PARTIAL DIFFERENTIAL EQUATIONS

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A funhouse where nothing is fun is just a house

– J-Lo

These notes were compiled by Assaf Bar-Natan and Ben Dees at the Canada/USA Mathcamp in the summer of 2019. For questions, corrections, or comments, send an email to assafbarnatan@gmail.com

Day 1: What is a PDE?

Time flies like an arrow
fruit flies like a banana

– Michael

1.1 Defining a PDE

A valuable text which we are referencing for these first few days of material is Evans's *Partial Differential Equations*.

Definition 1.1. A partial differential equation is any way of relating a function of several variables and arbitrary derivatives of that function.

For example, for a function of two variables $u(x, y)$, one example of a partial differential equation is

$$\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} = 1.$$

You will notice, from this definition of a partial differential equation, that there are a lot of them. We'll try not to worry too much about that—our focus will be on the PDEs that we can solve, not on the vast ocean that are totally unsolvable.

When solving PDEs, one often only cares about some subset of $\mathbb{R}^n$; e.g. if we're studying the weather in Oregon, maybe we decide to restrict our attention to, say, Oregon. However, the outside world still affects Oregon, so to kind of this into account we might want to solve this problem while setting certain constraints on the borders of Oregon. Then, we hope to figure out what will happen inside of Oregon just by knowing what happens on the boundary of Oregon.

The above process is called "choosing boundary conditions."

If you've done anything with differential equations in calculus, you might think that this is like choosing initial conditions for a differential equation, and is thus something you can just throw in later after solving the main problem. This is a nice intuition to think that you have.

It would be very, very hard for this intuition to be more wrong, but it might still be a comforting lie in these trying times.

1.2 Laplace's Equation

One important example of a PDE is **Poisson's equation**, which is

$$-\Delta u = f.$$

Here, we define $\Delta u = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} u$ in $\mathbb{R}^n$ (or on some open subset $U \subset \mathbb{R}^n$). In Poisson's equation, f is some given function on the (given) domain U, and we are trying to solve for the unknown function u.

The operator Δ is sometimes called the **Laplacian**; **Laplace's equation** is a special case of Poisson's equation where $f = 0$, i.e.

$$\Delta u = 0.$$

This has some important applications in physics and chemistry and so forth.

In the exercises, you'll see one way to approach this problem fairly directly, but it turns out that there are some very nice ways to characterize solutions to Poisson's equation that are rather abstract. So, let's get started with the abstract nonsense!

Suppose that we already had a solution to the PDE

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

for some open domain $U \subset \mathbb{R}^n$. (The assumption that the boundary data here is zero makes this a bit easier.)

Now, if we take any other (infinitely differentiable) function $g : U \to \mathbb{R}^n$ for which $g(x) = 0$ for $x \in \partial U$, we can compute the integral $\int_U gf$ (because f is some function that is given to us), which then lets us determine that:

$$\int_U gf = -\int_U g\Delta u = -\int_{\partial U} g(\nabla u \cdot \hat{n}) + \int_U \nabla g \cdot \nabla u = \int_U \nabla g \cdot \nabla u.$$

Here, we have integrated by parts and used the fact that $g = 0$ on the boundary of U.

Also, suppose that we had some other function v so that

$$\int_U gf = \int_U \nabla g \cdot \nabla v$$

for every infinitely differentiable g so that $g(x) = 0$ for $x \in \partial U$. We can quickly see that

$$0 = \int_U \nabla g \cdot \nabla(v - u)$$

for every such g, and then we see that u and v differ by a constant. If we then suppose that u, v agree on (say) the boundary of U, we see that they must be the same function. Moreover, integrating by parts the other way, we see that

$$\int_U gf = \int_U g(-\Delta v)$$

and from this, we conclude that $-\Delta v = f$ on U.

This leads us to the following remarkable fact: a function u solves the partial differential equation

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

if and only if for every infinitely differentiable function g so that $g(x) = 0$ on ∂U, it is the case that

$$\int_U gf = \int_U \nabla g \cdot \nabla u.$$

For reasons we will investigate more in days to come, the latter formulation (the one involving integrals) is called the **weak formulation of Poisson's equation**.

This has been a lot of telling you about a special case of one particular PDE, and one particular method that we might use to solve it. But that's not why you're in this course. You're in this course to solve PDEs, so at this point we've decided to just set you loose with a few PDEs to see how it goes for you! Good luck!

