Welcome to MAT135 LEC0501 (Assaf)

As you come in, ask your neighbours how their break was.

S10.1 – Using Polynomials in Clever Ways

Assaf Bar-Natan

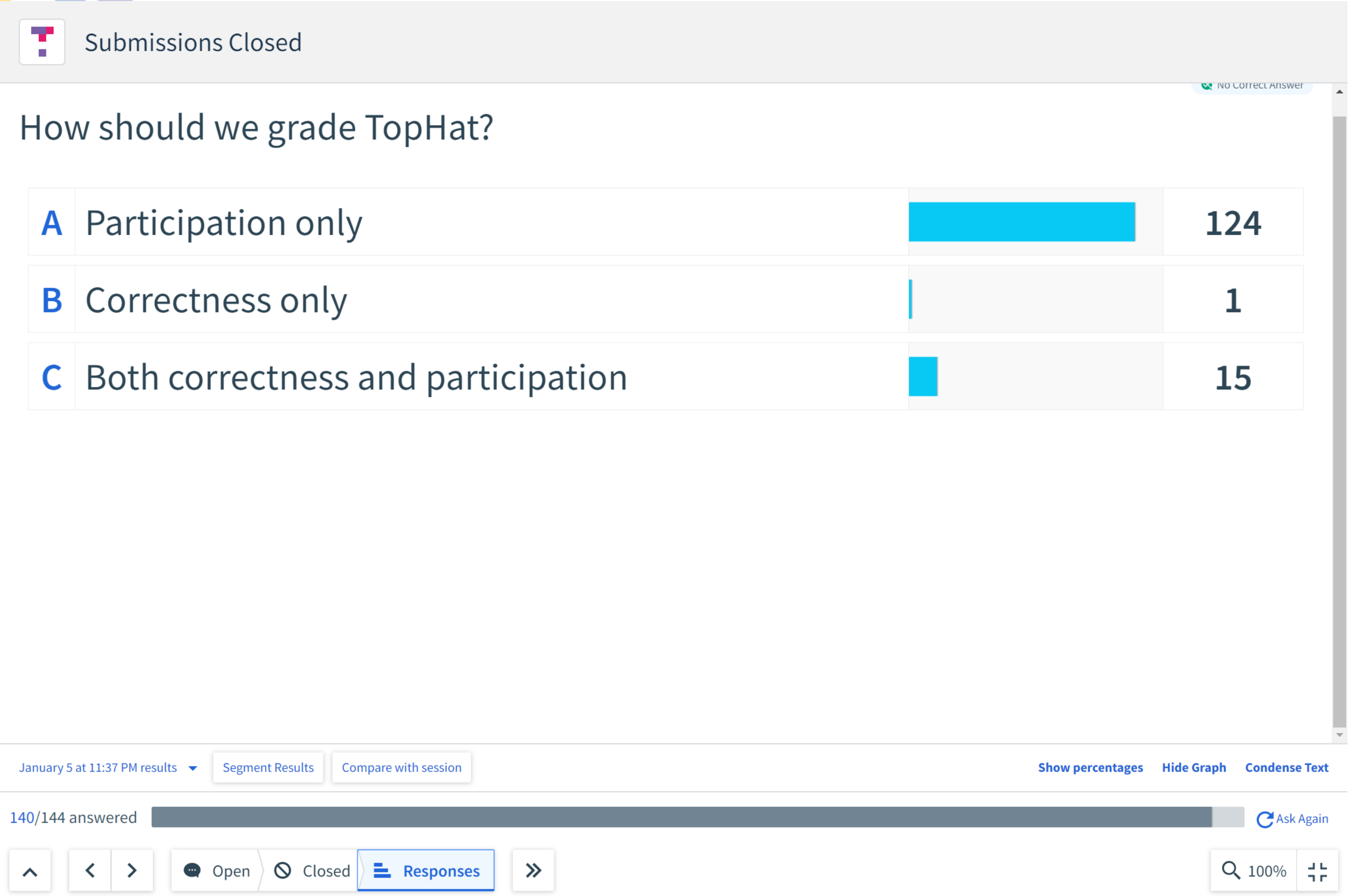
“ So this is me swallowing my pride
Standing in front of you saying I’m sorry for that night
And I go back to December all the time”

–“Back to December”, Taylor Swift

Jan. 6, 2020

Announcements

- Read the syllabus (it's on Quercus).
- My office hours: Mondays at 13:00, Wednesdays at 15:00, location: probably PG104
- Today: extra office hour after this class in PG104
- Download TopHat and purchase a subscription to it.



How should we grade TopHat?

A	Participation only	124
B	Correctness only	1
C	Both correctness and participation	15

140/144 answered

Math and Active Learning

Spend ten seconds to get into groups of three.

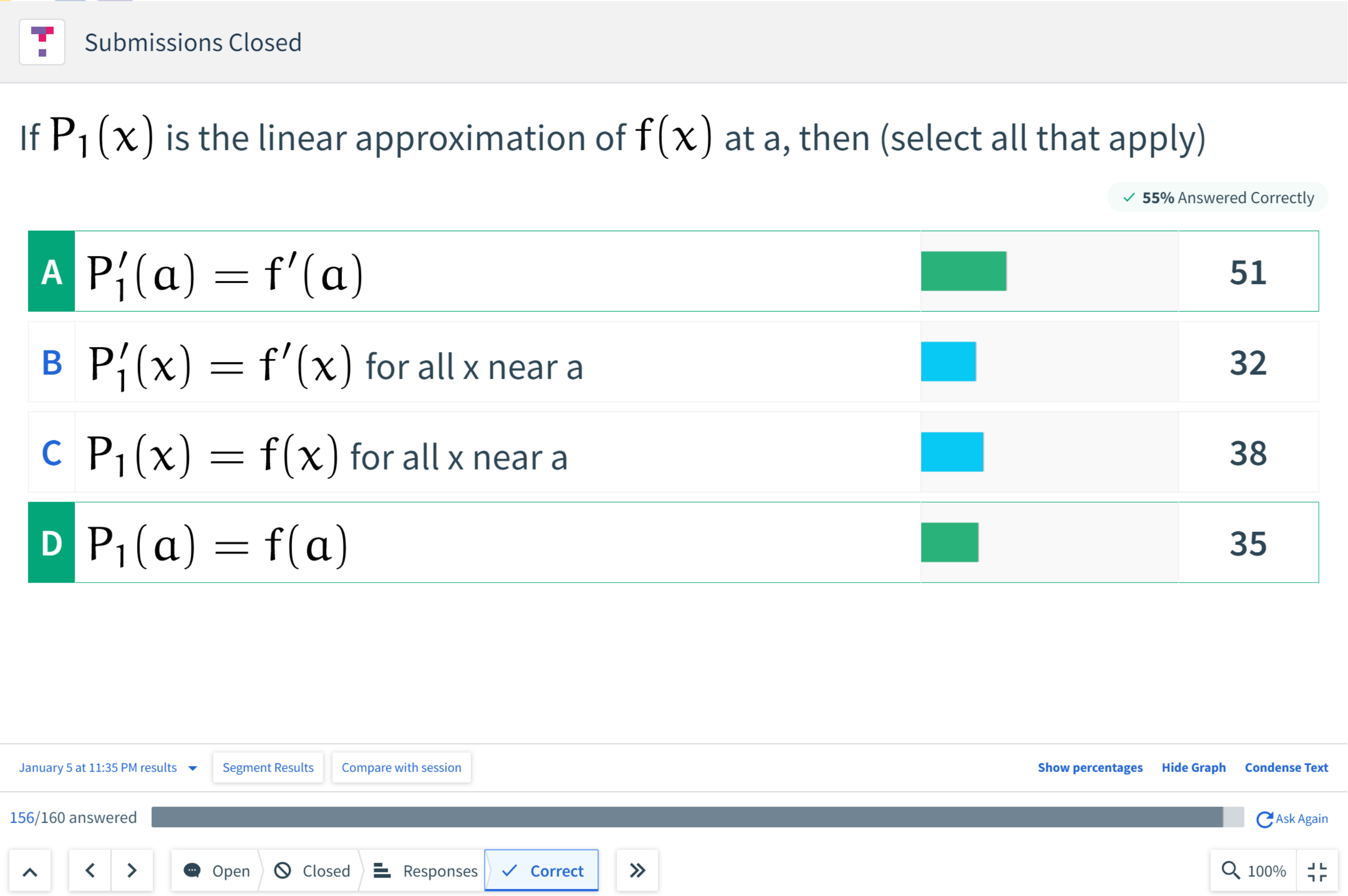
Math and Active Learning

Spend ten seconds to get into groups of three.
In your groups:

- Share names and contact information.
- Write down the main overarching **theme** of MAT135.

The Theme of MAT135

The main theme of MAT135 is that of the linear approximation. A “nice” looking function can be approximated by a line using the derivative



Submissions Closed

If $P_1(x)$ is the linear approximation of $f(x)$ at a, then (select all that apply)

✓ **55%** Answered Correctly

A	$P_1'(a) = f'(a)$	51
B	$P_1'(x) = f'(x)$ for all x near a	32
C	$P_1(x) = f(x)$ for all x near a	38
D	$P_1(a) = f(a)$	35

156/160 answered

Extending the Linear Approximation

What if instead of just requiring $f(a) = P(a)$ and $f'(a) = P'(a)$, we also required...

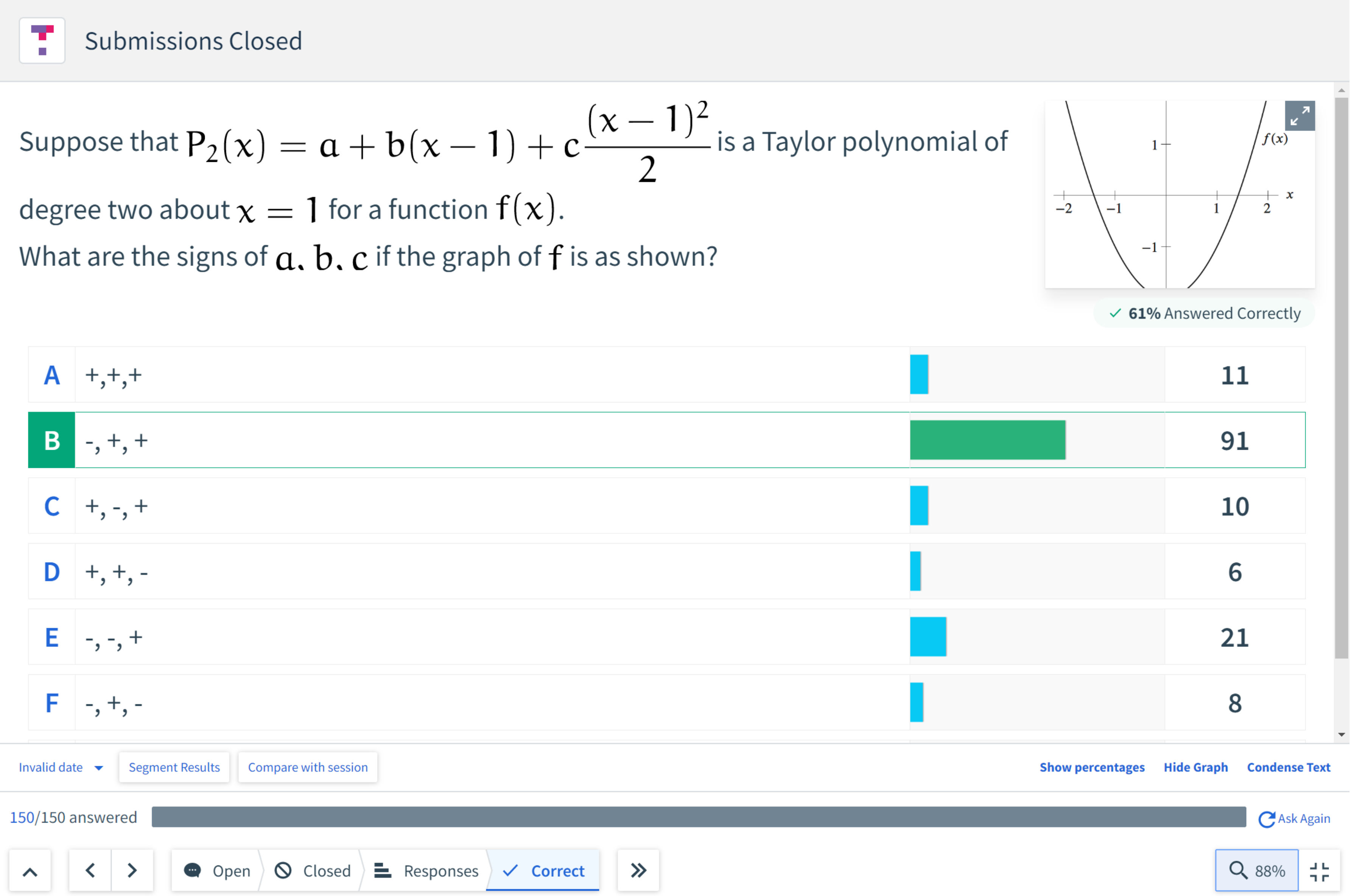
Extending the Linear Approximation

What if instead of just requiring $f(a) = P(a)$ and $f'(a) = P'(a)$, we also required...

$$f''(a) = P''(a)$$
$$f'''(a) = P'''(a)$$
$$\vdots$$

Takeaway

The main idea of approximating a function f around a point a using polynomials is to make the derivatives of f equal to the derivatives of the polynomial at a.



Submissions Closed

Suppose that $P_2(x) = a + b(x-1) + c\dfrac{(x-1)^2}{2}$ is a Taylor polynomial of degree two about $x = 1$ for a function $f(x)$.

What are the signs of a, b, c if the graph of f is as shown?

✓ **61%** Answered Correctly

A	+,+,+	11
B	-, +, +	91
C	+, -, +	10
D	+, +, -	6
E	-, -, +	21
F	-, +, -	8

150/150 answered

Additional Resources for This Chapter

- A good video by 3Blue1Brown
- Tutorials!
- The Math Learning Center (PG101)
- "Test Your Understanding" questions at the end of each chapter.
- Your peers! (This one is the best one)

Rainbow the Cat

Rainbow the kitten wants to compute the second degree polynomial approximation of $\cos(2x)$ around $x = 0$. He write:

$$\cos(2x) \approx 1 + (\underline{\quad}) \cdot x + (\underline{\quad}) \cdot x^2$$

but is unsure how to fill in these blanks.

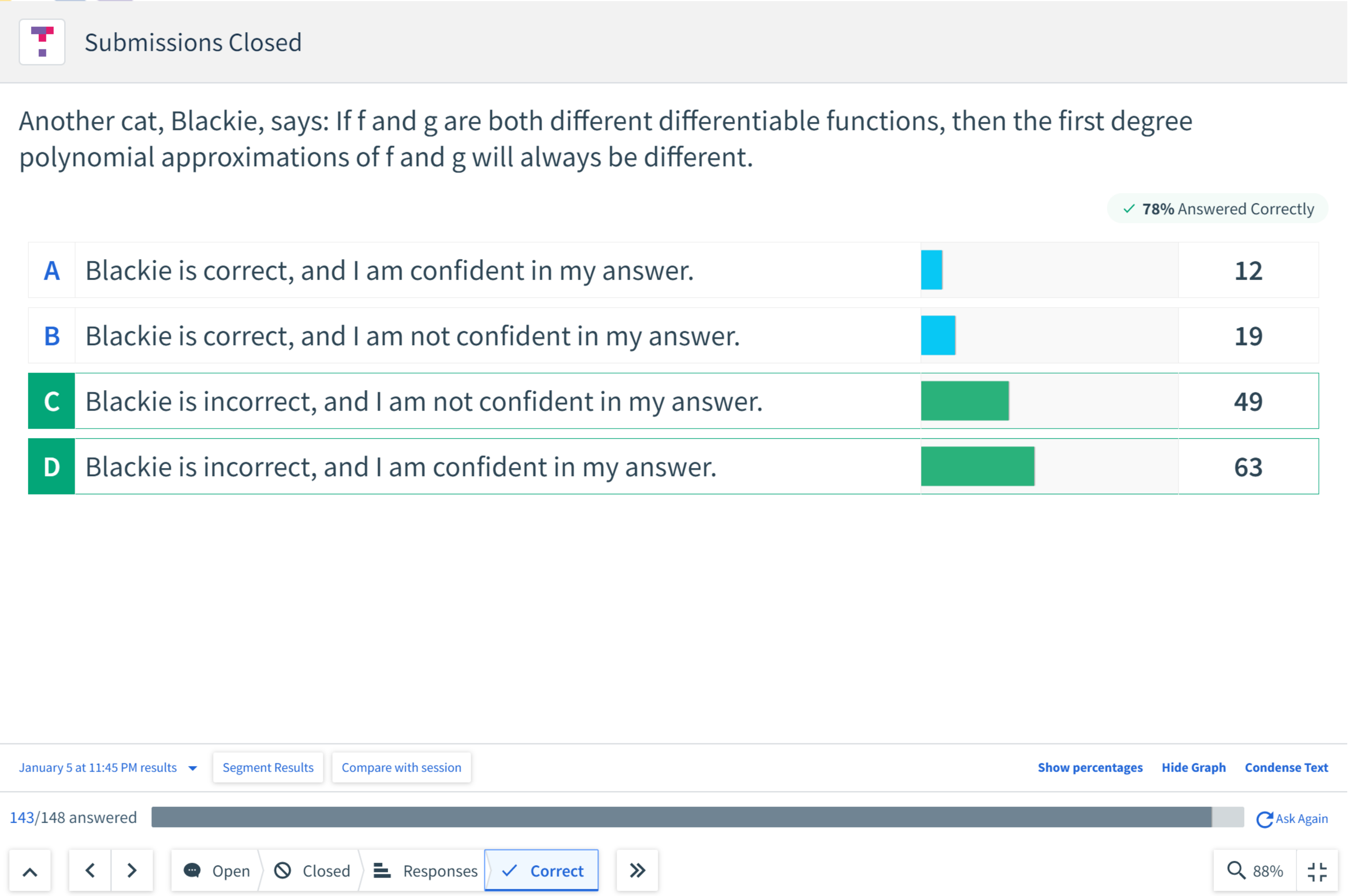
Rainbow the Cat

Rainbow the kitten wants to compute the second degree polynomial approximation of $\cos(2x)$ around $x = 0$. He write:

$$\cos(2x) \approx 1 + (\underline{\qquad}) \cdot x + (\underline{\qquad}) \cdot x^2$$

but is unsure how to fill in these blanks.

In your groups, fill in these blanks to give the second degree polynomial approximation of $\cos(2x)$ around $x = 0$.



Submissions Closed

Another cat, Blackie, says: If f and g are both different differentiable functions, then the first degree polynomial approximations of f and g will always be different.

✓ **78%** Answered Correctly

A	Blackie is correct, and I am confident in my answer.	12
B	Blackie is correct, and I am not confident in my answer.	19
C	Blackie is incorrect, and I am not confident in my answer.	49
D	Blackie is incorrect, and I am confident in my answer.	63

143/148 answered

Plans for the Future

For next time:

WeBWork 5.1-5.2 (worth marks!) and read sections 5.1&5.2

Things for you to check out:

- Course website: q.utoronto.ca
- Guide to Technology (on main website)
- Office hours calendar!
- Get a group together, order pizza, and read the syllabus!

Welcome to MAT135 LEC0501 (Assaf)

As you come in, introduce yourself to someone you haven't met yet.

S5.1&5.2 – Riemann Sums, Erors, and Areas

Assaf Bar-Natan

“ In the morning I’d awake
And I couldn’t remember
What is love and what is hate
The calculations error ”

–“ In The Morning of the Magicians ”, The Flaming Lips

Jan. 8, 2020

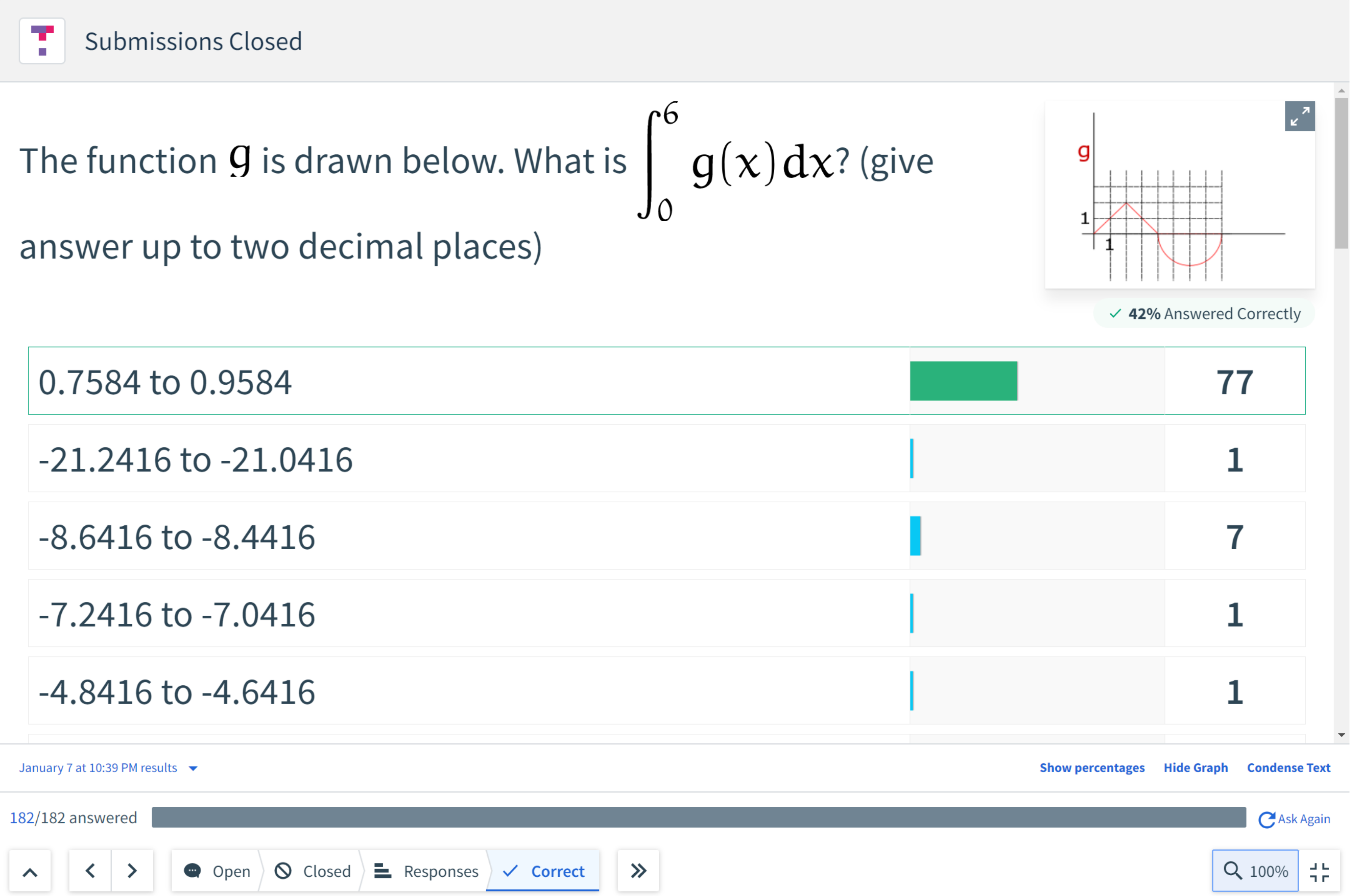
Announcements

- Read the syllabus (it's on Quercus).
- WeBWork is due the night before class
- We do not answer e-mails sent via WeBWork
- TopHat is graded by participation only. If it becomes meaningless, this will change!

Integrals and Areas

In your groups, write a sentence explaining the geometric interpretation of the expression:

$$\int_a^b f(x)dx$$



Submissions Closed

The function g is drawn below. What is $\int_0^6 g(x)\,dx$? (give answer up to two decimal places)

✓ **42%** Answered Correctly

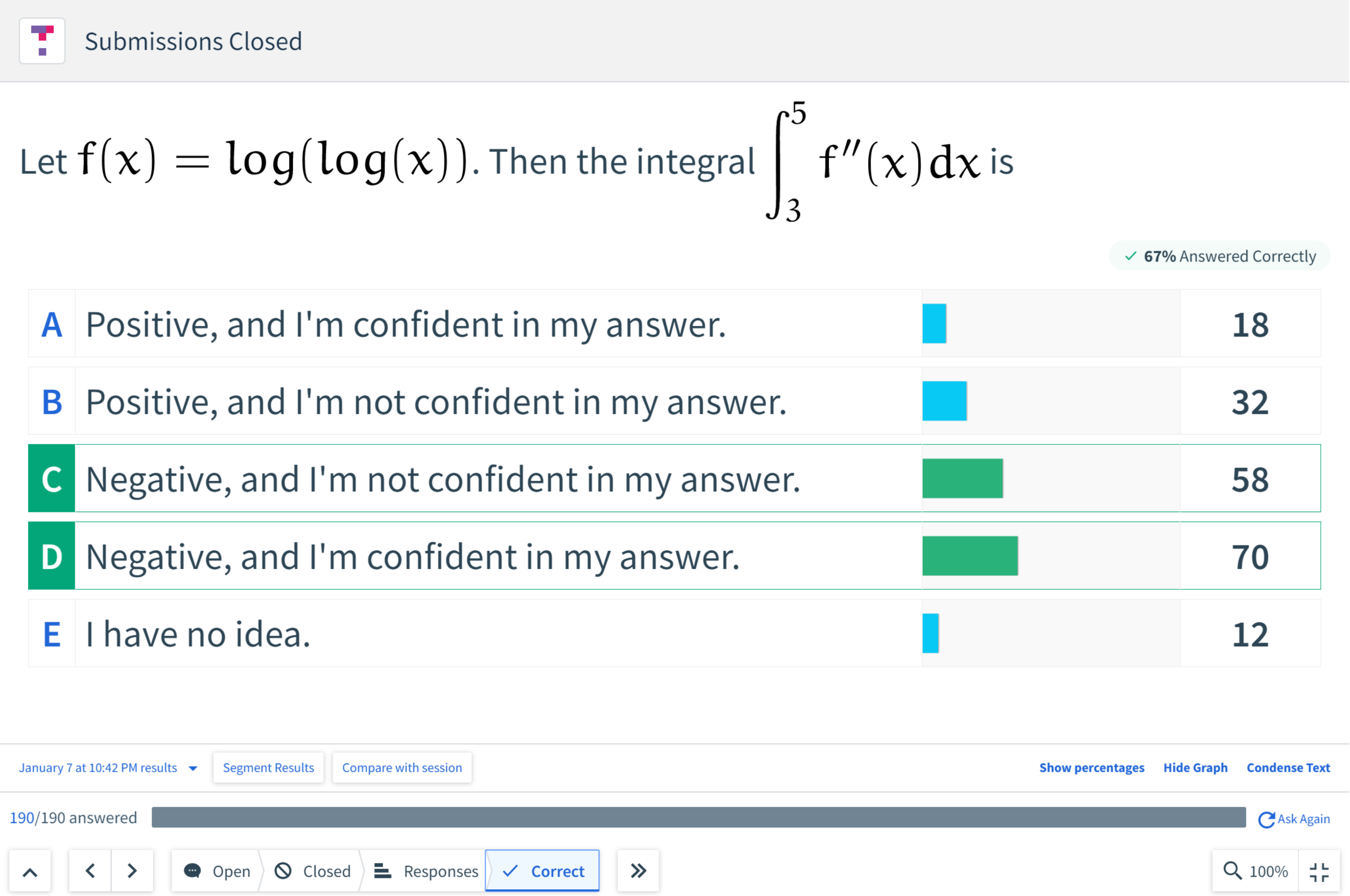
0.7584 to 0.9584	77
-21.2416 to -21.0416	1
-8.6416 to -8.4416	7
-7.2416 to -7.0416	1
-4.8416 to -4.6416	1

January 7 at 10:39 PM results

182/182 answered

Takeaway

The integral of a function between a and b is the signed area between the function and the x-axis.



Submissions Closed

Let $f(x) = \log(\log(x))$. Then the integral $\int_3^5 f''(x)\,dx$ is

✓ **67%** Answered Correctly

A	Positive, and I'm confident in my answer.	18
B	Positive, and I'm not confident in my answer.	32
C	Negative, and I'm not confident in my answer.	58
D	Negative, and I'm confident in my answer.	70
E	I have no idea.	12

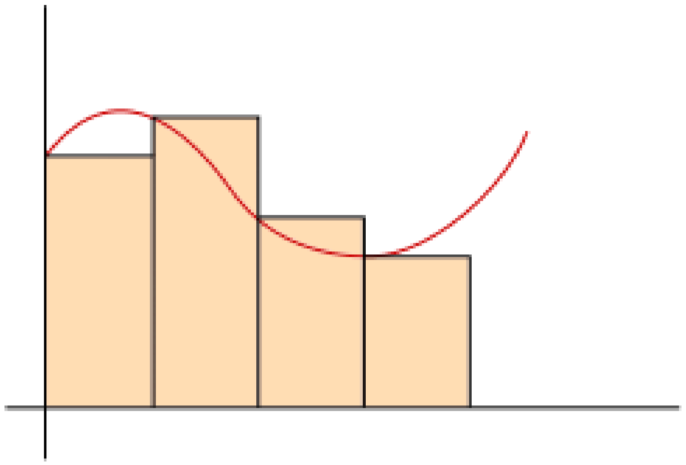
190/190 answered

Takeaway

The fundamental theorem can allow us to compute hard integrals in an instant. We just need to identify them as derivatives!

Computing Integrals – An Idea

- Draw the function
- Divide the interval
- Pick left- or right-rectangles
- Add up areas



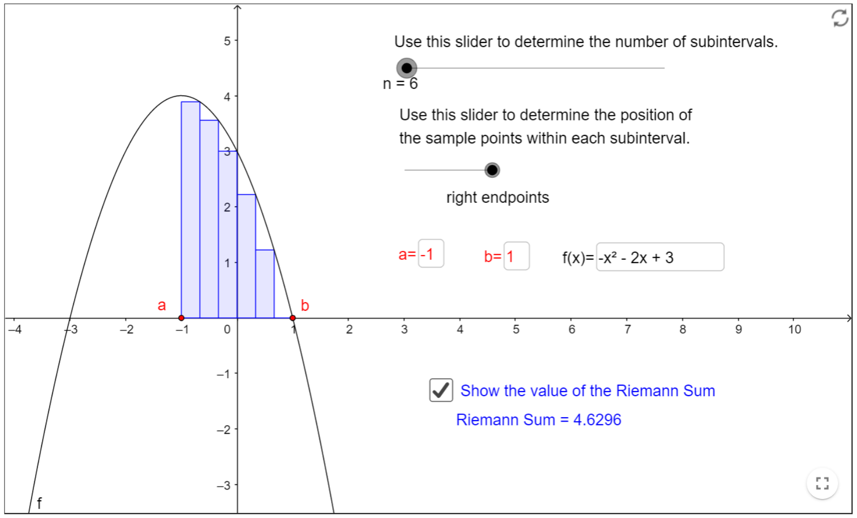
How does this work in practice?

Playing with Geogebra

In groups, spend five minutes playing around with the applet:

`https://www.geogebra.org/m/xJsZTG2i`

Playing with Geogebra



Use this slider to determine the number of subintervals.

n = 6

Use this slider to determine the position of the sample points within each subinterval.

right endpoints

a= -1 b= 1 f(x)= $-x^2 - 2x + 3$

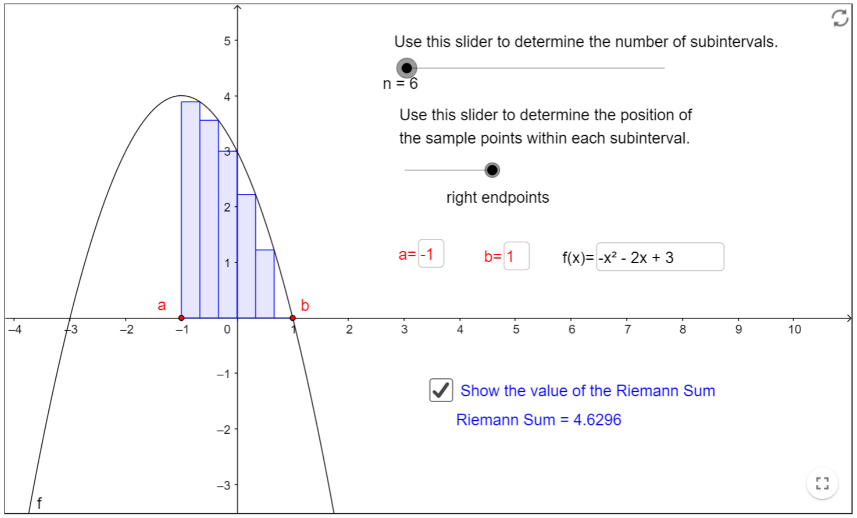
☑ Show the value of the Riemann Sum

Riemann Sum = 4.6296

For $n = 6$, the right Riemann sum is ($\Delta t = \frac{1}{3}$):

$$\Delta t(f(-\frac{2}{3}) + f(-\frac{1}{3}) + f(0) + f(\frac{1}{3}) + f(\frac{2}{3}) + f(1))$$

Playing with Geogebra



Use this slider to determine the number of subintervals.

