subspaces in $Z(\mathfrak{u}_\zeta^\vee)^{G^\vee}$.

Recall that $\mathfrak{u}_\zeta^\vee$ is a unimodular Hopf algebra, meaning that it contains a two-sided integral $\nu \in Z(\mathfrak{u}_\zeta^\vee)$ that is unique up to rescaling and such that $\nu x = \varepsilon(x)\nu$ and $x\nu = \varepsilon(x)\nu$ for any $x \in \mathfrak{u}_\zeta^\vee$, where $\varepsilon : \mathfrak{u}_\zeta^\vee \to \mathbb{C}$ is the counit. The Hopf algebra $\mathfrak{u}_\zeta^\vee$ is a left module over itself with respect to the Hopf adjoint action $\mathrm{ad}h(x) = \sum h_1 x S(h_2)$ for any $h, x \in \mathfrak{u}_\zeta^\vee$, where S is the antipode and we used the Sweedler's notation for the coproduct $\Delta h = \sum h_1 \otimes h_2$. It is easy to see that the space $\mathrm{ad}\nu(\mathfrak{u}_\zeta^\vee)$ is spanned by central elements.

Definition 3.11. *The space $\mathrm{ad}\nu(\mathfrak{u}_\zeta^\vee) \equiv Z_{\mathrm{Hig}}$ is an ideal in $Z(\mathfrak{u}_\zeta^\vee)$, called the Higman ideal.*

Recall also that $\mathfrak{u}_\zeta^\vee$ is a quasitriangular Hopf algebra with the invertible element $R \in \mathfrak{u}_\zeta^\vee \otimes \mathfrak{u}_\zeta^\vee$ such that the map

$$J : f \to m(f \circ S^{-1} \otimes \mathrm{id})(R_{21}R_{12})$$

sends the $K_{2\rho}$-twisted traces of $\mathfrak{u}_\zeta^\vee$-modules to $Z(\mathfrak{u}_\zeta^\vee)$. Moreover, it is an injective algebra homomorphism from the Grothendieck ring of $\mathfrak{u}_\zeta^\vee$ into the center.

Definition 3.12. *The homomorphic image of the Grothendieck ring under the map J is a subalgebra $Z_{\mathrm{HCh}} \subset Z(\mathfrak{u}_\zeta^\vee)$ called the Harish-Chandra center.*

It follows from Lusztig's tensor product theorem for simple modules over the big quantum group [27], that Z_{HCh} is isomorphic as an algebra to $\mathbb{C}[\Lambda]^W/\mathbb{C}[\ell\Lambda]^W$ [10], the algebra of characters of $\mathfrak{u}_\zeta^\vee$-modules, spanned by the W-symmetric functions with highest weights in $\Lambda/\ell\Lambda$, and that $\mathrm{dim} Z_{\mathrm{HCh}} = \ell^{r(\mathfrak{g}^\vee)}$.

Proposition 3.13. *([26], Proposition 2.26 and Theorem 4.3). The Higman ideal is spanned by the images of the characters of the projective $\mathfrak{u}_\zeta^\vee$-modules under the map J. The intersection of Z_{Hig} with each block of the center is one-dimensional and therefore we have $\mathrm{dim} Z_{\mathrm{Hig}} = \mathrm{Cat}_W(\ell, h)$.*

We have

$$Z_{\mathrm{Hig}} \subset Z_{\mathrm{HCh}} \subset Z(\mathfrak{u}_\zeta^\vee)^{G^\vee}$$

The second inclusion follows from the fact that the J-image of the Grothendieck ring also arizes as the specialization to $\mathfrak{u}_\zeta^\vee$ of the center of the big quantum group, where the action of $\mathfrak{g}^\vee$ is realized by the adjoint action of the l-th divided powers of the Chevalley generators, trivial on the central elements.

Now consider the model of $\overline{Z} \subset Z(\mathfrak{u}_\zeta^\vee)^{G^\vee}$ given by 3.7. Note that there exists exactly one regular orbit of W in $\mathbb{C}[Q/(h+1)Q]$, which decomposes as the regular representation of W: $O_{\mathrm{reg}} = \sum_{i \in I} d_i V_i$, where $\{V_i\}_{i\in I}$ is the complete set of inequivalent irreducible representations of W, and $d_i = \dim V_i$. On the other hand, $\mathbb{C}[Q/\ell Q]$ decomposes as $\sum_{i\in I} k_i V_i$ for some natural $k_i > 0, i \in I$. Now for each $i \in I$ there exists a unique $\bar{i} \in I$ such that $V_{\bar{i}} = V_i \otimes sgn$ and $d_i = d_{\bar{i}}$. Here sgn denotes the sign representation of W. We have that $V_j \otimes V_i$ contains sgn with multiplicity 1 if and only if $j = \bar{i}$. Then

$$[O_{\mathrm{reg}} \otimes \mathbb{C}[Q/\ell Q]]^{W_-} \cong [(\sum_{i\in I} d_i V_i) \otimes (\sum_{i\in I} k_i V_i)]^{W_-} \cong (\sum_{i \in I} k_i d_i (sgn)),$$

so that the dimension of this subspace equals to the dimension of $\mathbb{C}[Q/\ell Q]$, namely $\ell^{r(\mathfrak{g})}$. Therefore $\dim[O_{\mathrm{reg}} \otimes \mathbb{C}[Q/\ell Q]]^{W_-} = \dim Z_{\mathrm{HCh}}$.

