A zine dedicated to answering the age-old question:

> "1-bundle, 2-bundle,
> fibre bundle, vector bundle?"

7. The Hopf Fibration

Would you believe it if we told you that S^3 is a fibre bundle over S^2? You better believe it!

Example 7.1. Think of $S^3 \subset \mathbb{R}^4 = \mathbb{H}$ as the unit sphere in the *quaternions*. Let $p \in S^2 \subset \mathbb{R}^3 = \Im(\mathbb{H})$ be given, then we define:

$$\varphi(q) = qpq^{-1} \in \mathbb{H}$$

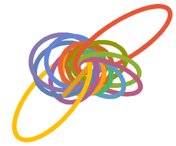
We leave it to you to verify that $\varphi(q) \in \Im(\mathbb{H})$, and has norm 1, meaning that $\varphi(q) \in S^2$.

Additionally,

$$\varphi^{-1}(p') = S^3 \cap \operatorname{span}_{\mathbb{R}}\{p + p', pp' - 1\}$$

is the intersection of a sphere in $\mathbb{R}^4$ with a 2-dimensional hyperplane, which is totes S^1. Thus, S^3 is a fibre bundle over S^2 with fibres S^1.

If we stereographically project $S^3 \to \mathbb{R}^3$, we can colour the fibres:



Section 3. Sections' section

The projection $\pi : E \to M$ sends an entire $\mathbb{R}^k$ to a point. This trivializes the vector space. Vector spaces want to be important, so they defined:

Definition. A *section* of a vector bundle E over M is a map $\sigma : M \to E$ such that $\pi \circ \sigma = \mathrm{id}$.

Imagine the possibilities! We can now associate a vector to every point in M. This is where the 🪄⭐☆ happens.

Example 2.1. If E is the *tangent bundle* (the smooth gluing of tangent spaces of a manifold M), then a section of E is a vector field on M.

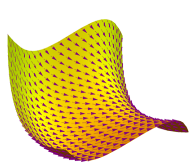
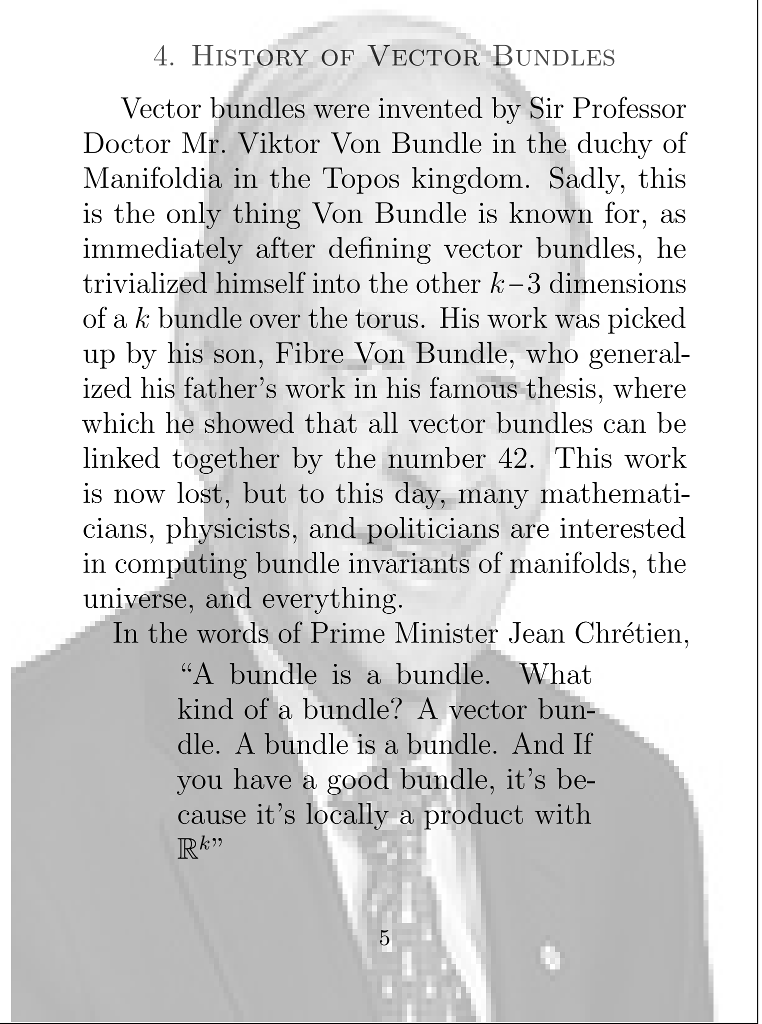