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right endpoints

a= -1 b= 1 f(x)= -x² - 2x + 3

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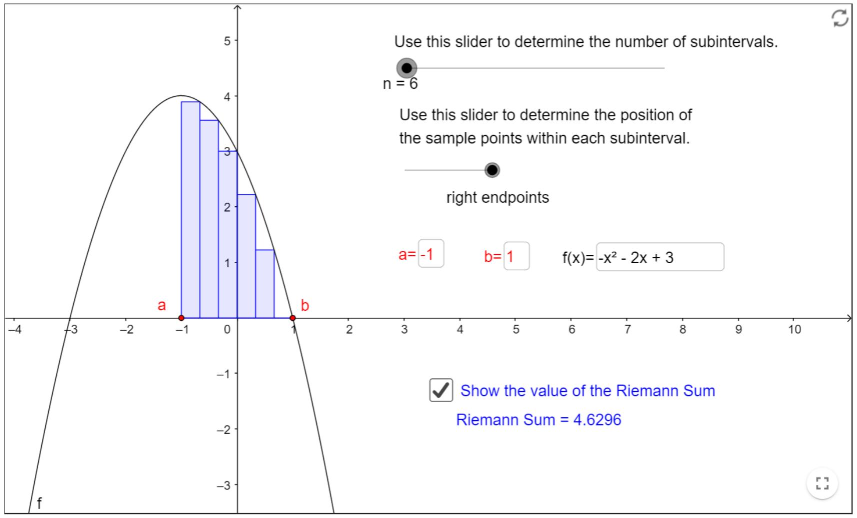
$$\Delta t(f(-\frac{2}{3}) + f(-\frac{1}{3}) + f(0) + f(\frac{1}{3}) + f(\frac{2}{3}) + f(1))$$

What is the left Riemann sum?

Playing with Geogebra



Use this slider to determine the number of subintervals.

$n = 6$

Use this slider to determine the position of the sample points within each subinterval.

right endpoints

$a = -1$ $b = 1$ $f(x) = -x^2 - 2x + 3$

☑ Show the value of the Riemann Sum

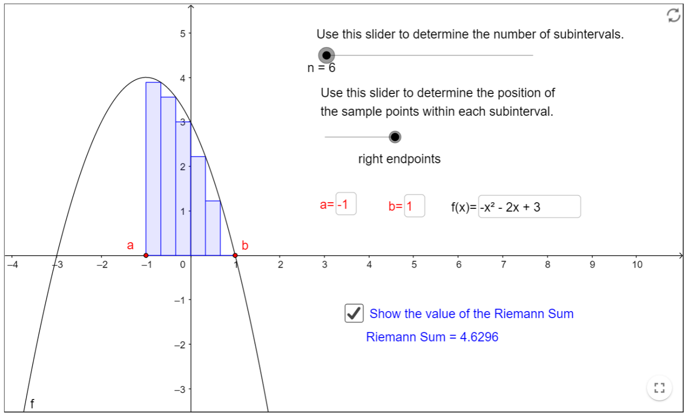
Riemann Sum = 4.6296

The integral is somewhere between the left and right Riemann sums:

$$\underline{\hspace{2cm}} \le \int_{-1}^{1} (-x^2 - 2x + 3)dx \le \underline{\hspace{2cm}}$$

Which Riemann sum goes where?

Playing with Geogebra



Use this slider to determine the number of subintervals.

n = 6

Use this slider to determine the position of the sample points within each subinterval.

right endpoints

a= -1 b= 1 f(x)= -x² - 2x + 3

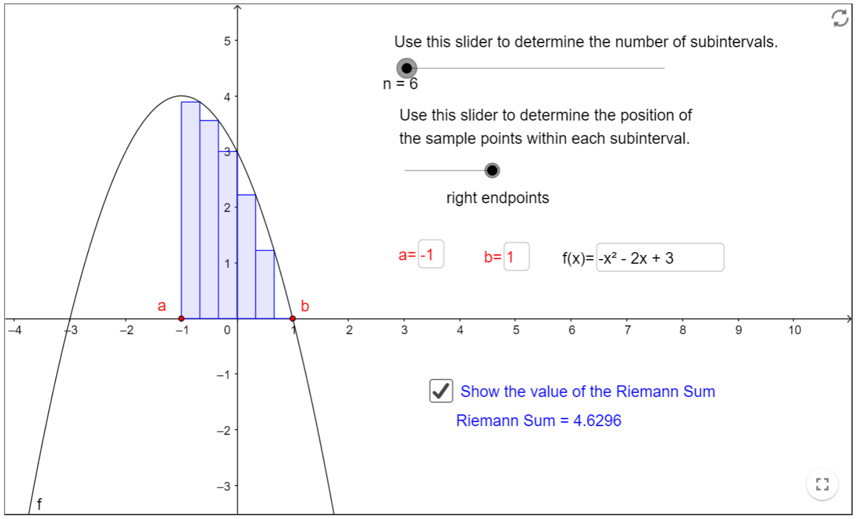
☑ Show the value of the Riemann Sum

Riemann Sum = 4.6296

$$R.H.S \leq \int_{-1}^{1} (-x^2 - 2x + 3)dx \leq L.H.S$$

Rainbow the cat wants to compute the area under the curve using a left-Riemann sum. He wants to know how far away from the true area his computation be.

Playing with Geogebra



Use this slider to determine the number of subintervals.

n = 6

Use this slider to determine the position of the sample points within each subinterval.

right endpoints

a= -1 b= 1 f(x)= -x² - 2x + 3

☑ Show the value of the Riemann Sum

Riemann Sum = 4.6296

We know:

$$R.H.S = \Delta t(f(-\frac{2}{3}) + f(-\frac{1}{3}) + f(0) + f(\frac{1}{3}) + f(\frac{2}{3}) + f(1))$$

$$L.H.S = (f(-1) + f(-\frac{2}{3}) + f(-\frac{1}{3}) + f(0) + f(\frac{1}{3}) + f(\frac{2}{3}))$$

What is $L.H.S - R.H.S$?

Playing with Geogebra

Q: Rainbow wants to compute the area under the curve $-x^2 - 2x + 3$ between $x = -1$ and $x = 1$. He wants his computation to fall within 0.02 of the true value. How many rectangles does He need?

Playing with Geogebra

Q: Rainbow wants to compute the area under the curve $-x^2 - 2x + 3$ between $x = -1$ and $x = 1$. He wants his computation to fall within 0.02 of the true value. How many rectangles does He need?

A: We know that the maximal error is $L.H.S - R.H.S$, which is given by $\Delta t(f(-1) - f(1))$. Plugging in values, we want:

$$0.02 \geq \Delta t \cdot 4$$

Playing with Geogebra

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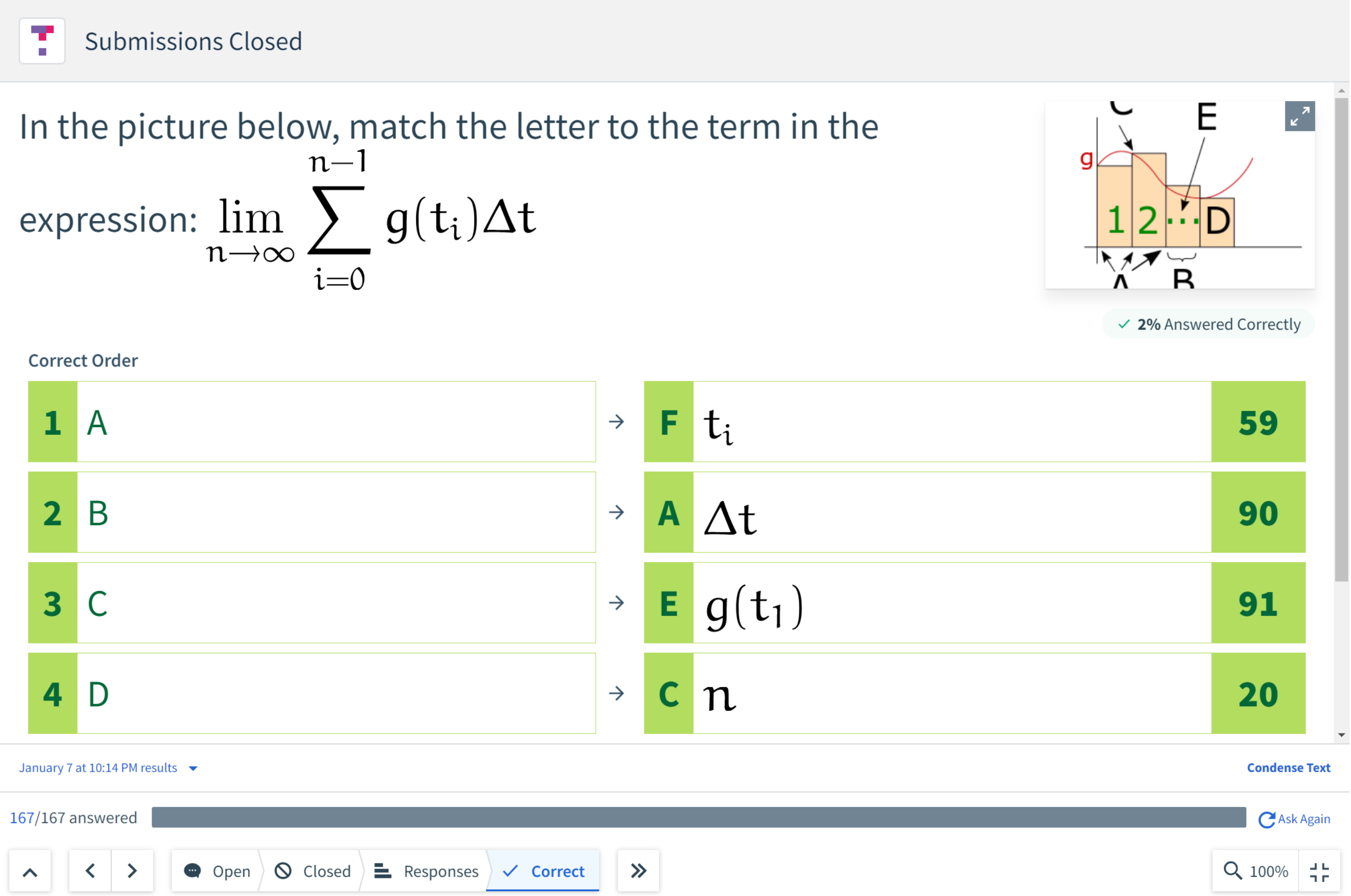
A: We know that the maximal error is $L.H.S - R.H.S$, which is given by $\Delta t(f(-1) - f(1))$. Plugging in values, we want:

$$0.02 \geq \Delta t \cdot 4$$

We know $\Delta t = \frac{2}{n}$, so to make $\Delta t < 0.005$, we need n to be at least 400.

Takeaway

When a function is monotonic, we have a good way to estimate the error between the left- and the right- Riemann sums



In the picture below, match the letter to the term in the expression: $\lim\limits_{n\to\infty}\sum\limits_{i=0}^{n-1} g(t_i)\Delta t$

✓ 2% Answered Correctly

Correct Order

1	A	→	F	t_i	59
2	B	→	A	Δt	90
3	C	→	E	$g(t_1)$	91
4	D	→	C	n	20

167/167 answered

One-Minute Explanation

Write a sentence explaining what happens to the left- and right-Riemann sums when we take the limit as $n \to \infty$.

One-Minute Explanation

Write a sentence explaining what happens to the left- and right-Riemann sums when we take the limit as $n \to \infty$.

> **“ When we take the limit as $n \to \infty$, the left and the right Riemann sums converge to the same thing. This is the signed area under the function, or, the definite integral.”**

Plans for the Future

For next time:

WeBWork 5.3 and read section 5.3

Welcome to MAT135 LEC0501 (Assaf)

Have you formed a study group yet?

S5.3 – The FUNdamental Theorem

Assaf Bar-Natan

“ F is for friends who do stuff together
U is for you and me
N is for anywhere and anytime at all
Down here in the deep blue sea ”

–“ F.U.N Song ”, Spongebob

Jan. 10, 2020

Ice-Cream Sandwich

Spend a minute to think about:

- Something in the chapter that you've mastered.
- Something in the chapter that was hard.

Ice-Cream Sandwich

Spend a minute to think about:

- Something in the chapter that you've mastered.
- Something in the chapter that was hard.
- Share with your group what made something click for you in this chapter.

Ice-Cream Sandwich

Spend a minute to think about:

- Something in the chapter that you've mastered.
- Something in the chapter that was hard.
- Share with your group what made something click for you in this chapter.

For me, the intuition for the F.T.C was something new that really made me understand what's going on.

Intuition for the F.T.C

Rainbow, Marzipan, Blackie, and Lexi are eating from the cat-dish, depleting the Christmas left-overs at a rate of $r(t)$ liters per minute. We'd like to find out how much the cats have eaten in the first five minutes of their feast.

Intuition for the F.T.C

Rainbow, Marzipan, Blackie, and Lexi are eating from the cat-dish, depleting the Christmas left-overs at a rate of $r(t)$ liters per minute. We'd like to find out how much the cats have eaten in the first five minutes of their feast.

- In the 30 seconds, they eat approximately ______ liters.

Intuition for the F.T.C

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- In the 30 seconds, they eat approximately ______ liters.
- In the next 30 seconds, they eat approximately ______ liters.

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- In the 30 seconds, they eat approximately ______ liters.
- In the next 30 seconds, they eat approximately ______ liters.
- ...

Intuition for the F.T.C

Rainbow, Marzipan, Blackie, and Lexi are eating from the cat-dish, depleting the Christmas left-overs at a rate of $r(t)$ liters per minute. We'd like to find out how much the cats have eaten in the first five minutes of their feast.

- In the 30 seconds, they eat approximately ______ liters.
- In the next 30 seconds, they eat approximately ______ liters.
- ...

Write an expression for the approximate amount of food the cats ate in five minutes. Use summation notation.

Intuition for the F.T.C

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They ate approximately:

Intuition for the F.T.C

Rainbow, Marzipan, Blackie, and Lexi are eating from the cat-dish, depleting the Christmas left-overs at a rate of $r(t)$ liters per minute. We'd like to find out how much the cats have eaten in the first five minutes of their feast.

They ate approximately:

$$\sum_{i=0}^{9} r\left(\frac{i}{2}\right) \cdot \frac{1}{2}$$

This looks like a Riemann sum!

Intuition for the F.T.C

Rainbow, Marzipan, Blackie, and Lexi are eating from the cat-dish, depleting the Christmas left-overs at a rate of $r(t)$ liters per minute. We'd like to find out how much the cats have eaten in the first five minutes of their feast.

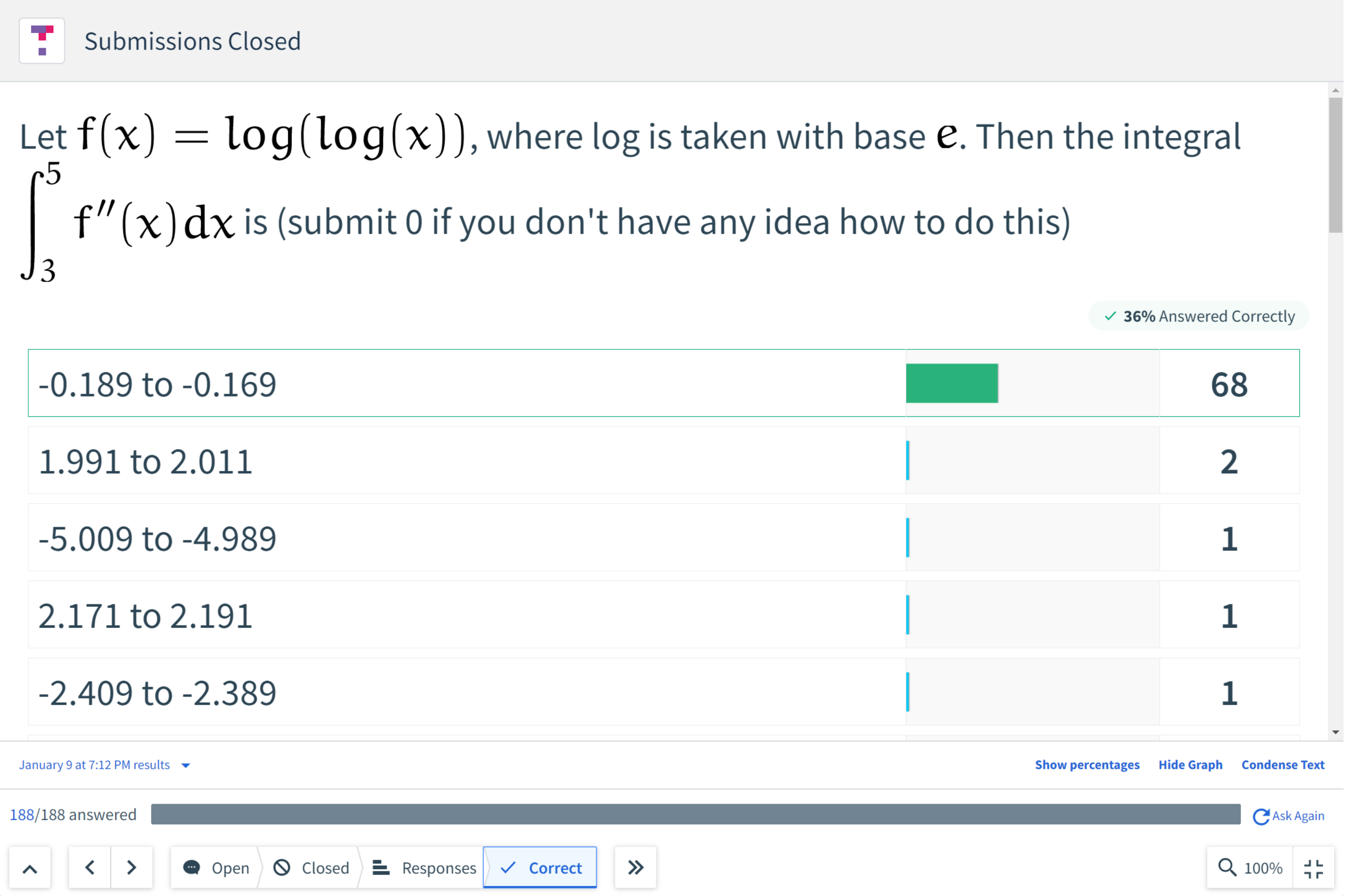
They ate approximately:

$$\sum_{i=0}^{9} r\left(\frac{i}{2}\right) \cdot \frac{1}{2}$$

This looks like a Riemann sum!
Write an expression for the exact amount of food the cats ate in five minutes.

Takeaway

If f is a differentiable function on an interval $[a, b]$ then $\int_a^b f'(x)dx = f(b) - f(a)$.



Submissions Closed

Let $f(x) = \log(\log(x))$, where log is taken with base e. Then the integral $\int_3^5 f''(x)\,dx$ is (submit 0 if you don't have any idea how to do this)

✓ **36%** Answered Correctly

-0.189 to -0.169	68
1.991 to 2.011	2
-5.009 to -4.989	1
2.171 to 2.191	1
-2.409 to -2.389	1

January 9 at 7:12 PM results

188/188 answered

Estimating using F.T.C

Rainbow, Marzipan, Blackie, and Lexi are eating from the cat-dish, depleting the Christmas left-overs at a rate of $r(t)$ liters per minute. This quantity is measured in the table below:

t	0	2	3	4	5
$r(t)$	0.5	0.3	0.2	0.1	0.05

Give your best upper **or** lower estimate for the total amount of food the cats ate in the first five minutes.

Estimating using F.T.C

Rainbow, Marzipan, Blackie, and Lexi are eating from the cat-dish, depleting the Christmas left-overs at a rate of $r(t)$ liters per minute. This quantity is measured in the table below:

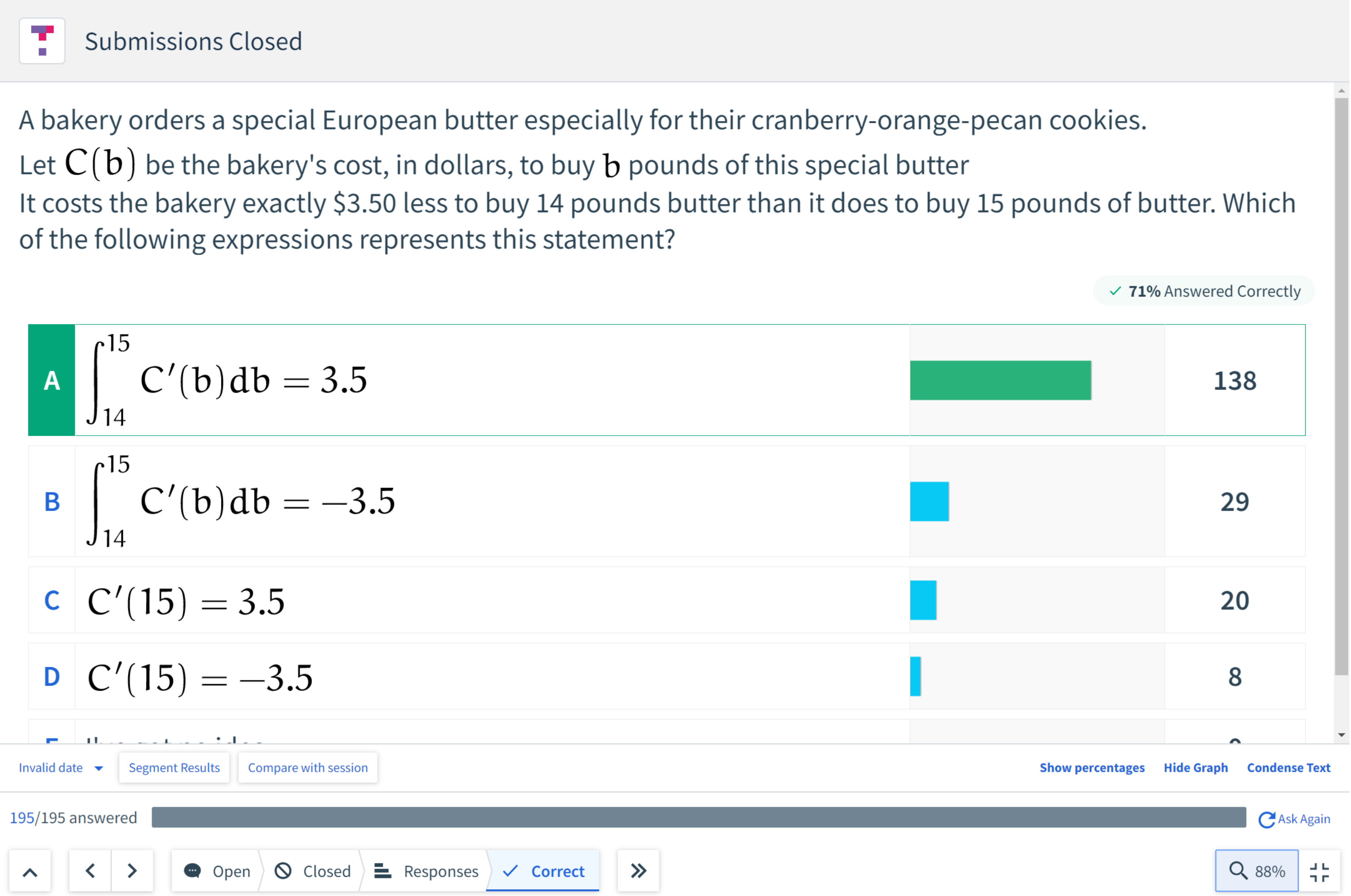
t	0	2	3	4	5
$r(t)$	0.5	0.3	0.2	0.1	0.05

Give your best upper **or** lower estimate for the total amount of food the cats ate in the first five minutes.

Find a group around you that estimated differently than you (ie, if you did a lower estimate, find a group who did an upper esimate), and explain to each other how you arrived at your estimates.

Takeaway

The fundamental theorem gives us a link between areas and rates!



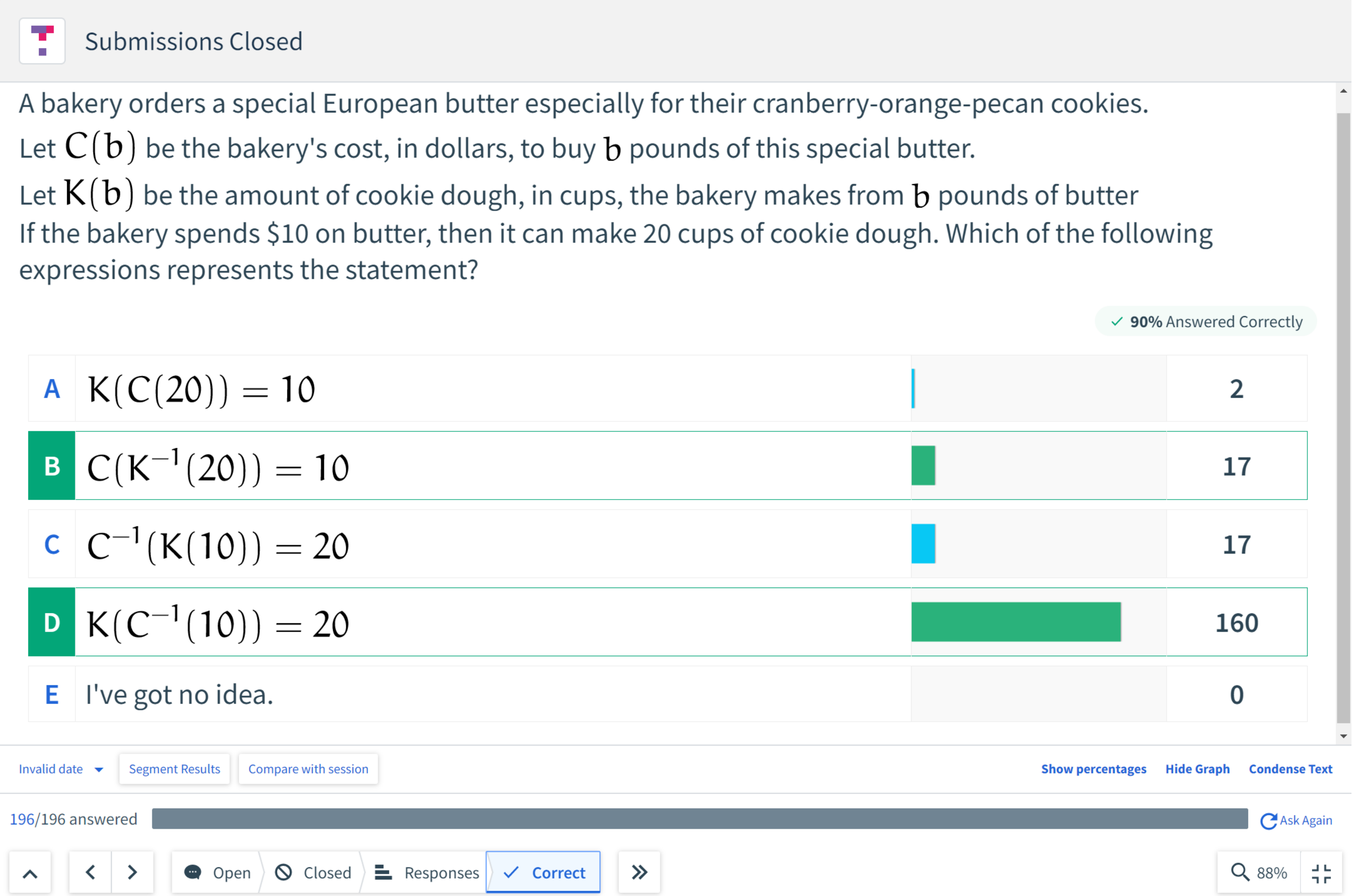
A bakery orders a special European butter especially for their cranberry-orange-pecan cookies.

Let $C(b)$ be the bakery's cost, in dollars, to buy b pounds of this special butter

It costs the bakery exactly \$3.50 less to buy 14 pounds butter than it does to buy 15 pounds of butter. Which of the following expressions represents this statement?

✓ **71%** Answered Correctly

A	$\int_{14}^{15} C'(b)db = 3.5$	138
B	$\int_{14}^{15} C'(b)db = -3.5$	29
C	$C'(15) = 3.5$	20
D	$C'(15) = -3.5$	8



Submissions Closed

A bakery orders a special European butter especially for their cranberry-orange-pecan cookies.

Let $C(b)$ be the bakery's cost, in dollars, to buy b pounds of this special butter.

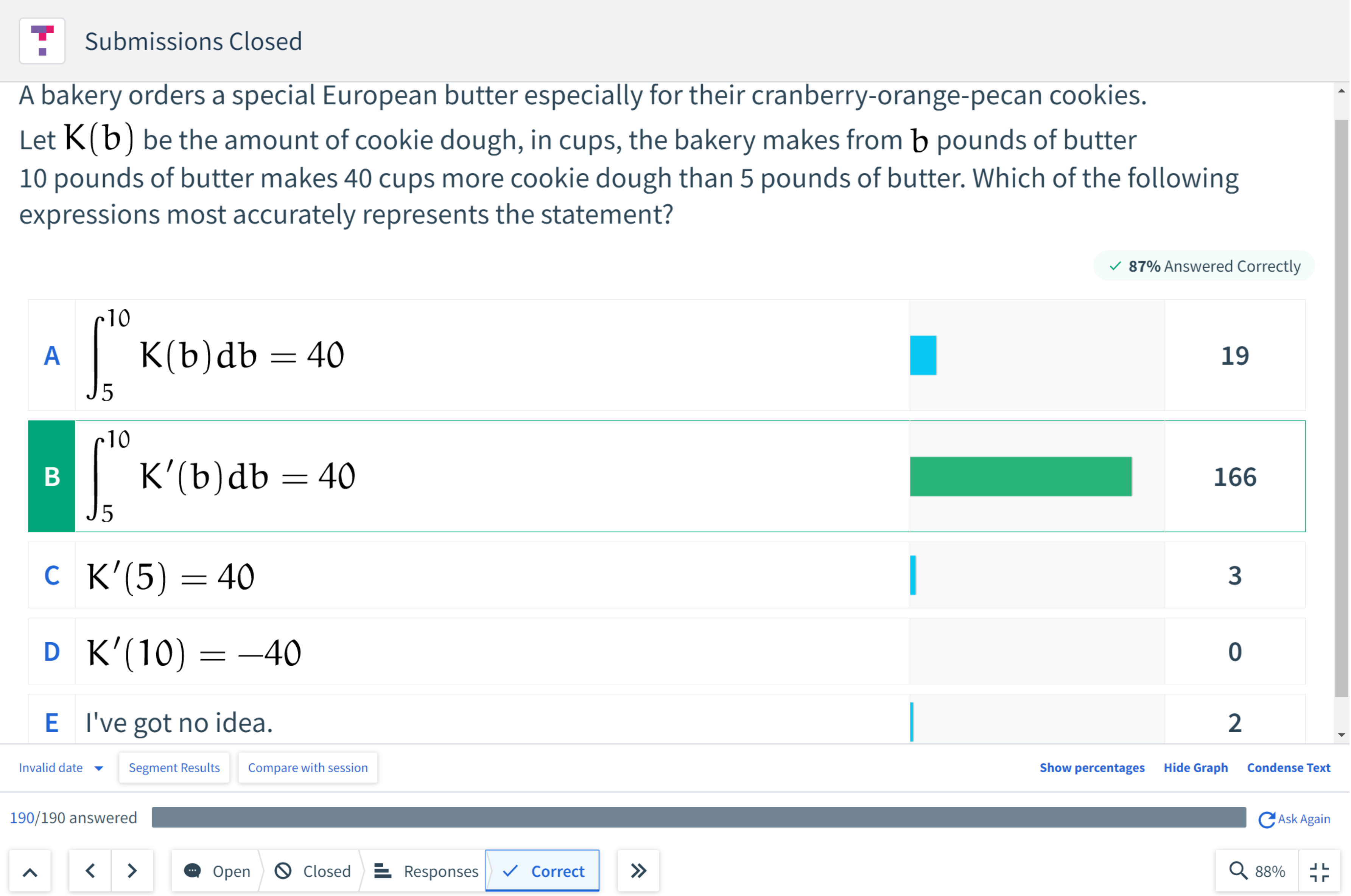
Let $K(b)$ be the amount of cookie dough, in cups, the bakery makes from b pounds of butter

If the bakery spends $10 on butter, then it can make 20 cups of cookie dough. Which of the following expressions represents the statement?