Further, there exists a natural subspace of dimension $\dim Z_{\mathrm{Hig}}$ in $[O_{\mathrm{reg}} \otimes \mathbb{C}[Q/\ell Q]]^{W_-}$. Lemma 2.3 implies that $\mathbb{C}[Q/(h+1)Q]$ contains the sign representation $sgn \subset O_{\mathrm{reg}}$ of W with multiplicity 1. Each of the $\mathrm{Cat}_W(\ell, h)$ orbits of W in $\mathbb{C}[Q/\ell Q]$ contains the trivial representation $triv$ with multiplicity 1. Therefore the subspace $\dim\, sgn \otimes triv^{\mathrm{Cat}_W(\ell,h)} = \dim\, Z_{\mathrm{Hig}}$ and we have

$$sgn \otimes triv^{\mathrm{Cat}_W(\ell,h)} \subset [O_{\mathrm{reg}} \otimes \mathbb{C}[Q/\ell Q]]^{W_-}.$$

Exchanging the roles of sgn and $triv$ also leads to an interesting subspace in $O_{\mathrm{reg}} \otimes \mathbb{C}[Q/\ell Q]]^{W_-}$. We have exactly one trivial representation $triv$ in O_{reg}. By Lemma 2.3 the sign representation in $\mathbb{C}[Q/\ell Q]$ has multiplicity $\mathrm{Cat}_W(\ell - h, h)$. Therefore the subspace $triv \otimes sgn^{\mathrm{Cat}_W(\ell-h,h)}$ in our model of the center has the dimension $\mathrm{Cat}_W(\ell - h, h)$, which is the dimension of the Verlinde quotient $Verl$ of the Harish-Chandra center of $\mathfrak{u}_\zeta^\vee$. Recall that the Verlinde quotient of the Grothendieck ring is spanned by the characters of the Weyl modules over $\mathfrak{u}_\zeta^\vee$ with highest weights running over the regular weights in the ℓ-dilated fundamental alcove $\mathfrak{a}_\ell$, and the multiplication is defined up to the ideal spanned by the linear combinations of Weyl characters symmetric with respect to any reflection in $\widetilde{W}_\ell$. The number of regular weights in $\mathfrak{a}_\ell$ is given by $\mathrm{Cat}_W(\ell-h, h)$.

We have

$$triv \otimes sgn^{\mathrm{Cat}_W(\ell-h,h)} \subset [O_{\mathrm{reg}} \otimes \mathbb{C}[Q/\ell Q]]^{W_-}$$

with $\dim\, triv \otimes sgn^{\mathrm{Cat}_W(\ell-h,h)} = \dim\, Verl$.

The Verlinde algebra admits a basis of representatives spanned by W-antisymmetric linear combinations Weyl characters ([1]), while the Higman ideal admits a basis of W-symmetric linear combinations of Weyl characters (([23], Proposition 4.3), in parallel with the appearence of $sgn^{\mathrm{Cat}_W(\ell-h,h)}$ and $triv^{\mathrm{Cat}_W(\ell,h)}$ respectively in our combinatorial model.

4 The bigrading in type A

In this section we explain how to get a bigrading on the space $H^*(\mathcal{F}l_{\mathbf{I}}^{\gamma})^{\widetilde{W}} \cong H^*(\mathcal{F}l_{\mathbf{I}}^{\gamma})^{\Lambda}$ for $G = GL_n$, which corresponds to the principal block of the center under Conjecture 1.2. There are in fact at least two geometric ways to do this. The first one is using the perverse filtration on the parabolic Hitchin fibration and the second one is using the "number of points" or "connected components" grading on the *positive part of the affine Springer fiber*, as studied in [11]. Using techniques of *loc. cit.* and assuming their Conjecture A, we can show that the bigraded structure as an S_n-module agrees with that of the diagonal coinvariant ring, as conjectured in [24].

On the other hand, the advantage of using the Hitchin fibration is that there is a natural "Lefschetz element" coming from the relatively ample determinant bundle on the parabolic Hitchin fibration. We conjecture that the $\mathfrak{sl}_2$-action on the center obtained this way coincides with the one given by the wedge product with the Poisson bivector field on the Springer resolution. Further, we conjecture the bigradings obtained in these two ways are the same (up to an explicit change of variables).

4.1 The parabolic Hitchin fibration

First, we want to construct a particular compactification of the singular curve given by $x^n + y^n = 0 \subset \mathbb{C}^2$, inside a Hirzebruch surface. Most importantly, this compactification will be irreducible, i.e. a spectral curve for the anisotropic locus of the Hitchin fibration, and have only an isolated singular point which is an ordinary n-uple point.

Let $\Sigma_r = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(r) \oplus \mathcal{O}_{\mathbb{P}^1})$ be the r-th Hirzebruch surface. The Picard group of Σ_r is generated by the zero section E_r and the class of a fiber F, with intersection form determined by $F^2 = 0, E_r^2 = -r$ and $E_r F = 1$.

Recall that there is a birational map from Σ_r to Σ_{r+1}, called an "elementary transform" (see [4, Chapter 3]), constructed as follows. We choose some fiber F, and consider the surface Σ_r', the blow-up of Σ_r at $p := F \cap E_r$. Let F', E_r' the strict transforms of F, E_r and $\tilde{E}$ be the exceptional divisor of this blow-up. Then we have

$$0 = F^2 = (F' + \tilde{E})^2 = (F')^2 + 2 - 1$$

hence F' is a (-1)-curve and can be contracted, the resulting surface being Σ_{r+1}. See Figure 4.1 for the toric picture, where the red line is the contracted curve.

