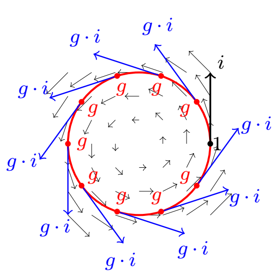
Lie Groups

Definition. A Lie group is a group that is also a manifold, for which inversion and group multiplication are smoooooooth.

Example 3. The unit circle $S^1 \subset \mathbb{C}$ is a Lie group with complex multiplication (the linear rotation map). More generally, if $\mathbb{K} = \mathbb{R}, \mathbb{C}$, then any matrix group over $\mathbb{K}$ is a Lie group.

Theorem 4. *If G is a Lie group, then G is parallelizable.*

Proof. Let $\sigma_1, \ldots, \sigma_n$ be a basis of $T_{\mathrm{id}}G$. Then define $\sigma_i(g) = D_{f_g}(\sigma_i)$, where $f_g(x) = gx$ is a diffeomorphism, and hence $\mathrm{Rk}(D_{f_g}) = \dim(G)$. □



S^1 and S^3 are Lie groups, but what about S^7?

Complexification

> "Complexification is the art of turning something real, tangible, and in proper order into something complex, hard to grasp, or even imaginary"

Definition. A **complex number** is a pair of real numbers (x, y), with multiplication[1]:

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - \bar{y}_2y_1, y_2x_1 + y_1\bar{x}_2)$$

We set $\overline{(x_1, x_2)} = (x_1, -x_2)$.

Definition. A **quaternion** is a pair of complex numbers (x, y), with multiplication:

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - \bar{y}_2y_1, y_2x_1 + y_1\bar{x}_2)$$

We set $\overline{((x_1, x_2), (x_3, x_4))} = ((x_1, -x_2), (-x_3, -x_4))$.

Definition. An ☠**ctonion** is a pair of quaternions (x, y), with multiplication:

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - \bar{y}_2y_1, y_2x_1 + y_1\bar{x}_2)$$

We set $\overline{(((x_1, x_2), (x_3, x_4)), ((x_5, x_6), (x_7, x_8)))} = (((x_1, -x_2), (-x_3, -x_4)), ((-x_5, -x_6), (-x_7, -x_8)))$

Nice 👍

[1] If a is real, $\bar{a} = a$.

The ☠ctonions

Let's decomplexify $\mathbb{R}^8$:

$e_0 = (1, 0)$ $e_1 = (i, 0)$ $e_2 = (0, 1)$ $e_3 = (0, i)$

$e_4 = (j, 0)$ $e_5 = (k, 0)$ $e_6 = (0, j)$ $e_7 = (0, k)$

Then use the ☠ product to compute the table:

$\mathbb{H}$?

×	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	$-e_6$	$-e_7$	e_4	e_5
e_3	e_3	e_2	$-e_1$	$-e_0$	$-e_7$	e_6	e_5	$-e_4$
e_4	e_4	$-e_5$	e_6	e_7	$-e_0$	e_1	$-e_2$	$-e_3$
e_5	e_5	e_4	e_7	$-e_6$	$-e_1$	$-e_0$	e_3	$-e_2$
e_6	e_6	e_7	$-e_4$	$-e_5$	e_2	$-e_3$	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	$-e_5$	e_4	e_3	e_2	e_1	$-e_0$

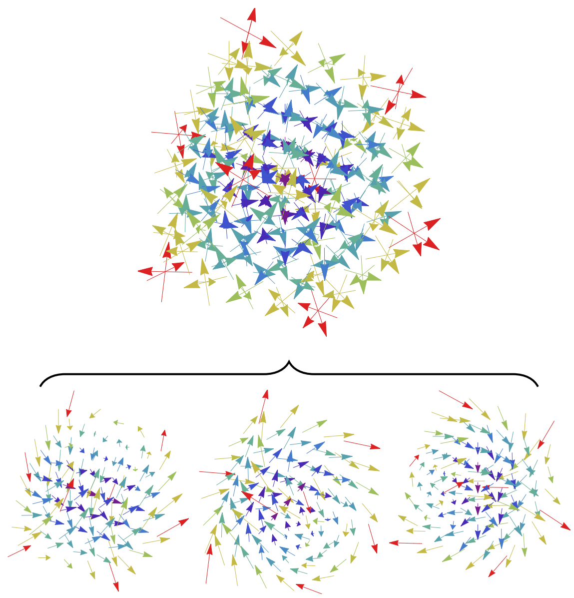
Change e_i to x_i to get 7 $\|\perp\|$ vector fields on S^7!

> By the way: S^7 is not a Lie group because $(e_1e_2)e_4 \neq e_1(e_2e_4)$. The ☠ctonions are not associative!

Why can't we go further to the **sedenions**?

📢 linear independence fails in S^{15}.

How to tell if your sphere is parallelizable



A politically neutral guide to things mathematicians wished were related to string theory

Say "Parallelizable" ten times fast

Definition. A manifold M^n is **parallelizable** if $TM \cong M \times \mathbb{R}^n$ (ie, TM is trivial).

The Möbius strip is a nontrivial line bundle over S^1. It is nontrivial because there is no way to unwind the twisty to make it cylinder.

Theorem 1. *An n-bundle E is trivial iff it admits n linearly independent nonzero sections.*

Proof. If $\sigma_1, \cdots, \sigma_n$ are such sections, then the map $f : M \times \mathbb{R}^n \to E$ defined by

$$f(p, (x_1, \cdots, x_n)) = (p, x_1\sigma_1(p) + \cdots + x_n\sigma_n(p))$$

is a well-defined smooth diffeomorphism, which is a linear isomorphism on the fibres, and hence is a bundle isomorphism.

The other direction is left as an exercise. □

Parallelization

Who cares? Me. Have a multiplication table:

×	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Do you care now? You should! Replace $1, i, j, k$ with x_1, x_2, x_3, x_4, and think of (x_1, x_2, x_3, x_4) as a vector on the unit sphere $S^3 \subset \mathbb{R}^4$.

x_1	x_2	x_3	x_4
x_2	$-x_1$	x_4	$-x_3$
x_3	$-x_4$	$-x_1$	x_2
x_4	x_3	$-x_2$	$-x_1$

This shows that S^3 is parallelizable. □

Decomplexification

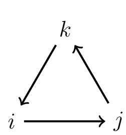
> "Decomplexification is the art of turning something horrible into something reasonable. In other words, it's just choosing a basis"

The complex numbers (resp. quaternions), denoted $\mathbb{C}$ (resp. $\mathbb{H}$) are 2 (resp. 4) dimensional real **algebras**! A basis makes understanding multiplication easier: If $z = (x, y) \in \mathbb{C}$ becomes $z = x + iy$, and $q = ((x_1, x_2), (x_3, x_4)) \in \mathbb{H}$ becomes $q = x_1 + ix_2 + jx_3 + kx_4$. Then:

$$ij = k,\ jk = i,\ ki = j$$
$$ji = -k,\ kj = -i,\ ik = -j$$

A mne·mon·ic: **Convoluded** !

The rules:



1. Players (+) and (−) chant "i, j, k!" and then choose one of $\pm i$, $\pm j$, $\pm k$.
2. Multiply the choices.
3. If the result has a (−) sign in front of it, then player (−) wins.
4. Argue about the order of multiplication.

Unwinding the Twisty to flat

For a manifold M^n to be parallelizable, we must find n independent nonvanishing vector fields. This is sometimes impossible, as shown by the Fundamental Theorem of S^2.

Example 2. The torus is parallelizable (see picture).