✓ **90%** Answered Correctly

A	$K(C(20)) = 10$	2
B	$C(K^{-1}(20)) = 10$	17
C	$C^{-1}(K(10)) = 20$	17
D	$K(C^{-1}(10)) = 20$	160
E	I've got no idea.	0

196/196 answered

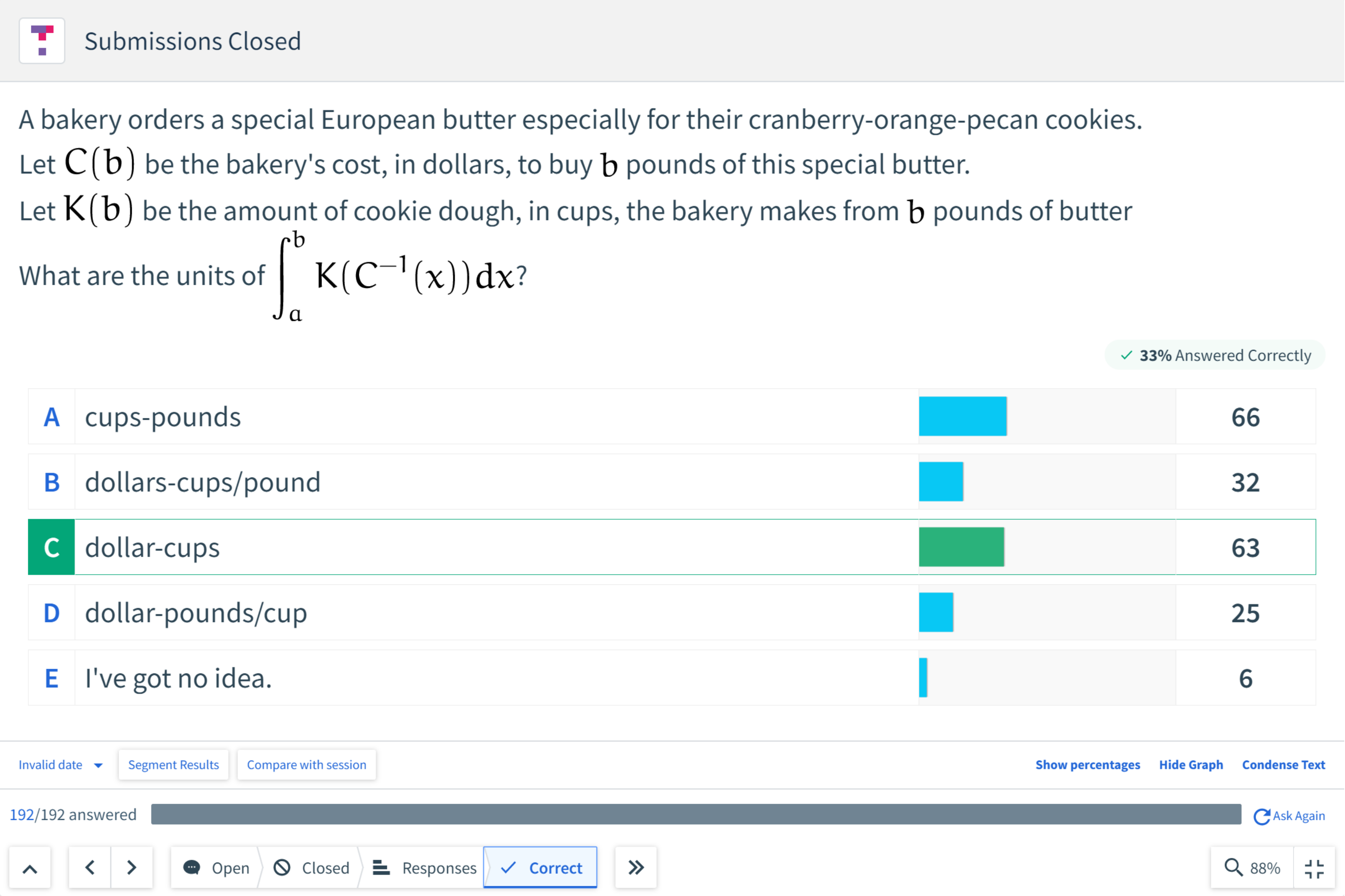


A bakery orders a special European butter especially for their cranberry-orange-pecan cookies. Let $K(b)$ be the amount of cookie dough, in cups, the bakery makes from b pounds of butter 10 pounds of butter makes 40 cups more cookie dough than 5 pounds of butter. Which of the following expressions most accurately represents the statement?

✓ 87% Answered Correctly

A	$\int_5^{10} K(b)db = 40$	19
B	$\int_5^{10} K'(b)db = 40$	166
C	$K'(5) = 40$	3
D	$K'(10) = -40$	0
E	I've got no idea.	2

190/190 answered



A bakery orders a special European butter especially for their cranberry-orange-pecan cookies.
Let $C(b)$ be the bakery's cost, in dollars, to buy b pounds of this special butter.
Let $K(b)$ be the amount of cookie dough, in cups, the bakery makes from b pounds of butter

What are the units of $\int_a^b K(C^{-1}(x))dx$?

✓ **33%** Answered Correctly

A	cups-pounds	66
B	dollars-cups/pound	32
✓ C	dollar-cups	63
D	dollar-pounds/cup	25
E	I've got no idea.	6

192/192 answered

Takeaway

When doing interpretation questions, work slowly, and watch for units!

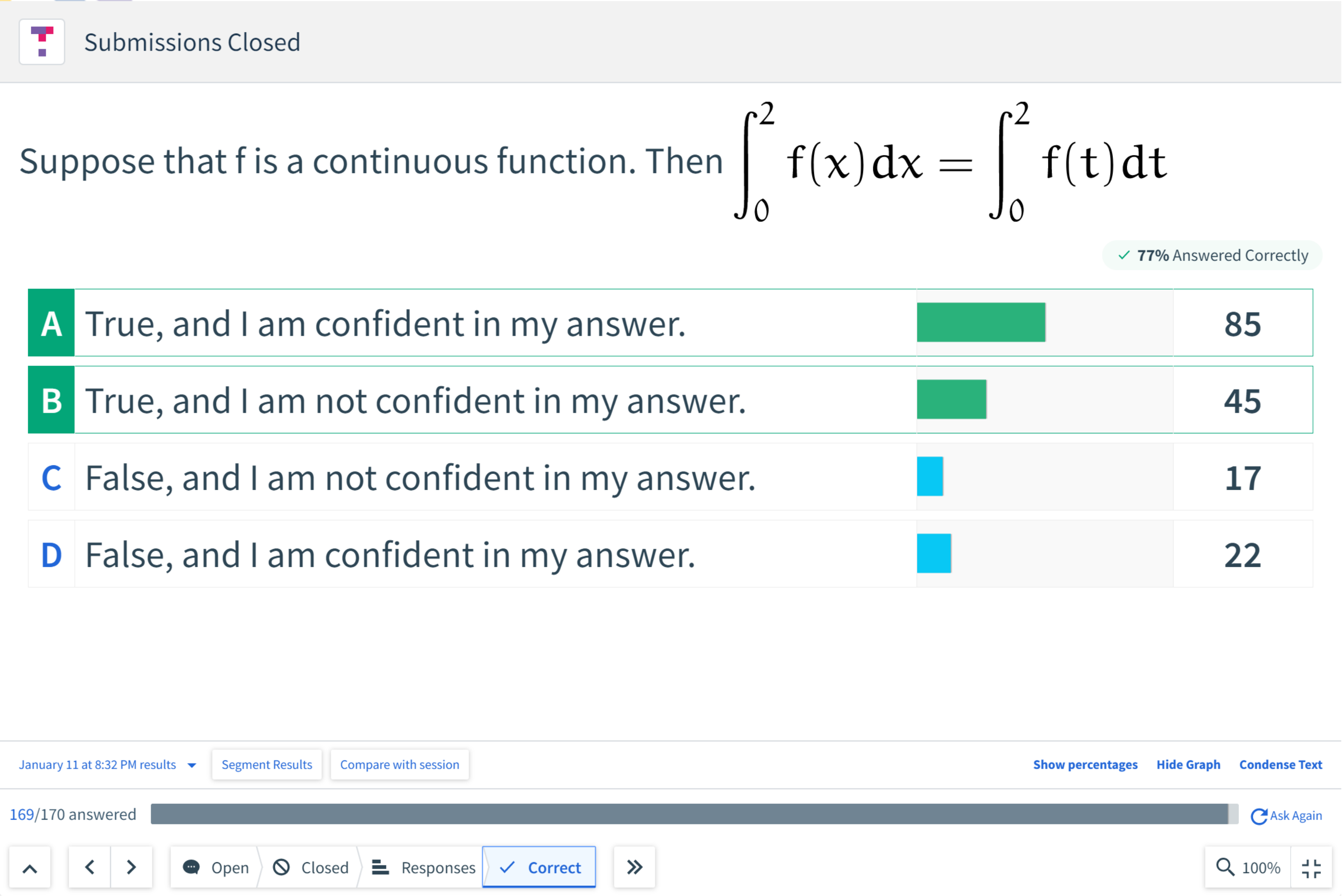
Plans for the Future

For next time:

WeBWork 5.4 and read section 5.4

Welcome to MAT135 LEC0501 (Assaf)

Share with your neighbour something you did during this rainy weekend.



Submissions Closed

Suppose that f is a continuous function. Then $\int_0^2 f(x)dx = \int_0^2 f(t)dt$

✓ 77% Answered Correctly

A	True, and I am confident in my answer.	85
B	True, and I am not confident in my answer.	45
C	False, and I am not confident in my answer.	17
D	False, and I am confident in my answer.	22

169/170 answered

S5.4 – Properties, Theorems, and Bounds on Definite Integrals

Assaf Bar-Natan

> “ On a tour of one-night stands my suitcase and guitar in hand
> And every stop is neatly planned for a poet and a one-man band...
> Homeward bound ”
>
> –“ Homeward Bound ”, Simon & Garfunkel

Jan. 13, 2020

Takeaway

In expressions like $\int_a^b f(x)dx$, the variable x is a dummy variable – It only is there to remind us that f is a function and that we are integrating with respect to its input.

Integration Theorems Round Robin

Get into groups of 3-4.

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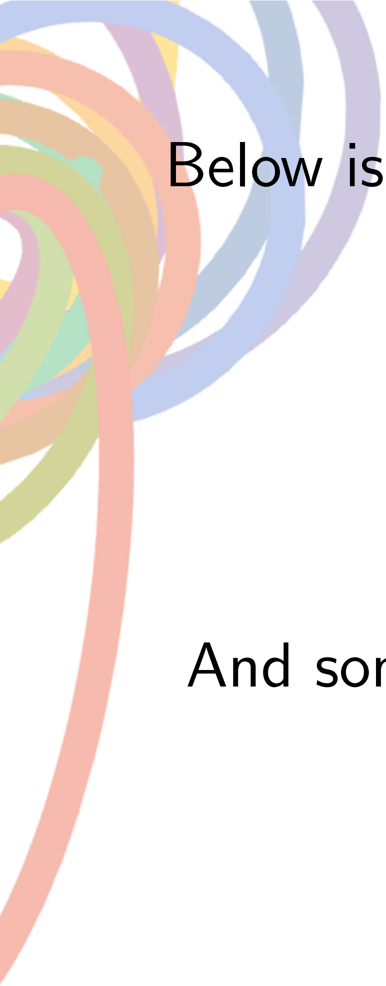
- Go around your group, and one by one state an integration theorem.

Integration Theorems Round Robin

Get into groups of 3-4.

- Go around your group, and one by one state an integration theorem.
- Go through the textbook, and make sure all of the theorems from chapter 5.4 have been stated.

Draw a Theorem



Below is a summary of some of the theorems from chapter 5.4:

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b f(x)dx$$

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$$

And some of the bounds:

$$m \leq f(x) \leq M \Rightarrow m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

$$f(x) \leq g(x) \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx$$

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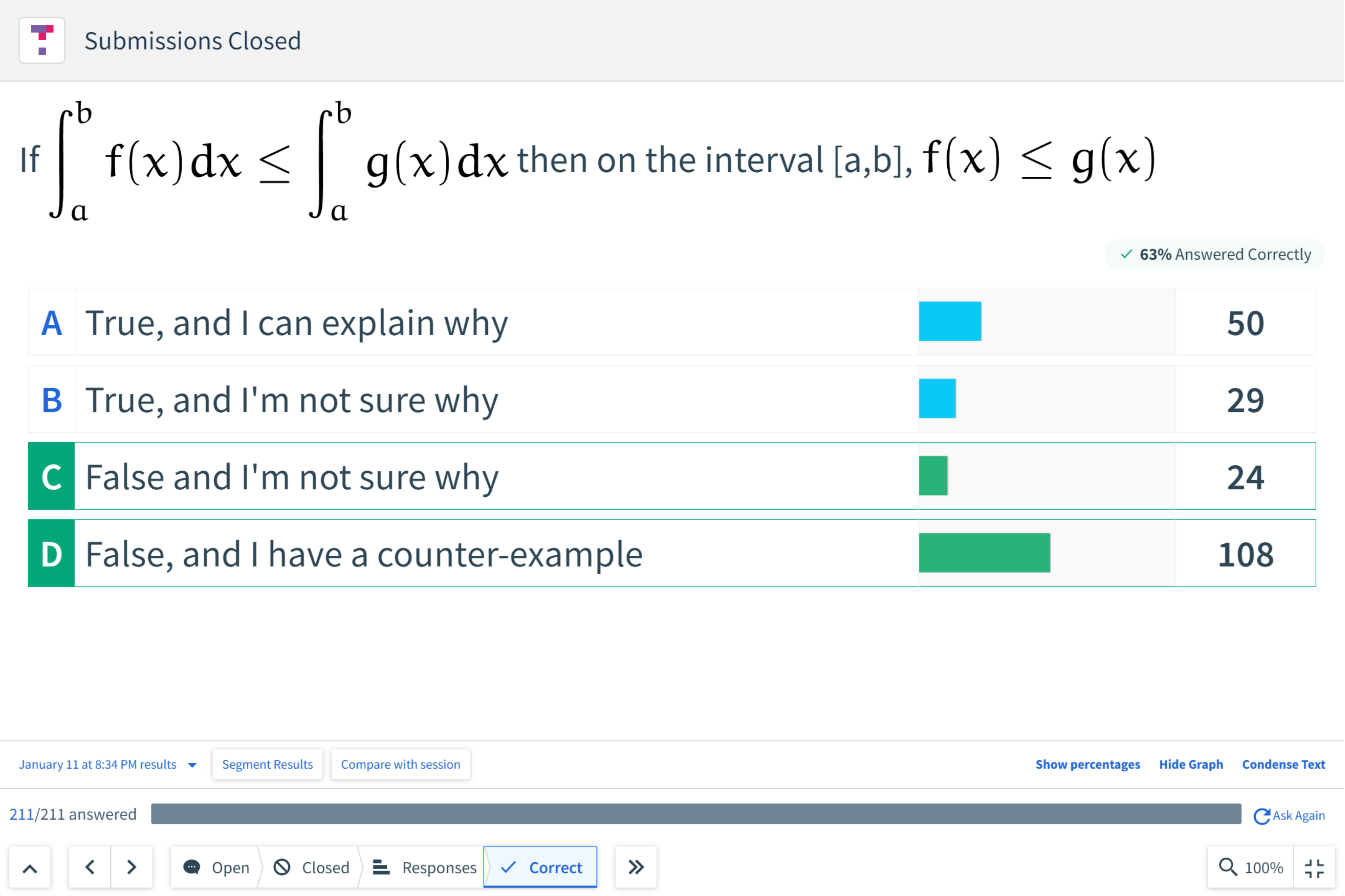
In your group, choose one of these theorems and one of these bounds, and draw a picture explaining why it's true.



Submissions Closed

If $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$ then on the interval [a,b], $f(x) \leq g(x)$

✓ **63%** Answered Correctly

A	True, and I can explain why	50
B	True, and I'm not sure why	29
C	False and I'm not sure why	24
D	False, and I have a counter-example	108

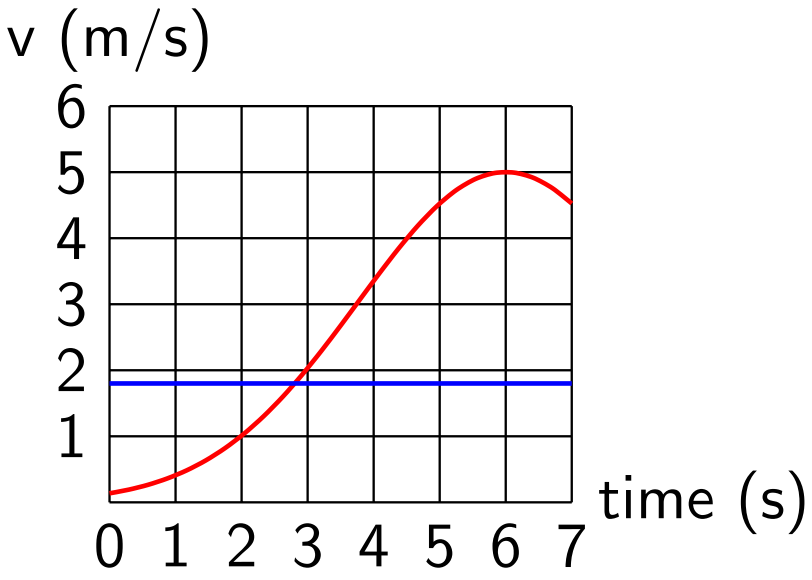
211/211 answered

Takeaway

If we know that $f(x) \leq g(x)$ **on** $[a, b]$**, then** $\int_a^b f(x)dx \leq \int_a^b g(x)dx$**. However, we cannot reverse this!**

An Application

Marzipan is chasing a mouse along the side of the barn. The mouse has a head start of about $1m$, and the velocities of Marzipan (red) and the mouse (blue) are plotted below:



Will Marzipan catch the mouse?

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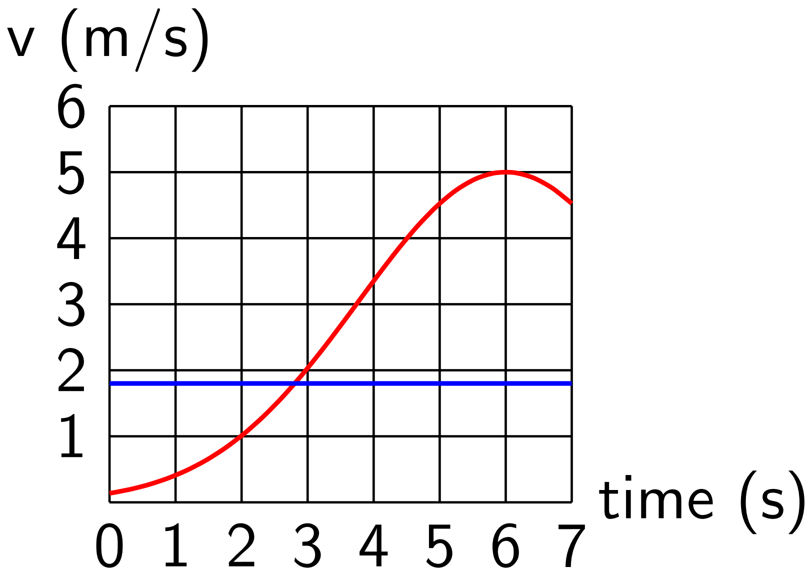
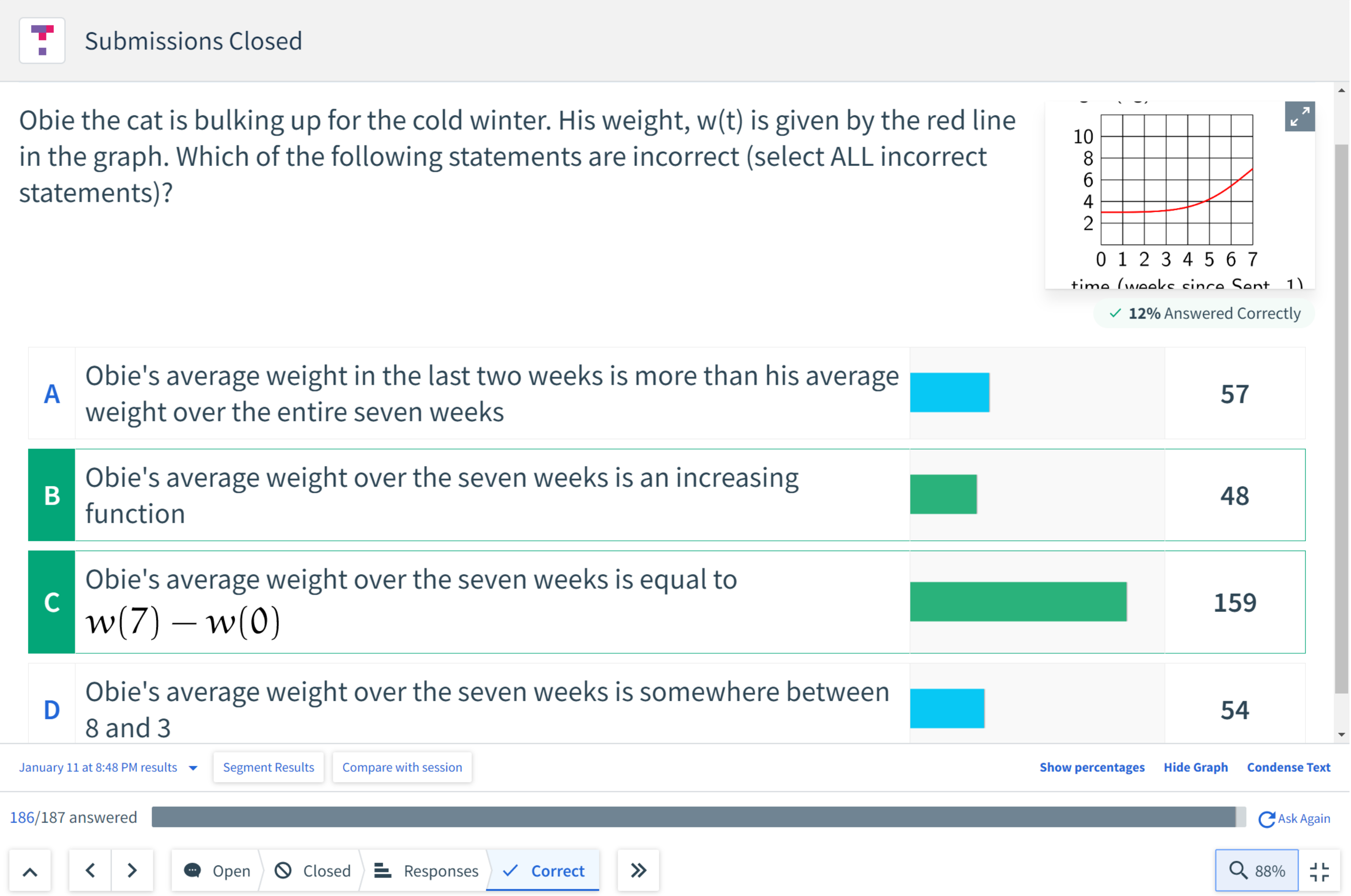


Will Marzipan catch the mouse? When?



Submissions Closed

Obie the cat is bulking up for the cold winter. His weight, w(t) is given by the red line in the graph. Which of the following statements are incorrect (select ALL incorrect statements)?

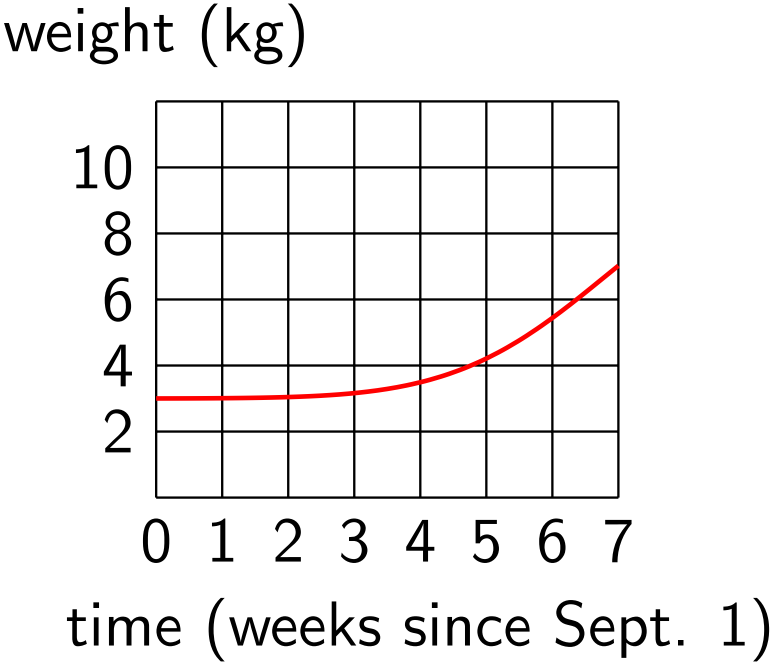
✓ **12%** Answered Correctly

A	Obie's average weight in the last two weeks is more than his average weight over the entire seven weeks	57
B	Obie's average weight over the seven weeks is an increasing function	48
C	Obie's average weight over the seven weeks is equal to $w(7) - w(0)$	159
D	Obie's average weight over the seven weeks is somewhere between 8 and 3	54

186/187 answered

Using Integrals to Estimate Averages

Obie's weight over the fall season is plotted below:



Estimate Obie's average weight during this time.

Plans for the Future

For next time:

WeBWork 6.1 and read section 6.1

S6.1 – New Technology – Antiderivatives

Assaf Bar-Natan (Replacing Josh Lackman)

> “ They took the credit for your second symphony
> Rewritten by machine on new technology
> And now I understand the problems you can see
> Oh, ah, oh! ”
>
> –“ Video Killed the Radio Star ”, The Buggles

Jan. 16, 2020

The Definition of an Antiderivative

If f and F are two functions, we say that F **is an antiderivative of** f if $F'(x) = f(x)$.

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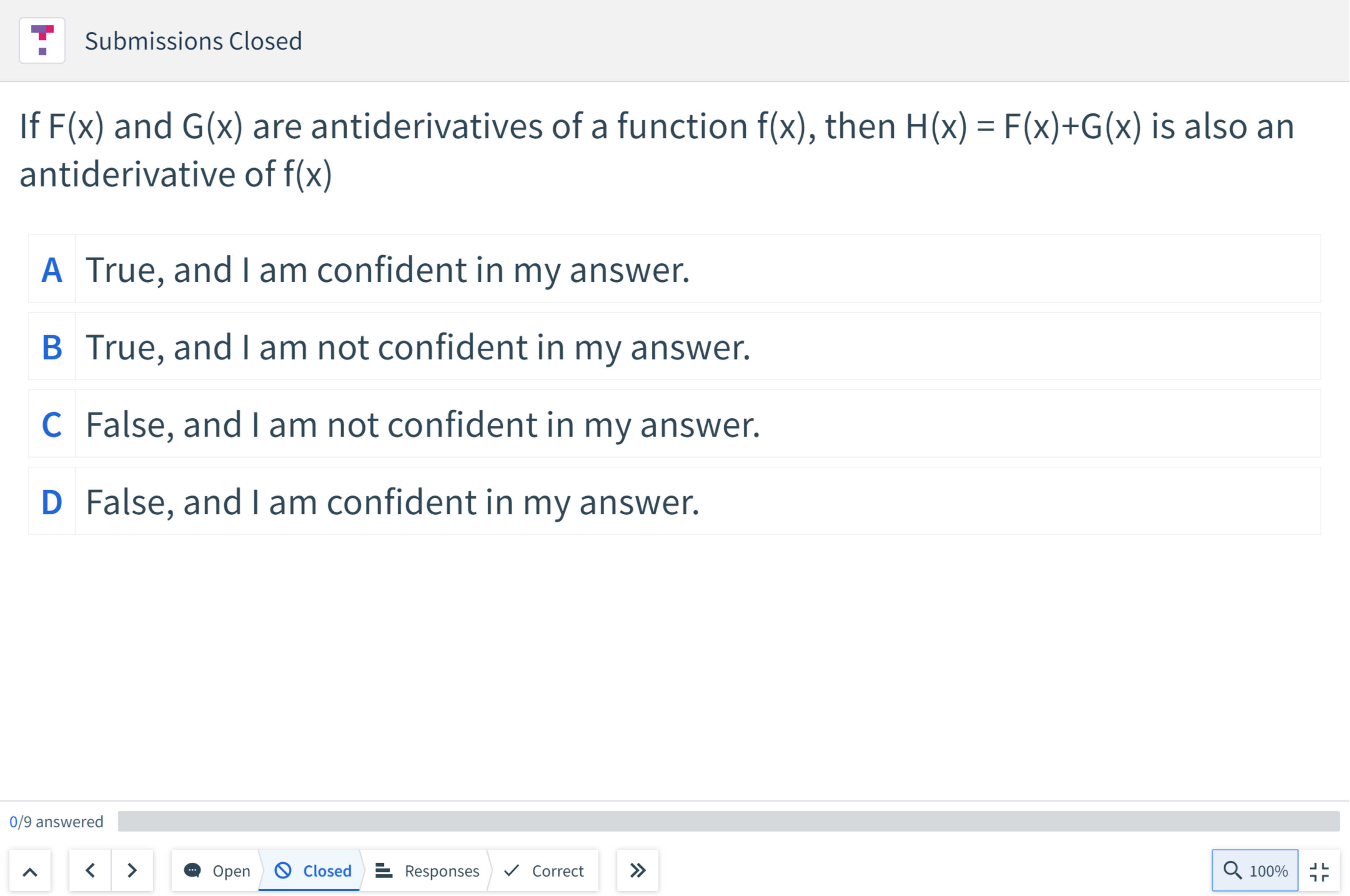
If f and F are two functions, we say that F **is an antiderivative of** f if $F'(x) = f(x)$.

For example: if $f(x) = 2x$ and $F(x) = x^2$, then $F(x)$ is an antiderivative of $f(x)$.



Submissions Closed

If F(x) and G(x) are antiderivatives of a function f(x), then H(x) = F(x)+G(x) is also an antiderivative of f(x)

A True, and I am confident in my answer.

B True, and I am not confident in my answer.

C False, and I am not confident in my answer.

D False, and I am confident in my answer.

0/9 answered

Takeaway

MAT136 tip: When you know the definition, use it instead of taking shortcuts.

Draw The Antiderivative

- Take out a sheet of paper, or borrow one from your neighbour.

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- Take out a sheet of paper, or borrow one from your neighbour.
- Draw a continuous function defined on $[0, 5]$ which is:
    1. Decreasing and linear on $[0, 2]$.
    2. Positive at 0 and negative at 2
    3. Equal to a positive constant between 4 and 5.

Make sure your axes are labelled!

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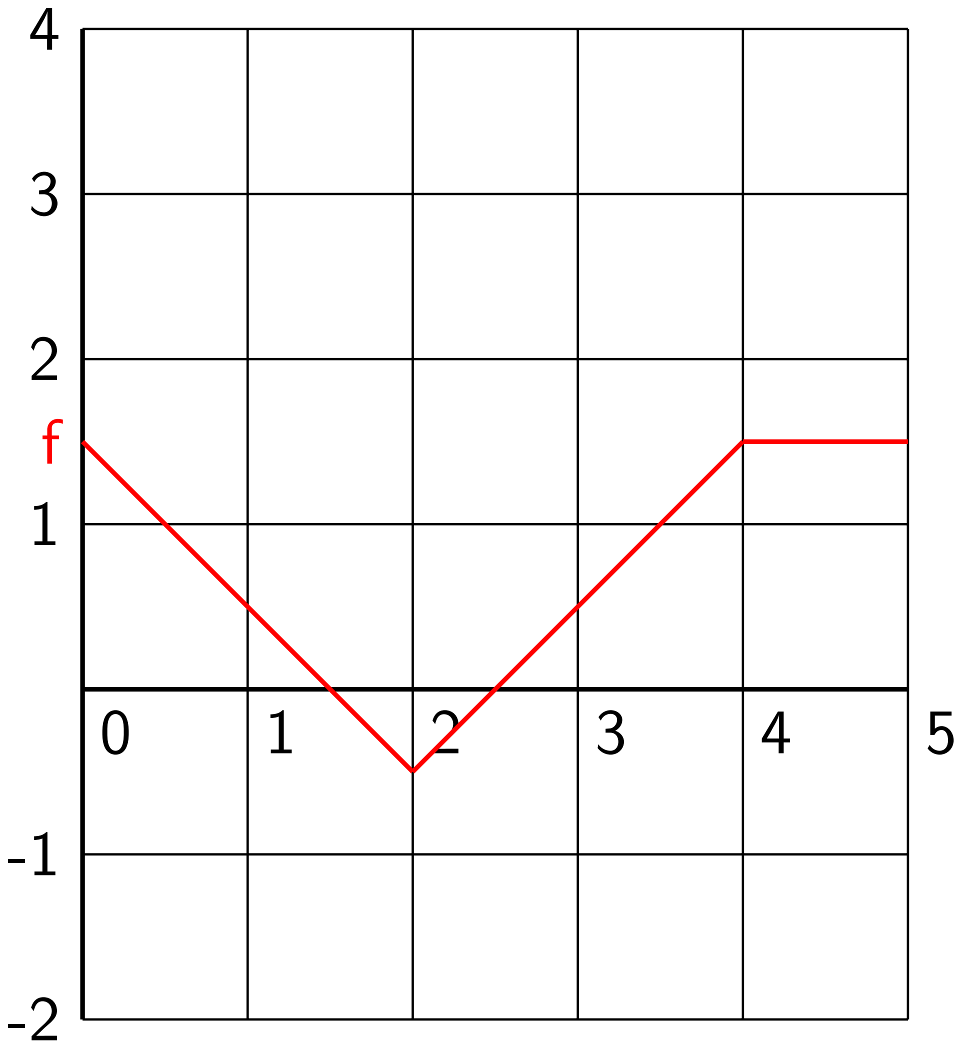
- Pass the papers back to your partner, and compare your answers. Explain what you drew.
- With your partner, pick a drawing, and draw on it an antiderivative of the original function that is different from the one you already drew

Draw The Antiderivative – My Drawing



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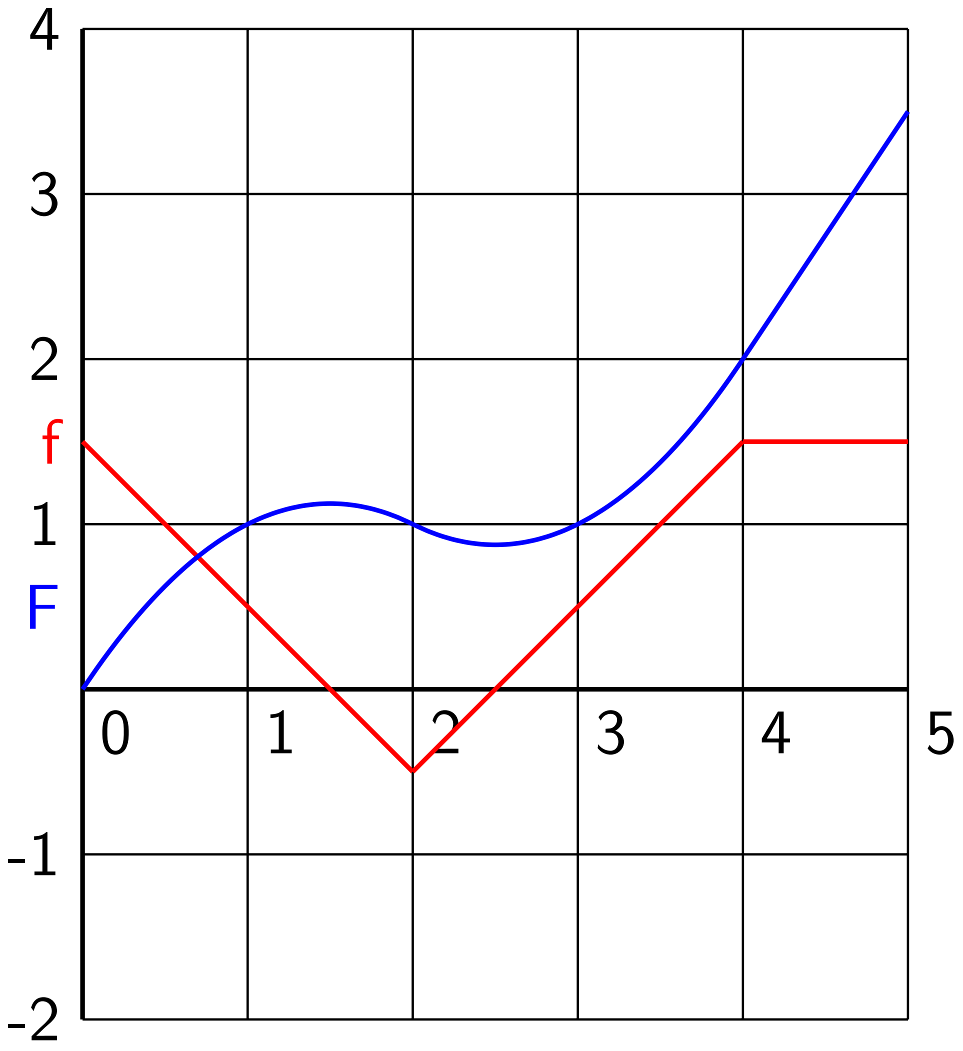