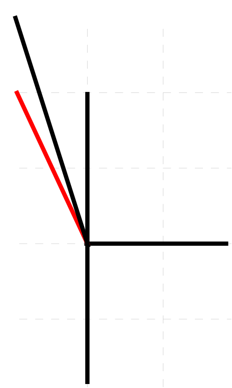

Figure 4.1: The toric blow-up and contraction giving a birational map $\Sigma_r \to \Sigma_{r+1}$.

Now we can prove :

Lemma 4.1. *For all $n \geq 0$, there is a curve $C \subset \Sigma_2$ such that C is irreducible, has a unique singular point, with singularity type $x^n + y^n = 0$.*

Proof. Let $C_2 \subset \mathbb{P}^2$ be a smooth curve of degree n, and Σ_1 be the blow-up of $\mathbb{P}^2$ at a point $a \notin C_2$. We denote by C_1 the strict transform of C_2. Consider a generic fiber F_0 and the corresponding elementary transform.

The strict transform C' of C_1 inside Σ_1' is isomorphic to C_1. Denote by C the image of C' under the contraction of F'. Since $F \cap C_1$ is given by n points, we see that C is analytically isomorphic to C_2 where n points have been glued transversally together, resulting in an ordinary n-uple point q. It's clear that $C\backslash\{q\}$ is smooth. Since $C\backslash\{q\}$ is connected, C is irreducible. □

Remark 4.1. Since C_2 is the normalization of C, the geometric genus of C is $g_g = \binom{n-1}{2}$. Since the blowdown introduces $\binom{n}{2}$ nodes to C', the arithmetic genus is $g_a = g_g + \binom{n}{2} = (n-1)^2$.

Definition 4.2. *Let $X/\mathbb{C}$ be a smooth projective curve, G a split reductive group, and $\mathcal{L}$ a line bundle on X with $\deg \mathcal{L} \geq g_X$. The* Hitchin moduli stack *is the functor*

$$\mathcal{M} : Sch_{\mathbb{C}} \to Grpd$$

sending

$$S \mapsto \{(E, \varphi) | E \text{ is a } G-\text{torsor over } S \times X, \varphi \in H^0(Ad(E) \otimes \mathcal{L})\}$$

Definition 4.3. *Let $X, G, \mathcal{L}$ be as above. The* parabolic Hitchin moduli stack *is the functor*

$$\widetilde{\mathcal{M}} : Sch_{\mathbb{C}} \to Grpd$$

sending

$$S \mapsto \{(E, \varphi, x, E_x) | (E, \varphi) \in \mathcal{M}, x \in X, E_x \text{ is a } B-\text{reduction along } \Gamma(x) \text{ of } E\}$$

Let D be a divisor so that $\mathcal{O}(D) = \mathcal{L}$. The Hitchin moduli stack can be interpreted as classifying sections

$$a : X \to \mathcal{O}(D) \times^{\mathbb{G}_m} [\mathfrak{g}/G]$$

see [31, Lemme 2.4.].

Definition 4.4. *The morphism*

$$\mathcal{M} \to \mathcal{A} := \bigoplus_{i=1}^{n} H^0(X, \mathcal{O}(d_i D))$$

sending a section a to its image in $\mathcal{O}(D) \times^{\mathbb{G}_m} \mathrm{Sym}(\mathfrak{t}^*)^W$ is called the Hitchin fibration*. The base $\mathcal{A}$ is called the* Hitchin base*. The composition*

$$\widetilde{\mathcal{M}} \to \mathcal{M} \times X \to \mathcal{A} \times X$$

is called the parabolic Hitchin fibration*.*

Let now $G = SL_n$ and $\mathcal{L}$ be a line bundle of degree ≥ 0 on $\mathbb{P}^1$. By the BNR correspondence [5], we may realize the curve C from Lemma 4.1, or rather its intersection with $\mathrm{Tot}(\mathcal{O}(2))$ as a spectral curve $\{\det(xI - \varphi) = 0\}$ for the Hitchin fibration

$$\mathcal{M} \to \mathcal{A}$$

associated to the data of $\mathbb{P}^1, G, \mathcal{L}$. Let $a \in \mathcal{A}$ be such that C is the associated spectral curve. Note that we in fact have $a \in \mathcal{A}^{ani} \subset \mathcal{A}^{\heartsuit}$, the locus where the spectral curves are irreducible, resp. reduced (we will not need a more general definition of $\mathcal{A}^{ani}$ or $\mathcal{A}^{\heartsuit}$ here, for that see [31, § 6.1]).

The relationship to the affine Springer fibers considered in this paper is as follows. The curve C may be chosen so that the unique singularity is over $0 \in X$. Its local form corresponds to $\gamma = st \in \mathfrak{g}(\mathcal{K})$ as before, for $s = \mathrm{diag}(1, \rho, \ldots, \rho^{n-1})$ where ρ is a primitive n:th root of unity. Let $(a, 0) \in \mathcal{A}^{\heartsuit} \times X$. Then [36, Proposition 2.4.1] says that

$$\mathcal{P}_a \times^{P_0^{red}(J_a)} \mathcal{F}l_{\mathbf{I}}^{\gamma} \to \widetilde{\mathcal{M}}_a \tag{4.1}$$

is a homeomorphism of stacks.

