Traffic

I want to ride my bicycle
I want to ride my bike
I want to ride my bicycle
I want to ride it where I like

– Queen, "Bicycle Race"

Definition 1. A **congestion game** consists of:

- A set E of congestible elements
- n players
- For each player i, a finite set of strategies, $S_i \subset \mathcal{P}(E)$.
- For each $e \in E$ and strategy list $(P_1, \ldots, P_n)$, a load $x_e = \#\{i : e \in P_i\}$.
- A delay function $d : E \times \mathbb{N} \to \mathbb{R}$

In such a game, we say that a player experiences a delay of

$$\text{delay}(i) = \sum_{e \in P_i} d(e, x_e)$$

Exercise 1. Define the **social welfare function**, ie, the total delay in the system, given a list of strategies.

Definition 2. A **Nash Equilibrium** is a list of strategies $(P_1, \ldots, P_n)$ such that no player can reduce their delay by changing strategies (provided all other players keep their strategies the same)

Exercise 2. We define the **potential function** $\Phi : S_1 \times \ldots \times S_n \to \mathbb{R}$ by:

$$\Phi(P_1, \ldots, P_n) = \sum_{e \in E} \sum_{k=1}^{x_e} d(e,k)$$

Exercise 3. Prove that if player i moves from strategy P_i to Q_i, then the change in this player's delay function is equal to the change in Φ.

Exercise 4. Prove that a strategy list that minimizes Φ is a Nash equilibrium

Exercise 5. (harder) Prove that the following algorithm terminates in a Nash equilibrium:

- Let player 1 choose a strategy while all the others watch
- Let player 2 choose a strategy after player 1 while all the others watch
- and so on until all the players have chosen a strategy

Exercise 6. Find an example where the social welfare function is not minimized at a Nash equilibrium.