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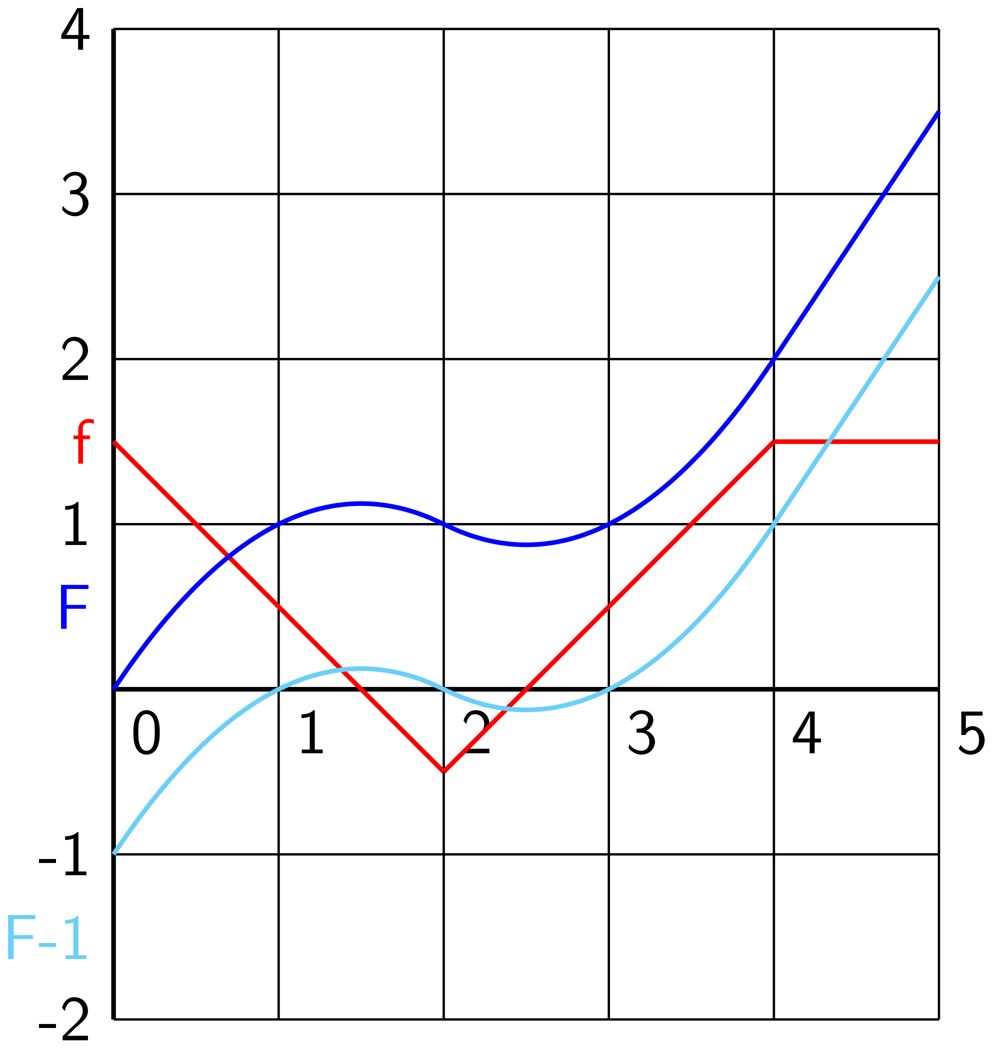

Takeaway

If F is an antiderivative of f, then $F + c$ is an antiderivative of f for any constant c

Summarizing What We Know

Feature of function at a point	Feature of an antiderivative at that point
positive	
negative	
x-intercept	
increasing	
decreasing	
maximum	
minimum	

Summarizing What We Know

Feature of function at a point	Feature of an antiderivative at that point
positive	increasing
negative	decreasing
x-intercept	critical point
increasing	concave up
decreasing	concave down
maximum	inflection point
minimum	inflection point

Takeaway

In the same way that we sketch a function's derivative, we can reverse the process to sketch the antiderivative.

Antiderivatives and the F.T.C

Recall that if F is a differentiable function on an interval $[a, b]$, and $F' = f$, then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

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Recall that if F is a differentiable function on an interval $[a, b]$, and $F' = f$, then:

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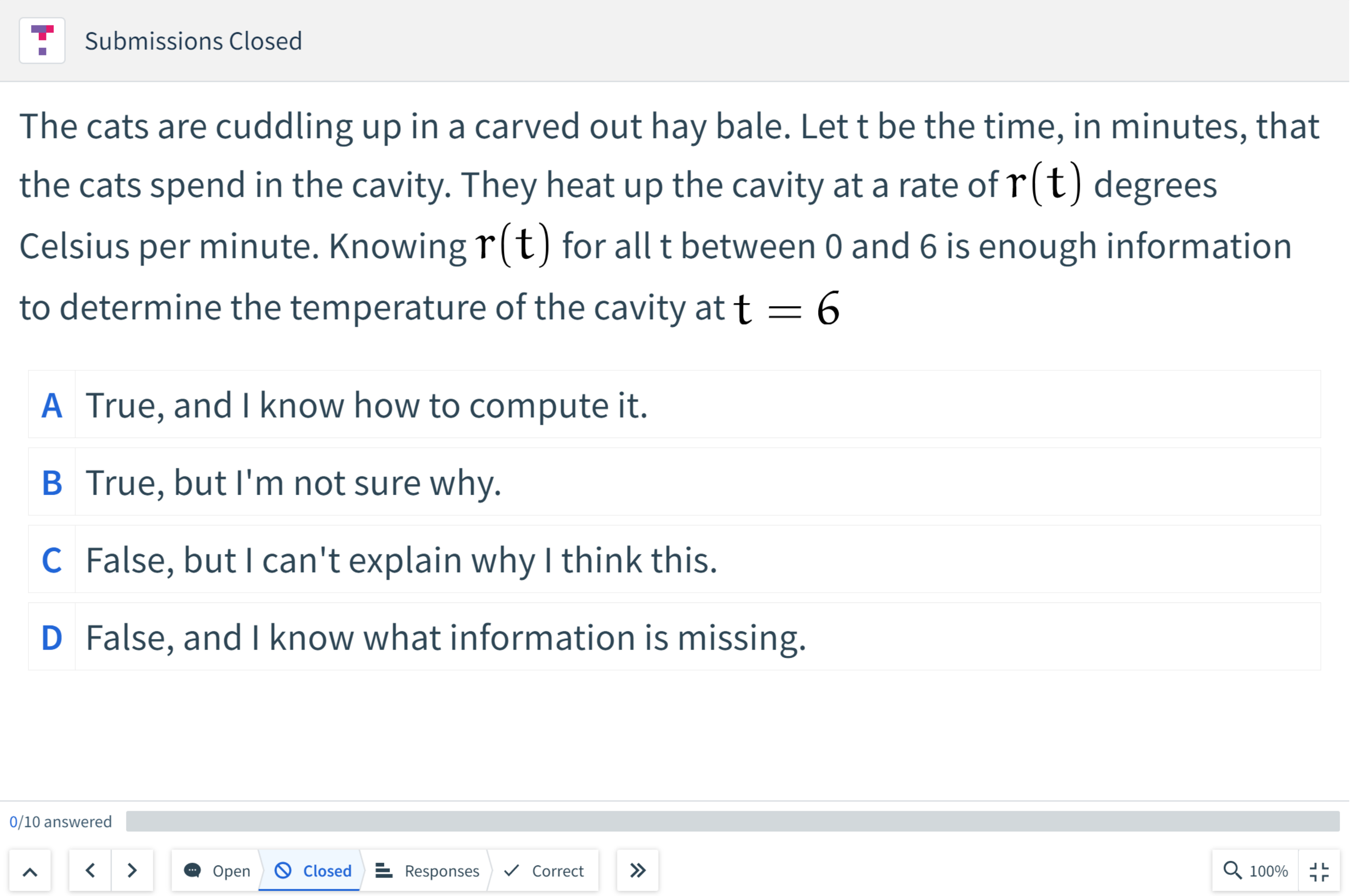
Knowing the antiderivative allows us to compute definite integrals easily.



Submissions Closed

The cats are cuddling up in a carved out hay bale. Let t be the time, in minutes, that the cats spend in the cavity. They heat up the cavity at a rate of $r(t)$ degrees Celsius per minute. Knowing $r(t)$ for all t between 0 and 6 is enough information to determine the temperature of the cavity at $t = 6$

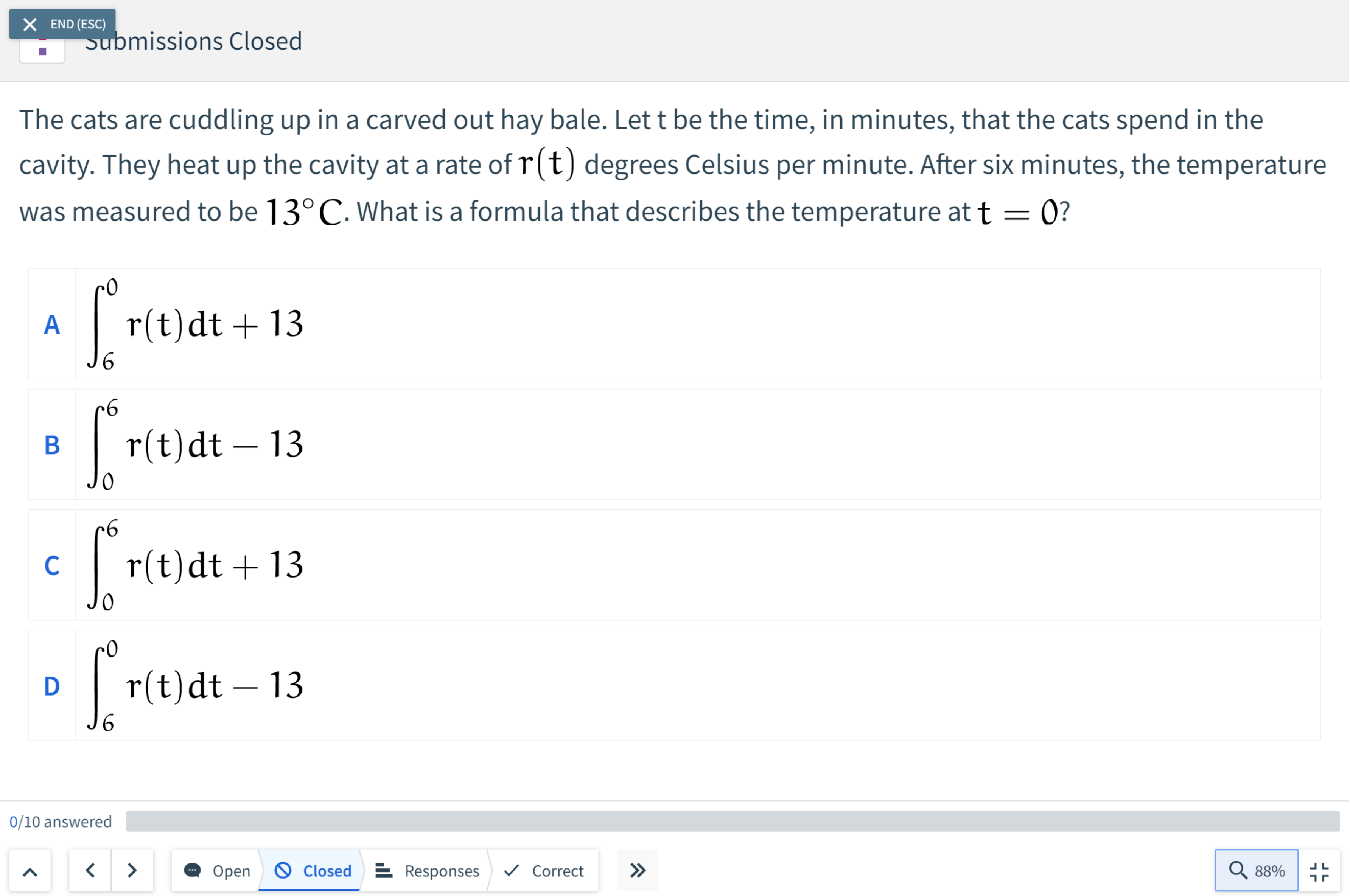
A True, and I know how to compute it.

B True, but I'm not sure why.

C False, but I can't explain why I think this.

D False, and I know what information is missing.

0/10 answered



The cats are cuddling up in a carved out hay bale. Let t be the time, in minutes, that the cats spend in the cavity. They heat up the cavity at a rate of $r(t)$ degrees Celsius per minute. After six minutes, the temperature was measured to be $13°C$. What is a formula that describes the temperature at $t = 0$?

A $\int_6^0 r(t)dt + 13$

B $\int_0^6 r(t)dt - 13$

C $\int_0^6 r(t)dt + 13$

D $\int_6^0 r(t)dt - 13$

Plans for the Future

For next time:

WeBWork 6.2 and read section 6.2

S6.1 – Analyzing Antiderivatives Algebraically

Assaf Bar-Natan

“ Now the teacher would say to learn your algebra
But I’d bring home C’s and D’s
How could I make an A when there’s a swingin’ maid
On the left and on the right and in the back and the front of me? ”

–“ Straight A’s in Love ”, Johnny Cash

Jan. 17, 2020

WeBWork Reflection

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- Look around you for someone who is not in a group, and invite them to your group.

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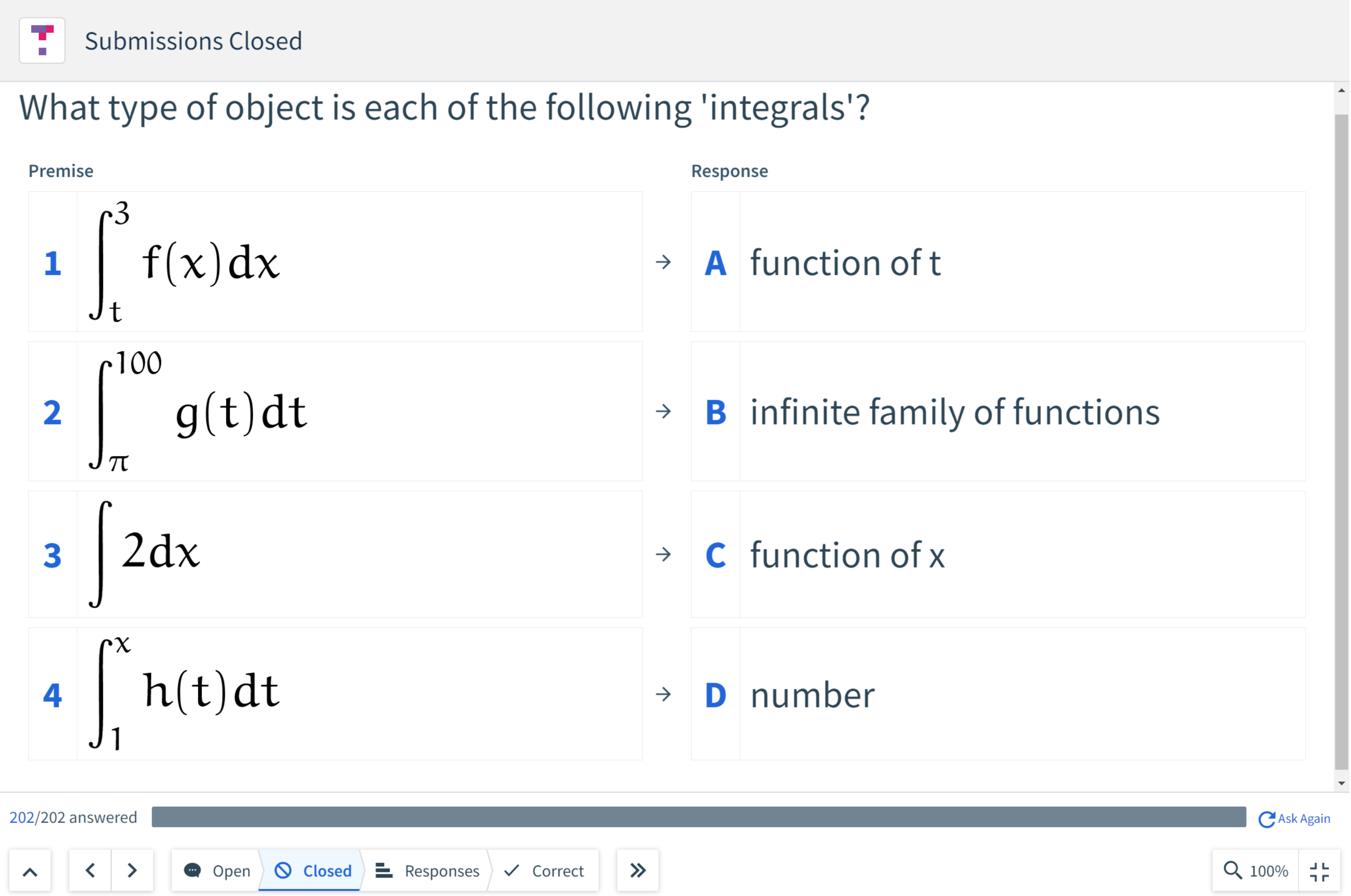
- Look around you for someone who is not in a group, and invite them to your group.
- Pick a WeBWork question from this section that you struggled with.
- Share with your group your progress or how you solved it.

Takeaway

MAT136 tip: WeBWork questions are hard! Help each other!



Submissions Closed

What type of object is each of the following 'integrals'?

Premise			Response	
1	$\int_{t}^{3} f(x)\,dx$	→	A	function of t
2	$\int_{\pi}^{100} g(t)\,dt$	→	B	infinite family of functions
3	$\int 2dx$	→	C	function of x
4	$\int_{1}^{x} h(t)\,dt$	→	D	number

202/202 answered

Cats and Hay-Bales

The cats (Marzipan, Obie, Blackie, and Roy) are cuddling up in a carved out hay bale. Let t be the time, in minutes, that the cats spend in the cavity. They heat up the cavity at a rate of $3e^{-0.2t}$ degrees Celsius per minute. After six minutes, the temperature was measured to be $13^\circ C$.

- Write an expression for the temperature two minutes after the cats jumped into the cavity.

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- Find the antiderivative of $3e^{-0.2t}$.

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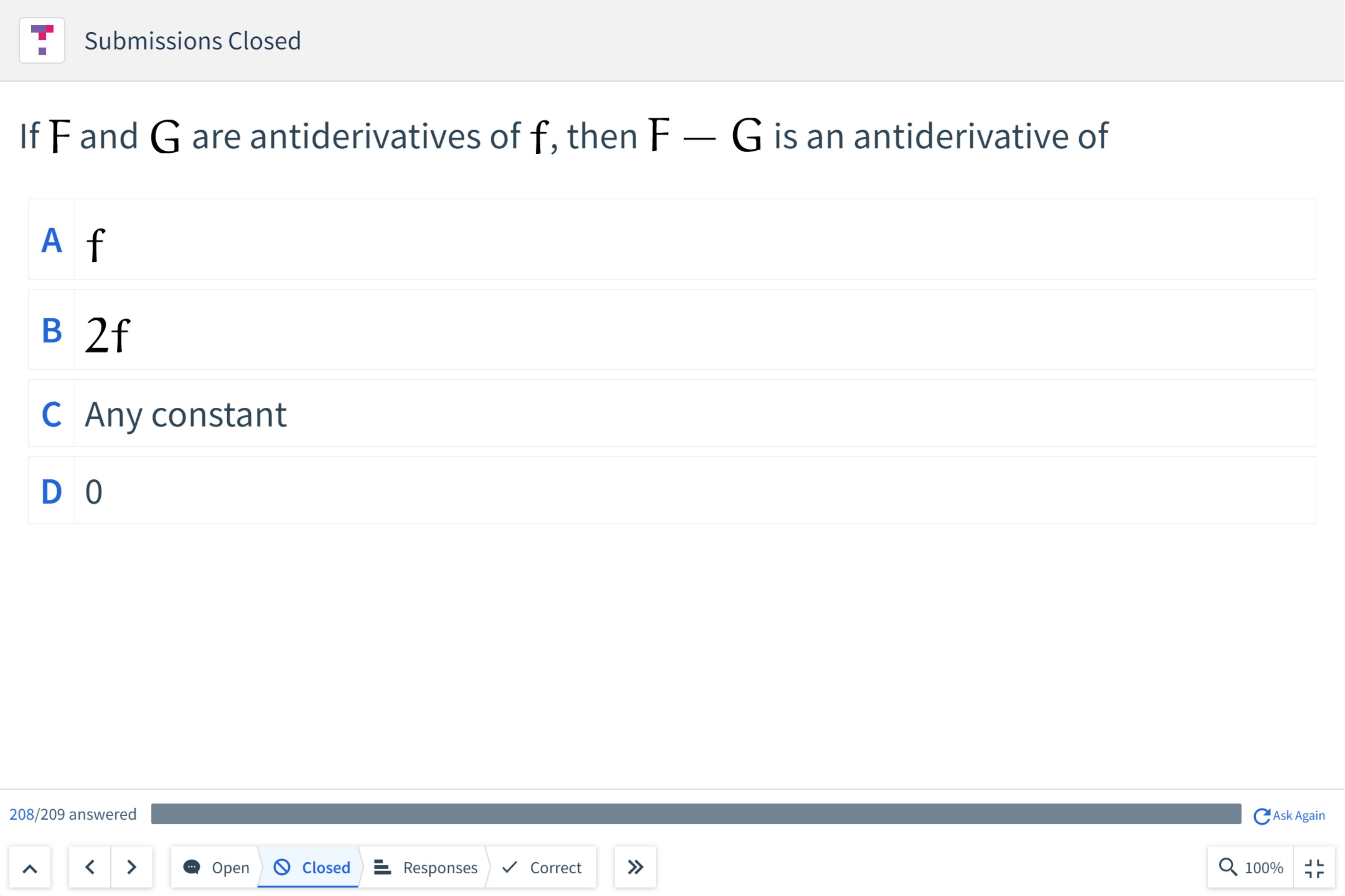
- Write an expression for the temperature two minutes after the cats jumped into the cavity.
- Find the antiderivative of $3e^{-0.2t}$.
- What was the temperature when $t = 2$?

Solution

- $T(2) = 13 - \int_2^6 3e^{-0.2t}dt$
- $\int 3e^{-0.2t}dt = 3\int e^{-0.2t}dt = \frac{3e^{-0.2t}}{-0.2}$

Takeaway

For any function, f, and a co-ordinate (x, y), there is a single antiderivative F, for which $F(x) = y$.



Submissions Closed

If F and G are antiderivatives of f, then $F - G$ is an antiderivative of

A f

B $2f$

C Any constant

D 0

208/209 answered

Punctuated Lecture – Finding All Antiderivatives

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If F and G are antiderivatives of f, then $F - G$ is an antiderivative of 0.

This means that $F - G$ is constant. **Why?**

Punctuated Lecture – Finding All Antiderivatives

If F and G are antiderivatives of f, then $F - G$ is an antiderivative of 0. So $F - G$ is constant.

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If F and G are antiderivatives of f, then $F - G$ is an antiderivative of 0. So $F - G$ is constant.

What does this tell us about any other antiderivative of f?

Cats and Logs

Mia and Obie are having a fight. Both want to compute $\int \frac{1}{5x}$.

Mia says:	Obie says:
"I can pull out $\frac{1}{5}$, and use $\frac{d}{dx}\log(\lvert x\rvert) = \frac{1}{x}$ to get that every antiderivative of $\frac{1}{5x}$ is of the form $\frac{1}{5}\log(\lvert x\rvert) + C$."	"When I compute the derivative of $\frac{1}{5}\log(\pi\lvert x\rvert)$, I get $\frac{1}{5x}$, so $\frac{1}{5}\log(\pi\lvert x\rvert)$ is an antiderivative of $\frac{1}{5x}$ that doesn't fit your pattern."

Who is right?

Solution

Both are right, because if we apply logarithm rules, we get:

$$\frac{1}{5}\log(\pi|x|) = \frac{1}{5}\log(|x|) + \frac{1}{5}\log(\pi)$$

which is of the form that Mia wanted.

Plans for the Future

For next time:

WeBWork 6.3 and read section 6.3

S6.3 – Differential Equations and Motion

Assaf Bar-Natan

> “’Cause you can’t stop the motion of the ocean or the sun in the sky
> You can wonder, if you wanna, but I never ask why
> And if you try to hold me down, I’m gonna spit in your eye and say
> That you can’t stop the beat! ”
>
> –“ You can’t Stop The Beat ”, Hairspray

Jan. 20, 2020

Example: The S.I.R Model of Infection

The cats are getting sick. Let t be the time, in days, since the illness outbreak, and let:

- N be the total number of cats
- $S(t)$ be the number of cats susceptible to the disease
- $I(t)$ be the number of cats infected with the disease
- $R(t)$ be the number of cats who recovered from the disease

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The S.I.R model says that I, S, and R satisfy:

$$\frac{dS}{dt} = -\beta \frac{I(t)S(t)}{N}$$
$$\frac{dI}{dt} = \beta \frac{I(t)S(t)}{N} - \gamma I(t)$$
$$\frac{dR}{dt} = \gamma I(t)$$

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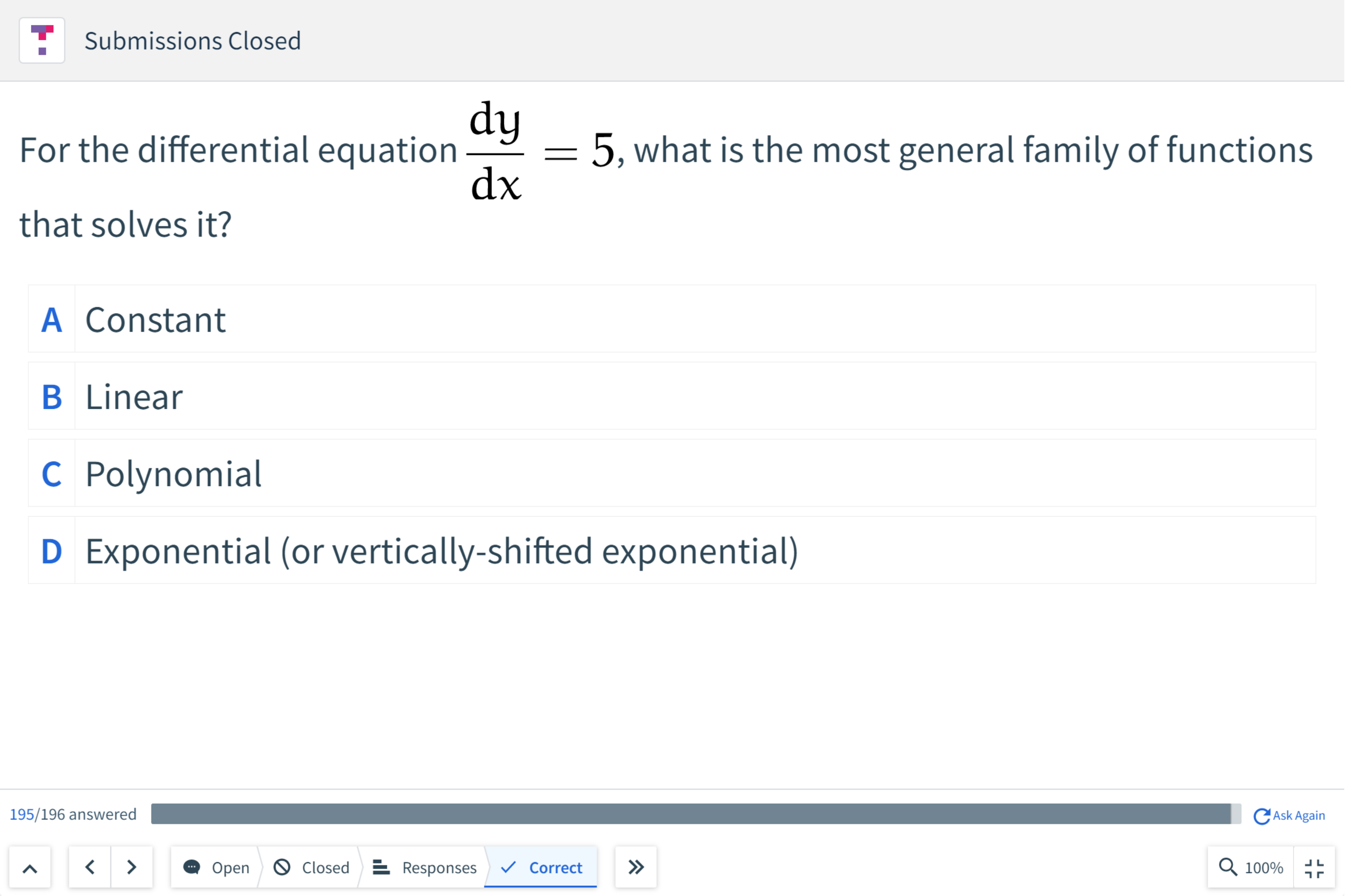
Very difficult goal: Find the functions S, I, and R

Use these equations to show that $\frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{t} = 0$. What does this tell us about $S + I + R$?

Takeaway

Differential equations appear in unlikely places, and their solutions have important real-world reprecussions.

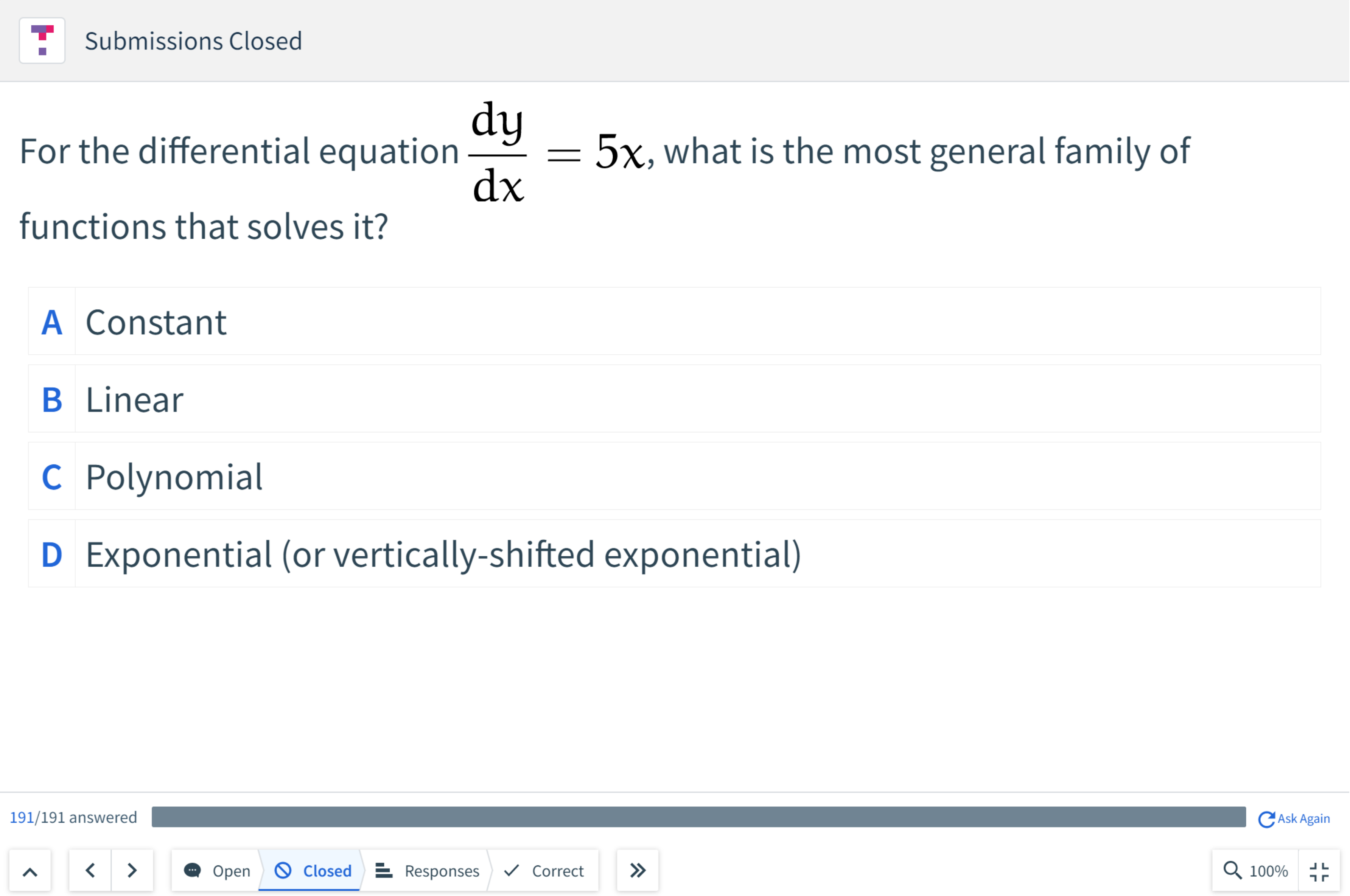


Submissions Closed

For the differential equation $\frac{dy}{dx} = 5$, what is the most general family of functions that solves it?