Here $\mathcal{P}_a$ is the generalized Picard stack, $P_0^{red}(J_a)$ the reduced quotient of the local Picard stack at 0. Modding out by $\mathcal{P}_a$, the left-hand side of Eq. (4.1) simplifies to $\mathcal{F}l_{\mathbf{I}}^{\gamma}/P_0^{red}(J_a)$. By taking $\gamma = st$ for $s \in \mathfrak{t}^{reg}$ as above, it is easy to compute by hand in this case that $P_0(J_a) = T(\mathbb{C}) \times \Lambda$ where T is the diagonal torus in GL_n and $\Lambda = X^*(T) \cong \mathbb{Z}^n$ is the lattice part of the centralizer.

Modifying the proof of [31, Proposition 4.13.1] slightly, we can write the following variant of Eq. (4.1):

$$\widetilde{\mathcal{M}}_a/\mathcal{P}_a^{\flat} \cong \mathcal{F}l_{\mathbf{I}}^{\gamma}/\Lambda \tag{4.2}$$

where $\mathcal{P}_a^{\flat}$ is the Picard group of the normalization of C as in [31, 4.7.3].

The upshot of this analysis is that we may define the *perverse filtration* on $H^*(\mathcal{F}l_{\mathbf{I}}^{\gamma}/\Lambda)$. Namely, if $\pi : \widetilde{\mathcal{M}} \to \mathcal{M} \times \{0\} \to \mathcal{A}^{ani}$ denotes the restriction of the parabolic Hitchin fibration to the locus of irreducible spectral curves and with the parabolic reduction at $0 \in X$, $\pi_*\mathbb{C}$ acquires a filtration from the t-structure on the base as

$$P_{\leq i} := \mathrm{im}({}^p\tau_{\leq i}\pi_*\mathbb{C} \to {}^p\tau_{\leq i+1}\pi_*\mathbb{C}).$$

Restricting to the stalk at a, we get a filtration $P_{\leq i}$ on $H^*(\widetilde{\mathcal{M}}_a/\mathcal{P}_a^{\flat}) \cong H^*(\mathcal{F}l_{\mathbf{I}}^{\gamma}/\Lambda)$. By results of Maulik-Yun [28] this filtration is independent of the choice of deformation of C used here (we only require the total space to be smooth and a codimension estimate on the base, handled in this case by [31]). See also [28, Section 3.1.3.].

We make the following conjecture, which holds for $G = SL_2$.

Conjecture 4.5. *As bigraded vector spaces*

$$\overline{\mathrm{DR}}_n \cong \mathrm{gr}^P H^*(\mathcal{F}l_{\mathbf{I}}^{\gamma})^{\Lambda} \tag{4.3}$$

Proposition 4.6. *The conjecture 4.5 is true for $G = SL_2$.*

Proof. In this case, the two vector spaces are equal to $\mathbb{C}^3$, hence we just need to check that the gradings agree. The affine Springer fiber $\mathcal{F}l_{\mathbf{I}}^{\gamma}$ can be identified with an infinite chains of $\mathbb{P}^1$, and the lattice action is obtained by translation by 2 ([37]). Hence the quotient $X_0 = \mathcal{F}l_{\mathbf{I}}^{\gamma}/\Lambda$ is isomorphic to an elliptic curve with a singularity of type I_2 (i.e two $\mathbb{P}^1$ glued transversally twice). By the discussion before, this curve also appears as a spectral curve inside a cotangent bundle of $\mathbb{P}^1$, hence its compactified Jacobian is a Hitchin fiber inside the corresponding Hitchin fibration. Since X_0 has arithmetic genus 1, it is isomorphic to its own compactified Jacobian. It follows by versality of the Hitchin map in this case that the restriction of this fibration to a generic line is simply a smoothing of X_0, say $f : X \to L = \mathbb{C}$. Let $L^* = L\backslash\{0\}$. By the decomposition theorem, we have

$$f_*\underline{\mathbb{C}}_X = \underline{\mathbb{C}}_L \oplus \underline{\mathbb{C}}_L[-2] \oplus \underline{\mathbb{C}}_0[-2] \oplus \mathscr{L}[-1]$$

where $\mathscr{L}$ is the rank 2 local system on L^* given by the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. The pure part is given by $\underline{\mathbb{C}}_L \oplus \underline{\mathbb{C}}_L[-2] \oplus \underline{\mathbb{C}}_0[-2]$. The perverse degree are $-2, 0, 2$. Up to renormalisation, we obtained the same bigrading as the diagonal coinvariants in this case. □

4.2 The Lefschetz element

Let $\mathcal{L}_{det}$ be the determinant line bundle on $\mathcal{M}$. The iterated cup product by $c_1(\mathcal{L}_{det})$ induces a map

$$\cup c_1(\mathcal{L}_{det})^{g_a - i} : {}^pH^{\dim \mathcal{A}+i}\pi_*\mathbb{C} \to {}^pH^{\dim \mathcal{A}+2g_a-i}\pi_*\mathbb{C}$$

and therefore maps

$$\cup c_1(\mathcal{L}_{det}) : \mathrm{gr}^P H^*(\mathcal{F}l_{\mathbf{I}}^\gamma/\Lambda) \to \mathrm{gr}^P H^*(\mathcal{F}l_{\mathbf{I}}^\gamma/\Lambda)$$

of bidegrees $(a, b) = (2, 2)$, where a is the cohomological degree and b is the perverse degree.

