



Have no fear, and let me be your guide. I will teach you about



3. A BAKER'S STORY

A baker wants to bake the perfect cookie: a cookie of 100% chocolate. She bakes it, and sets her cookie to cool. In the ☾, the 👑 of fairies sends a mischievous fairy upon the cookie, to ⚡ it with her ✂️*☆☆ that turns chocolate into cookie. The next ⚙️, the baker finds that her perfect cookie is now $1 - 1 = \square$ chocolate! She decides to outsmart the 👑 by splitting her 🍪 chocolate cookie into two pieces. The 👑 decides to send two fairies to ⚡ the halves with their ✂️*☆☆. However, being simple creatures of *☆☆, neither fairy knows which half the other ⚡ and each ⚡ the halves randomly. The next day, the baker wakes up to find that her 🍪 cookie is now $(1 - \frac{1}{2})^2 = \square$ chocolate! She decides outsmart the 👑 by splitting her 🍪 chocolate cookie into three pieces...



Whoa what's this? We only learned that e was a few pages ago and now we're doing differential equations?

Remark. This also works when replacing a matrix with any linear operator. This is how physicists solve Schrödinger's equation. Meaning that $f' = Ae^{Ax}v$ solves the system, and meets the initial condition $f(0) = e^{0A}v = I^n v = v$.

$$\frac{d}{dx} e^{Ax} v = e^{Ax} \left(\frac{d}{dx} v + A e^{Ax} v \right) = A e^{Ax} v$$

Let $f(0) = v \in \mathbb{R}^n$ be the initial condition. Then: functions of a single variable.

of constants and $f = (f_1, \dots, f_n)$ is a vector of AKA $\frac{d}{dx} f = Af$, where $A = (a_{ij})$ is a matrix

$$\begin{pmatrix} f_1' \\ f_2' \\ \vdots \\ f_n' \end{pmatrix} = \begin{pmatrix} a_{11}f_1 + \dots + a_{1n}f_n \\ a_{21}f_1 + \dots + a_{2n}f_n \\ \vdots \\ a_{n1}f_1 + \dots + a_{nn}f_n \end{pmatrix}$$

Example 2. Consider the system of linear differential equations:

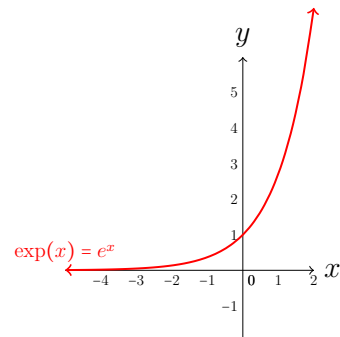
EQUATIONS

8. e^x -TRA-ORDINARY DIFFERENTIAL

4. THE FUNCTION OF THE FUNCTION e^x

Yes, we defined e^{ax} as a series, or as some limit of a sequence, but that's ok. It's still a function, with some nice properties.

- (1) e^x is a monotonically increasing positive function from \mathbb{R} to \mathbb{R} .
- (2) $e^{x+y} = e^x e^y$, $e^0 = 1$ (homomorphism!)
- (3) If you made a slide in the shape of the plot of $y = e^x$, you will never stop sliding leftward.



But most importantly:

Theorem. $\frac{d}{dx} e^x = e^x \Leftrightarrow \frac{d}{dx} e^{cx} = ce^{cx}$ by ⚙️ rule).

Proof. Use the first definition, and differentiate term by term. □

“log(x), being the inverse function of e^x , has always been controversial. Some say that log makes them feel 🤔🤔”

CAPTAIN’S LOG DAY 1: e-LOG-ICAL MUSINGS

- >How does a number theorist drown?
- >loglogloglog
- >What do you get when you integrate one over cabin with respect to cabin?
- >log cabin
- >No, it’s a houseboat, you forgot to add the c!
- >No, it’s a priceless yacht, you disregarded the absolute value