1.3 Exercises

You are in a pitch black room, with an unknown number of partial differential equations. You do not know whether the partial differential equations eat campers.[1] You do not know whether the partial differential equations have solutions.[2]

1. You crash into a partial differential equation as it wanders across the room. It looks like

$$\frac{\partial u}{\partial t} + b_1 \frac{\partial u}{\partial x_1} + \cdots + b_n \frac{\partial u}{\partial x_n} = 0$$

where $u(x, t)$ is defined for values of $x \in \mathbb{R}^n$ and values of $t \in [0, \infty)$. All of the b_i are constants.

You ask for the partial differential equation's name, and it tells you that it is called the **transport equation**. It hands you the initial condition $u(x, 0) = g(x)$, where $g(x)$ is a function from $\mathbb{R}^n \to \mathbb{R}$. It asks you to keep this initial condition secret and safe.

   (a) You decide to help solve the partial differential equation. It tells you that there are some curves in spacetime[3] that have a very special meaning to it, but it can't remember what they are. You try to figure out what these curves are.

   (b) Now that you know the curves, figure out the solution to this PDE in terms of its secret initial condition $g(x)$. (You did keep that safe, didn't you?)

[1] Some of them do.

[2] Some of them do. Probably. We hope.

[3] i.e. functions from $\mathbb{R}$ into $\mathbb{R}^n \times [0, \infty)$; the latter we think of as being "spacetime" because there's the "space" component ($\mathbb{R}^n$) and the "time" component, "$[0, \infty)$".

2. The transport equation introduces you to its cousin, who looks like

$$\frac{\partial u}{\partial t} + b_1 \frac{\partial u}{\partial x_1} + \cdots + b_n \frac{\partial u}{\partial x_n} = cu.$$

Again, this partial differential equation lives on the spacetime domain $\mathbb{R}^n \times [0, \infty)$, just like the transport equation does.

The cousin also hands you an initial condition, $u(x, 0) = h(x)$. You decide to help the cousin in much the same way that you helped the transport equation. You hope that they're pretty similar.

3. You meet a mysterious figure who shows you a **partial differential equation.**

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$

They give you the two functions $f : U \to \mathbb{R}$ and $g : \partial U \to \mathbb{R}$.

The mysterious figure offers you a **glorious bargain** for **power** and **glory**.

You ask it (them?) "Power and glory?"

They clarify (The figure clarifies?) **Well, more like energy, and the glory is kind of optional, if I'm being honest.**

You agree to the bargain, and the figure tells you of the **energy functional**:

$$I[w] = \int_U \left(\frac{1}{2} |\nabla w|^2 - wf \right)$$

where w is such that $w = g$ on ∂U. We call such functions w "admissible."

In return, they want to know things about the solution to their partial differential equation.

(a) First, as a warm-up, prove that if u, w are vectors, then

$$|u \cdot v| \le \frac{1}{2} \left(|u|^2 + |v|^2 \right).$$

(From this, it is possible to prove something called the Cauchy-Schwarz Inequality, which is important sometimes. The mysterious figure doesn't seem to be interested in Cauchy or Schwarz. Maybe you are, though.)

(b) You imagine that u is a solution to the mysterious figure's PDE. You then notice that $-\Delta u - f = 0$ on all of U, and consider

$$\int_U (-\Delta u - f)(u - w)$$

for any admissible function w.

You use integration by parts and the inequality from part (a) to show that if u is a solution to this PDE, and if w is any admissible function, then $I[u] \le I[w]$.

(c) On the other hand, you imagine that u is such that $I[u] \leq I[w]$ for every admissible function w. You decide to invent the calculus of variations.

To do this, you imagine that v is any infinitely differentiable function $v : U \to \mathbb{R}$, so that $v(x) = 0$ on ∂U. You notice that for any $\tau \in \mathbb{R}$, $u + \tau v$ is an admissible function.

You decide to define $i(\tau) = I[u + \tau v]$. You inspect the derivative of this function at $\tau = 0$.

(You remember, here, that integration by parts is your FRIEND in this pitch black room full of PDEs and mysterious strangers.)

(d) You conclude that u is a solution to the mysterious figure's PDE if and only if u minimizes $I[w]$.

4. You meet a mysterious figure who says: SOLVE POISSON'S EQUATION IN $\mathbb{R}^n$ THE HARD WAY. NO MORE ABSTRACT NONSENSE. NO MORE MESSING AROUND.