We will now prove that $c_1(\mathcal{L}_{det})$ coincides with a certain polynomial in the ring of diagonal coinvariants, under Conjecture 4.5. On the other hand, under the bigraded isomorphism of the principal block of the center $Z(\mathfrak{u}_\zeta^0)$ with sheaf cohomology groups of the Springer resolution, we can hope that $c_1(\mathcal{L}_{det})$ coincides with the Poisson bivector field on the Springer resolution as explained in more detail in Conjecture 4.15.

Theorem 4.7. *Under the identification Eq.* (4.3)*, the element* $c_1(\mathcal{L}_{det}) \in \mathrm{gr}^P H^*(\mathcal{F}l_{\mathbf{I}}^\gamma)^\Lambda$ *corresponds up to a nonzero scalar to the "Haiman determinant"* $\Delta_{(n-1,1)} \in DR_n$ *given by*

$$\Delta_{(n-1,1)} = \det(y_i^{p_j} x_i^{q_j})_{1 \le i,j \le n}$$

where $(p_1, q_1), \ldots, (p_n, q_n)$ *is any ordering of* $(0, 0), (0, 1), \ldots, (0, n-1), (1, 0) \in \mathbb{Z}^2_{\ge 0}$

Proof. Since the relevant bigraded piece $(n-1, n-1)$ is 1-dimensional, contains $\Delta_{(n-1,1)}$ and $c_1(\mathcal{L}_{det})$ is nonzero, we are done. □

Finally, note that by the Jacobson-Morozov theorem, the nilpotent action of $e = \cup c_1(\mathcal{L}_{det})$ extends to an $\mathfrak{sl}_2$-triple (e, f, h) acting on $\mathrm{gr}^P H^*(\mathcal{F}l_{\mathbf{I}}^\gamma/\Lambda)$. By [28, Conjecture 2.17.] the Jacobson-Morozov filtration induced by $c_1(\mathcal{L}_{det})$ on $H^*(\mathcal{F}l_{\mathbf{I}}^\gamma/\Lambda)$ is opposite to the perverse filtration. It is clear that the Jacobson-Morozov filtration induced by $\Delta_{(n-1,1)}$ on the diagonal coinvariants induces the filtration by antidiagonals.

4.3 The positive part of the affine Springer fiber

We now recall some results from [11]. We will use symmetric functions in two sets of variables X, Y, see Section 3.1. Using the standard plethystic notation, see for example [18], we write $f[XY]$ for the result of substituting $p_k(X)$ by $p_k(X)p_k(Y)$ in the expansion of $f \in \mathrm{Sym}_{q,t}$ in the basis of the p_λ. Recall from Section 3.1 that we have defined $\mathrm{Frob}_{q,t}$ to be the bigraded Frobenius character of a $\mathbb{Z}^2$-graded S_n representation.

Next, suppose $\gamma = st$ for $s \in \mathfrak{t}^{reg}$ as before and $G = GL_n$. Let Gr_G^+ be the so called positive part of the affine Grassmannian, consisting of lattices $\Lambda \subset \mathcal{K}^n$ contained in the standard lattice $\mathcal{O}^n$. The positive part of the affine flag variety, $\mathcal{F}l_{\mathbf{I}}^+$ is defined as the preimage of Gr_G^+ under the natural projection. The positive part of $\mathcal{F}l_{\mathbf{I}}^\gamma$ is defined to be

$$\mathcal{F}l_{\mathbf{I}}^{\gamma,+} := \mathcal{F}l_{\mathbf{I}}^+ \cap \mathcal{F}l_{\mathbf{I}}^\gamma$$

Its equivariant Borel-Moore homology $H_*^T(\mathcal{F}l_{\mathbf{I}}^{\gamma,+})$ is bigraded by the connected component $t^i \in \pi_0(\mathcal{F}l_{\mathbf{I}}^+) = \mathbb{Z}$ and the (half of the) cohomological grading $q^j \in \mathbb{Z}$ and carries two bigraded S_n-actions, one from the Springer action and one from the monodromy as s moves in a family. In [11], these are called the *star* and the *dot* actions, respectively. In [9], they are called the Springer action and the equivariant centralizer-monodromy actions, respectively.

The following is [11, Theorem A].

Theorem 4.8 (Carlsson-Mellit)**.** *The Frobenius character of $H_*^T(\mathcal{F}l_{\mathbf{I}}^{\gamma,+})$ for the $S_n \times S_n$-action is given by*

$$\mathrm{Frob}_{q,t,X,Y}(H_*^T(\mathcal{F}l_{\mathbf{I}}^{\gamma,+})) = q^{-k\binom{n}{2}}\omega_X \nabla^k e_n\left[\frac{XY}{(1-q)(1-t)}\right]$$

Compare this also to [13, Conjecture 3.7.], proved in [9, Remark 7.3.].

