The Weierstrass $\wp$ Function

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Day 1: Uniform Convergence on Compact Sets

Someone told me long ago
There's a calm before the storm
I know it's been comin' for some time

– Creedence Clearwater Revival, "Have You Ever Seen The Rain?"

Definition 1. A set $\Omega \subset \mathbb{C}$ is called **open** if for every $x \in \Omega$ there exists some $r > 0$ for which $\{z \in \mathbb{C} : |x - z| < r\} \subset \Omega$.

Definition 2. Let $f : \Omega \to \mathbb{C}$. We say that f is **holomorphic (analytic) on** Ω if f satisfies the Cauchy-Riemann equations on Ω. We denote the **set of holomorphic functions on** Ω by $\mathcal{H}(\Omega)$.

Theorem 3. *(Moreira's Theorem) Let Ω be an open set. A function $f : \Omega \to \mathbb{C}$ is holomorphic if and only if for any closed loop around a simply-connected region $R \subset \Omega$, we have*

$$\oint_{\partial R} f(z)dz = 0$$

Definition 4. A subset $X \subset \mathbb{C}$ is called **closed** if $\Omega = X^c$ is open. A subset $X \subset \mathbb{C}$ is called **compact** if X is closed and bounded.

Definition 5. We say that a sequence $\{f_n\} \subset \mathcal{H}(\Omega)$ **converges uniformly on compact sets to** $f : \Omega \to \mathbb{C}$ if for every compact set $K \subset \Omega$, and $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, and $z \in K$, we have $|f_n(z) - f(z)| < \varepsilon$.

Definition 6. If $\{f_n\} \subset \mathcal{H}(\Omega)$, we say that $\sum_{n=1}^{\infty} f_n$ **converges uniformly on compact sets to** f if the partial sums, $s_N = \sum_{n=1}^{N} f_n$ converge uniformly on Ω to f.

Exercise 1. Prove that if $\{f_n\} \subset \mathcal{H}(\Omega)$, and $f_n \to f$ uniformly on compact sets, then for any simple closed curve $\gamma \subset \Omega$, we have

$$\oint_{\gamma} f(z)dz = \lim_{n\to\infty} \oint_{\gamma} f_n(z)dz$$

Theorem 7. *If* $\{f_n\} \subset \mathcal{H}(\Omega)$ *converges uniformly on compact sets to* f*, then* $f \in \mathcal{H}(\Omega)$*, and* $\{f_n'\}$ *converge uniformly to* f'*.*

Homework Problems

Exercise 2. Show that if $\{f_n\} \subset \mathcal{H}(\Omega)$, and $f = \sum_{n=1}^{\infty} f_n$ converges uniformly on compact sets, then

$$f' = \lim_{N\to\infty} \sum_{n=1}^{N} f_n'$$

Exercise 3. Determine whether the following sets are open, closed, or neither:

$$\mathbb{C} \setminus \mathbb{Z} \qquad \{z : \|z\| \leq 1\} \qquad \{z : -1 < \operatorname{Im}(z) \leq 1\} \qquad \mathbb{C} \setminus \{a + bi : a, b \in \mathbb{Z}\}$$

Exercise 4. Show that, away from problematic points, the series

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$$

converges pointwise. (Bonus) Show that this series also converges uniformly on compact sets.

Exercise 5. We define $g(z) = \frac{1}{z} + \sum_{n\neq 0} \left(\frac{1}{z-n} + \frac{1}{n}\right)$. Manipulate the general term in the sum to find the problem points, and show that this sum converges away from them. (Bonus) Show that this series converges uniformly on compact sets.

Exercise 6. Compute $g'(z)$, using theorems from above. Keep your answer as a series.

Exercise 7. Assume $\{f_n\} \subset \mathcal{H}(\Omega)$ converge uniformly on compact sets to f, and each f_n vanishes nowhere in Ω. Prove that either f vanishes nowhere, or that $f \equiv 0$. You should use the fact[1] that if f has a zero of order k at z_0, then

$$k = \frac{1}{2\pi i} \oint_\gamma \frac{f'(z)}{f(z)} dz$$

Where γ is a small counter-clockwise circle around z_0.

Exercise 8. Prove that if $\{f_n\} \subset \mathcal{H}(\Omega)$ converge uniformly on compact sets to f, and each f_n is injective, then f is either injective or constant. Find an example of either case.

[1] try proving this yourself, or talk to me at TAU

Day 2: Meromorphic Functions

I see a bad moon a-rising
I see trouble on the way
I see earthquakes and lightnin'
I see bad times today

– Creedence Clearwater Revival, "Bad Moon Rising"

Definition 8. A quotient of two holomorphic functions is called **meromorphic**. We say that a meromorphic function f **blows up at** z_0 if $\lim_{z\to z_0}|f(z)| = \infty$. We say that f **blows up on** X if f blows up on some $z_0 \in X$.