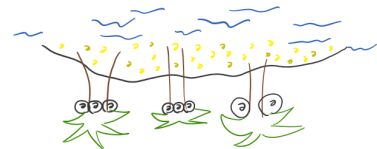


FIGURE 1. log for scale. Scale pictured on previous page

2.7182818284590452353602874713526624977572470936999959
 57496696762772407663035354759457138217852516642742746
 639193200305992182.4 PROPERTIES OF e 3342952605956307
 381323286279434907632338298807531952510190115738341879
 30702154 Ok, so we have this number, but what is it 8477
 411853743120177093775183828573231882660626
 13313845830007520449338265602976067371132007093287091
 27443704709086845959150192810897463
 77211 is a good approximation for π for sufficiently 45490
 59879 bad approximations of good. Succinctly: 95763514822
 08269895193668033182528869398496465105820939239829488
 7933203625094431173012 $e \approx 3 \approx \pi$ 6140397019837679320683
 282376464804295311802328782509819455815301756717361332
 06981 **Theorem.** e is irrational 598888519345807273866738
 58942 *Proof.* If $e = \frac{a}{b}$, then: 279212998068058257492796104841984443634632
 4496848755023362427041978623209002160990235304369941
 84914631409343 $x = b! \left(\frac{a}{b} - \sum_{n=0}^b \frac{1}{n!} \right) = a(b-1)! - \sum_{n=0}^b \frac{b!}{n!} \in \mathbb{Z}$ 16253152096188707016768
 39642437 $\frac{b!}{n!} \in \mathbb{Z}$ 161310713855503540212340784
 70417189861068739696552126715468895703503540212340784
 98193 But then by definition of $e = \sum_{n=0}^{\infty} \frac{1}{n!}$: 2474501585390473
 0419957777093503 $x = b! \left(\sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^b \frac{1}{n!} \right) = \sum_{n=b+1}^{\infty} \frac{b!}{n!} > 0$ 419972515088887666403555707162
 2684471625 $x = b! \left(\sum_{n=0}^{\infty} \frac{1}{n!} - \sum_{n=0}^b \frac{1}{n!} \right) = \sum_{n=b+1}^{\infty} \frac{b!}{n!} > 0$ 1236677194
 3252786753985589 $\frac{b!}{n!} > 0$ 595638023637016
 21120 Next, note that if $n > b$, then $\frac{b!}{n!} < \frac{1}{(b+1)^{n-b}}$, so: 244352918363721
 4174023889344124796357437026375529444337998016125492
 27850925778256 $0 < x < \sum_{n=b+1}^{\infty} \frac{1}{(b+1)^{n-b}} = \sum_{k=1}^{\infty} \frac{1}{(b+1)^k} = \frac{1}{b} \leq 1$ 5664462772516401
 9105900491644998 $\frac{1}{(b+1)^k} = \frac{1}{(b+1)^k}$ 1851851956532
 44258698294695930801915298721172556347546396447910145
 9040903862984967912874088703548938586717479854857757
 57320568128849205413340539220001137863300945560688166
 7400169842055804033637953764520304024322566135278369
 51177883863874439662532249850654995886234281899707733
 27617178392803494650143455889707194258639877275471096

1. HOW CAN OUR e 'S BE REAL IF OUR NUMBERS AREN'T REAL?

If real numbers were an ocean, e would be the tropical island where there's nothing to eat but coconuts.



Definition. For any $\mathfrak{R} \in \mathbb{R}$, we define: $e^{\mathfrak{R}} = \sum_{i=0}^{\infty} \frac{\mathfrak{R}^i}{i!}$

This definition sucks, let's try again: **Definition.** We define:

$$e = \lim_{\mathfrak{R} \rightarrow \infty} \left(1 + \frac{\mathfrak{R}}{1} \right)$$

and its side-kick:

$$e^{\mathfrak{R}} = \lim_{\mathfrak{R} \rightarrow \infty} \left(1 + \frac{\mathfrak{R}}{\mathfrak{R}} \right)^{\mathfrak{R}}$$

Shit, we accidentally defined the same thing. *Proof.* Expand for finite \mathfrak{R} . \square

7. e IS WAY MORE THAN \mathbb{R}

Nothing prevents x from being another kind of number. Another number? Different numbers? Other numbers? Complex numbers. **Theorem.** Let $a + bi = z \in \mathbb{C}$. Then $e^z = e^a e^{bi} = e^a (\cos(b) + i \sin(b))$

Proof. To prove $e^z = e^a e^{bi}$, we have to use multi-normal theorem on the first definition, like in the real case. To prove $e^{bi} = \cos(b) + i \sin(b)$, exponentiate and use Taylor's formulas for sin and cos. \square

Example 1. We can take the exponent of a square matrix to get another matrix. It turns out that if $A \in \text{Mat}(n)$, then $e^{-A} = I_n$, so e^A is always invertible. However, some of the exponent rules don't work. If $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then: $e^A e^B \neq e^{A+B}$

Well, some things, like being able to take powers, sums, division over \mathbb{Q} , convergence...