Corollary 4.9.

$$\mathrm{Frob}_{q,t,X,Y}(H_*(\mathcal{F}l_{\mathbf{I}}^{\gamma,+})) = q^{-k\binom{n}{2}}\omega_X \nabla^k e_n\left[\frac{XY}{(1-t)}\right]$$

Proof. Since $\mathcal{F}l_{\mathbf{I}}^{\gamma,+}$ is equivariantly formal, the generators of $H_*^T(pt)$ form a regular sequence in $H_*^T(\mathcal{F}l_{\mathbf{I}}^{\gamma,+})$. Now apply [19, Lemma 3.6.]. □

Next, note that the positive part of the lattice Λ, i.e. $\Lambda^+ \cong \mathbb{Z}^+_{\geq 0}$ acts on $\mathcal{F}l_{\mathbf{I}}^{\gamma,+}$, and by [22] we have

$$\mathcal{F}l_{\mathbf{I}}^{\gamma,+}/\Lambda_+ \cong \mathcal{F}l_{\mathbf{I}}^{\gamma}/\Lambda$$

Further, from the explicit description as the module called "M" in [11], we see

$$H_*^T(\mathcal{F}l_{\mathbf{I}}^{\gamma}) \cong H_*^T(\mathcal{F}l_{\mathbf{I}}^{\gamma,+}) \otimes_{\mathbb{C}[\Lambda^+]} \mathbb{C}[\Lambda]$$

as $\mathbb{C}[\Lambda]$-modules.

Using the degeneration of the Cartan-Leray spectral sequence for the Λ^+ and Λ-actions on $\mathcal{F}l_{\mathbf{I}}^{\gamma,+}$, resp. $\mathcal{F}l_{\mathbf{I}}^{\gamma}$, we have

Lemma 4.10.
$$H_*(\mathcal{F}l^{\gamma}_{\mathbf{I}}/\Lambda) = \bigoplus_i \mathrm{Tor}_i^{\mathbb{C}[\Lambda^+]}(H_*(\mathcal{F}l^{\gamma,+}_{\mathbf{I}}), \mathbb{C})$$

Suppose we wanted to kill the lattice action instead. Indeed, since $(H_*(\mathcal{F}l^{\gamma,+}_{\mathbf{I}})_{\Lambda^+})^* \cong H^*(\mathcal{F}l^{\gamma,+}_{\mathbf{I}})^{\Lambda}$, the bigraded Frobenius characters under S_n are the same (up to contragredient). Note that the coinvariant space is by definition $H_*(\mathcal{F}l^{\gamma,+}_{\mathbf{I}})_{\Lambda^+} = \mathrm{Tor}_0^{\mathbb{C}[\Lambda^+]}(H_*(\mathcal{F}l^{\gamma,+}_{\mathbf{I}}), \mathbb{C})$, so inherits a second grading from $H_*(\mathcal{F}l^{\gamma,+}_{\mathbf{I}})$.

Again by [19, Lemma 3.6.]

$$\mathrm{Frob}_{q,t,X,Y}(H^T_*(U)) = \omega_X \nabla^k e_n \left[\frac{XY}{(1-q)}\right]$$

where the LHS is the *equivariant* Borel-Moore homology of a certain open fundamental domain U of the lattice action defined in [11, Definition 6.9.]. It has only even dimensional nontrivial cohomology groups, as is implied from the formula. But interestingly, this does not mean that it is equivariantly formal, and indeed this space will have nontrivial odd usual Borel-Moore homology groups.

Finally, Eq. (4) of *loc. cit.* implies that

$$\sum_{i\geq 0}(-1)^i \mathrm{Frob}_{q,t,X,Y}(\mathrm{Tor}_i^{\mathbb{C}[\Lambda^+]}(H_*(U), \mathbb{C})) = \omega_X \nabla^k e_n\,[XY]\,.$$

The main conjecture of [11] essentially says that the Tor_i groups that appear on the left contain only those nontrivial representations χ_λ of the "dot" or equivariant-monodromy S_n-action for $i = \iota(\lambda')$, ι being a certain combinatorial statistic from the nabla positivity conjecture of *loc. cit.*

As we know by the Shuffle Theorem of [14], $\nabla^k e_n[X]$ is the character of the diagonal coinvariants, and is the result of substituting $p_k(Y) = 1$ in $e_n[XY]$, in other words taking the trivial component of the representation of the "dot" action. By [11, Conjecture A], this is the same as the Tor_0 part, and so corresponds to tensoring out both $\mathbf{x}$ and $\mathbf{y}$ $H^T_*(\mathcal{F}l^{\gamma,+}_{\mathbf{I}})$ over $\mathbb{C}[\Lambda^+] \otimes \mathbb{C}[\mathfrak{t}] \cong \mathbb{C}[\mathbf{x}, \mathbf{y}]$, without including higher derived functors.

Combining the above remarks, we have

Theorem 4.11. *Suppose [11, Conjecture A] is true. Then*

$$\mathrm{Frob}_{q,t}(H^*(\mathcal{F}l^{\gamma}_{\mathbf{I}})^{\Lambda}) = \omega_X \nabla e_n$$

i.e. the bigraded structure of the Λ-invariants, as an S_n-representation, coincides with the sign-twist of the diagonal coinvariants.

