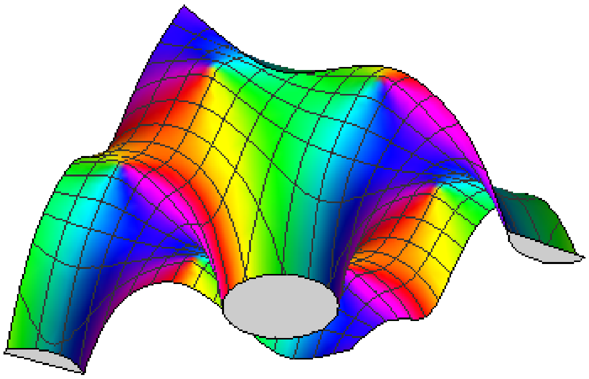
Simple ℂomplex Analysis



"A non-analytic zine about analytic analysis."

Proof of Theorem 3:

Suppose for contradiction that f fails to map to some two points, say z_0 and z_1. By composing f with a linear fractional transformation, we may assume that $z_0 = 0$ and $z_1 = 1$. Choose three points on the unit circle and connect them by hyperbolic lines intersecting the unit circle to make a hyperbolic "triangle" T. By the ℝiemann mapping theorem, there exists an analytic isomorphism $h : \mathbb{H} \to T$ from the half plane to the triangle extending analytic isomorphically to the boundary, sending a certain three real segments to the sides of T. Analytically continuing h will map the reflected half-plane regions onto the reflected hyperbolic triangles. Thus, h can be extended along any curve γ in $\mathbb{C} - \{z_0, z_1\}$ and the continuation's image will stay within the unit disc. There exists some z on which $f(z) \in \mathbb{H}$, and some disc that f maps into $\mathbb{H}$. Then along any path, $h \circ f$ can be analytically continued along a sequence of discs covering the path. The extension's value is independent of the path, so it is well defined. But the image of the continuation to $\mathbb{C}$ of $h \circ f$ is contained in the unit disc, so it is bounded and hence constant. But this implies that f is constant, a contradiction. □

Joke 1. Why is Canada a meromorphic function? Because it goes from ℂ to ℂ and has a pole.

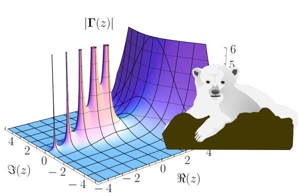
Joke 2. Why did the mathematician name his dog ℂauchy? Because he left a residue at every pole.

Joke 3. What is the contour integral around Africa? Zero; all the Poles are in Europe

Joke 4. What is a pirate's favorite set? You would think it's ℝ but it's really the ℂ.

Joke 5. Why were communists obsessed with complex analysis? Because $\mathbb{C}'s$ the means of production

Joke 6. What did ℂauchy and ℝiemann say years after they retired? Long time no ℂ.



Claim.

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} = \pi$$

Proof: The integral above is identical to

$$Im(\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx)$$

which is, by theorem 2,

$$Im(\lim_{R\to\infty} \int_{C_R} \frac{e^{i\zeta}}{\zeta} d\zeta + \lim_{\epsilon\to 0} \int_{C_\epsilon} \frac{e^{i\zeta}}{\zeta} d\zeta)$$

where C_R and C_ϵ are semicircles oriented clockwise and counterclockwise, respectively. Plugging in the path formula for C_R, the former integral is

$$\lim_{R\to\infty} Im(i \int_0^{\pi} e^{iRcost - Rsint} dt)$$

, which is bounded in absolute value by the integral over the absolute value:

$$\lim_{R\to\infty} \int_0^{\pi} e^{-Rsint} dt = 0$$

. Hence the limit as $R \to \infty$ of the former integral is 0. Plugging in the path formula for the latter integral we obtain

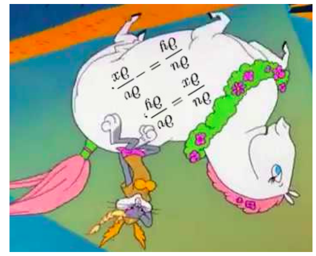
$$Im(i \lim_{\epsilon\to 0} \int_0^{\pi} e^{i\epsilon cost - \epsilon sint} dt) = Im(i \int_0^{\pi} dt) = \pi$$