- **A** Constant
- **B** Linear
- **C** Polynomial
- **D** Exponential (or vertically-shifted exponential)

195/196 answered



For the differential equation $\frac{dy}{dx} = 5x$, what is the most general family of functions that solves it?

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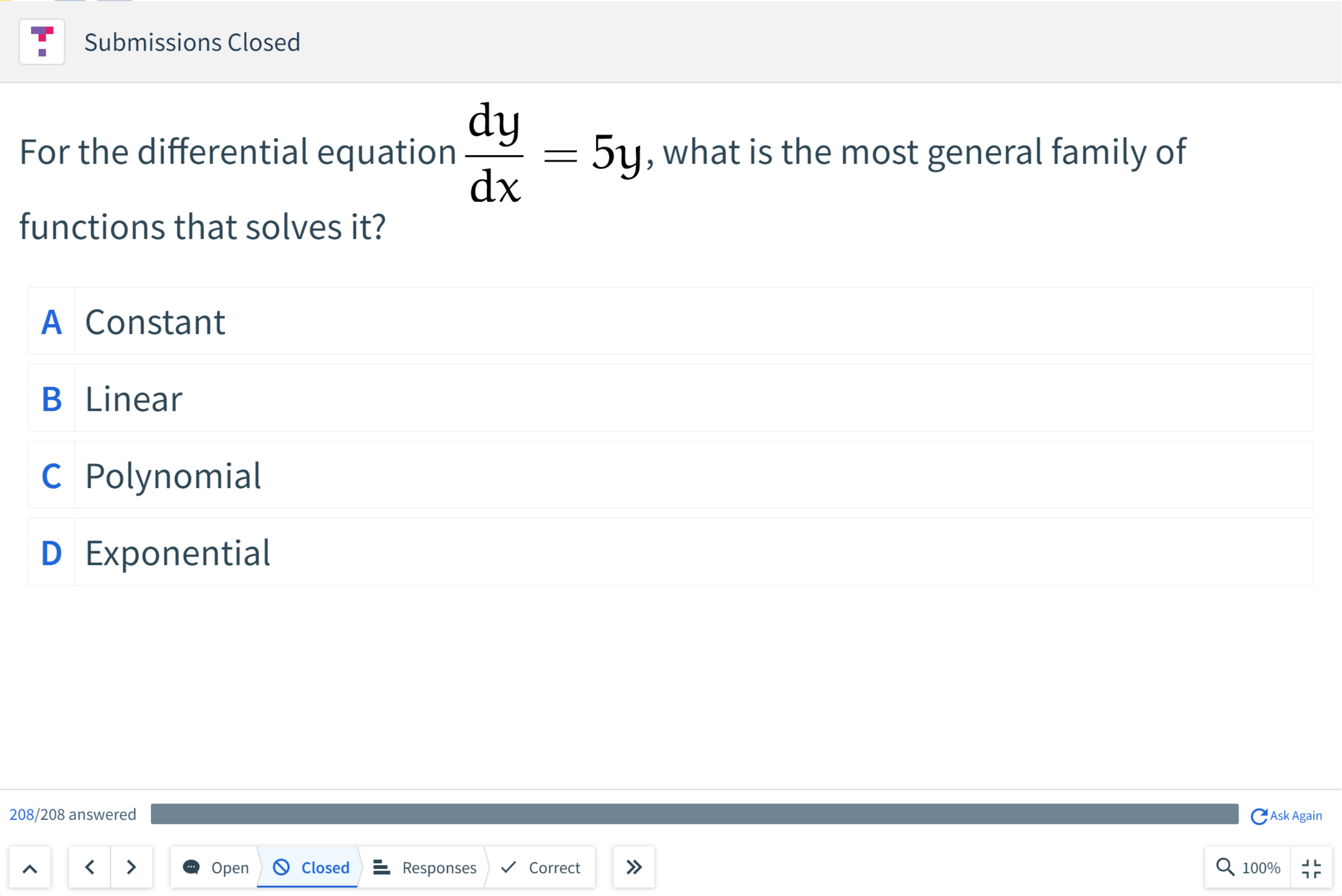


Submissions Closed

For the differential equation $\frac{dy}{dx} = 5y$, what is the most general family of functions that solves it?

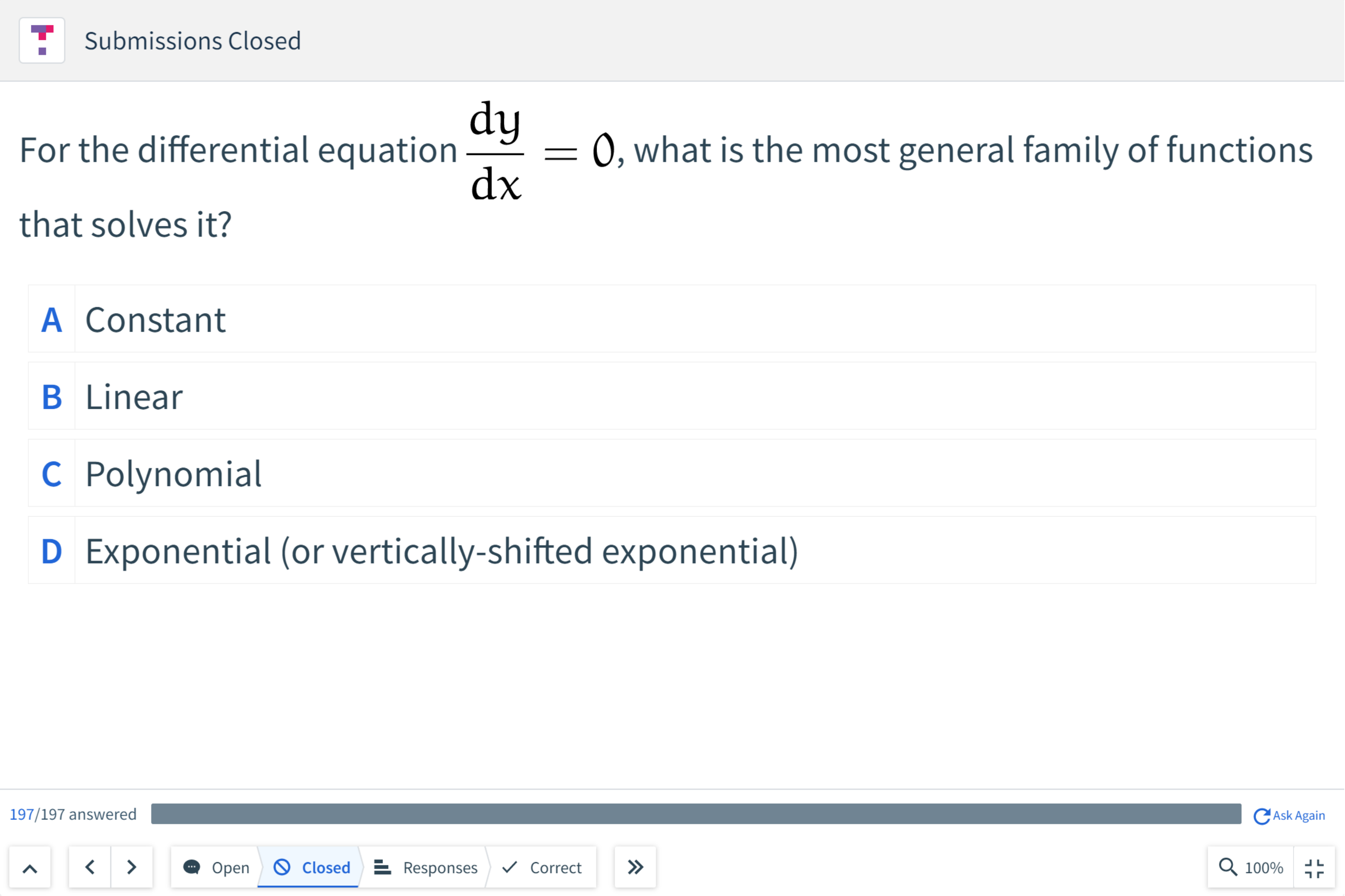
A Constant

B Linear

C Polynomial

D Exponential

208/208 answered



Submissions Closed

For the differential equation $\frac{dy}{dx} = 0$, what is the most general family of functions that solves it?

- A Constant
- B Linear
- C Polynomial
- D Exponential (or vertically-shifted exponential)

197/197 answered

Cats Jumping

Roy sneaks up on Blackie, and surprises him with a loud meow. Blackie jumps straight into the air at a speed of $3m/s$.

1 min. Write a differential equation that involves Blackie's velocity (in m/s) while he's in the air.

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1 min. What is a family of functions that satisfy the above equation? $v(t) = -9.8t + C$

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1 min. What is a family of functions that satisfy the above equation? $v(t) = -9.8t + C$

1 min. What is the appropriate constant to choose?

Cats Jumping

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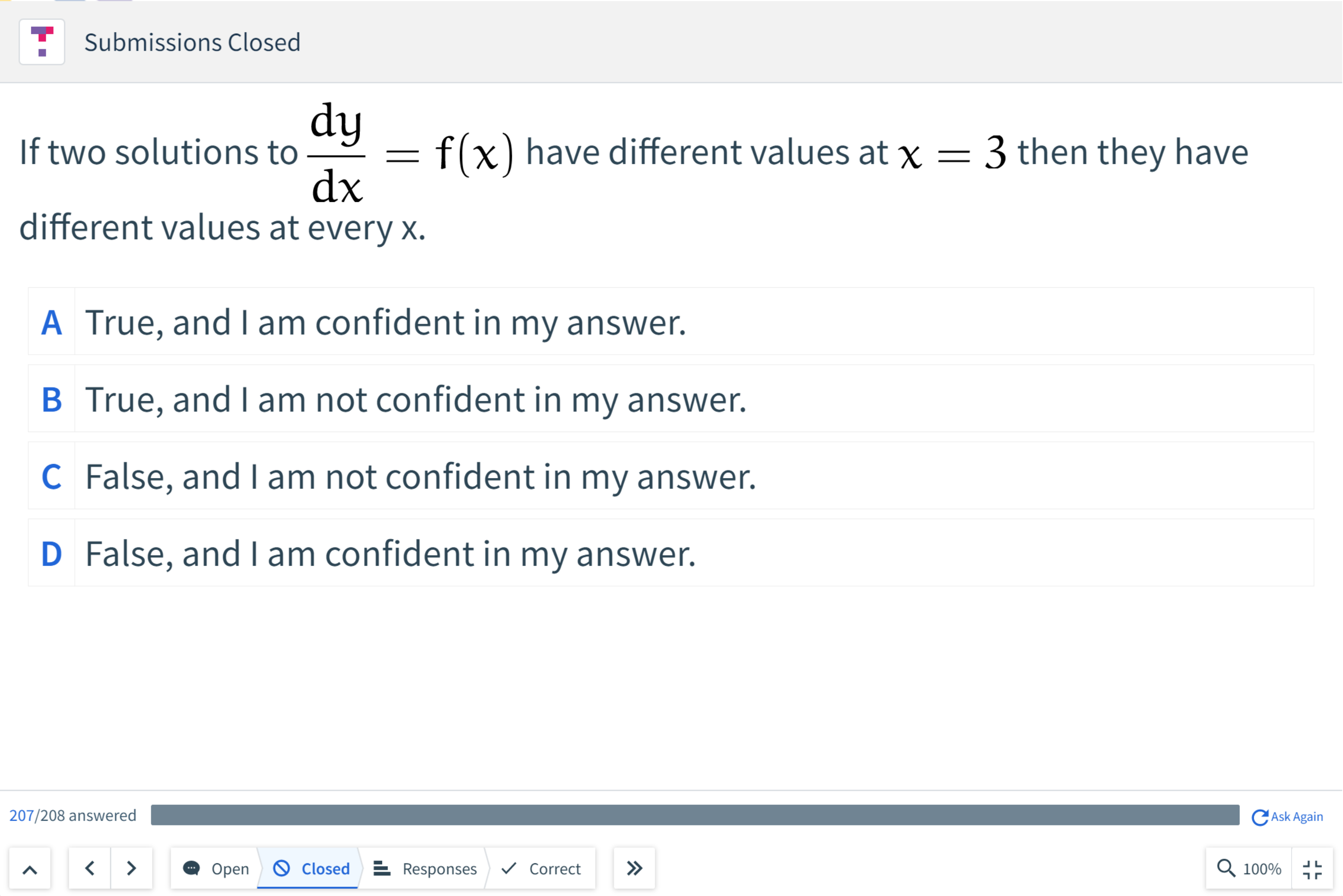
1 min. What is the appropriate constant to choose? $C = 3$ because $v(0) = 3m/s$



Submissions Closed

If two solutions to $\frac{dy}{dx} = f(x)$ have different values at $x = 3$ then they have different values at every x.

A True, and I am confident in my answer.

B True, and I am not confident in my answer.

C False, and I am not confident in my answer.

D False, and I am confident in my answer.

207/208 answered

Cats Jumping

Roy sneaks up on Blackie, and surprises him with a loud meow. Blackie jumps straight into the air at a speed of $3m/s$. We know that Blackie's velocity, $v(t) = 3 - 9.8t$, measured in m/s.

1 min. Write a differential equation that involves Blackie's height above the ground (in m) while he's in the air.

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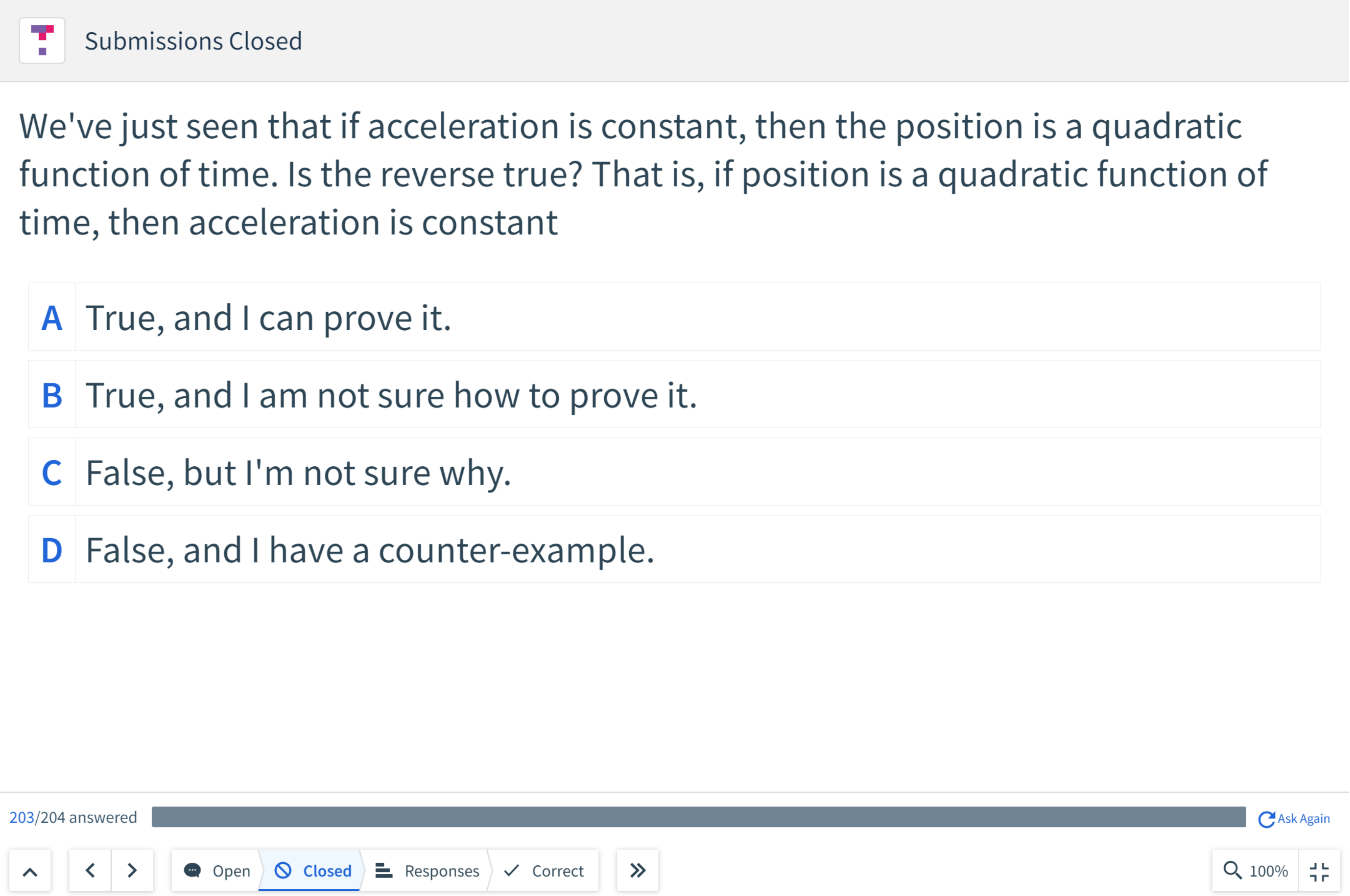
1 min. What is the appropriate constant to choose? $D = 0$ because Blackie starts on the ground.



Submissions Closed

We've just seen that if acceleration is constant, then the position is a quadratic function of time. Is the reverse true? That is, if position is a quadratic function of time, then acceleration is constant

- **A** True, and I can prove it.
- **B** True, and I am not sure how to prove it.
- **C** False, but I'm not sure why.
- **D** False, and I have a counter-example.

203/204 answered

Cats Jumping

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Cats Jumping

Roy sneaks up on Blackie, and surprises him with a loud meow. Blackie jumps straight into the air at a speed of $3m/s$.

Spend one minute writing a list of steps (from the start of the question to its finish) outlining how you could compute how high Blackie jumps.

PCats Jumping – The Steps

- Read the question
- Write the differential equation
- Find a family of solutions to the differential equation
- Find the right constants, and narrow down the family to one function
- Repeat the last three steps until we have the desired function (in our case, it was the height function)
- Optimize

Plans for the Future

For next time:

WeBWork 6.4 and read section 6.4

Welcome to MAT135 LEC0501 (Assaf)

Now is a good time to think about the midterm!

S6.4 – The Other Fundamental Theorem – The Construction Theorem

Assaf Bar-Natan

“Try to change.
I try to change.
I make a list of all the ways to change my ways.
But I stay the same,
I stay the s-ame.”

–“Try To Change”, Mother Mother

Jan. 22, 2020

Functions Defined by Integrals

We've encountered many functions in MAT135, and in this course:

- Some functions are given as tables, verbally, or as a graph...
- Some functions are defined algebraically or geometrically: trigonometric, polynomials, exponents..
- Some functions are inverses or compositions of others: $\sin(e^{3x+5})$, $\log(x)$, $\log^2(x)$...

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- Some functions are inverses or compositions of others: $\sin(e^{3x+5})$, $\log(x)$, $\log^2(x)$...

Today: Functions defined as integrals of other functions:

$$f(x) = \int_a^x g(t)dt$$

where a is some constant.

Functions Defined by Integrals

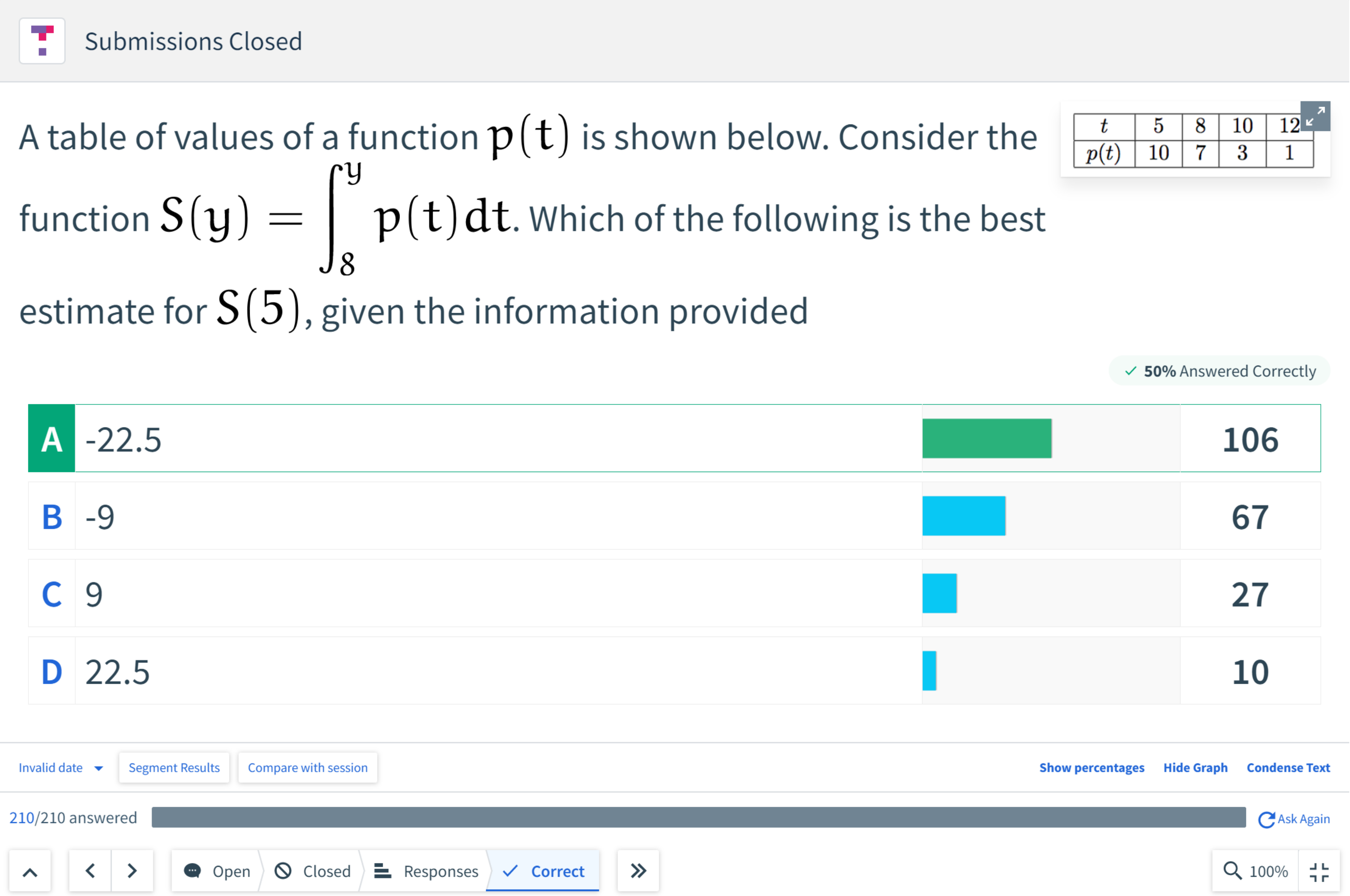
Some examples:

$$Si(x) = \int_0^x \frac{\sin(t)}{t} dt$$

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$li(x) = \int_0^x \frac{1}{\log(t)} dt$$

(log is the natural logarithm here)



Submissions Closed

A table of values of a function $p(t)$ is shown below. Consider the function $S(y) = \int_8^y p(t)\,dt$. Which of the following is the best estimate for $S(5)$, given the information provided

t	5	8	10	12
$p(t)$	10	7	3	1

✓ **50%** Answered Correctly

A	-22.5	106
B	-9	67
C	9	27
D	22.5	10

210/210 answered

What's The Difference

Let's say that we have a function, $f(x)$. In groups, write an explanation of the difference between:

- A definite integral of f.
- The antiderivatives of f.

- A function defined by an integral of f.

What's The Difference

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- A function defined by an integral of f.

Hint: think about the definitions!

What's The Difference

Let's say that we have a function, $f(x)$. In groups, write an explanation of the difference between:

- A definite integral of f. **This is a number.**
- The antiderivatives of f. **This is a family of functions whose derivative is f.**
- A function defined by an integral of f. **This is a function defined by an expression of the form $\int_a^x f(t)dt$.**

Hint: think about the definitions!

The Construction Theorem

Let $f(t)$ be a continuous function defined everywhere, and we will write $F(x) = \int_a^x f(t)dt$.

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Write the limit definition of the derivative of F

The Construction Theorem

Let $f(t)$ be a continuous function defined everywhere, and we will write $F(x) = \int_a^x f(t)dt$. We have:

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

We can rewrite this as:

$$F'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)dt$$

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Explain why we can do this to your neighbour

The Construction Theorem

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Draw a picture representing the integral above as a small rectangle. What does the limit above equal to?

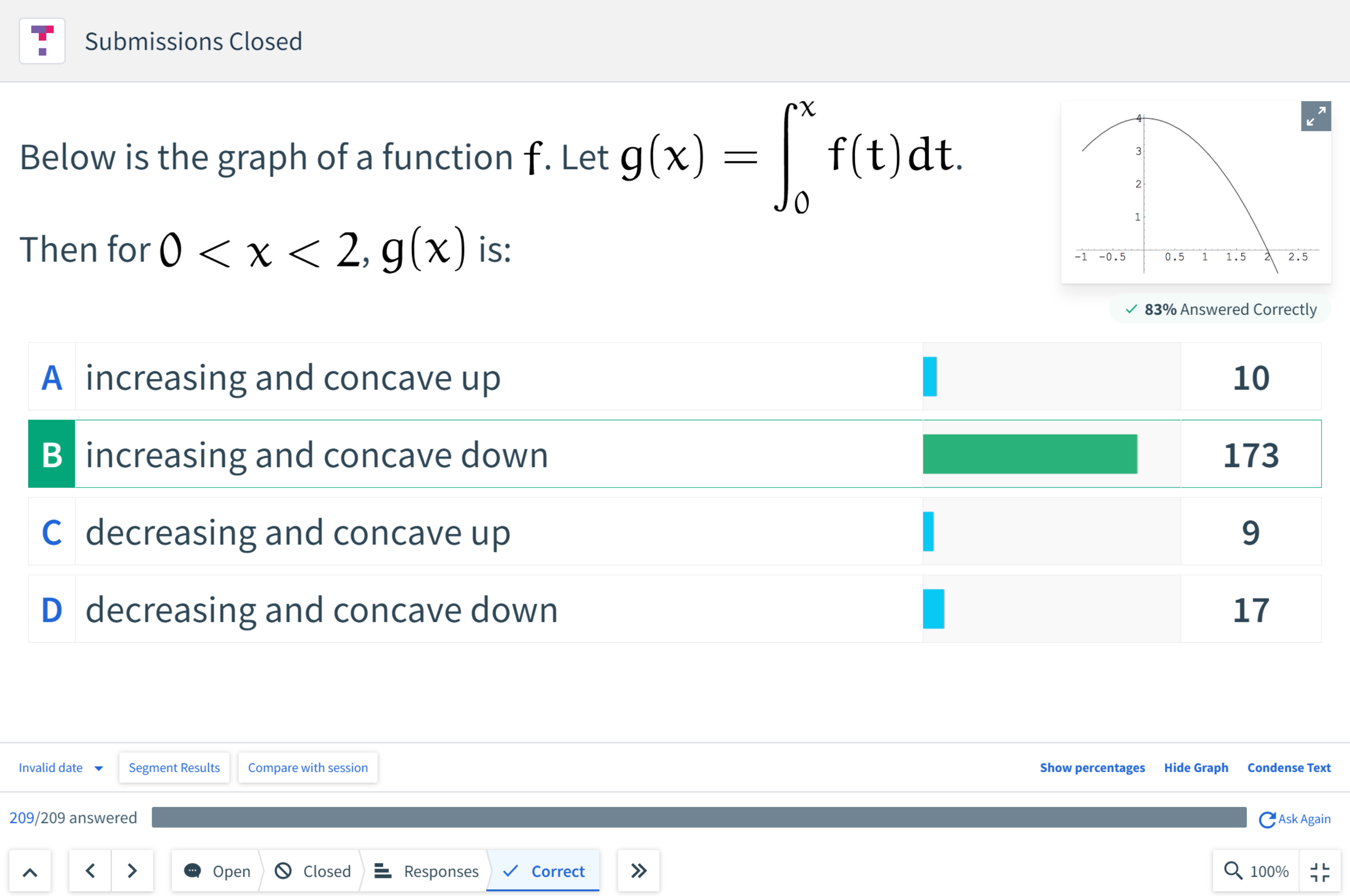
The Construction Theorem

Theorem

(Construction Theorem, or, the Second Fundamental Theorem of Calculus)
If f is continuous, then the function defined by the integral $F(x) = \int_a^x f(t)dt$ *satisfies* $F'(x) = f(x)$.

Takeaway

Functions defined by integrals are antiderivatives of the integrands



Submissions Closed

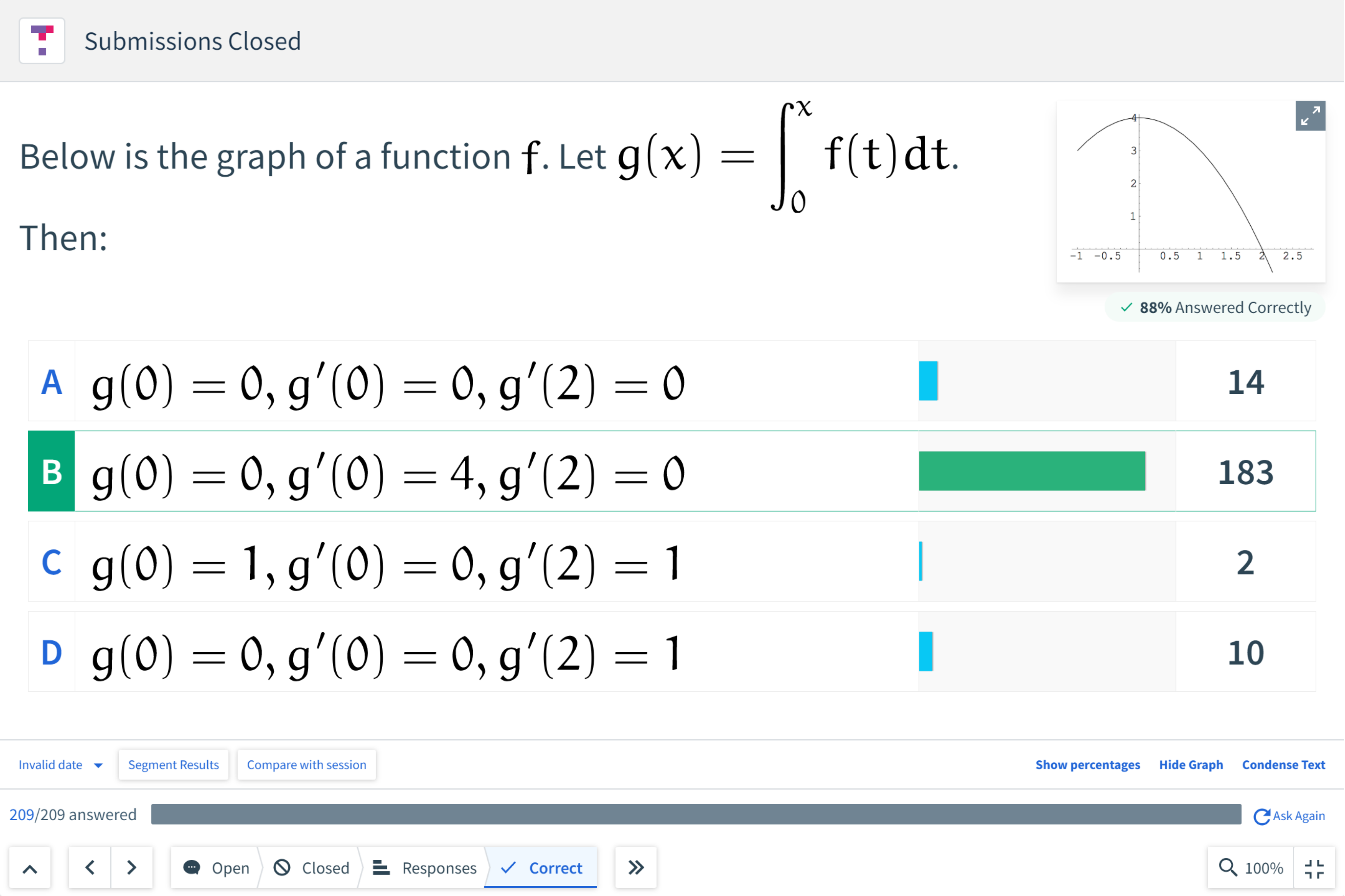
Below is the graph of a function f. Let $g(x) = \int_0^x f(t)\,dt$.

Then for $0 < x < 2$, $g(x)$ is:

✓ 83% Answered Correctly

A	increasing and concave up	10
B	increasing and concave down	173
C	decreasing and concave up	9
D	decreasing and concave down	17

209/209 answered



Submissions Closed

Below is the graph of a function f. Let $g(x) = \int_0^x f(t)dt$.

Then:

✓ 88% Answered Correctly

A	$g(0) = 0, g'(0) = 0, g'(2) = 0$	14
B	$g(0) = 0, g'(0) = 4, g'(2) = 0$	183
C	$g(0) = 1, g'(0) = 0, g'(2) = 1$	2
D	$g(0) = 0, g'(0) = 0, g'(2) = 1$	10

209/209 answered

Hard Derivatives

We define:

$$F(x) = \int_5^{e^x} \frac{\sin(t)}{t} dt$$

Our goal is to find $F'(x)$.

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Our goal is to find $F'(x)$.

- Use $Si(x)$, and the net change theorem to write $F(x)$ explicitly.
- Use differentiation rules to compute $F'(x)$
- Use the construction theorem to simplify

Bonus: replace $\frac{\sin(t)}{t}$ with $\sin(t^3)$. How does your solution change?