Similarly, orthogonality of the bases $\{m_\lambda\}$ and $\{h_\lambda\}$ gives

Corollary 4.12. *For any $\lambda \in \Lambda/\widetilde{W}_\ell$, we have*

$$\mathrm{Frob}_{q,t}(H^*(\mathrm{fl}^{\gamma}_{\mathbf{P}_\lambda})^{\Lambda}) = \mathrm{Frob}_{q,t}(H^*(\mathcal{F}l^{\gamma}_{\mathbf{I}})^{\Lambda}\mathbf{e}^-_\lambda) = \langle \omega\nabla e_n, h_\lambda\rangle m_\lambda.$$

4.4 The $\mathfrak{sl}_2$ symmetry of the center

Let $P \subset G$ be a parabolic subgroup of G, and $X = G/P$ the partial flag variety. Set $\widetilde{N}_P = T^*X$, the cotangent bundle of X. The following result is proven in [25]:

Theorem 4.13. *Let $\lambda \in \Lambda$ be a weight singular with respect to the shifted action of $\widetilde{W}$, and P a parabolic subgroup of G whose Weyl group $W_P \subset W$ stabilizes a weight in the $\widetilde{W}$-orbit of λ. Then the Hochschild cohomology of the singular block $\mathfrak{u}_\zeta^\lambda$ is given by the $\mathbb{C}^*$-equivariant Hochschild cohomology of $\widetilde{N}_P$:*

$$\mathrm{HH}^\bullet(\mathfrak{u}_\zeta^\lambda) = \oplus_{i+j+k=\bullet} H^i(\widetilde{N}_P, \wedge^j T\widetilde{N}_P)^k,$$

where the degree k comes from the $\mathbb{C}^*$ action by dilation on the fibers of $\widetilde{N}_P$. In particular,

$$Z(\mathfrak{u}_\zeta^\lambda) = \oplus_{i+j+k=0} H^i(\widetilde{N}_P, \wedge^j T\widetilde{N}_P)^k.$$

The statement is based on the derived equivalence of categories between certain category of representations of quantum groups at roots of unity and a derived category of $G \times \mathbb{C}^*$ equivariant coherent sheaves over the Springer resolution (see [2]).

The variety $\widetilde{N}_P$ is naturally symplectic and the action of G on $\widetilde{N}_P$ is Hamiltonian.

In particular, we have the following result [25]:

Proposition 4.14. *The space $H^0(\widetilde{N}_P, \wedge^2 T\widetilde{N}_P)^{-2}$ is one-dimensional, spanned by the Poisson bivector field τ, that is dual to the canonical holomorphic symplectic form $w \in H^0(\widetilde{N}_P, \wedge^2 T^*\widetilde{N}_P)^{-2}$. The exterior product with τ and contraction with w defines an $\mathfrak{sl}_2$ action on the total Hochschild cohomology of $\widetilde{N}_P$. This gives for any $j = 0, 1, \ldots n$ an isomorphism of vector spaces*

$$\tau^{n-j}\wedge : \quad H^i(\widetilde{N}_P, \wedge^j T\widetilde{N}_P)^k \to H^i(\widetilde{N}_P, \wedge^{2n-j} T\widetilde{N}_P)^{k+2j-2n}.$$

Combining Theorem 4.13 and Conjecture 1.2, we also have the following conjecture.

Conjecture 4.15. *There is a bigraded algebra isomorphism*

$$\oplus_{i+j+k=0} H^i(\widetilde{N}, \wedge^j T\widetilde{N})^k \cong H^*(\mathcal{F}l_{\mathbf{I}}^\gamma)^\Lambda$$

where the bigrading on the right is explained in Section 4.3. Alternatively,

$$\oplus_{i+j+k=0} H^i(\widetilde{N}, \wedge^j T\widetilde{N})^k \cong \mathrm{gr}^P H^*(\mathcal{F}l_{\mathbf{I}}^\gamma)^\Lambda$$

Moreover, the element τ on the left should correspond up to a scalar to the polynomial $\Delta_{(n-1,1)}$ introduced in Theorem 4.7 on the right, or in the second version equivalently to $c_1(\mathcal{L}_{det})$.

In particular, combined with Theorem 4.11 this Conjecture would imply [24, Conjecture 4.9(3)].

4.5 A degeneration of spectral curves

To propose yet another model for the center in type A, we will study the elliptic homogeneous affine Springer fibers of slope $(n+1)/n$ associated to the elements $\gamma_{n+1/n}$ introduced in Eq. (2.2) and their relation to $\mathcal{F}l_{\mathbf{I}}^{\gamma}$, where γ is as in the introduction. It is known by e.g. [12] that

$$\mathrm{gr}^P H^*(\mathcal{F}l^{\gamma_{n+1/n}}) \cong \overline{\mathrm{DR}}_n$$

We will construct a a family of irreducible spectral curves $C_t \subset \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(2))$, such that the associated family of parabolic Hitchin fibers models the degeneration of affine Springer fibers of slope n/n to the one of slope $(n+1)/n$. One can then ask whether the specialization map from the cohomology of the total family (which turns out to be just that of the central fiber) gives an injection to the cohomology of the special fiber, respecting the perverse filtration.

Theorem 4.16. *There exists a family of irreducible curves $\mathcal{C} \to \mathbb{A}^1$, arising as a restriction of the Hitchin system to a line on the Hitchin base, so that the spectral curve $C_t, t \in \mathbb{A}^1$ will have two singular points: one "constant" (i.e independent of t) singular point with equation $y^n = z^{n-1}$, and another singular point of the form $y^n = tx^n + x^{n+1}$.*

Remark 4.2. Let us notice that if $t \neq 0$, this singular point is isomorphic to the singularity $y^n + x^n = 0$. Indeed, in $\mathbb{C}[[x, y]]$ the equation can be written $(t + x)^{-1}y^n = x^n$. Taking a n-th root of the unit $(t + x)$, say a, we can use the coordinate change $Y = a^{-1}y$ and $X = x$ to get $Y^n = X^n$. Hence, around this second singular point we are degenerating the singularity $y^n = x^n$ into the singularity $y^n = x^{n+1}$.