You realize that just solving Poisson's equation is hard and decide to solve Laplace's equation instead.

You realize that solving Laplace's equation is hard and decide to guess that there's some solution $u(\mathbf{x})$ that only depends on the magnitude of the vector $\mathbf{x} \in \mathbb{R}^n$.

That is, you guess that there's a solution of the form $u(\mathbf{x}) = v(r)$, where $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, and $v : \mathbb{R} \to \mathbb{R}$ is some function.

(a) You suppose that u is such a solution to Laplace's equation $\Delta u = 0$, and figure out the corresponding differential equation that v satisfies.

(b) You solve for v, up to choices of a few constants.

(c) You notice that, when you compute Δu, it's actually only zero everywhere but zero, and that it explodes at 0. You decide to write down $\Delta u = \delta$.

(d) You remember someone telling you that if you define $\delta : \mathbb{R}^n \to \mathbb{R}$ by $\delta(x) = 0$ when $x \neq 0$ and $\int_{\mathbb{R}^n} \delta = 1$, then for any function $f : \mathbb{R}^n \to \mathbb{R}$, it is the case that

$$\int_{\mathbb{R}^n} \delta(y - x) f(x) dx = f(y).$$

You ask yourself whether you should check this.

(e) You define

$$s(y) = \int_{\mathbb{R}^n} u(y - x) f(x) dx$$

and then compute $\Delta s(y)$. You see that $\Delta s(y) = f(y)$ and realize that you have solved Poisson's equation.

5. You find some equations that not only don't have solutions, but that don't even make sense. Maybe that means that they don't have solutions? This doesn't make sense.

   (a) You think about Laplace's equation $\Delta u = 0$ on $\mathbb{R}^2$, with the boundary condition

   $$u = 0, \frac{\partial u}{\partial x_2} = \frac{1}{n}\sin(nx_1) \quad \text{on } \{x_2 = 0\}$$

   You see that a solution is:

   $$u = \frac{1}{n^2}\sin(nx_1)\sinh(nx_2)$$

   This makes you feel vaguely strange. Could it be the fish in your shoes? Was it the potions you drank earlier? Was that cryptic puzzle a PDE, a poem or a curse? You feel a sense of impending doom. Why is this?

   What happens to u as $n \to \infty$? What do the initial conditions look like, and does the solution make sense? Can you make sense of your feeling of impending doom?

   (b) You make tea. After a while, you notice that your hot cup of tea is getting hotter, not colder. Is this OK?

   You have entered the realm of the **backwards heat equation**. Can you escape before your tea sucks all the heat out of the room? The PDE whose lair you have blundered into looks like

   $$\frac{\partial u}{\partial t} + \Delta u = 0.$$

   i. You simplify to the case where $n = 1$, that is, consider the partial differential equation

   $$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0.$$

   You imagine that the solution is simple, looking like $u(x,t) = a(x)b(t)$.
   Try to learn as much as you can about a and b before your tea boils you.

   ii. You eventually notice that $u(x,t) = \frac{1}{n}e^{n^2 t}\cos(nx)$ is one option for such a solution. As n gets very large, what happens to the maximum value of $u(x,0)$? What happens to the maximum value of $u(x,t)$?

   (c) You imagine that u solves:

   $$\Delta u - u + \|u\|^{2\sigma} u = 0$$

   You think that u is a real-valued function, of course.

   i. You realize that u satisfies

   $$\int_{\mathbb{R}^n} |\nabla u|^2 dx + \int_{\mathbb{R}^n} |u|^2 dx = \int_{\mathbb{R}^n} |u|^{2\sigma+2} dx$$

ii. (harder) (Pohozaev's identity)

$$(n-2)\int_{\mathbb{R}^n} |\nabla u|^2 dx + n\int_{\mathbb{R}^n} |u|^2 dx = \frac{n}{\sigma+1}\int_{\mathbb{R}^n} |u|^{2\sigma+2} dx$$

for $n \geq 3$

iii. You realize that there are no solutions u such that $\int_{\mathbb{R}^n}(|u|^2 + |\nabla u|^2)$ is finite if the nonlinearity (i.e. the exponent σ) satisfies $\sigma > \frac{2}{n-2}$.

Day 2: Where is a PDE?

How can mirrors be real if our i's j's and k's aren't real?