Figure 3. A section of the tangent bundle of a section of a surface



4. History of Vector Bundles

Vector bundles were invented by Sir Professor Doctor Mr. Viktor Von Bundle in the duchy of Manifoldia in the Topos kingdom. Sadly, this is the only thing Von Bundle is known for, as immediately after defining vector bundles, he trivialized himself into the other $k-3$ dimensions of a k bundle over the torus. His work was picked up by his son, Fibre Von Bundle, who generalized his father's work in his famous thesis, where which he showed that all vector bundles can be linked together by the number 42. This work is now lost, but to this day, many mathematicians, physicists, and politicians are interested in computing bundle invariants of manifolds, the universe, and everything.

In the words of Prime Minister Jean Chrétien,

> "A bundle is a bundle. What kind of a bundle? A vector bundle. A bundle is a bundle. And If you have a good bundle, it's because it's locally a product with $\mathbb{R}^k$"

1. What's a vector bundle and where are my glasses?

Definition. Let M be an m-dimensional manifold. A *vector bundle* of rank k (sometimes called a k-bundle) is a manifold E, equipped with a projection $\pi : E \to M$ such that:

(1) $\pi^{-1}(x)$ is a vector space $\cong \mathbb{R}^k$.
(2) For every $x \in M$ there exists a neighbourhood $U \subset M$ and a homeomorphism $\varphi_a : U \times \mathbb{R}^k \to \pi^{-1}(U)$ for which $(\pi \circ \varphi) = \mathrm{id}$ and $v \to \varphi(x, v)$ is a linear isomorphism.

A *line bundle* is a vector bundle of rank 1.

A vector bundle E of rank k is called *trivial* if $E \cong M \times \mathbb{R}^k$. We think of a vector bundle as a smooth way to glue vector spaces to M.

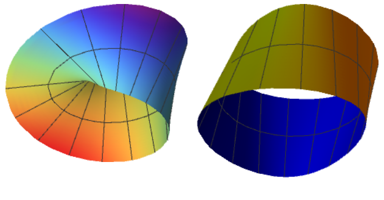


Figure 1. Trivial and non-trivial 1-bundles over S^1

2. Vector Bundle Pop Quiz

Joke 1. What do you get when you cross a baby with a manifold?

Solution. A bundle of joy (see fig. 2). ☐

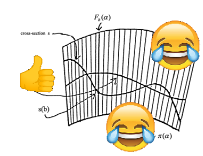


Figure 2. A vector bundle that makes you happy

Joke 2. What happens when you cross a citrus fruit with a bull?

Solution. A trivial lime bundle over a torus. ☐

Joke 3. What do you call a vector bundle that makes you groan?

Solution. A vector pundle. ☐

Joke 4. What do you call a violently shaking bunny?

Solution. The Hop vibration (see p.8). ☐

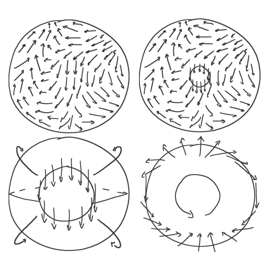
5. The Harry Potter Ball Theorem

Some manifolds have nice sections on their tangent bundles. Some don't. Some manifolds don't have any interesting sections; they are just points.

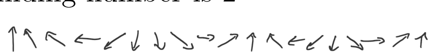
Theorem 5.1. *If s is a section of the tangent bundle of S^2, then there is a point $x \in S^2$ such that $s(x) = (x, 0)$.*

In other words, every vector field on S^2 vanishes at some point.

Proof. Invert the sphere:



The winding number is 2



and nonvanishing vector fields have winding number 0. ☐

6. Other Fibrations

Why restrict to vector spaces?

Definition. A *fibre bundle* with fibre manifold F over an n-dimensional manifold M is a manifold E, with a surjective map of manifolds $\pi : E \to M$ such that the fibres of π are isomorphic to the manifold F, and locally, the manifold E looks like the manifold $\mathbb{R}^n \times F$.
Manifold Manifold Manifold Manifold Manifold.

Remark. This definition also works for general topological spaces, not just manifolds.

Example 6.1. The torus is the trivial fibre bundle over S^1 with fibre S^1, since the torus is $S^1 \times S^1$.

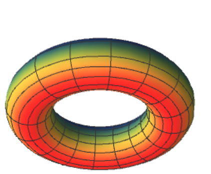


Figure 4. The torus is a (trivial) fibre bundle.