Proof. We construct the family of spectral curves realizing this degeneration as follows: let $E \subset \Sigma_1$ be the exceptional section inside the first Hirzebruch surface, and $F \subset \Sigma_1$ some fiber, which we will call "the fiber at infinity". Let $U = \Sigma_1 \backslash (E \cup F)$. Take coordinates x, y on U such that the straight lines $x =$ constant are the fibers of the projection $U \subset \Sigma_1 \to \mathbb{P}^1$. Let us consider the curve $\widehat{C}_t \subset U$ given by the equation $y^n = t + x$. The effect of a positive elementary transform $\varphi : \Sigma_r \dashrightarrow \Sigma_{r+1}$ is given by the change of variables $u = y/x, v = x$.

Hence the strict transforms of $\widehat{C}_t$ (inside $\varphi(U)$) have local equation given by $u^n = tv^n + v^{n+1}$, giving the desired degeneration. Now let us describe the singular point at infinity (i.e compute the closure of these curves inside Σ_2), and prove that C_t is irreducible for all $t \in \mathbb{A}^1$.

First, we claim that the closure of $\widehat{C}_t$ doesn't intersect E. Indeed, recall that Σ_1 is the blow-up of $\mathbb{P}^2$ at a point. Hence, it's enough to take the closure of the preimage of $\widehat{C}_t$ inside $\mathbb{P}^2$ (call this curve $\tilde{C}_t$) and check that $\tilde{C}_t$ doesn't intersect the center of the blow-up. On U, we have coordinates x, y, that form a dense open of $\mathbb{P}^2$ (recall that U and E are disjoint by definition). Because $U \cong \mathbb{A}^2$, we can take homogeneous coordinates $[x : y : z]$ on $\mathbb{P}^2$. The fiber at infinity is given by $z = 0$ and U is given by $z = 1$. The fiber $x = 0$ and $z = 0$ both contains the center of the blow-up which is therefore $[0 : 1 : 0]$.

The closure of $\tilde{C}_t$ has equation $y^n = tz^n + xz^{n-1}$, which clearly doesn't contain $[0 : 1 : 0]$.
Since the elementary transforms are isomorphism outside the exceptional locus, it follows that the closure of C_t coincide with the closure of $\tilde{C}_t$ inside $\mathbb{P}^2$,

i.e the curve with equation $y^n = tz^n + xz^{n-1}$. The only point at infinity is $[1:0:0]$, and has local equation $y^n = tz^n + z^{n-1}$ as claimed. To check that C_t is irreducible, it's enough to check that C_t is irreducible on the chart $x \neq 0$. On this chart, C_t is isomorphic to the curve given by $y^n = z^{n-1}$, which is irreducible. □

Consider the associated family of parabolic Hitchin fibers, which is a restriction of the family in 4.4 to a line. Using Eq. (4.1), we note that the only affine Springer fibers contributing to the cohomology are the ones coming from the singularities described above. We will ignore the one which is constant, for there is an injective map in cohomology sending the cohomology classes $\alpha \in H^*(\mathcal{F}l_{\mathbf{I}}^{\eta}/\Lambda_\eta)$ of interest to

$$\alpha \otimes 1 \in H^*(\mathcal{F}l_{\mathbf{I}}^{\eta}/\Lambda_\eta) \otimes H^*(\mathcal{F}l_{\mathbf{I}}^{\gamma_{n-1/n}}) \cong H^*(\widetilde{\mathcal{M}}_a)$$

where η is either γ or $\gamma_{n+1/n}$.

In particular, by Theorem 4.16, we get a pullback map $i^* : H^*(\mathcal{F}l_{\mathbf{I}}^{\gamma_{n+1/n}}) \to H^*(\mathcal{F}l_{\mathbf{I}}^{\gamma}/\Lambda)$. Now, both of these spaces are endowed with the perverse filtration, as the family comes by restriction of the Hitchin fibration and on the locus of interest the map is proper, so that the decomposition theorem applies. Note however that it is unclear how this filtration compares to that induced by the t-structure on $\mathbb{A}^1$, as the pullback along the inclusion to the base is in general only right t-exact.

Remark 4.3. It seems likely that the map i^* is injective and its image is exactly $H^*(\mathcal{F}l_{\mathbf{I}}^{\gamma})^{\Lambda}$. Moreover, the map respects the perverse filtration. Note that as the map is a pullback in cohomology, it automatically respects the multiplicative structure.

The only supporting evidence for this remark is that these properties are true for $G = SL_2$, where they are easy to check. In general, we observe that $\mathcal{F}l_{\mathbf{I}}^{\gamma_{n+1/n}}$ has only even-dimensional cohomology as it is paved by affines. It is also known that it has $n!$ components, as does $\mathcal{F}l_{\mathbf{I}}^{\gamma}/\Lambda$. On the level of top cohomology, it is clear that the map is injective, but in general it seems hard to control the associated vanishing-nearby cycles-central fiber exact sequence in cohomology.

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