– Susan

Today, we want to learn why functional analysis is amazing. To do this, we integrate by parts.

Definition 2.1. For a function $u : U \to \mathbb{R}$, where $U \subset \mathbb{R}^n$, we say that another function v is the **weak derivative of u with respect to x_i** if

$$\int_U \frac{\partial \phi}{\partial x_i} u = -\int_U \phi v$$

for every infinitely differentiable function ϕ so that $\phi = 0$ on the boundary.

Similarly, if u has weak derivatives $v_1, \ldots, v_n$ with respect to $x_1, \ldots, x_n$, we call the vector of functions $\mathbf{v} = (v_1, \ldots, v_n)$ the **weak gradient** of u, and we often will write $|\mathbf{v}| = \sqrt{|v_1|^2 + \cdots + |v_n|^2}$.

Definition 2.2. Denote by $H^1(U)$ the space of functions $u : U \to \mathbb{R}$ that have a weak gradient $\mathbf{u} : U \to \mathbb{R}^n$, and are such that

$$\int_U \left(|u|^2 + |\mathbf{u}|^2\right) < \infty.$$

This space is a kind of space that is sometimes called a **Sobolev space**.

We note, at this point, that $H^1(U)$ is an inner product space, where we define the inner product of two elements u, v (with weak derivatives $\mathbf{u}, \mathbf{v}$, respectively) to be:

$$\langle u, v \rangle := \int_U \big(uv + \mathbf{u} \cdot \mathbf{w}\big).$$

This inner product induces a norm on the space $H^1(U)$, to be specific, the norm of this space is $\|u\| = \big(\int_U |u|^2 + |\mathbf{u}|^2\big)^{1/2}$.

and this norm makes $H^1(U)$ into a **Hilbert space**, which is a rather technical definition that we will not work with explicitly here.[4] (See if you can find Viv's notes from last week if you want more information on this).

The really, really important fact about Hilbert spaces that we do actually need is the

[4] There's one story that Hilbert, on hearing the phrase "Hilbert space" for the first time, asked the question "Pardon me, but could you please explain exactly what a 'Hilbert space' is?"

(Riesz Representation Theorem) If H is a Hilbert space where we write the inner product of two elements x, y as $\langle x, y\rangle$, and if $f : H \to \mathbb{R}$ is a continuous linear function, then there is a unique $x \in H$ so that

$$f(w) = \langle x, w\rangle$$

for every $w \in H$.

To see this, first note that if f is zero, this is clearly true (take $x = 0$). Otherwise, the kernel of f is a proper closed subspace of H, so there is some $y \in H$ perpendicular to this kernel; we can choose this y so that $\|y\| = 1$.

Then, if $w \in H$, we see that

$$f\Big(w - \frac{f(w)}{f(y)}y\Big) = f(w) - \frac{f(w)}{f(y)}f(y) = 0$$

so that $w - \frac{f(w)}{f(y)}y \in \ker(f)$. Now, since y is perpendicular to this kernel, we see that letting $x = f(y)y$, we have that

$$\begin{aligned} 0 = \Big\langle x, w - \frac{f(w)}{f(y)}y\Big\rangle &= \langle x, w\rangle - \Big\langle f(y)y, \frac{f(w)}{f(y)}y\Big\rangle \\ &= \langle x, w\rangle - f(w)\langle y, y\rangle \end{aligned}$$

and we conclude that $\langle x, w\rangle = f(w)$ for any $w \in H$, as we wished to show.

We also need a fact about Sobolev spaces, which we will state without proof:

(Poincaré's Inequality) Suppose that $u \in H^1(U)$ is a function that vanishes on the boundary of U, with weak derivative $\mathbf{u}$. Assume that U is bounded. Then it is the case that

$$\int_U |u|^2 \leq C \int_U |\mathbf{u}|^2$$

where the constant C does *not* depend on the choice of function: it only depends on the domain U. In particular, we can bound the full $H^1(U)$ norm by $(C+1)\int_U |\mathbf{u}|^2$.

Now, we remember the way we had of solving Poisson's equation yesterday, and we decide to generalize this.

