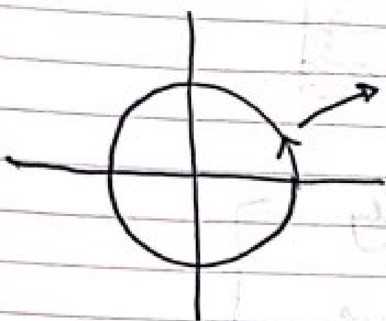
July 8, 2019

Weierstrass $\wp$ - function
Day 1 notes

Day outline

- Explain the problem with the course & complex analysis
- Show the $\sin(x)$ bar-trick
- Try to do the same with a periodic function on $\mathbb{C}$
- Liouville theorem $\Rightarrow$ we should use $\mathbb{C}P^2$

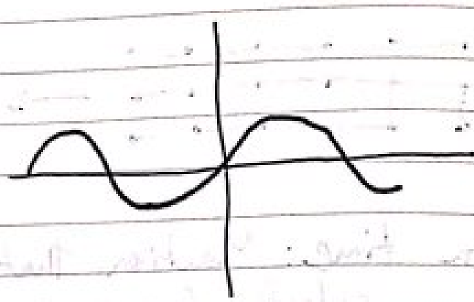
~~Theorem~~: $\sin(x)$ bar-trick



$(\cos(x), \sin(x))$

$\uparrow$

$= \frac{d}{dx}\sin(x)$



$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ is a periodic function with period 2π.

$\cos(x) = \frac{d}{dx}\sin(x)$ is periodic as well. When $\sin = 0, \cos = 1$ and satisfy

$$(x, y) = (\cos(\theta), \sin(\theta)) \Rightarrow x^2 + y^2 = 1$$

$\nwarrow$ circle!

Doing the same in $\mathbb{C}$:

Define $\wp$ to be periodic in both directions. i, 1
ie

$$\wp(z+i) = \wp(z+1) = \wp(z)$$

Problem: $\wp$ holomorphic $\Rightarrow$ $\wp$ constant $\Rightarrow$ NOT interesting.

So $\wp : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ to use $\mathbb{C}P^2$ to make things work out.

July 10, 2019

Weierstrass ℘-function

Day 2 notes

Day outline:
- Go over homework
- Define meromorphic & problem points
- State analyticity results for $\mathcal{H}(\Omega)$
- $\left(\frac{\pi}{\sin \pi z}\right)^2$ example

Useful computations to write on board:

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} = z - \frac{z^3}{3!} + z^2(\cdots)$$

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots = 1 + z + z^2(\cdots)$$

$$\sum_{n=0}^{\infty} \frac{1}{n^2+k^2} \cong \int_0^{\infty} \frac{1}{x^2+k^2}\,dx = \frac{1}{k}\arctan\left(\frac{x}{k}\right)\Big|_0^{\infty} = \frac{\pi}{2k} \to 0$$

↑ Integral test

$$|\sin(z)| = |\sin(x)|^2 + |\sinh(y)|^2 \quad \text{if } z = x + yi$$

$$\frac{\pi^2}{\sin^2 \pi z} = \frac{\pi^2}{(\pi z - \frac{1}{6}\pi^3 z^3 + \cdots)^2} \cancel{= \frac{\pi^2}{\pi^2 z^2} \frac{1}{(1 - \frac{1}{6} z\pi^2 + \cdots)^2}}$$

$$\cancel{= \frac{\pi^2}{\pi^2 z^2} \frac{1}{1 - \frac{1}{3}\pi^2 z^2 + z^4(\cdots)}}$$

$$\cancel{= \frac{1}{z^2}\left(1 + \frac{1}{3}\pi^2\right.}$$

$$= \frac{\pi^2}{(\pi^2 z^2 - \frac{1}{3}\pi^4 z^4 + z^6(\cdots))} = \frac{\pi^2}{\pi^2 z^2} \cdot \frac{1}{(1 - \frac{1}{3}\pi^2 z^2 + z^6(\cdots))}$$

$$= \frac{1}{z^2}\left(1 - \frac{1}{3}\pi^2 z^2 + z^6(\cdots)\right) = \frac{1}{z^2} - \frac{\pi^2}{3} + z^4(\cdots)$$

July 11, 2019

Weierstrass $\wp$ function
Day 3 notes

Day 2 reflection:
- Too much detail verbally, not enough written
- Don't give handouts before class, but after.
- Finish thoughts linearly
- Ask students to complete details verbally.

Today
- Finish proof of $\sum \frac{1}{(z-n)^2} = \left(\frac{\pi}{\sin \pi z}\right)^2$
- Define $\wp(z)$.
- Show that $\wp$ parametrizes a cubic

Note: $\wp(z) - \frac{1}{z^2}$ is holomorphic around 0

so ~~$\wp(z) = \frac{1}{z^2} + a_0 + a_1 z + a_2 z^2 + \cdots$~~ later

$a_1 = 0$ because $\wp$ is even

$a_0 = 0$ because $\wp(z) - \frac{1}{z^2} = 0$ at $z=0$.