Hard Derivatives

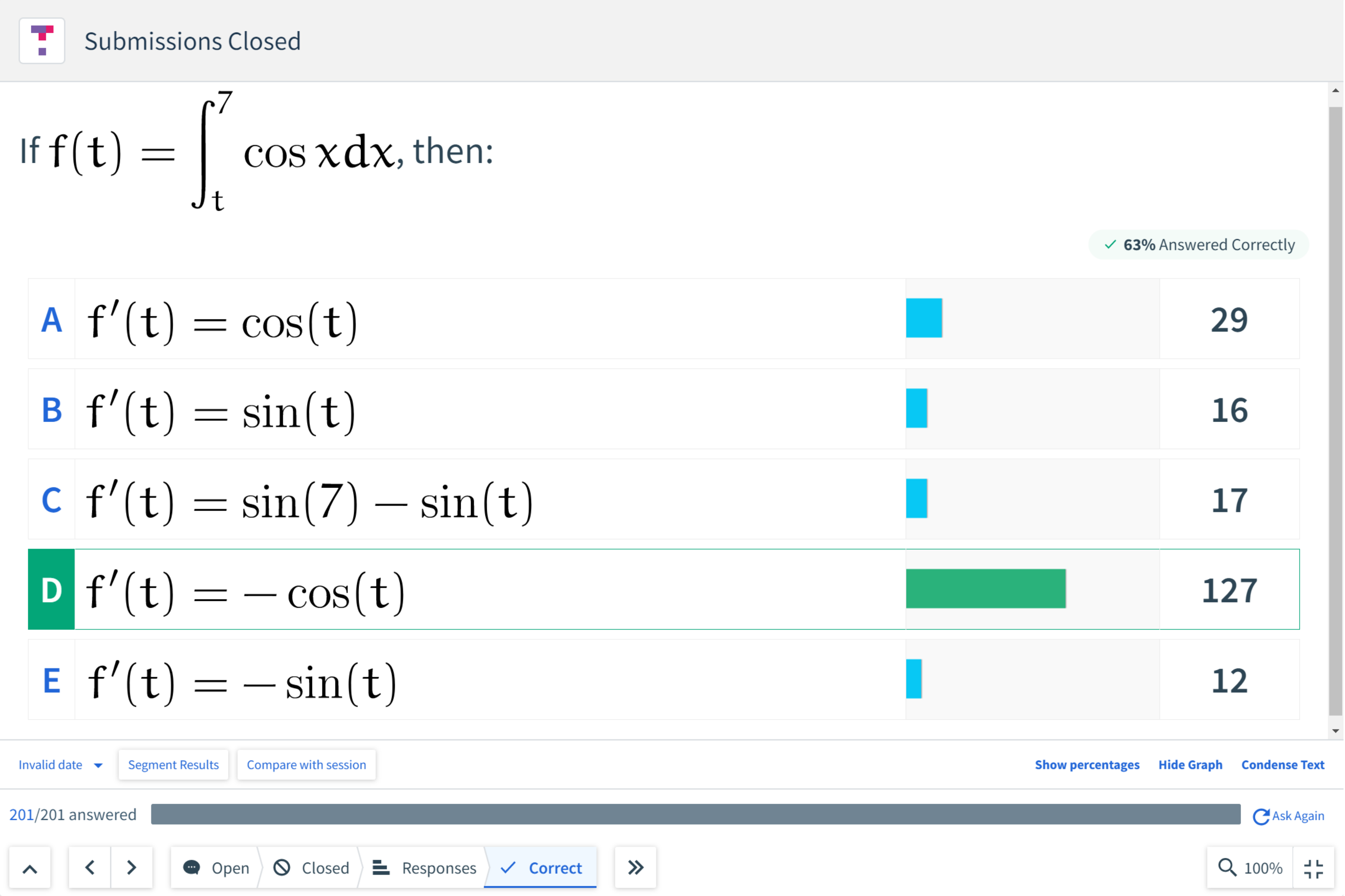
We define:

$$F(x) = \int_5^{e^x} \frac{\sin(t)}{t} dt$$

Our goal is to find $F'(x)$.

- Use $Si(x)$, and the net change theorem to write $F(x)$ explicitly. $F(x) = Si(e^x) - Si(5)$
- Use differentiation rules to compute $F'(x)$. **By the chain rule:** $F'(x) = Si'(e^x) \cdot e^x$
- Use the construction theorem to simplify. **Since** $Si(x)$ **is an antiderivative of** $\frac{\sin(x)}{x}$**, we get:** $F'(x) = \frac{\sin(e^x)}{e^x} e^x = \sin(e^x)$

Bonus: replace $\frac{\sin(t)}{t}$ with $\sin(t^3)$. How does your solution change?
We get $\sin(e^{3x})e^x$



Submissions Closed

If $f(t) = \int_t^7 \cos x dx$, then:

✓ 63% Answered Correctly

A	$f'(t) = \cos(t)$	29
B	$f'(t) = \sin(t)$	16
C	$f'(t) = \sin(7) - \sin(t)$	17
D	$f'(t) = -\cos(t)$	127
E	$f'(t) = -\sin(t)$	12

201/201 answered

Plans for the Future

For next time:
WeBWork 7.1 and read section 7.1

Welcome to MAT135 LEC0501 (Assaf)

Critical Incident Questionnaire:

`https://tinyurl.com/Unit1CIQ`

If you've done this, here's two challenging integrals (answers next week):

$$\int \sin(e^t)dt$$

$$\int \sqrt{\tan(x)}dx$$

S7.1 – Integration Methods – Substitution

Assaf Bar-Natan

“You don’t have to feel like a waste of space
You’re original, cannot be replaced.”

–“Firework”, Katy Perry

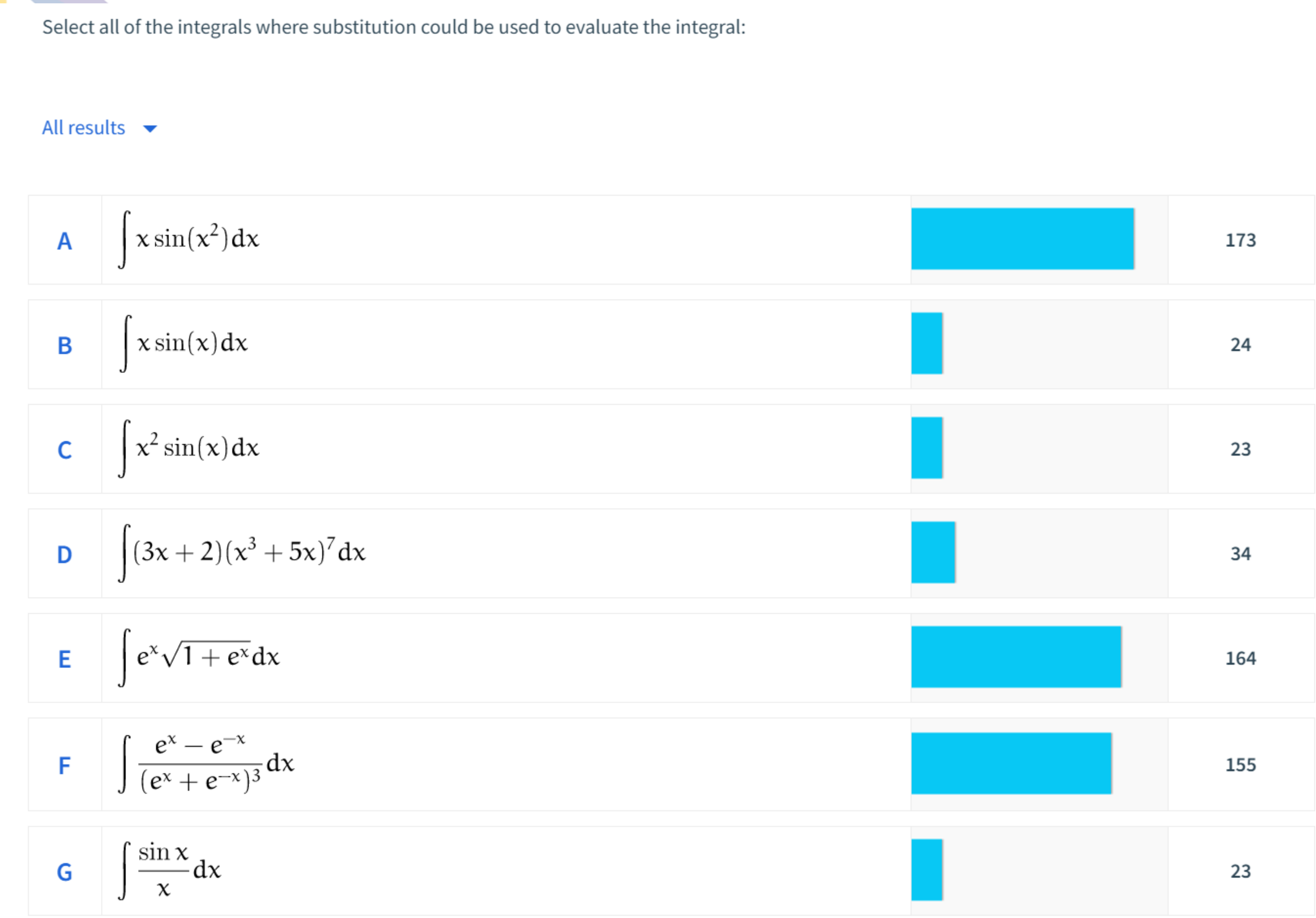
Jan. 24, 2020

Reading Comprehension

The substitution technique tells us that if F is an antiderivative of f, then ______ is an antiderivative of $f(g)g'$.

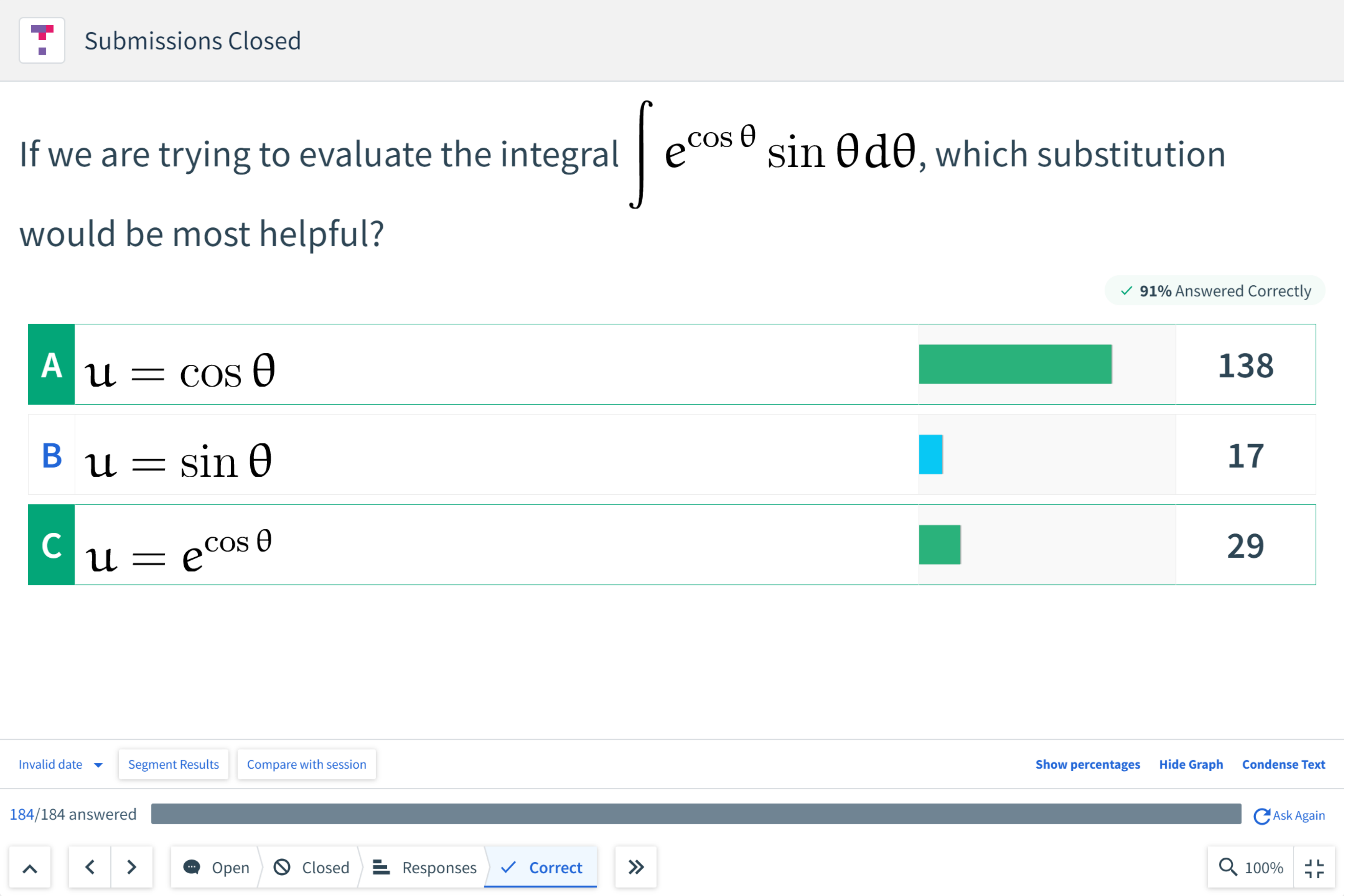
Takeaway

When faced with an integral that has a function g inside another function, try a substitution.



Select all of the integrals where substitution could be used to evaluate the integral:

A	$\int x\sin(x^2)dx$	173
B	$\int x\sin(x)dx$	24
C	$\int x^2\sin(x)dx$	23
D	$\int (3x+2)(x^3+5x)^7dx$	34
E	$\int e^x\sqrt{1+e^x}dx$	164
F	$\int \frac{e^x-e^{-x}}{(e^x+e^{-x})^3}dx$	155
G	$\int \frac{\sin x}{x}dx$	23



Submissions Closed

If we are trying to evaluate the integral $\int e^{\cos\theta}\sin\theta\, d\theta$, which substitution would be most helpful?

✓ **91%** Answered Correctly

A	$u = \cos\theta$	138
B	$u = \sin\theta$	17
C	$u = e^{\cos\theta}$	29

184/184 answered

A Simple Substitution

Find an antiderivative, F of $\int e^{\cos\theta} \sin\theta d\theta$, with $F(0) = 0$.

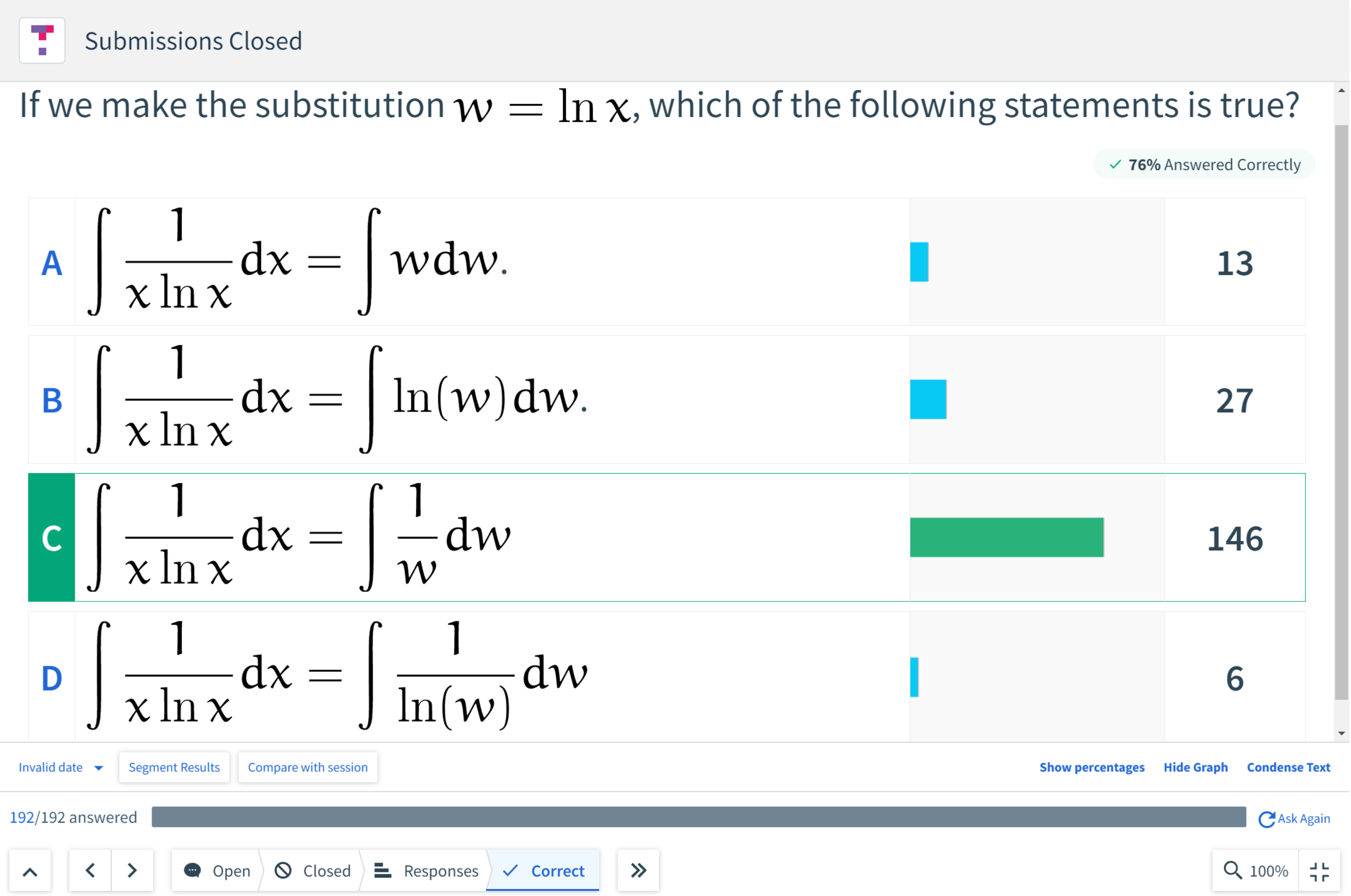
A Simple Substitution

Find an antiderivative, F of $\int e^{\cos\theta}\sin\theta d\theta$, with $F(0)=0$.
We substitute $\cos(\theta)=u$. Then $\frac{du}{d\theta}=-\sin(\theta)$. Thus,

$$\int e^{\cos\theta}\sin(\theta)d\theta=\int e^{u(\theta)}(-u'(\theta))d\theta=-e^{u(\theta)}$$

Thus, all of the antiderivatives of $e^{\cos\theta}\sin(\theta)$ are of the form $-e^{\cos\theta}+C$. To find the appropriate C, we plug in $\theta=0$, and solve, to get:

$$F(\theta)=-e^{\cos\theta}+e$$



Submissions Closed

If we make the substitution $w = \ln x$, which of the following statements is true?

✓ **76%** Answered Correctly

A	$\int \frac{1}{x\ln x}dx = \int w dw.$	13
B	$\int \frac{1}{x\ln x}dx = \int \ln(w) dw.$	27
C	$\int \frac{1}{x\ln x}dx = \int \frac{1}{w} dw$	146
D	$\int \frac{1}{x\ln x}dx = \int \frac{1}{\ln(w)} dw$	6

192/192 answered

Substituting Back

Compute:

$$\int \frac{1}{x(\log(x))^2}$$

Where log is the natural logarithm.

Substituting Back

Compute:

$$\int \frac{1}{x(\log(x))^2}$$

Where log is the natural logarithm.

$$\int \frac{1}{x(\log(x))^2} dx = \frac{-1}{\log(x)} + C$$

We can verify this by differentiating.

Spot the Error

Blackie is sitting in the yard, lying on his back in the sun, computing integrals. That's just what cats do. He writes:

Spot the Error

Blackie is sitting in the yard, lying on his back in the sun, computing integrals. That's just what cats do. He writes:

> To compute $\int_1^4 \sqrt{1+\sqrt{x}}dx$, I will let $w = 1+\sqrt{x}$, so $\frac{dw}{dx} = \frac{1}{2\sqrt{x}}$. Thus,
>
> $$dx = dw(2\sqrt{x}) = dw(2(w-1))$$
>
> Plugging this in, I get:
>
> $$\begin{aligned}\int_1^4 \sqrt{1+\sqrt{x}}dx &= \int_1^4 \sqrt{w}(2(w-1))dw \\ &= \int_1^4 (2w^{3/2} - 2w^{1/2})dw \\ &= \left[2\frac{2}{5}w^{5/2} - 2\frac{2}{3}w^{3/2}\right]_1^4\end{aligned}$$

Takeaway

When substituting in a definite integral, don't forget to change your bounds!

Lexi and Obie and Mouse

Lexi, the tail-less cat, is meowing at Obie for stealing her mouse. As her mouth opens, the size, S, of the opening changes as a function of time: $S = g(t)$.

Lexi and Obie and Mouse

Lexi, the tail-less cat, is meowing at Obie for stealing her mouse. As her mouth opens, the size, S, of the opening changes as a function of time: $S = g(t)$.

Let m be the volume of the meow. Denote by $\frac{dm}{dS} = f(S)$, and let Δm be the change in meow volume between $1s$ and $2s$.

Lexi and Obie and Mouse

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Let m be the volume of the meow. Denote by $\frac{dm}{dS} = f(S)$, and let Δm be the change in meow volume between $1s$ and $2s$.

Fill in the following:

$$\Delta m = \int_{\square}^{\square} f(g(t))g'(t)dt$$

$$\Delta m = \int_{\square}^{\square} f(s)ds$$

$$\Delta m = \int_{\square}^{\square} dm$$

Lexi and Obie and Mouse

$$\Delta m = \int_1^2 f(g(t))g'(t)dt$$

$$\Delta m = \int_{g(1)}^{g(2)} f(s)ds$$

$$\Delta m = \int_{m(g(1))}^{m(g(2))} dm$$

Plans for the Future

For next time:

WeBWork 7.2 and read section 7.2

Welcome to MAT135 LEC0501 (Assaf)

What is the integral $\int \frac{1}{\textbf{cabin}} d\textbf{cabin}$?

Two challenging integrals from last week:

$$\int \sin(e^t)dt = Si(e^x) + C$$

For $\int \sqrt{\tan(x)}$, substitute $u = \tan(x)$ to get:

$$\int \frac{\sqrt{u}}{u^2+1} = ???$$

This is very hard. Further developments next week.

S7.2 – Integration Methods – Integration by Parts

Assaf Bar-Natan

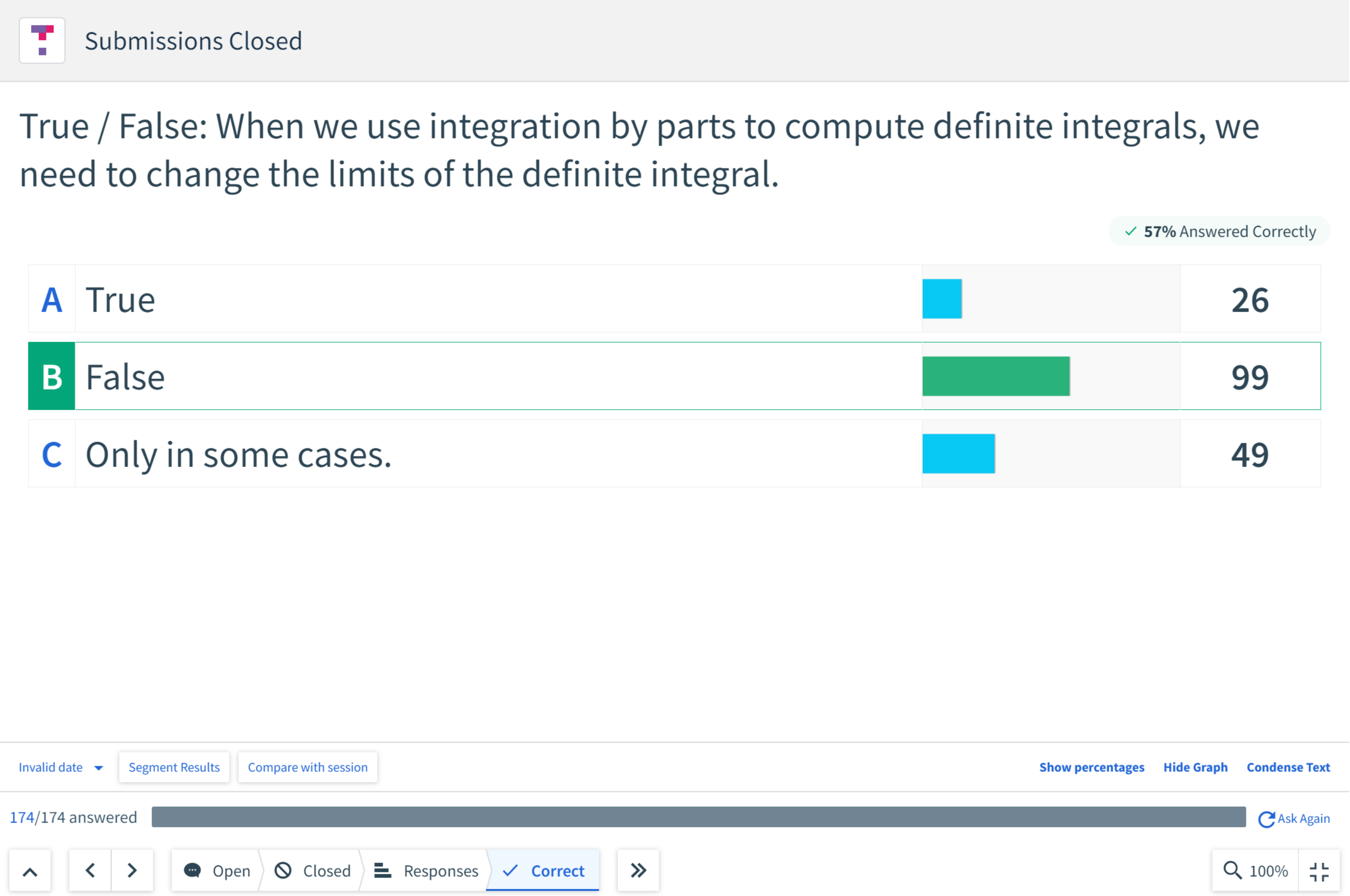
"Sometimes I lie awake, night after night
Coming apart at the seams
Eager to please, ready to fight
Why do I go to extremes?"

–"Why Do I Go To Extremes", Billy Joel

Jan. 27, 2020

Reading Comprehension

- The differentiation rule that gives us integration by parts is the ______ rule.
- The integration by parts technique tells us that $\int uv' dx =$ ______ $-$ ______.



Submissions Closed

True / False: When we use integration by parts to compute definite integrals, we need to change the limits of the definite integral.

✓ 57% Answered Correctly

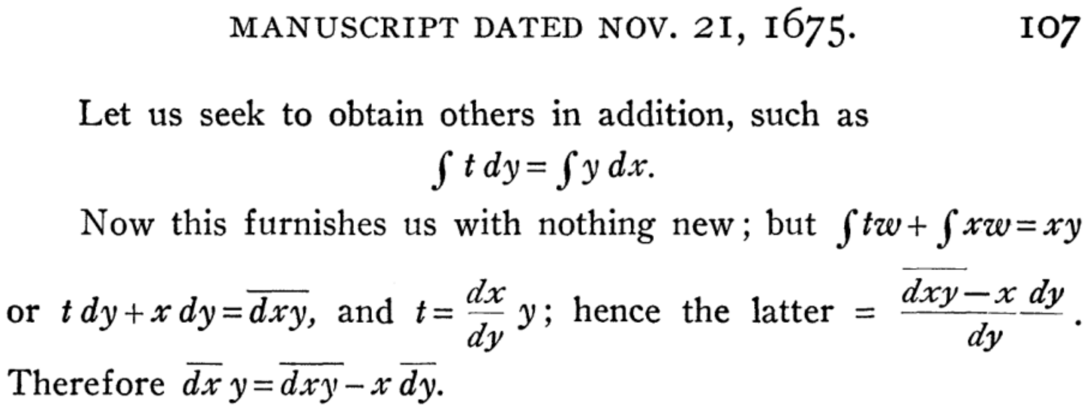
A	True	26
B	False	99
C	Only in some cases.	49

174/174 answered

Takeaway

When faced with an integral that is a product of functions, try integration by parts. (Sometimes this will work for other functions too)

Leibniz and The Product Rule



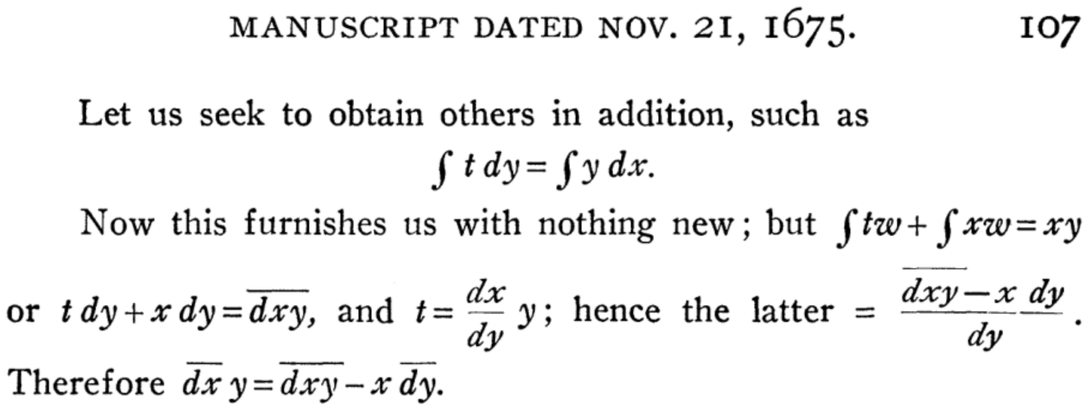
MANUSCRIPT DATED NOV. 21, 1675. 107

Let us seek to obtain others in addition, such as

$$\int t\,dy = \int y\,dx.$$

Now this furnishes us with nothing new; but $\int tw + \int xw = xy$ or $t\,dy + x\,dy = \overline{dxy}$, and $t = \dfrac{dx}{dy}\,y$; hence the latter $= \dfrac{\overline{dxy} - x\,dy}{dy}$. Therefore $\overline{dx}\,y = \overline{dxy} - x\,\overline{dy}$.

Leibniz and The Product Rule



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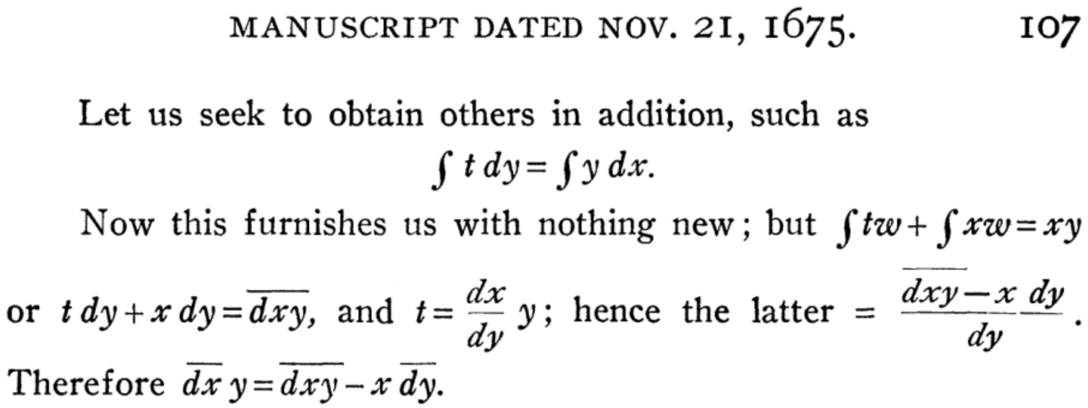
- dx means “the derivative of x”
- $\overline{xy}$ means (xy)

Leibniz and The Product Rule



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Therefore $\overline{dx}\,y = \overline{dxy} - x\,\overline{dy}$.

- dx means "the derivative of x"
- $\overline{xy}$ means (xy)

Leibniz used the fundamental theorem and integration by parts to conclude the product rule!

Building Fur – I.B.P Example

A cold snap hits the cats, and Mia's body starts building up her fur at a rate of $f(t)$ pounds per day. If $f(t) = 0.5 * t^2 e^{-t}$, how much hair has she built up after ten days?

Building Fur – I.B.P Example

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- Integrate by parts. What should be u? What should be v?
- Does this simplify the question?

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We wish to compute:

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gives:

$$\int_0^{10} 0.5t^2e^{-t}dt = \left[0.5t^2(-e^{-t})\right]_0^{10} + \int_0^{10} te^{-t}dt$$

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$$\begin{aligned}\int_0^{10} 0.5t^2 e^{-t} dt &= \left[0.5t^2(-e^{-t})\right]_0^{10} + \int_0^{10} te^{-t} dt \\ &= \left[0.5t^2(-e^{-t})\right]_0^{10} + \left[t(-e^{-t})\right]_0^{10} + \int_0^{10} e^{-t} dt\end{aligned}$$

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Does this simplify the question enough to solve?