Suppose that L is a differential operator of the form

$$Lu = -\Big[\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\Big(a_{ij}(x)\frac{\partial u}{\partial x_i}\Big)\Big]$$

where $a_{ij}(x) : U \to \mathbb{R}$ are some functions that are all bounded on U. (To recover the Laplacian, set $a_{ij}(x)$ to be 1 if $i = j$ and 0 otherwise).[5]

We say that L is a **uniformly elliptic operator** if for every $x \in U$, $a_{ij}(x) = a_{ji}(x)$, and if there is a constant $\theta > 0$ so that for every $x \in U$, for every vector $\xi = (\xi_1, \ldots, \xi_n)$, it is the case that

$$\theta|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j.$$

[5] In more generality, we can allow there to be some terms involving first-order derivatives of u and the function u itself, but this complicates the computations greatly.

For a uniformly elliptic operator L, you will see that there are solutions to the PDE

$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

We first recall the weak formulation of Poisson's equation, which inspires us to concoct a weak formulation of our new PDE.

In particular, multiplying everything by some $\varphi \in H^1(U)$ that vanishes on ∂U and integrating by parts, we say that u is a **weak solution** to the above partial differential equation if

$$\int_U \Big[\sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \Big] = \int_U f\varphi$$

for every such φ.

2.4 Exercises

1. From the definition of a weak derivative, argue that if a function u has two weak derivatives on U (with respect to x_i) v, w, then v and w are equal. (Technically, these functions need only be equal "almost everywhere," in a measure theory sense)

2. Prove that if u has a weak derivative with respect to x_i on the domain U, and if η is an infinitely differentiable function equal to 0 on ∂U, then the function ηu also has a weak derivative with respect to x_i, and we can compute this derivative by the product rule.[6]

3. Prove that if we take $U = B(0,1)$, the open ball of radius 1 about the origin of $\mathbb{R}^n$, and consider the function $u(x) = \frac{1}{|x|^\alpha}$.

   For which n, α is $u \in H^1(U)$?

4. Use the Riesz Representation Theorem to prove that the partial differential equation on this page, above, has a weak solution as long as $\int_U |f|^2 < \infty$.

   (a) First, show that if we define

   $$B(u,v) = \int_U \Big[\sum_{i,j=1}^n a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \Big]$$

   then $B(u,v)$ defines an inner product on $H^1(U)$.

[6] In fact, other rules with derivatives can be stated and remain valid for weak derivatives. These include the linearity of differentiation (which is perhaps easier than these two exercises), the chain rule (which is rather tedious), and (most interestingly) the relation between differentiation and the convolution. This last fact is particularly useful when one wishes to approximate a function in $H^1(U)$ by smooth functions. With a bit of work, one can even state and prove something like the Fundamental Theorem of Calculus, in the case of a one-dimensional domain.

(Recall that B is an inner product if $B(u,v) = B(v,u)$, if $B(u, kv+w) = kB(u,v)+B(u,w)$, and if $B(u,u) > 0$ for any $u \neq 0$. For measure theory fans, "$u \neq 0$" should be interpreted as "it is not the case that $u = 0$ almost everywhere." As a hint, you can use the uniform ellipticity, and the Poincaré Inequality to show something a little bit stronger than the third condition)

(b) Show that the function defined by $\varphi \mapsto \int_U f\varphi$ is linear and continuous, to satisfy the conditions of the Riesz Representation Theorem.

(Hint: bound the integral using Cauchy-Schwarz, and then use the Poincaré Inequality and the assumption of uniform ellipticity to bound $\int_U |v|^2 \leq CB(v,v)$. At the end, you should be able to show continuity fairly directly)

(c) Apply the Riesz Representation Theorem to conclude that the PDE has a weak solution, as desired.

5. In a similar vein to the previous problem, suppose that we define an operator

$$Lu := -\Big[\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\Big(a_{ij}(x)\frac{\partial u}{\partial x_i}\Big)\Big] + c(x)u$$

where the coefficients $a_{ij}(x)$ are still symmetric and uniformly elliptic, and where $c(x) \geq 0$.

Argue that there is still a weak solution to this partial differential equation. What part of this argument will go wrong if we remove the assumption that $c(x) \geq 0$?

Exercises should be: 1) prove that weak derivatives do exactly what we expect 2) Prove that the function from Evans is in W1,p 3)

Day 3: When is a PDE?

Early bird gets the worm,
late worm misses the bird

– Marisa

4 Purposefully Dissecting Equations

Today we will beat up, abuse, and dissect solutions of PDEs just by using the equations they satisfy. We will see that a lot of rich structure can be gained without even approaching a solution to the PDE. The first thing we will examine is the **heat equation**. We also introduce new notation! $\frac{\partial u}{\partial z} = u_z$ for any variable!