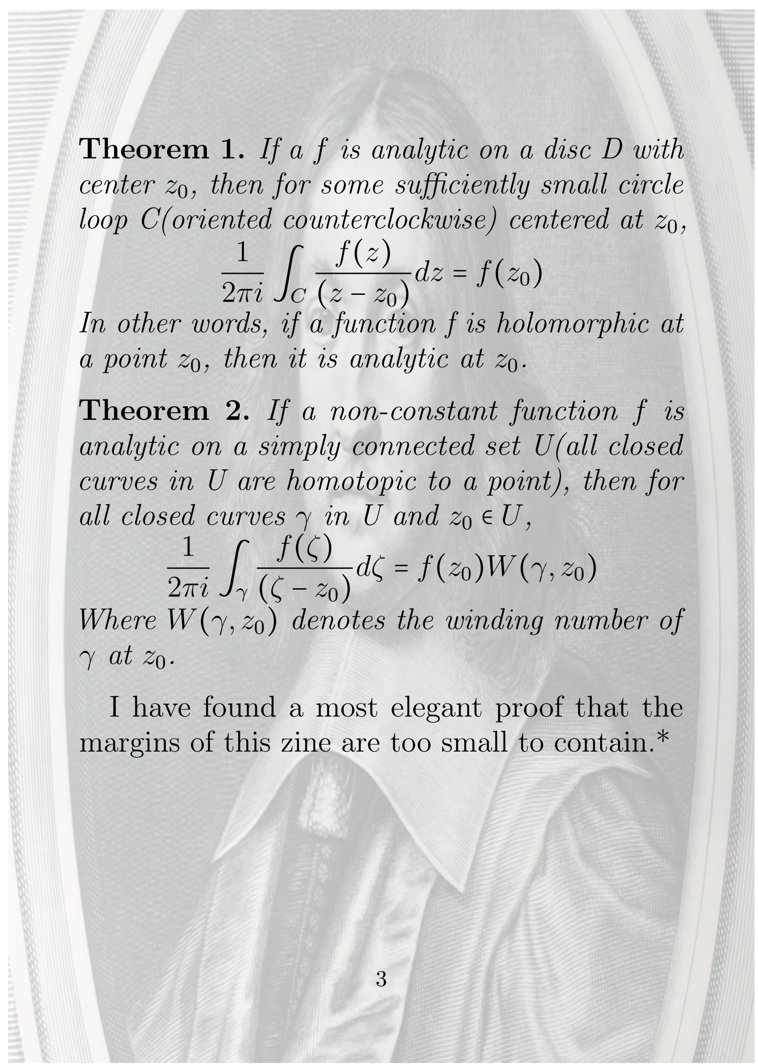
, finishing the proof. □

The Birth of Complex Analysis - Abridged Version

The following tale has been communicated to the authors through the first hand account of a passing brick. Though the brick has long crumbled after being thrown into a Wells Fargo drive thru, the authors still managed to obtain a full account through the use of a ouija board and some algebraic manipulations.

One day ℂauchy and ℝiemann were taking a stroll along the beach. As the ℂ lapped against their feet, they caught sight of Plump Pony prancing wildly across the sand."A more appalling spectacle I have not seen," squealed poor ℂauchy, "we must help out the poor chap". Plump Pony confessed a passionate yearning to advance mammalkind's knowledge of functions of a complex variable. Inspired by Plump Pony, and with the help of a few friends, the two mathematicians would go on to usher in a new era for mathematics.





Theorem 1. *If a f is analytic on a disc D with center z_0, then for some sufficiently small circle loop C(oriented counterclockwise) centered at z_0,*

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)}dz = f(z_0)$$

In other words, if a function f is holomorphic at a point z_0, then it is analytic at z_0.

Theorem 2. *If a non-constant function f is analytic on a simply connected set U(all closed curves in U are homotopic to a point), then for all closed curves γ in U and $z_0 \in U$,*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z_0)}d\zeta = f(z_0)W(\gamma, z_0)$$

Where $W(\gamma, z_0)$ denotes the winding number of γ at z_0.

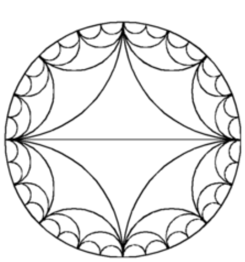
I have found a most elegant proof that the margins of this zine are too small to contain.*

An analytic continuation of our friends' travels

On the request of the Plump Pony, ℂauchy and ℝiemann meet with a magician who is said to know all tricks with machinery. "ℙicard", the magician says, "any card." ℂauchy conceals an ace of spades as ℝiemann goes for his card. "Halt," a kid who looks to be the magician's slaps ℝiemann's hand with a glove, "you have identified the maximum number of values that an entire function can omit in its image."

Theorem 3. *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Then $f(\mathbb{C})$ is either $\mathbb{C}$ or $\mathbb{C}$ with a singleton removed.*

I have found a most elegant proof that the margins of this zine are too small to contain.** "The complex numbers can be affine thing", said ℝiemann.



*Proof of Theorem 2. We use the following equivalent statement: For a function f that is holomorphic on a simply connected set U, the integral over f of any curve γ in U is zero. The statement is equivalent because

$$\int_\gamma \frac{f(\zeta)}{\zeta - z_0}d\zeta = \int_\gamma \frac{f(z_0)}{\zeta - z_0}d\zeta + \int_\gamma g(\zeta)d\zeta$$

where g is some analytic function on U. The former term of the right expression becomes the right expression of Theorem 1, and the latter vanishes when the equivalent statement is proven. If Theorem 1 holds, and given some analytic function on U, f, plugging in $(z-z_0)f$ into the theorem statement proves the equivalent statement. We aim to prove the equivalent statement; let $g : U \times U \to \mathbb{C}$ be defined by

$$g(z, w) = \frac{f(z) - f(w)}{z - w}$$

Note that it is continuous on $U \times U$ and analytic with respect to w on U when we fix z. Define the entire function $h : \mathbb{C} \to \mathbb{C}$ by $\int_\gamma g(w, \zeta)d\zeta$ on U and $h(w) = \int_\gamma f(\zeta)(\zeta - w)^{-1}d\zeta$ on the set for which $W(\gamma, w) = 0$. The definitions coincide: the difference between the two definitions on the intersection of the potentially conflicting sets is $\int_\gamma \frac{f(w)}{\zeta - w}d\zeta = f(w)W(\gamma, w) = 0$ for w in both U and the set on which $W(\gamma, w) = 0$. Because γ is a compact (and hence bounded) curve, the set for which the winding number of γ is nonzero is a bounded set and the latter definition for h approaches to zero as $w \to \infty$. Hence h is a bounded entire function which is constant. By the limit, h is identically zero, completing the proof. □