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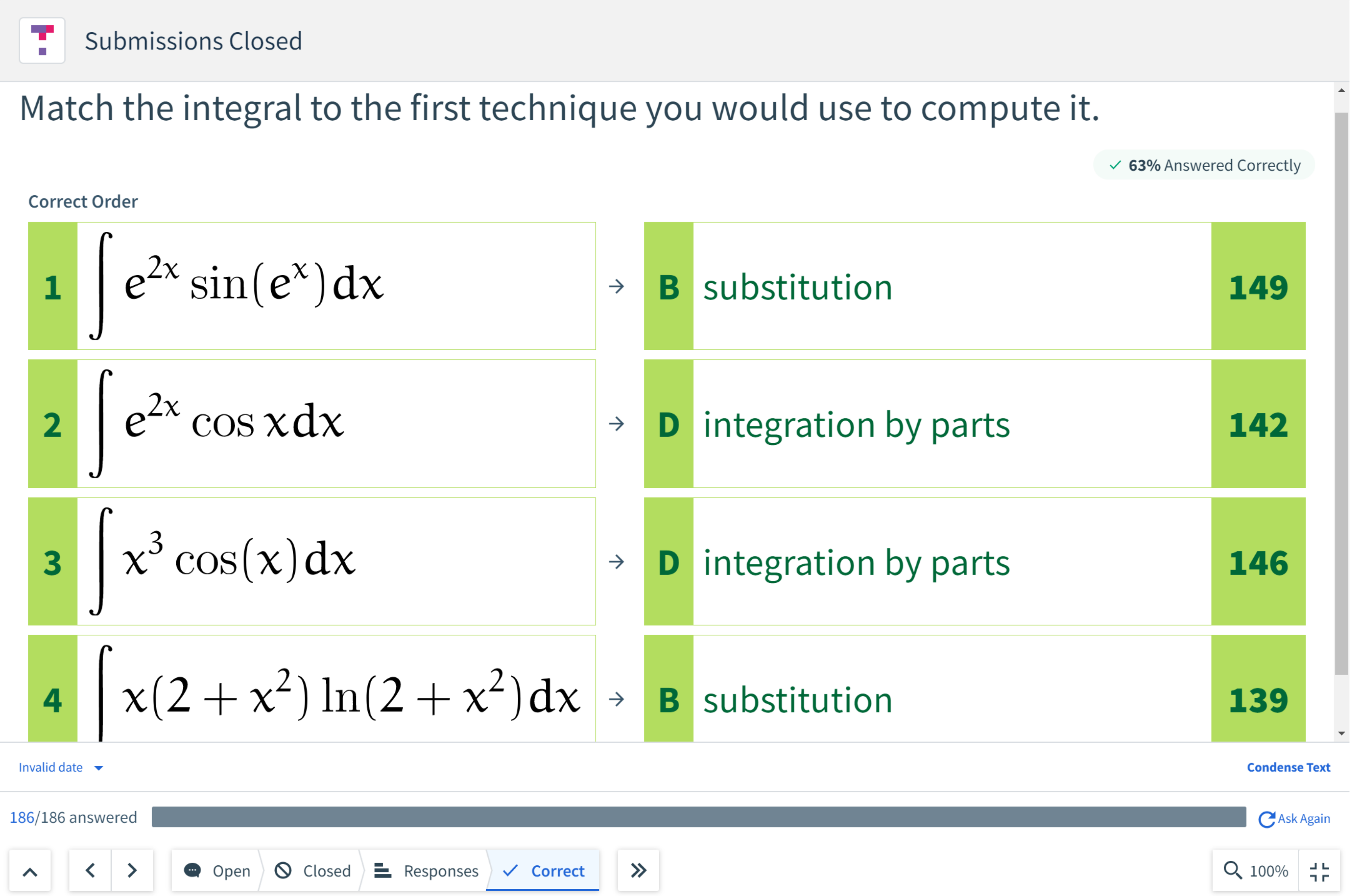
$$\begin{aligned}\int_0^{10} 0.5t^2e^{-t}dt &= \left[0.5t^2(-e^{-t})\right]_0^{10} + \int_0^{10} te^{-t}dt \\ &= \left[0.5t^2(-e^{-t})\right]_0^{10} + \left[t(-e^{-t})\right]_0^{10} + \int_0^{10} e^{-t}dt\end{aligned}$$

Does this simplify the question enough to solve? Evaluating the last integral, and plugging in the bounds gives $\int_0^{10} 0.5t^2e^{-t}dt \approx 0.997$.



Submissions Closed

Match the integral to the first technique you would use to compute it.

✓ **63%** Answered Correctly

Correct Order

1	$\int e^{2x}\sin(e^{x})\,dx$	→	B	substitution	149
2	$\int e^{2x}\cos x\,dx$	→	D	integration by parts	142
3	$\int x^{3}\cos(x)\,dx$	→	D	integration by parts	146
4	$\int x(2+x^{2})\ln(2+x^{2})\,dx$	→	B	substitution	139

Invalid date

Condense Text

186/186 answered

Takeaway

Integration by parts is useful when there is a product of functions, and we want one of them to “disappear”.

dETAILS Mnemonic

$$\int uv'dx = uv - \int vu'dx$$

Here is a mnemonic for what functions to use for v' (read backwards for what functions to use as u)

- d erivative function (ie, the v' in $\int uv' = uv - \int u'v$)

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- A lgebraic (ie polynomials, ratios of polynomials)
- I nverse trigonometric
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- **S** pecial functions (like $Si(x)$)

Bonus: find $\int xSi(x)dx$.

Integration by Parts – Functions Given Strangely

Let's say we have two functions, f, and g. g is given as a table of values, and f is given as a formula. Which of the following are easy to estimate?

- $f \cdot g$ evaluated at some point
- $\int_a^b f'g dx$
- $\int_a^b fg' dx$
- $\int_a^b f'g' dx$

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- $\int_a^b f'g dx$
- $\int_a^b fg' dx$
- $\int_a^b f'g' dx$

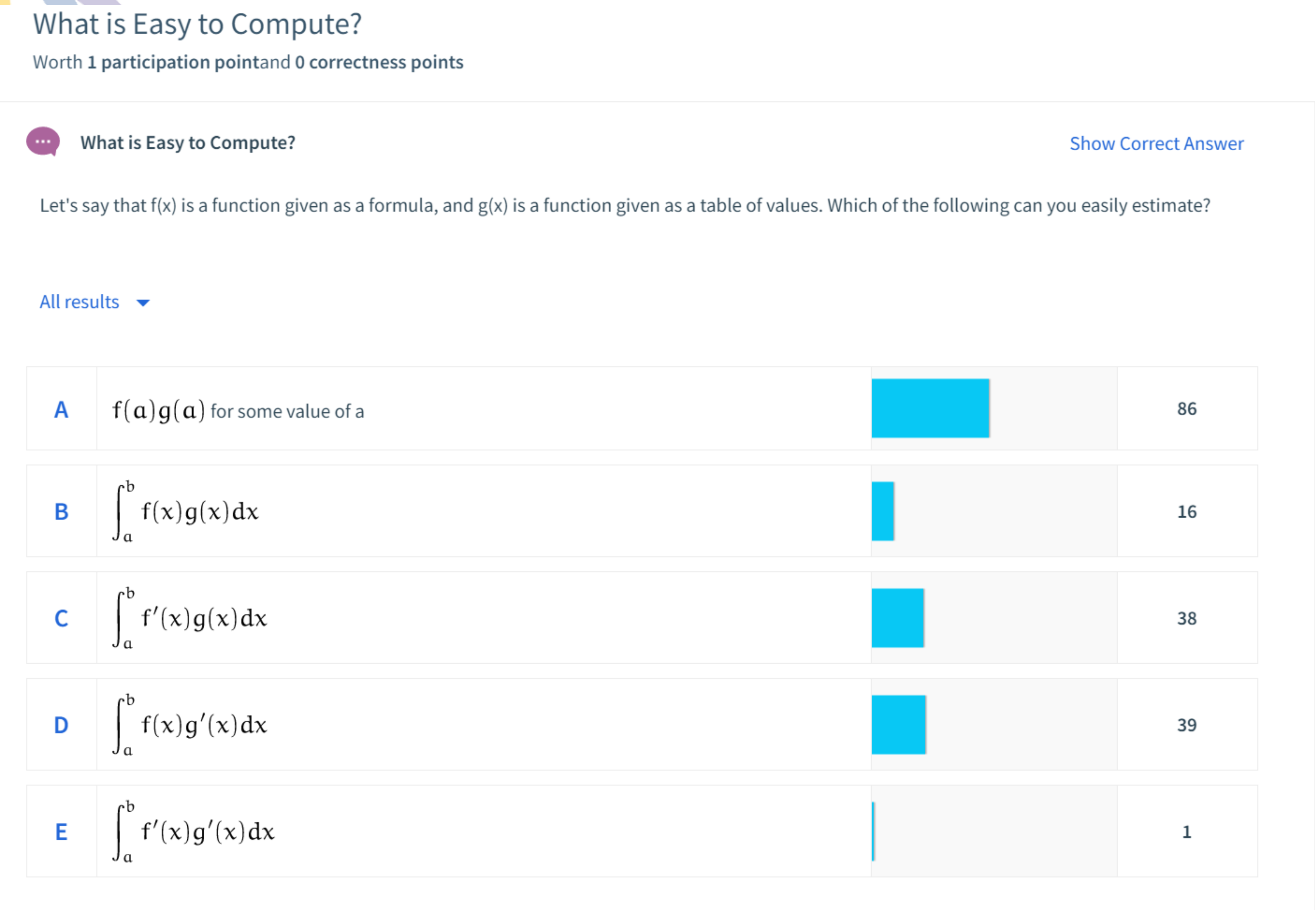
Hint: For which of these integrands can you write a table of values?

Integration by Parts – Functions Given Strangely

Let's say we have two functions, f, and g. g is given as a table of values, and f is given as a formula.

$$\int_a^b f(x)g'(x) = [fg]_a^b - \int_a^b f'(x)g(x)dx$$

We can now write a table for $f'(x)$, for $g(x)$, and $f'(x)g(x)$, and estimate the integral on the right.



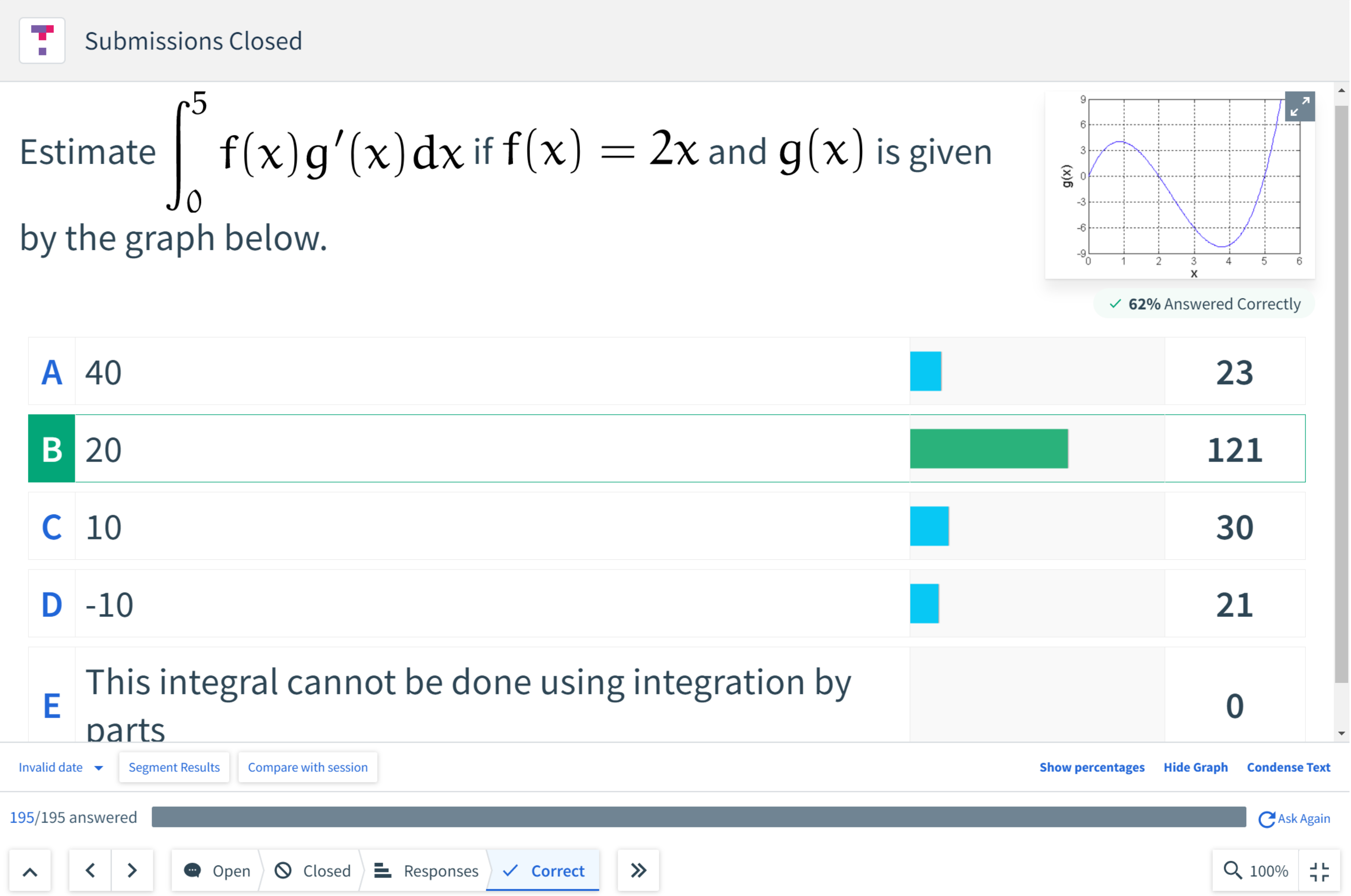
What is Easy to Compute?

Worth **1 participation point**and **0 correctness points**

What is Easy to Compute?

Let's say that f(x) is a function given as a formula, and g(x) is a function given as a table of values. Which of the following can you easily estimate?

A	$f(a)g(a)$ for some value of a	86
B	$\int_a^b f(x)g(x)dx$	16
C	$\int_a^b f'(x)g(x)dx$	38
D	$\int_a^b f(x)g'(x)dx$	39
E	$\int_a^b f'(x)g'(x)dx$	1



Submissions Closed

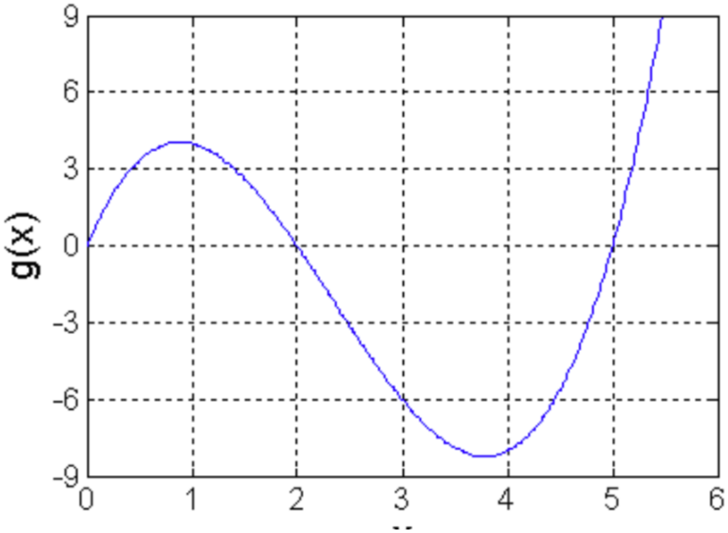
Estimate $\int_0^5 f(x)g'(x)\,dx$ if $f(x) = 2x$ and $g(x)$ is given by the graph below.

✓ **62%** Answered Correctly

A	40	23
B	20	121
C	10	30
D	-10	21
E	This integral cannot be done using integration by parts	0

195/195 answered

Graphical Estimation



$$\int_0^5 f(x)g'(x) = f(5)g(5) - f(0)g(0) - \int_0^5 g(x)f'(x)dx$$

$$= -\int_0^5 2g(x)$$

Spot The Error

The Calculus Cats find a note on the floor. It reads:

> To compute $\int \tan(x)dx$, we integrate by parts.
>
> $$u = \frac{1}{\cos(x)} \qquad v' = \sin(x)$$
> $$u' = \tan(x)\sec(x) \qquad v = -\cos(x)$$
>
> so
>
> $$\int \tan(x)dx = \int uv'dx = uv - \int vu'dx = -1 + \int \tan(x)$$
>
> Simplifying, we get $0 = -1$.

The cats are stressed by this, to say the least. Can you help them?

Spot The Error

The Calculus Cats find a note on the floor. It reads:

To compute $\int_{\pi/6}^{\pi/4} \tan(x)dx$, we integrate by parts.

$$u = \frac{1}{\cos(x)} \qquad v' = \sin(x)$$
$$u' = \tan(x)\sec(x) \qquad v = -\cos(x)$$

so

$$\int_{\pi/6}^{\pi/4} \tan(x)dx = \int_{\pi/6}^{\pi/4} uv'dx = uv - \int_{\pi/6}^{\pi/4} vu'dx = -1 + \int_{\pi/6}^{\pi/4} \tan(x)$$

Simplifying, we get $0 = -1$.

The cats are even more stressed by this. Can you help them?

Plans for the Future

For next time:

Integration and Computer Algebra Systems. No reading, but bring a computer, and review what we've done!

Welcome to MAT135 LEC0501 (Assaf)

Challenge: compute the integral:

$$\int Si(x)dx$$

The integral from last class: $\int xSi(x)dx$. Integrate by parts, letting $u = Si(x)$ and $v' = x$. This gives:

$$\int xSi(x)dx = \frac{x^2}{2}Si(x) - \frac{1}{2}\int x\sin(x)dx$$

Integrating by parts again yields:

$$\int xSi(x)dx = \frac{x^2}{2}Si(x) - \sin(x) + x\cos(x) + C$$

Computer Algebra Systems & Taylor Approximations

Assaf Bar-Natan

> “It’s automated computer speech
> It’s automated computer speech
> It’s a Casio on a plastic beach
> It’s a Casio”
>
> –“Plastic Beach”, Gorillaz

Jan. 29, 2020

Functions Defined by Integrals

Recall: we can define functions using integrals. For example:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$F(x) = \int_0^x (1+t)e^t \sqrt{1+t^2 e^{2t}} dt$$

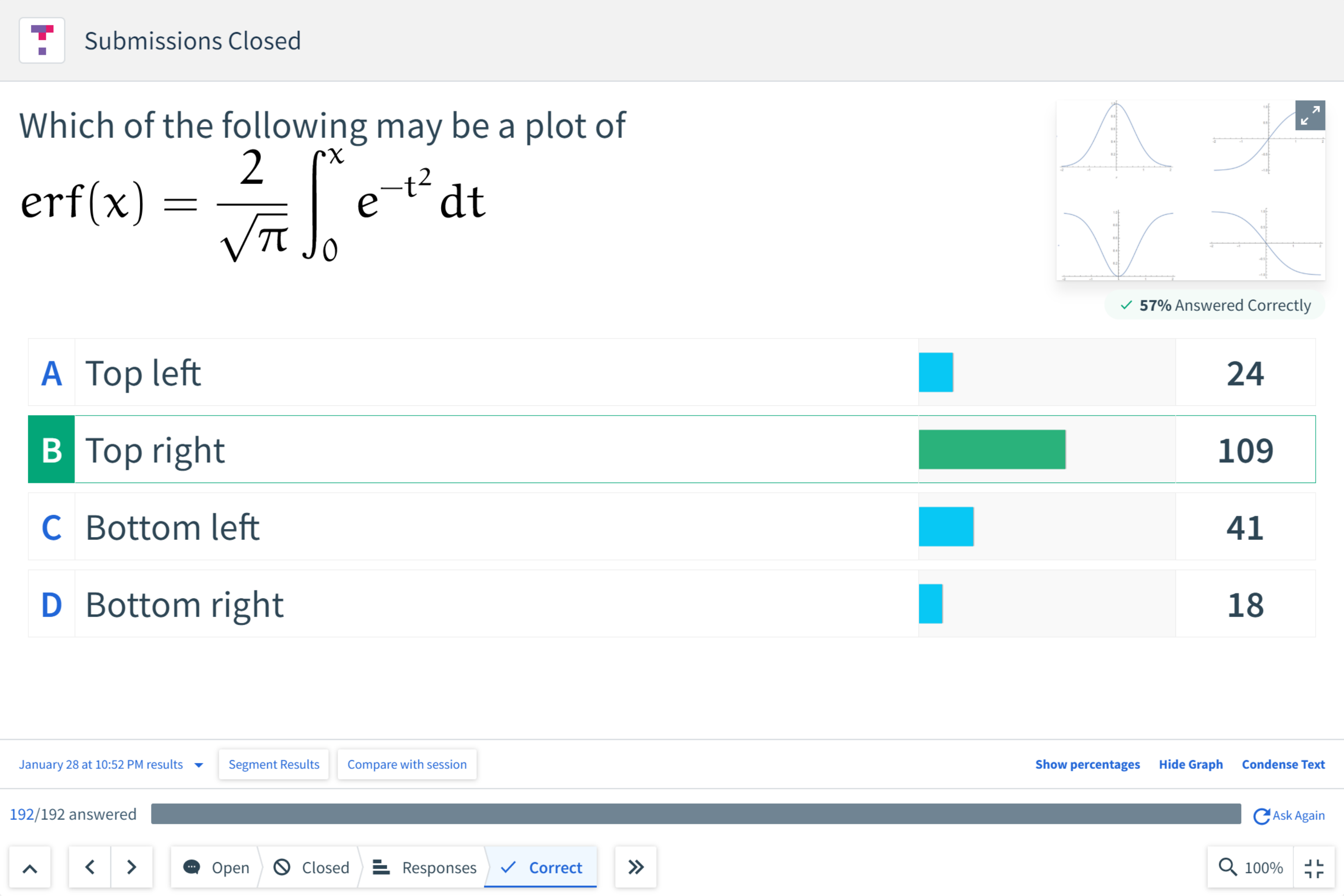
Functions Defined by Integrals

Recall: we can define functions using integrals. For example:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$F(x) = \int_0^x (1+t)e^t \sqrt{1+t^2e^{2t}} dt$$

Today: explore how to work with these functions and with computer algebra systems.



Submissions Closed

Which of the following may be a plot of

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

✓ **57%** Answered Correctly

A	Top left	24
B	Top right	109
C	Bottom left	41
D	Bottom right	18

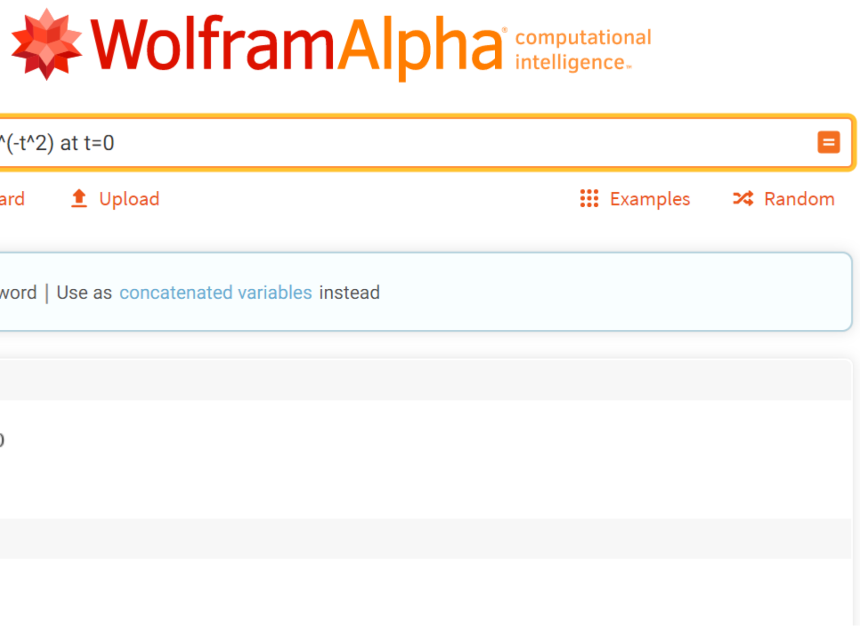
192/192 answered

Taylor Approximations Using C.A.S

The third-order Taylor approximation of a function, f around 0 is given by:

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3$$

WolframAlpha can be used to compute derivatives quickly!



4th derivative of e^(-t^2) at t=0

Assuming "at" is a word | Use as concatenated variables instead

Input interpretation:

$\frac{\partial^4 e^{-t^2}}{\partial t^4}$ where $t = 0$

Result:

12

Use the third-order Taylor approximation of $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ to estimate $erf(0.5)$.

Use the third-order Taylor approximation of $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ to estimate $erf(0.5)$.

$$erf(x) \approx \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} \right)$$

$$erf(05) \approx 0.517$$

Compute $erf(0.5)$ directly using WolframAlpha.

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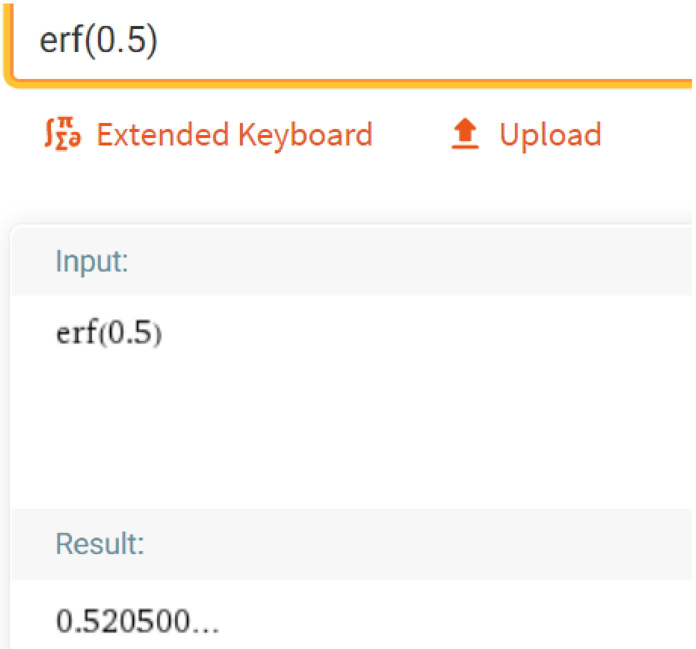
Compute $erf(0.5)$ directly using WolframAlpha.



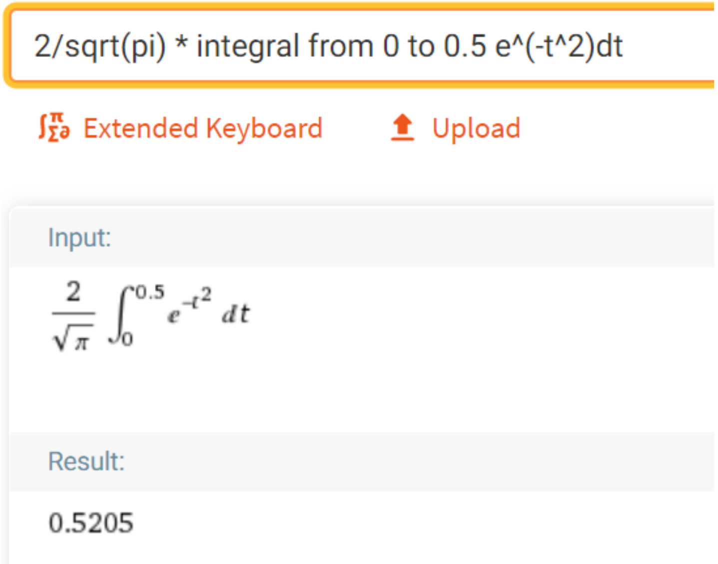
erf(0.5)

Input:

erf(0.5)

Result:

0.520500...



2/sqrt(pi) * integral from 0 to 0.5 e^(-t^2)dt

Input:

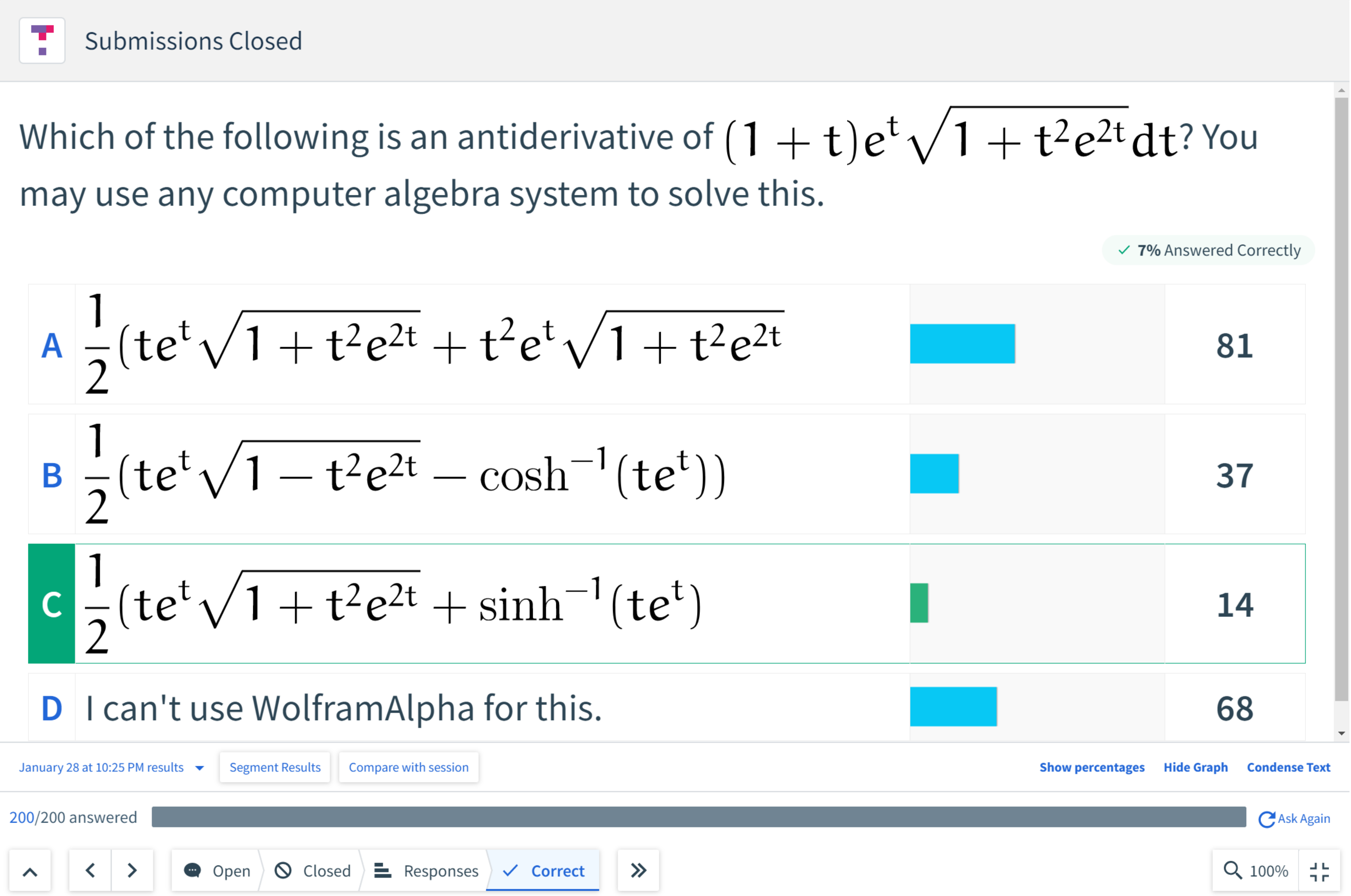
$$\frac{2}{\sqrt{\pi}} \int_0^{0.5} e^{-t^2} dt$$

Result:

0.5205

Takeaway

Computer algebra systems can do some of the work for us, even if we have to stitch it together at the end.



Submissions Closed

Which of the following is an antiderivative of $(1+t)e^t\sqrt{1+t^2e^{2t}}dt$? You may use any computer algebra system to solve this.

✓ **7%** Answered Correctly

A	$\frac{1}{2}(te^t\sqrt{1+t^2e^{2t}}+t^2e^t\sqrt{1+t^2e^{2t}}$	81
B	$\frac{1}{2}(te^t\sqrt{1-t^2e^{2t}}-\cosh^{-1}(te^t))$	37
C	$\frac{1}{2}(te^t\sqrt{1+t^2e^{2t}}+\sinh^{-1}(te^t)$	14
D	I can't use WolframAlpha for this.	68

200/200 answered

What Went Wrong?

WolframAlpha could not solve:

$$\int (1+t)e^t\sqrt{1+t^2e^{2t}}dt$$

The integral was too complex. What can we do?