Definition 9. Let $\{f_n\}$ be a sequence of meromorphic functions on an open $\Omega \subset \mathbb{C}$. We say that $\sum_{n=1}^{\infty} f_n$ **converges uniformly** (or **uniformly and absolutely**) on $X \subset \Omega$ if only finitely many f_n's have poles in X, and if the rest of the f_n's form a uniformly (or uniformly and absolutely) converging series on X.

Lemma 10. *If $f_n : \Omega \to \mathbb{C}$ are meromorphic, and $\sum f_n$ converges uniformly on compact sets, then on every compact set $X \subset \Omega$, only a finite number of f_n's blow up on X.*

Definition 11. If $\Omega \subset \mathbb{C}$ is open, and $f_n : \Omega \to \mathbb{C}$ are a sequence of meromorphic functions converging on a compact subset $X \subset \Omega$, we define

$$\sum f_n = \sum_{n<n_0} f_n(z) + \sum_{n\geq n_0} f_n(z)$$

Where n_0 is chosen such that f_n does not blow up on X if $n > n_0$.

Exercise 1. Set $f = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$. Show that f is even and periodic. That is, show that $f(z+2\pi) = f(z)$ and $f(-z) = f(z)$.

Theorem 12. *We have:*

$$\left(\frac{\pi}{\sin(\pi z)}\right)^2 = \sum_{n=-\infty}^{\infty} \frac{1}{(n-z)^2}$$

Definition 13. Let $e_1, e_2 \in \mathbb{C}$ not be real multiples of each other. We define the **group of periods**, denoted by Γ, by $\Gamma = \{me_1 + ne_2 : m, n \in \mathbb{Z}\}$.

Definition 14. Let Γ be a group of periods. We define:

$$\wp(z) = \frac{1}{z^2} + \sum_{w \in \Gamma, w \neq 0} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right)$$

Homework Problems

Exercise 2. Prove that if Γ is a group of periods, then

$$\sum_{w\in\Gamma, w\neq 0} \frac{1}{|w|^3}$$

converges.

Exercise 3. Use this to prove that $\wp$ converges uniformly on compact sets.

Exercise 4. Prove that $\wp$ is even, and conclude that $\wp'$ is odd

Exercise 5. Prove that $\wp$ and $\wp'$ are doubly periodic

Exercise 6. Show that for any $w \in \Gamma$, $\frac{1}{(z-w)^2} - \wp(z)$ does not have a problem point at $z = w$.

Exercise 7. Show that $\wp(z) - \frac{1}{z^2}$ is holomorphic at 0, and vanishes at $z = 0$.
Hint: can you think of a way to write this as a series?

Exercise 8. Write

$$\wp(z) = a_0 + \sum_{n=1}^{\infty} a_n z^n + \sum_{n=-1}^{-\infty} a_{-n} z^{-n}$$

Show that $a_0 = 0$, and that $a_{-n} = 0$ unless $n = 2$. What is a_{-2}?

Exercise 9. By explicitly differentiating $\wp$, find a formula for a_n as above.

Day 3: The Weierstrass $\wp$

I went down Virginia
Seekin' shelter from the storm
Caught up in the fable
I watched the tower grow

– Creedence Clearwater Revival, "Who'll Stop The Rain"

Recall from last time:

Theorem 15.

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2} = \left(\frac{\pi}{\sin(\pi z)}\right)^2$$

and the convergence is uniform on compact sets.

Proof. We showed that $g(z) = f(z) - \frac{\pi^2}{\sin^2(\pi z)}$ is holomorphic around 0 and is periodic in the real direction. We wish to show that $g \equiv 0$.

Lemma 16. *g is bounded.*

Lemma 17. $g = 0$

This proves the theorem. $\square$

Definition 18. Let $e_1, e_2 \in \mathbb{C}$ not be real multiples of each other. We define the **group of periods**, denoted by Γ, by $\Gamma = \{me_1 + ne_2 : m, n \in \mathbb{Z}\}$.

Definition 19. Let Γ be a group of periods. We define:

$$\wp(z) = \frac{1}{z^2} + \sum_{w \in \Gamma, w \neq 0} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

Lemma 20. *$\wp$ is doubly periodic*

Lemma 21. *$\wp$ is even, and $\wp - \frac{1}{z^2}$ is holomorphic.*

Theorem 22. *We can write $\wp(z) = \frac{1}{z^2} + a_2 z^2 + a_4 z^4 + \cdots$.*

Homework Problems

Some of these problems are repeats of yesterday, which now make more sense considering the material we covered.

Exercise 1. Prove that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

Exercise 2. Identify the function (from the first problem set) up to a constant:

$$f(z) = \frac{1}{z} + \sum_{n\in\mathbb{Z}, n\neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right)$$

Use the fact that this function is odd to show that this constant must be 0.

