OCTONIONS

Quaternions are pairs of complex numbers, likewise octonions are pairs of quaternions.

given $q_1 = a_1 + b_1 i + c_1 j + d_1 k$ $q_2 = a_2 + b_2 i + c_2 j + d_2 k$
$\theta = (q_1, q_2)$.

$$\boxed{\begin{aligned} &\text{Let } \theta_1 = (p_1, q_1) \quad \theta_2 = (p_2, q_2) \\ &\theta_1 \cdot \theta_2 = (p_1 p_2 - \bar{q}_2 q_1,\ q_2 p_1 + q_1 \bar{p}_2) \end{aligned}}$$

Let: $e_0 = (1,0)$ $e_4 = (j,0)$
$e_1 = (i,0)$ $e_5 = (k,0)$
$e_2 = (0,1)$ $e_6 = (0,j)$
$e_3 = (0,i)$ $e_7 = (0,k)$

An octonion can be written as:

$$\theta = x_0 e_0 + x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7$$

QUATERNIONS

Quaternions are pairs of complex numbers

Given $z = a+bi$, $w = c+di$, then $q = (z, w)$

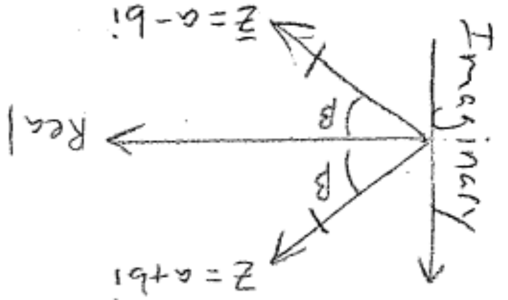
$$\boxed{\begin{aligned} &\text{Let } p = (p_1, p_2) \\ &q = (q_1, q_2) \\ &q \cdot p = (q_1 p_1 - \bar{p}_2 q_2,\ p_2 q_1 + q_2 \bar{p}_1) \end{aligned}}$$

Quaternions can be written as
$q = a + bi + cj + dk$
where:

$i = (i,0)$ $k = (0,i)$

$j = (0,1)$ $1 = (1,0)$

The complex conjugate of $z = a+bi$ is $\bar{z} = a-bi$.



Complex reciprocal of $z = a+bi$:

$$\frac{1}{z} = \frac{\bar{z}}{z \cdot \bar{z}} = \frac{a-bi}{a^2+b^2}$$

Norm of a complex number $z = a+bi$:

$$\|z\| = \sqrt{z \cdot \bar{z}} = \sqrt{a^2+b^2}$$

FUN FACT:
Multiplying by i corresponds to a rotation in the 90°

OCTONION MULTIPLICATION TABLE

X	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	$-e_6$	$-e_7$	e_4	e_5
e_3	e_3	e_2	$-e_1$	$-e_0$	$-e_7$	e_6	e_5	$-e_4$
e_4	e_4	$-e_5$	e_6	e_7	$-e_0$	e_1	$-e_2$	$-e_3$
e_5	e_5	e_4	e_7	$-e_6$	$-e_1$	$-e_0$	e_3	$-e_2$
e_6	e_6	e_7	$-e_4$	$-e_5$	e_2	$-e_3$	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	$-e_5$	e_4	e_3	e_2	e_1	$-e_0$

NOT ASSOCIATIVE

$$(e_1 e_2) e_4 = e_3 e_4 = -e_7$$
$$e_1 (e_2 e_4) = e_1 (-e_6) = e_7$$

It turns out:

$$\boxed{i^2 = j^2 = k^2 = ijk = -1}$$

$\times$

	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

They are not commutative!

eg: $ij = k$, $ji = -k$ $\quad ij \neq ji$

SOME PROPERTIES

1) Conjugate: $\bar{q} = a - bi - cj - dk$
2) norm: $\|q\| = \sqrt{a^2+b^2+c^2+d^2}$
3) reciprocal: $q^{-1} = \frac{\bar{q}}{q\bar{q}} = \frac{\bar{q}}{\|q\|^2}$
4) $\|qp\|^2 = qp\,\overline{qp} = qp\bar{p}\bar{q} = \|q\|^2\|p\|^2$

pull out the real constant of $\|p\|^2$

FOUR SQUARE THEOREM

If $m^2 = a^2+b^2+c^2+d^2$ and $n = t^2+u^2+v^2+w^2$ then mn is a sum of 4 squares (natural numbers) $(a, b, c, d, t, u, v, w \in \mathbb{Z})$

$m = \|a+bi+cj+dk\|^2 = \|q_1\|^2$ with integer coefficients

$n = \|t+ui+vj+wk\|^2 = \|q_2\|^2$

$mn = \|q_1\|^2\|q_2\|^2 = \|q_1q_2\|^2$

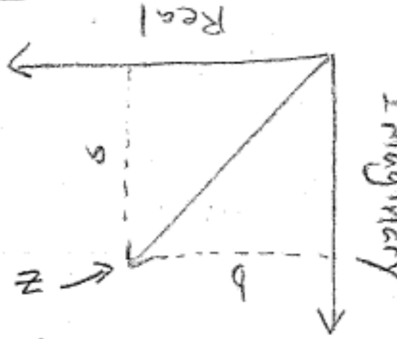
$\therefore q_1q_2$ is a quaternion

$\therefore mn$ can be written as the sums of 4 squares.

COMPLEX NUMBERS

A complex number is a pair of 2 real numbers; one as x coordinate, one as y coordinate, We can write them as:

$z \equiv a+bi$



Think of i as $\sqrt{-1}$ in the sense that $i^2 = -1$

Addition: $(a+bi)+(c+di) = (a+c)+(b+d)i$

Multiplication: $(a+bi)(c+di) = (ac-bd)+(bc+ad)i$

Complex numbers can also be written as ordered pairs:

$a+bi = (a, b)$

The sphere is the set of vectors x in $\mathbb{R}^n$ of norm 1, $\|x\| = \langle x, x\rangle = 1$.
A vector y is tangent to a point x on the sphere if $\langle x, y\rangle = 0$.
A vector field on the sphere S^n is a continuous function such that $\langle f(x), x\rangle = 0$

For example

1) On S^1, in the complex numbers, $\langle xi, x\rangle = 0$, so $f(x) = xi$ is a vector field on S^1.

2) On S^3, in the quaternions, $\langle xi, x\rangle = 0$, $\langle xj, x\rangle = 0$, $\langle xk, x\rangle = 0$ so $f(x) = xi$, or xj or xk are vector fields on S^3.

3) On S^7, in the octonions, $\langle xe_i, x\rangle = 0$ for $1 \le i \le 7$
$f(x) = xe_i$ are 7 vector fields on S^7

Note: All of the above vector fields are orthogonal to one another.