4.1 The Maximum Principle

You walk confidently into the dungeon, but miss a step, and set off a trap. An iron rod starts rolling towards you, melting everything in its path. You think long and hard about what to do, but can only recall your undergraduate physics training, and, as you roll your concentration check, the only thing you brain can muster up is the heat equation.

$$u_t = \Delta u$$
$$u(x,0) = g(x)$$

In the back of your head, you panic. The log is rolling towards you faster and faster. You know that this equation governs the heat distribution on the iron rod as a function of time. But where should you let the rod hit you to minimize damage?

Definition 4.1. We define $U_T = (0,l) \times (0,T)$ and

$$\Gamma_T = \bar{U}_T \setminus U_T = \{0\} \times [0,T] \cup \{l\} \times [0,T] \cup [0,l] \times \{0\}$$

You have a feeling that the maximum heat will be on the edges of the rod, or at its start point. You want to prove this formally.

Theorem 4.2. *Let $U = (0,l)$ be any interval. Assume that $u \in C^2(U_T)$ is a solution to the heat equation with initial condition $g(x)$. Then*

$$\max_{U_T \cup \Gamma_T} u(x,t) = \max_{\Gamma_T} u(x,t)$$

Throughout these exercises, we let $M = \max_{\Gamma_T} u$.

Exercise 4.1. Define $v(x,t) = u(x,t) + \varepsilon x^2$, and prove that $v_t - v_{xx} < 0$

Exercise 4.2. Prove that v cannot have a local maximum on U_T

Exercise 4.3. Prove that on U_T, $v(x,t) \leq M + \varepsilon l^2$

Exercise 4.4. Prove that for any $\varepsilon > 0$, and on U_T, $u(x,t) \leq M + \varepsilon(l^2 - x^2)$, and conclude that $u(x,t) \leq M$

You choose to stand in the center of the rod, and it deals (rolls dice) 20 hitpoints of damage. You are burnt, but can continue the dungeon.

4.2 Decay

The rod rolls away into the darkness. You muse about your fortune, and try to think about what would have happened if the rod was infinite. What would happen to all the heat? A cryptic rune tells you to consider the L^2 norm, because it's the best norm, and everything is beautiful.

Definition 4.3. Let $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function of x and t. We define the L^2-norm of u by:

$$\|u\|_{L^2}^2 = \int_{\mathbb{R}} u(t,x)^2 dx$$

Exercise 4.5. Let u be a solution to the heat equation $u_{xx} - u_t = 0$. Prove that $\|u\|_{L^2}$ is non-increasing.
Hint: Show that $\|u\|_{L^2}^2$ *is non-increasing.*

In other words, the quantity $\|u\|_{L^2}$ is not conserved as time goes forward. This spooks you. You take a deep breath. Everything will be ok. The dungeon crawl continues.

5 Symmetry

You wander into a dark narrow hallway full of water. Suddenly, torches on both sides light up, and reveal the horrors on the other end. An army of undead goblins. They lumber towards you in a grotesque gait, dragging half-limbs, each goblin different from the next. These aren't just ordinary differential goblins, these are partial differential goblins.

An idea comes into your head. You want to throw a bomb into the water between you and the goblins. You hope that the created burst will wash them away with the resultant wave which satisfies the **wave equation**:

$$\left(\frac{\partial}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u = 0$$

with initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = 0$.

Exercise 5.1. Use difference of squares to factor the above equation

Exercise 5.2. Solve each of the factors, and conclude that you, too, will be swept away.

6 Conservation Laws

Realizing this, you start running back to the room with the iron rod, and trigger a magical trap. You start to dizzify, and the world around you spins. Sickness overwhelmes you. This feeling is not unpleasant. Is it? What are you? Where is time? Where am I here? A loop of questions that lasts forever in your mind. You escape the eyes. The i's are back. Confined to shrink in melodic hallucination. A sharp metallic taste in your tongue, and suddenly clarity. You have entered the

QUANTUM REALM

Oops! This realm is governed by the **Schrödinger Equation**:

$$\frac{u_t}{i} = \Delta u$$
$$u(x, 0) = \varphi(x)$$

Fear. Strange. Confusion. Yet somehow, conservation. Something is right. At least integration by parts still works.