Lemma: $\wp$ is even. Proof: every term in the defn is symmetric with another one

Lemma: $\wp$ is doubly periodic

Proof: $\wp'(z) = -2 \sum_{\omega \in \Gamma} \frac{1}{(z-\omega)^3}$ is odd

so $\wp'(z+e_1) - \wp'(z) = 0$ So

$\wp(z+e_1) - \wp(z) = \text{constant}$

but $\wp\left(-\frac{e_1}{2} + e_1\right) - \wp\left(-\frac{e_1}{2}\right) = 0$

Thm: $\wp(z) = \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + \cdots$

We compute:

$$\wp'(z) = -2\sum_{\omega\in\Gamma}\frac{1}{(z-\omega)^3}$$

~~8a~~ -or-

$$\wp'(z) = \frac{-2\cdot 1}{z^3} + 2a_2 z + 4a_4 z^3 + \cdots$$

$$(\wp'(z))^2 = \frac{4}{z^6} - \frac{8a_2}{z^2} - 16a_4 + z^2(\cdots)$$

$$(\wp(z))^3 = \frac{1}{z^6} + 3\frac{a_2}{z^2} + 3a_4 + z^2(\cdots)$$

$$(\wp'(z))^2 - 4\wp(z)^3 = -\frac{20a_2}{z^2} - 28a_4 + z^2(\cdots)$$

$$\frac{1}{z^2} = \wp(z) - z^2(\cdots) \quad \text{so}$$

$$(\wp'(z))^2 = -20a_2\wp(z) + 4\wp(z)^3 + \underbrace{z^2(\cdots)}_{h}$$

h is holomorphic at 0, doubly periodic so constant.
at $z=0$, $h=0$ so $h\equiv 0$

$$\boxed{y^2 = 4x^3 - 20a_2 x - 28a_4 \quad (x,y) = (\wp, \wp')}$$

Scanned with CamScanner

July 13, 2019

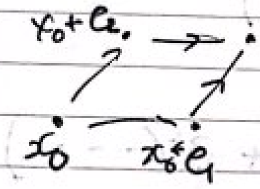
Weierstrass $\wp$-function

Today's goal: prove that $4x^3 - 20a_2 x - 28a_4 = y^2$ is parametrized bijectively by $(x,y) = (\wp, \wp')$

Tomorrow's goal: Show that ___ is a torus, show addition formula, discuss Abel's theorem

Prop.: If f is a nonconstant meromorphic function on $\mathbb{C}$ with group of periods Γ then

Q: how many poles? w/ multiplicity?



# of zeroes of f = # of poles of f

z^2: 2 zeroes

$\frac{1}{z^2}$: 2 poles

Proof: $Z - P = \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,dz = 0$ by periodicity of $\frac{f'}{f}$ Q: why?

Q: why? cancel cancel

Thm: $4x^3 - 20a_2 - 28a_4$ has 3 distinct roots

Moreover, for all $(x,y) \in \mathbb{C}^2$ lying on the curve, there is a unique z s.t. $x = \wp(z)$, $y = \wp'(z)$

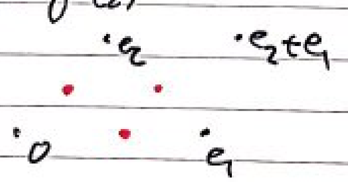
Proof

(1) $4x^2 - 20a_2 - 28a_4$ has 3 distinct roots:

Every value of $\wp(z)$ is taken exactly twice

" " " $\wp'(z)$ " " " thrice Q: how many times?

Consider:



$e_1/2$, $e_2/2$, $e_1+e_2/2$

$$\boxed{\wp'(e_1/2) \underset{\text{odd}}{=} -\wp'(-e_1/2) \underset{\text{periodic}}{=} -\wp'(e_1/2)}$$ Same for $e_1+e_2/2$

Q?

so $e_1/2$, $e_1+e_2/2$ are the three roots of $\wp'$

WTS that $\wp(e_1/2)$, $\wp(e_2/2)$, $\wp(e_1+e_2/2)$ are distinct roots of $4x^3-20a_2x-28a_4$

Q?

$\boxed{\text{Why roots?}}$ by the equation w/ $\wp'$, $\wp$!

Comment: be careful with the word "root"

$\wp(e_1/2)$ is taken twice, but since $\wp'(e_1/2)=0$ it follows that $-\wp(e_1/2)-\wp(z)$ has derivative 0 at $e_1/2$ so $e_1/2$ is a double root!

So $\wp(e_1/2)$ is taken exactly once.

(2) uniqueness

Now if $z_0 \in \mathbb{C}$ with $2z \notin \Gamma$ then $\wp(z_0)$ is taken twice, at $z_0, -z_0$ at what points?