Exercise 3. Prove that if Γ is a group of periods, then

$$\sum_{w\in\Gamma, w\neq 0} \frac{1}{|w|^3}$$

converges.

Exercise 4. Use this to prove that $\wp$ converges uniformly on compact sets.

Exercise 5. By explicitly differentiating $\wp$, find a formula for a_n in the expansion of $\wp$ done in class.

Day 4: The Cubic Equation

I set out on the road
Seekin' my fame and fortune
Lookin' for a pot of gold
Thing got bad things got worse

– Creedence Clearwater Revival, "Lodi"

Today's goal is to show that $\wp$ and $\wp'$ parametrize the cubic equation

$$y^2 = 4x^3 - 20a_2x - 28a_4$$

Definition 23. A period parallelogram is:

Lemma 24. *If f is a nonconstant meromorphic function on $\mathbb{C}$ with periods e_1 and e_2, then the number of zeros of f is equal to the number of poles of f in any period parallelogram which does not have a pole or a zero on its boundary.*

Theorem 25. $4x^3 - 20a_2a - 28a_4$ *has three distinct roots.*

Theorem 26. *For all $(x, y) \subset \mathbb{C}^2$ satisfying $y^2 = 4x^3 - 20a_2x - 28a_4$, there is a unique z such that $x = \wp(z)$ and $y = \wp'(z)$.*

Theorem 27. *The curve $y^2 = 4x^3 - 20a_2x - 28a_4$ is smooth in $\mathbb{C}^2$.*

Homework Problems

Exercise 1. We define the **complex projective space** $\mathbb{C}P^n$ by:

$$\mathbb{C}P^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$$

Where we identify $x \sim y$ if $y = \lambda x$ for some $\lambda \in \mathbb{C}$. Show that we can think of $\mathbb{C}P^1$ as the regular 2-sphere.

Definition 28. We think of points in $\mathbb{C}P^n$ in **projective coordinates**, $[x_0, x_1, \ldots, x_n]$, where $(x_0, \ldots, x_n)$ represents the points in $\mathbb{C}^{n+1}$, and the square brackets means we are thinking of them as points in $\mathbb{C}P^n$.

Exercise 2. Try writing all of the points in $\mathbb{C}P^2$ in the form $[x, y, 1]$. What goes wrong? What point is missing?

Exercise 3. Show that the set defined by $x^3 - y^2 = 0$ is not smooth (try looking at it in $\mathbb{R}$)

Exercise 4. Show that the set defined by $y^2 - 4x^3 - a_2x - 28a_4 = 0$ is locally the graph of a smooth function (and hence is a manifold).
Hint: Use the implicit function theorem

The next two exercises will probably be done in Mark's class tomorrow as homework or in class. You might want to just take them as fact, or read their proofs in any complex analysis book.

Exercise 5. Assume that f is a ratio of polynomials, and γ is a curve that does not have any poles or zeros of f on it. Let $\{\alpha_i\}$ be the zeroes of f and $\{\beta_j\}$ be the poles of f in the region bounded by γ. Show that

$$\int_\gamma \frac{1}{2\pi i} \frac{zf'(z)}{f(z)} dz = \sum_i \alpha_i - \sum_j \beta_j$$

Exercise 6. (hard) Show that the above also works for any function f.

Exercise 7. Let f be any doubly periodic function with group of periods Γ. Use the above formula to show that in any period parallelogram, the sums of the roots of f is equal to the sum of the poles of f up to an element of Γ. In other words, if $\{\alpha_i\}$ are the roots of f, and $\{\beta_j\}$ are the poles of f, show that

$$\sum_i \alpha_i + \sum_j \beta_j \in \Gamma$$

Day 5: The Torus

Cruisin' on through the junction, I'm flyin' 'bout the speed of sound
Noticin' peculiar function, I ain't no roller coaster show me down

– Creedence Clearwater Revival, "Sweet Hitchhiker"

Definition 29. We define the **complex projective space** by

$$\mathbb{C}P^n = (\mathbb{C}^n \setminus \{0\})/ \sim$$

Where we say $x \sim y$ if there exists some $\lambda \in \mathbb{C}$ for which $x = \lambda y$.

Lemma 30. *Let $u, v \in \mathbb{C}$. Then the vectors*

$$\begin{pmatrix} \wp'(u) \\ \wp'(v) \\ \wp'(w) \end{pmatrix}, \begin{pmatrix} \wp(u) \\ \wp(v) \\ \wp(w) \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

are linearly dependent over $\mathbb{C}$.

Theorem 31. *(Addition theorem) We have:*

$$\wp(u+v) = -\wp(u) - \wp(v) + \frac{1}{4}\left(\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right)^2$$