Exercise 6.1. Prove that if $u : \mathbb{R}^n\mathbb{R} \to \mathbb{C}$ is a solution to the Schrödinger equation, then the following quantities are conserved (ie, their time derivative is 0):

- The mass[7]: $\|u\|_{L^2}^2 = \int_{\mathbb{R}^n} |u|^2 dx$
- The momentum:$I_j = \text{Im}\left(\int_{\mathbb{R}^n}(u_j\bar{u})dx\right)$
- The energy[8]: $\frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx$

(you will need a lot of paper for this)

Exercise 6.2. Show that if $u(x, t)$ satisfies the Schrödinger equation, then so does:

- $u(x, t + t_0)$ for some constant t_0
- $u(x - x_0, t)$ for some constant x_0
- $e^{i\theta}u(x, t)$ for some θ
- $u(Ax, t)$ where A is any rotation of $\mathbb{R}^n$

By Noether's theorem these symmetries give rise to other conserved quantities. You decide that the dungeon has tortured you enough today. You return to the local town to restock on supplies. These other conserved quantities will have to wait until you are higher level.

[7] Hint: multiply the equation by $\bar{u}$ and you should know the rest.
[8] Multiply the equation by $\bar{u}_t$

Day 4: How is a PDE?

Assaf, you're strange, stop asking me to say things for your dumb classes. I wouldn't use the word dumb. This quote is a lie.

– Lizka

5 Exponents

No nonsense. No games. Just solving PDEs. Let's go.

Exercise 5.1. Use the Taylor series of e^x to compute:

$$(e^{t\frac{d}{dx}} f)(x)$$

Exercise 5.2. What PDE does $g(x,t) = e^{t\frac{d}{dx}} f(x)$ solve? For what initial condition?

Exercise 5.3. A **linear differential operator** L is a linear combination of spacialderivatives (ie, derivatives with respect to x-co-ordinates). Write a solution to

$$u_t = Lu$$
$$u(x,0) = f(x)$$

in the same way as in the last two problems.

Exercise 5.4. You might be bothered by convergence of the above series. You're right! There are some convergence problems. Use an exercise from the first day to show that our method does not work for all times and initial conditions when solving the reverse heat equation: $u_t = -u_{xx}$,

Fact 5.1. *Let $U \subset \mathbb{R}^n$ be some bounded domain, then there exists a sequence of positive real numbers $\lambda_1, \lambda_2, \ldots$ and functions $u_1, u_2, \ldots$ such that*

$$\begin{aligned} \Delta u_n &= \lambda_n u_n \\ u_n &= 0 \quad \text{on } \partial U \end{aligned}$$

Moreover, these functions give an orthonormal basis for $L^2(U)$ with the inner product $g, h = \int_U ghdx$.

Exercise 5.5. Let $g(x)$ be some arbitrary function in $L^2(U)$. Find a solution to $\Delta u = u_t$ with initial condition $u(x, 0) = g(x)$ by expanding $g(x)$ in the basis $u_1, u_2, \ldots$

6 Fourier Methods in PDEs

Definition 6.1. The **Fourier transform** of a function $f : \mathbb{R} \to \mathbb{R}$ is defined to be:

$$F(f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx$$

Exercise 6.1. Prove that $F(f \cdot g) = f * g$, where

$$f * g(x) = \int_{-\infty}^{\infty} f(x) g(y - x) dx$$

Conclude that $f * g = g * f$

Fact 6.2. *The Fourier transform has an inverse, given by*

$$F^{-1}(g) = \sqrt{2\pi} \int_{-\infty}^{\infty} g(\xi) e^{i\xi x} d\xi$$

It is a waste of time to try to prove that this is indeed the inverse, but you should immediately notice that $F^{-1}(fg) = f * g$.

Exercise 6.2. Prove that

$$F(f_x)(\xi) = i\xi F(f)(\xi)$$

Exercise 6.3. Show that if u solves $u_t = u_{xx}$ with $u(x, 0) = g(x)$, then

$$\frac{\partial}{\partial t} F(u) = -\xi^2 F(u)$$

What should be the initial condition?

Exercise 6.4. Solve the above ODE for $F(u)$.

Fact 6.3. *The Fourier transform of* $e^{-x^2/2}$ *is* $\frac{1}{\sqrt{2\pi}} e^{-\xi^2/2}$.

Exercise 6.5. Perform a coordinate transformation on the above fact to find a function $h(x, t)$ for which $u(x, t) = g(x) * h(x, t)$