Q: $\wp(z) - z_0$ has a root: $\boxed{\text{why?}}$

~~taking~~ If $x = \wp(z_0) = \wp(-z_0)$ then
~~$y=\wp$~~ $y = \wp'(z_0)$ and $\wp'(z_0) = -y$

so the points $(\underset{\text{fixed}}{x}, \underset{\text{varies}}{y})$ on the curve are obtained on z_0 and $-z_0$ for which the y value is different

$\square$

Thm: $4x^3-20a_2x-28a_4$ is smooth

Proof: 3 distinct roots!

If time: define $\mathbb{C}P^2 = \mathbb{C}^3 \setminus \{0\} / x \sim \lambda x$, $\lambda \in \mathbb{C}$
+ coordinates on $\mathbb{C}P^2$

$[x:y:1] \cup [x:1:z] \cup [0:y:z]$

July 13, 20[illegible]

Weierstrass $\wp$ function
Day 5 notes

We have a curve $4x^3 - 20a_2x - 28a_4 = y^2$ which is perfectly parametrized by $(x,y) = (\wp, \wp')$

Problem points: Γ !

Definition: $\mathbb{C}P^n = \mathbb{C}^{n+1}\setminus\{0\}/\sim \quad x \sim y$ if $\exists \lambda\ x = \lambda y$

$$\mathbb{C}P^n = \{[x_0, \ldots, x_n] : x_n \in \mathbb{C}\}$$

where $[x_0, \ldots, x_n] = [1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]$

Eg: $\mathbb{C}P^2 = [x, y, 1] \cup [x, 1, t] \cup [1, y, t]$

$= [x, y, 1]$ ~~$\cup [0,1,0] \cup [$~~

~~Consider $X' = (ty^2 = 4x^3 - 20a_2xt - 28a_4t^3)$~~

Notice: In $\mathbb{C}P^2$, $X^* \hookrightarrow \mathbb{C}P^2$ by $(x,y) \to [x:y:1]$

but $[x:y:t] = [\frac{x}{t} : \frac{y}{t} : 1] \to (\frac{x}{t}, \frac{y}{t})$

So we better have (x, y, t) and $(\frac{x}{t}, \frac{y}{t}, 1)$ on the curve.

Homogenize

$$ty^2 = 4x^3 - 20a_2xt - 28a_4t^3.$$

There is also the points at $t = 0$: $4x^3 = 0$

So there is also the point $[0:y:0] = [0:1:0]$

So define $X' = [0:1:0] \cup \{[x:y:t] \mid ty^2 = 4x^3 - 20a_2xt - 28a_4t\}$

We extend the map ~~[illegible]~~ $\mathbb{C}/\Gamma \xrightarrow[\wp,\wp']{f} X$ by setting $f(0) = [0:1:0]$

Why is this reasonable? ~~[illegible]~~ Compactification at ∞.

So: $\mathbb{C}/\Gamma \longrightarrow X'$ biholomorphism so

$X' = $ Torus!

<u>The Addition formula</u>

If $u, v \in \Gamma$ then $(\wp(u), \wp'(u)) \neq (\wp(v), \wp'(v))$
so $\exists\, a, b, c$ s.t

$$0 = c\wp'(u) + a\wp(u) + b$$
$$0 = c\wp'(v) + a\wp(v) + b$$

~~$f_{ab} = cx + ay + b$~~
$f_{ab} = c\wp'(z) + a\wp(z) + b$ has triple pole
so has 3 zeroes that add up to smthing in Γ:

$$u + v + \underbrace{(-u-v)}_{3^{rd}\text{ root}} \in \Gamma$$

So $\begin{pmatrix} \wp'(u) \\ \wp'(v) \\ \wp'(\underset{-v-u}{\underline{v+u}}) \end{pmatrix}, \begin{pmatrix} \wp(u) \\ \wp(v) \\ \wp(\underset{-v-u}{\underline{v+u}}) \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ are lin. dep.

ie: the line passing through 2 points on X' also hits a third.

Let $(\wp(u), \wp'(u))$, $(\wp(v), \wp'(v))$ be 2 points, lying on $y = \lambda x + \eta$

plug in: $4x^3 - 20a_2x - 28a_4 - \lambda^2x^2 - 2\lambda\eta x - \eta^2$

$$\parallel$$

$$c(x - x_1)(x - x_2)(x - x_3)$$

$$\uparrow \qquad \uparrow \qquad \uparrow$$
$$\wp(u) \qquad \wp(v) \qquad \wp(u+v)$$

So by equating coeff. $-\lambda^2 = -4(x_1 + x_2 + x_3)$

So:

$$\wp(u+v) = -\wp(u) - \wp(v) + \frac{\lambda^2}{4}$$

$$\lambda = \frac{\text{rise}}{\text{run}} = \left(\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} \right)$$

So

$$\wp(u+v) = -\wp(u) - \wp(v) + \frac{1}{4}\left(\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)} \right)$$