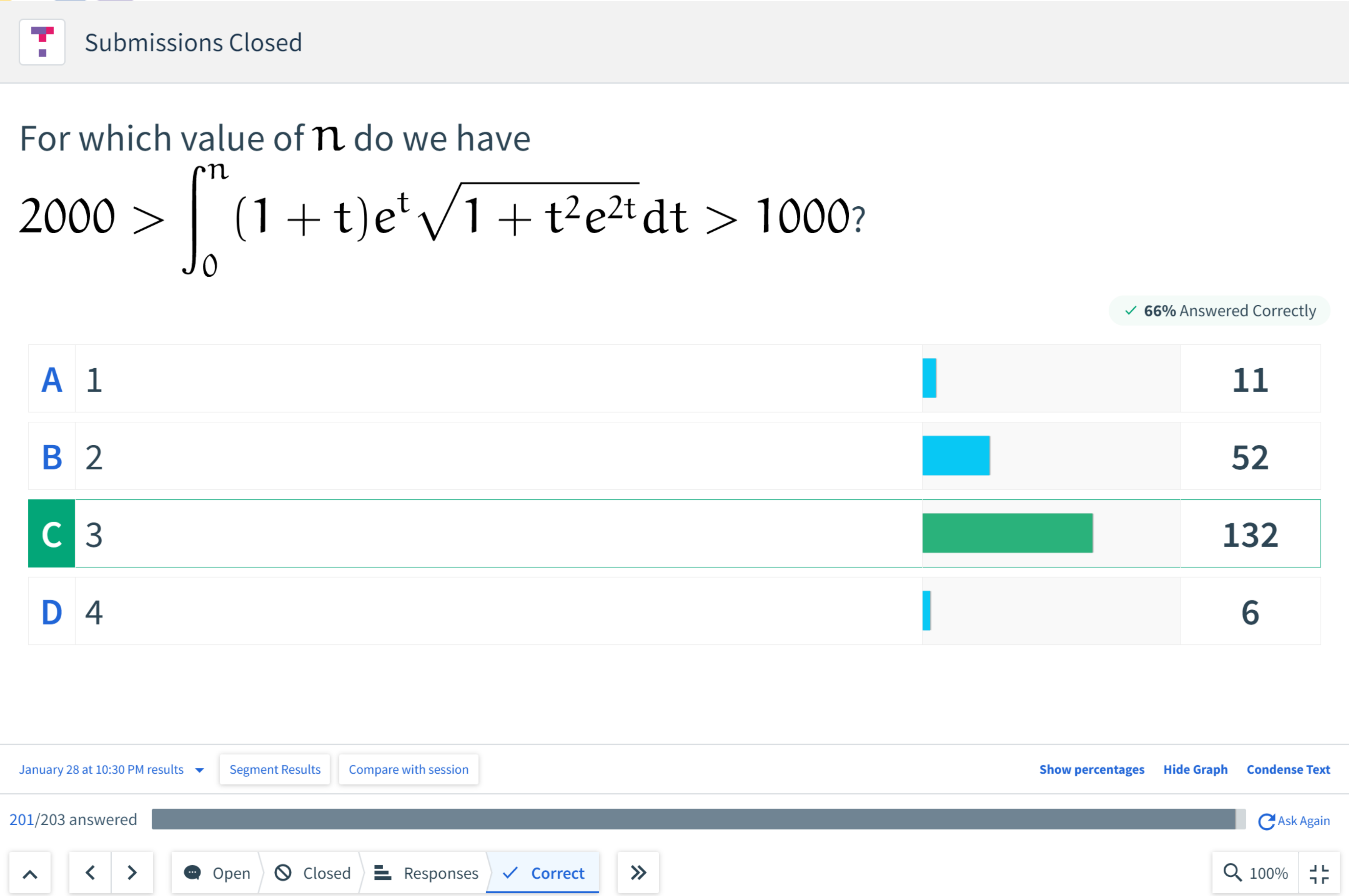
What Went Wrong?

WolframAlpha could not solve:

$$\int (1+t)e^t\sqrt{1+t^2e^{2t}}dt$$

The integral was too complex. What can we do?

- Ask for a definite integral instead.
- Use a Taylor polynomial to estimate the integrand
- Change the input to something nicer



Submissions Closed

For which value of n do we have

$$2000 > \int_0^n (1+t)e^t\sqrt{1+t^2e^{2t}}\,dt > 1000?$$

✓ **66%** Answered Correctly

A	1	11
B	2	52
C	3	132
D	4	6

201/203 answered

Takeaway

Computer algebra systems can do definite integrals like it's nobody's business. Remember: it's just sums!

Using Taylor Polynomials on the Integrand

Idea: use a Taylor polynomial to approximate $(1+t)e^t\sqrt{1+t^2e^{2t}}$ around $t=0$, then integrate that. If the polynomial and the function are close, then their integrals will be close too.

Using Taylor Polynomials on the Integrand

Idea: use a Taylor polynomial to approximate $(1+t)e^t\sqrt{1+t^2e^{2t}}$ around $t=0$, then integrate that. If the polynomial and the function are close, then their integrals will be close too.

- Find the Taylor polynomial of the function
- Use it to estimate the function for small values of x
- Find an antiderivative of the Taylor polynomial

Estimating Integerals With Taylor Polynomials

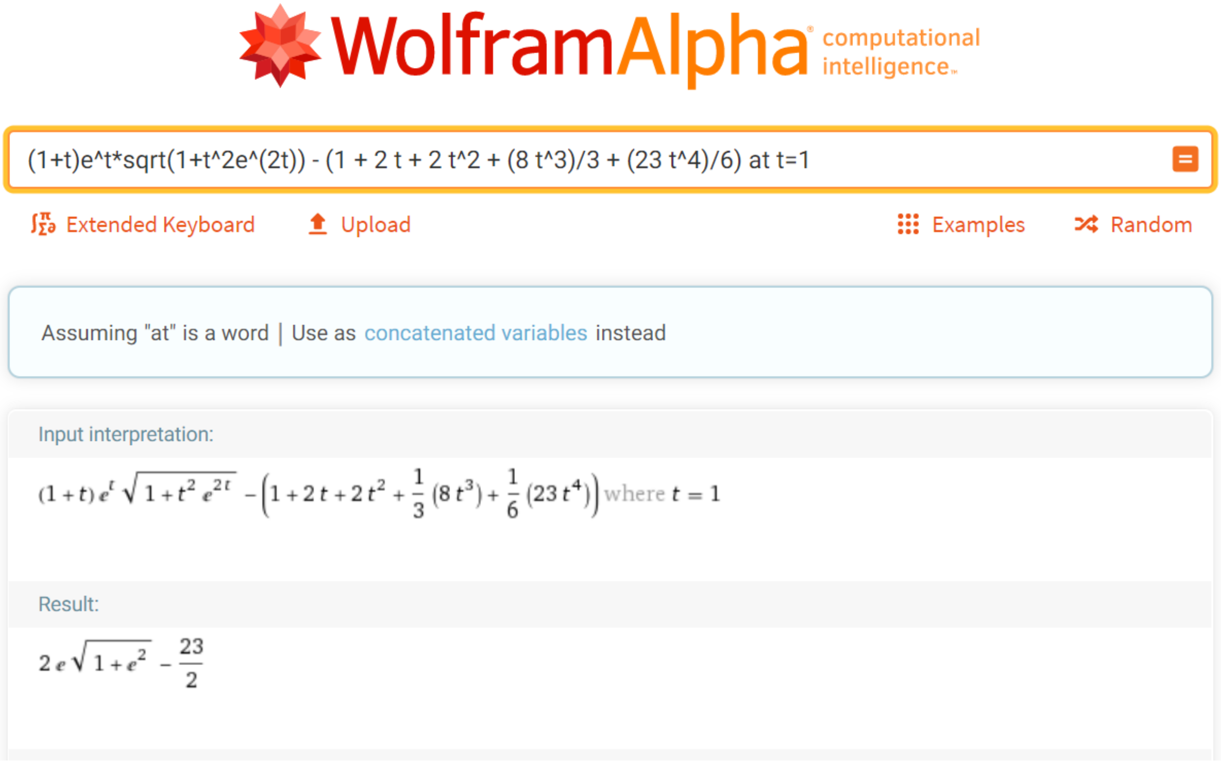
Use WolframAlpha to compute the Taylor polynomial of $(1+t)e^t\sqrt{1+t^2e^{2t}}$ around $t=0$ to fourth order.

Estimating Integerals With Taylor Polynomials

Use WolframAlpha to compute the Taylor polynomial of $(1+t)e^t\sqrt{1+t^2e^{2t}}$ around $t=0$ to fourth order.

$$T_3(t) = 1 + 2t + 2t^2 + 8\frac{t^3}{3} + 23\frac{t^4}{6}$$

Estimating Integerals With Taylor Polynomials



(1+t)e^t*sqrt(1+t^2e^(2t)) - (1 + 2 t + 2 t^2 + (8 t^3)/3 + (23 t^4)/6) at t=1

Assuming "at" is a word | Use as concatenated variables instead

Input interpretation:

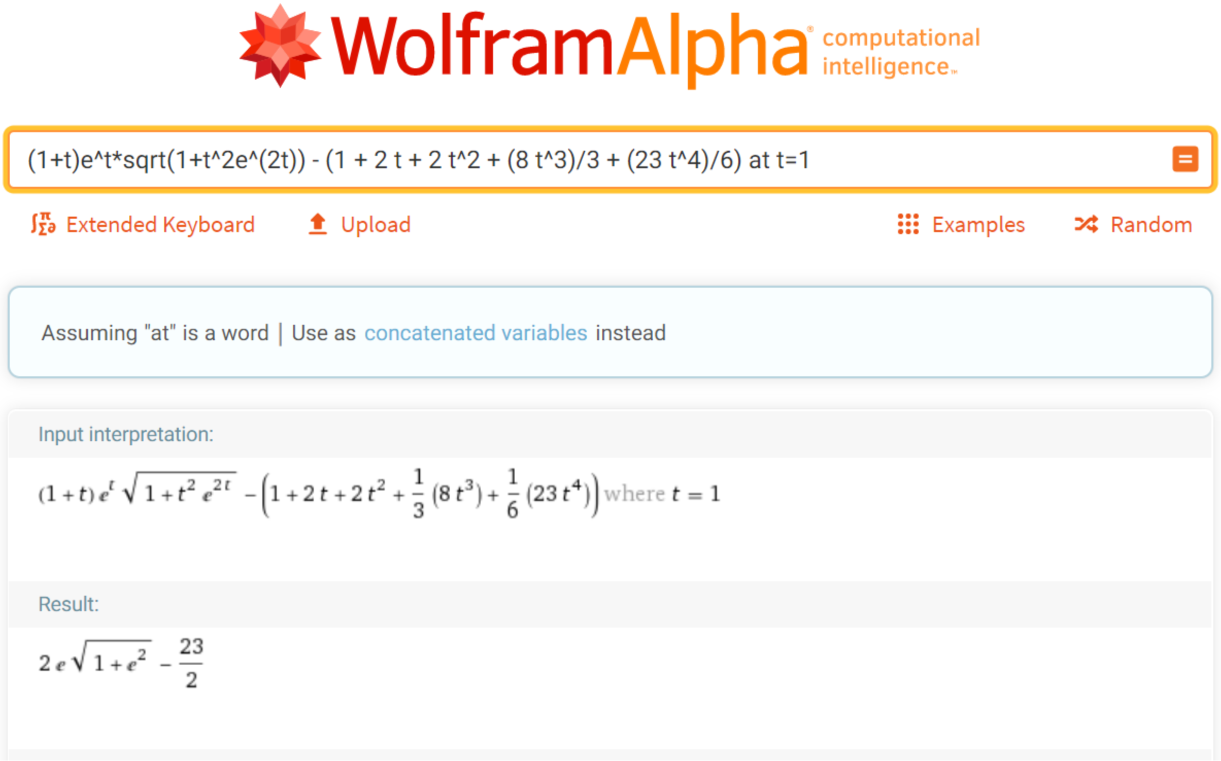
$$(1+t)e^t\sqrt{1+t^2e^{2t}} - \left(1+2t+2t^2+\frac{1}{3}(8t^3)+\frac{1}{6}(23t^4)\right) \text{ where } t=1$$

Result:

$$2e\sqrt{1+e^2}-\frac{23}{2}$$

This error ends up being approximately 4.2.

Estimating Integerals With Taylor Polynomials



```
(1+t)e^t*sqrt(1+t^2e^(2t)) - (1 + 2 t + 2 t^2 + (8 t^3)/3 + (23 t^4)/6) at t=1
```

Assuming "at" is a word | Use as concatenated variables instead

Input interpretation:

$$(1+t)e^t\sqrt{1+t^2e^{2t}}-\left(1+2t+2t^2+\frac{1}{3}(8t^3)+\frac{1}{6}(23t^4)\right) \text{ where } t=1$$

Result:

$$2e\sqrt{1+e^2}-\frac{23}{2}$$

This error ends up being approximately 4.2.
Estimate $\int_0^1 (1+t)e^t\sqrt{1+t^2e^{2t}}dt$ using the Taylor approximation you found. How good is this approximation?

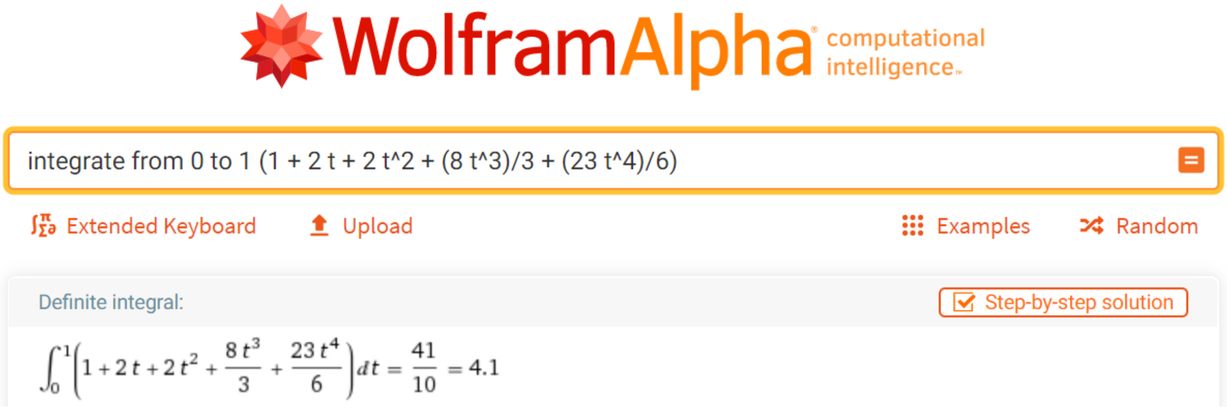
Estimating Integerals With Taylor Polynomials

$$\int_0^1 (1+t)e^t\sqrt{1+t^2e^{2t}}dt \approx \int_0^1 T_3(t)dt$$

$$= \int_0^1 \left(1+2t+2t^2+8\frac{t^3}{3}+23\frac{t^4}{6}\right) dt$$

Estimating Integrals With Taylor Polynomials

$$\int_0^1 (1+t)e^t\sqrt{1+t^2e^{2t}}dt \approx \int_0^1 T_3(t)dt$$
$$= \int_0^1 \left(1+2t+2t^2+8\frac{t^3}{3}+23\frac{t^4}{6}\right) dt$$



```
integrate from 0 to 1 (1 + 2 t + 2 t^2 + (8 t^3)/3 + (23 t^4)/6)
```

Definite integral:

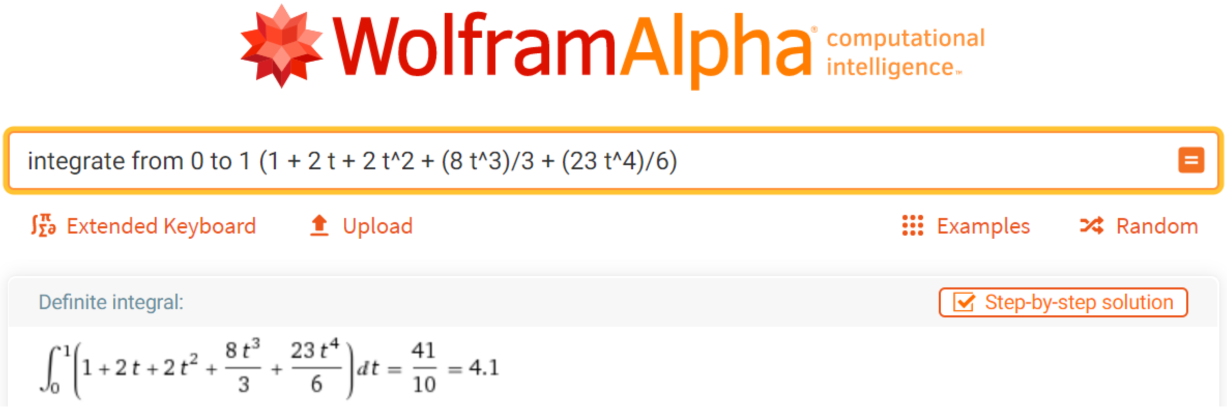
$$\int_0^1\left(1+2t+2t^2+\frac{8t^3}{3}+\frac{23t^4}{6}\right)dt=\frac{41}{10}=4.1$$

What is the true value of the integral?

Estimating Integrals With Taylor Polynomials

$$\int_0^1 (1+t)e^t\sqrt{1+t^2e^{2t}}dt \approx \int_0^1 T_3(t)dt$$

$$= \int_0^1 \left(1+2t+2t^2+8\frac{t^3}{3}+23\frac{t^4}{6}\right) dt$$



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Definite integral:

$$\int_0^1\left(1+2t+2t^2+\frac{8t^3}{3}+\frac{23t^4}{6}\right)dt=\frac{41}{10}=4.1$$

What is the true value of the integral? 4.78

Simplifying the Integral with Substitution

We wish to compute:

$$\int (1+t)e^t\sqrt{1+t^2e^{2t}}dt$$

Make the substitution $u = te^t$.

Simplifying the Integral with Substitution

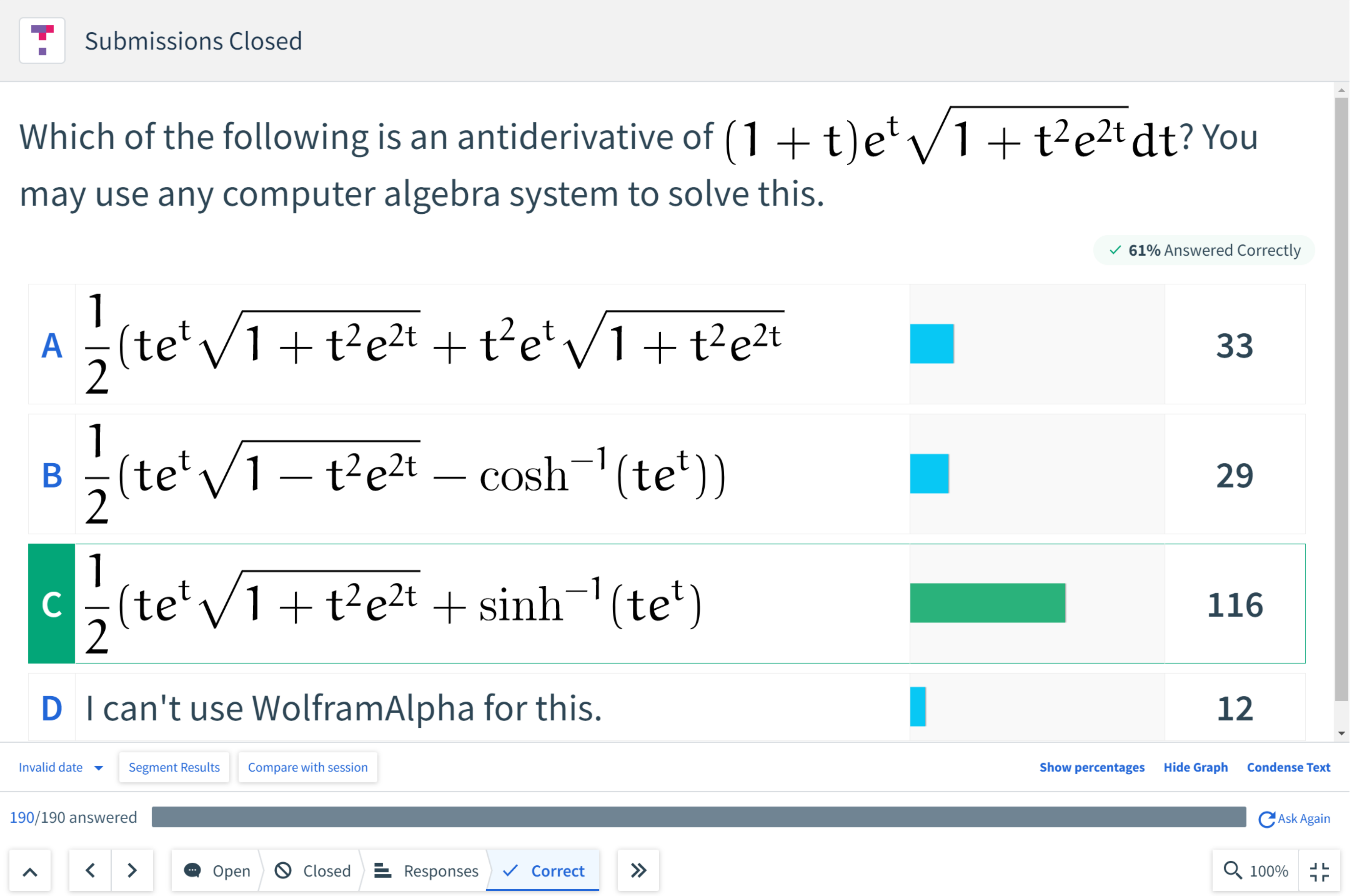
We wish to compute:

$$\int (1+t)e^t\sqrt{1+t^2e^{2t}}dt$$

Make the substitution $u = te^t$. The integral then becomes:

$$\int \sqrt{1+u^2}du$$

Plug this integral into a computer algebra system



Which of the following is an antiderivative of $(1+t)e^t\sqrt{1+t^2e^{2t}}dt$? You may use any computer algebra system to solve this.

✓ **61%** Answered Correctly

A	$\frac{1}{2}(te^t\sqrt{1+t^2e^{2t}}+t^2e^t\sqrt{1+t^2e^{2t}}$	33
B	$\frac{1}{2}(te^t\sqrt{1-t^2e^{2t}}-\cosh^{-1}(te^t))$	29
C	$\frac{1}{2}(te^t\sqrt{1+t^2e^{2t}}+\sinh^{-1}(te^t)$	116
D	I can't use WolframAlpha for this.	12

190/190 answered

Plans for the Future

For next time:

WeBWork 7.6 and read section 7.6

Welcome to MAT135 LEC0501 (Assaf)

The Borwein integrals:

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\,dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5}\,dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\cdots\frac{\sin(x/13)}{x/13}\,dx = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\cdots\frac{\sin(x/15)}{x/15}\,dx = \frac{467807924713440738696537864469}{935615849440640907310521750000}\pi$$

$$\approx \frac{\pi}{2} - 2.31 \times 10^{-11}$$

Improper Integrals – Going to Infinity

Assaf Bar-Natan

“Out all night, sun’s too bright
Though I’m blind, it’ll be all right
Going to infinity
What does it mean?
Infinity”

–“What Does it Mean?”, The Flaming Lips

Jan. 30, 2020

Reading Comprehension – Fill in Blanks

An integral $\int_a^b f(t)dt$ is an improper intergral when _____ are infinite or when the _____ is infinite.

Reading Comprehension – Fill in Blanks

The faster $f(t)$ decreases as ______, the more likely that $\int_a^\infty f(t)dt$ ____________

Reading Comprehension – Fill in Blanks

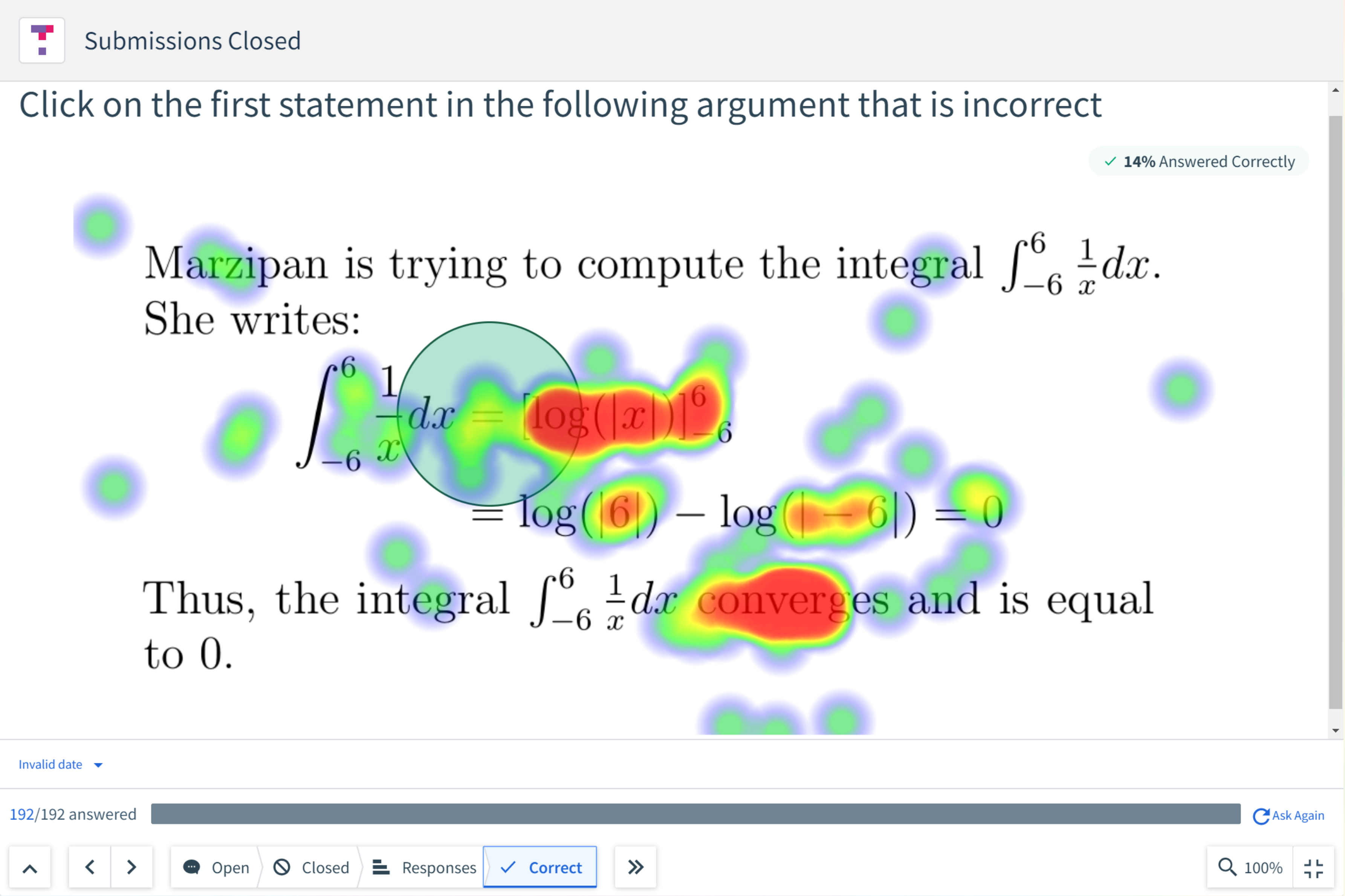
An improper integral is defined as a ______ of definite integrals.

Reading Comprehension – Fill in Blanks

Suppose that $\lim_{x\to b} f(x) = \infty$**. If** $\lim_{x\to b} \int_a^x f(t)dt$ ______**, we define** $\int_a^b f(t)dt$ **by** ____________**. Otherwise, we say that** $\int_a^b f(t)dt$ ______**.**

Reading Comprehension – Fill in Blanks

If $\lim_{x\to\infty}\int_a^x f(t)dt$ ______, **we define** $\int_a^\infty f(t)dt$ **by** ______, **and we say that** $\int_a^\infty f(t)dt$ ______.



Submissions Closed

Click on the first statement in the following argument that is incorrect

✓ **14%** Answered Correctly

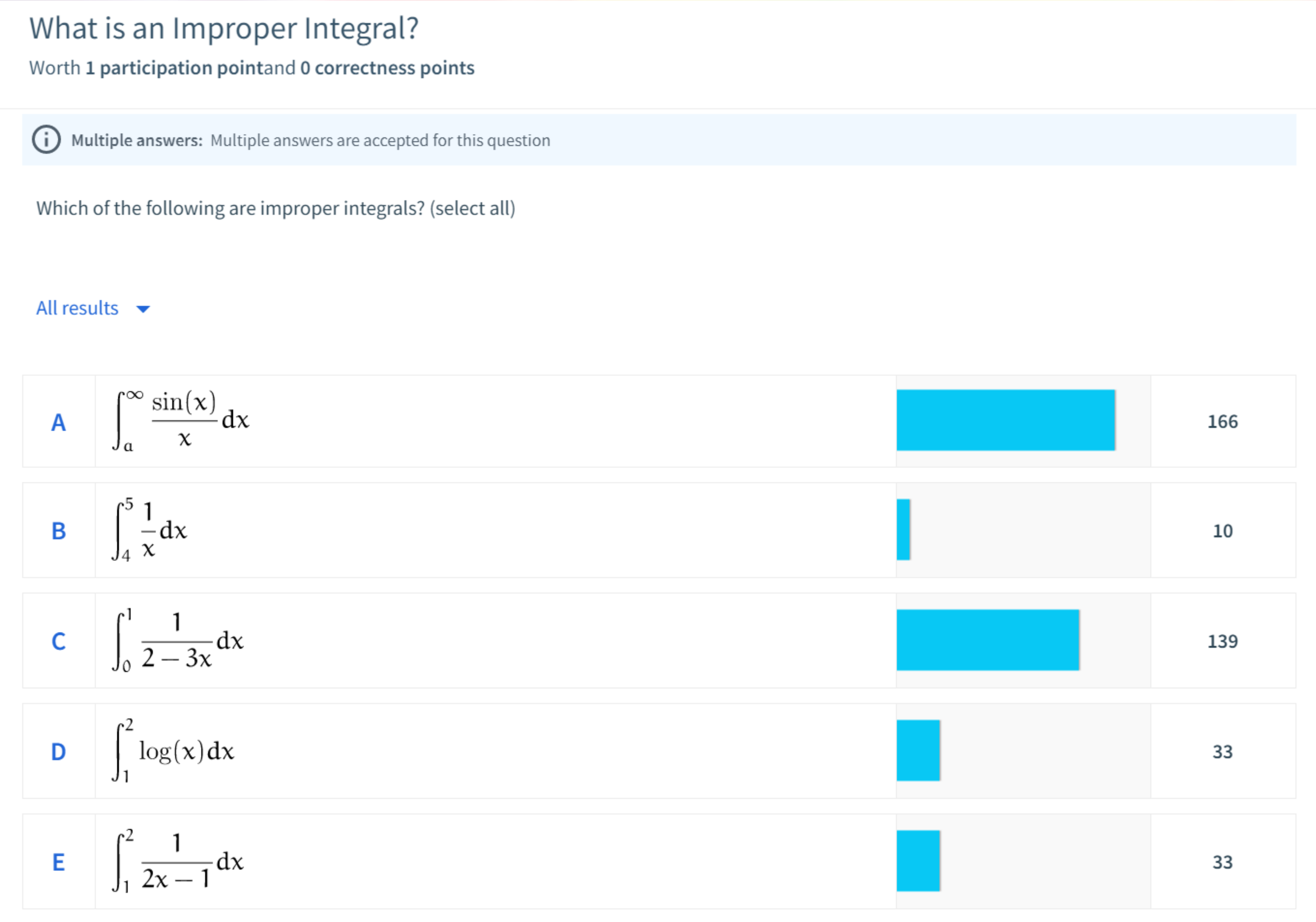
Marzipan is trying to compute the integral $\int_{-6}^{6} \frac{1}{x}dx$. She writes:

$$\int_{-6}^{6} \frac{1}{x}dx = [\log(|x|)]_{-6}^{6}$$
$$= \log(|6|) - \log(|-6|) = 0$$

Thus, the integral $\int_{-6}^{6} \frac{1}{x}dx$ converges and is equal to 0.

Takeaway

The fundamental theorem only works when the integrand is continuous. If f is infinite between the bounds, the integral is improper!



What is an Improper Integral?

Worth **1 participation point**and **0 correctness points**

ⓘ **Multiple answers:** Multiple answers are accepted for this question

Which of the following are improper integrals? (select all)

All results ▾

A	$\int_a^{\infty} \frac{\sin(x)}{x} dx$	166
B	$\int_4^5 \frac{1}{x} dx$	10
C	$\int_0^1 \frac{1}{2-3x} dx$	139
D	$\int_1^2 \log(x) dx$	33
E	$\int_1^2 \frac{1}{2x-1} dx$	33

An Example

We will determine if $\int_{-6}^{6} \frac{1}{x} dx$ converges.

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Write a list of steps you should take to determine this.

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Write a list of steps you should take to determine this.

- Split the integral into two improper integrals
- Turn each integral into a limit
- Take the limit
- Do the limits converge?

An Example – Splitting the Integral

- **Split the integral into two improper integrals**
- Turn each integral into a limit
- Take the limit
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An Example – Splitting the Integral

- **Split the integral into two improper integrals**
- Turn each integral into a limit
- Take the limit
- Do the limits converge?

The function $f(x) = \frac{1}{x}$ goes to ∞ when $x \to 0$, so we should split the integrals there.

$$\int_{-6}^{6} \frac{1}{x} dx = \int_{-6}^{0} \frac{1}{x} dx + \int_{0}^{6} \frac{1}{x} dx$$

Now, we should solve each of these as an improper integral.

An Example – Turning it Into a Limit

- Split the integral into two improper integrals
- **Turn each integral into a limit**
- Take the limit
- Do the limits converge?

$$\int_{-6}^{6} \frac{1}{x} dx = \int_{-6}^{0} \frac{1}{x} dx + \int_{0}^{6} \frac{1}{x} dx$$

An Example – Turning it Into a Limit

- Split the integral into two improper integrals
- **Turn each integral into a limit**
- Take the limit
- Do the limits converge?

$$\int_{-6}^{6} \frac{1}{x} dx = \int_{-6}^{0} \frac{1}{x} dx + \int_{0}^{6} \frac{1}{x} dx$$

We need to check if the following limits exist:

$$\lim_{b \to 0^-} \int_{-6}^{b} \frac{1}{x} dx$$

$$\lim_{a \to 0^+} \int_{a}^{6} \frac{1}{x} dx$$

An Example – Taking the Limit

- Split the integral into two improper integrals
- Turn each integral into a limit
- **Take the limit**
- Do the limits converge?

We need to check if the following limits exist:

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An Example – Taking the Limit

- Split the integral into two improper integrals
- Turn each integral into a limit
- **Take the limit**
- Do the limits converge?

We need to check if the following limits exist:

$$\lim_{b\to 0^-}\int_{-6}^{b}\frac{1}{x}dx$$

$$\lim_{a\to 0^+}\int_{a}^{6}\frac{1}{x}dx$$

We compute:

$$\lim_{b\to 0^-}\int_{-6}^{b}\frac{1}{x}dx = \lim_{b\to 0^-}(\log(b)-\log(|-6|)) = -\infty$$

$$\lim_{a\to 0^+}\int_{a}^{6}\frac{1}{x}dx = \lim_{a\to 0^+}(\log(6)-\log(|a|)) = \infty$$

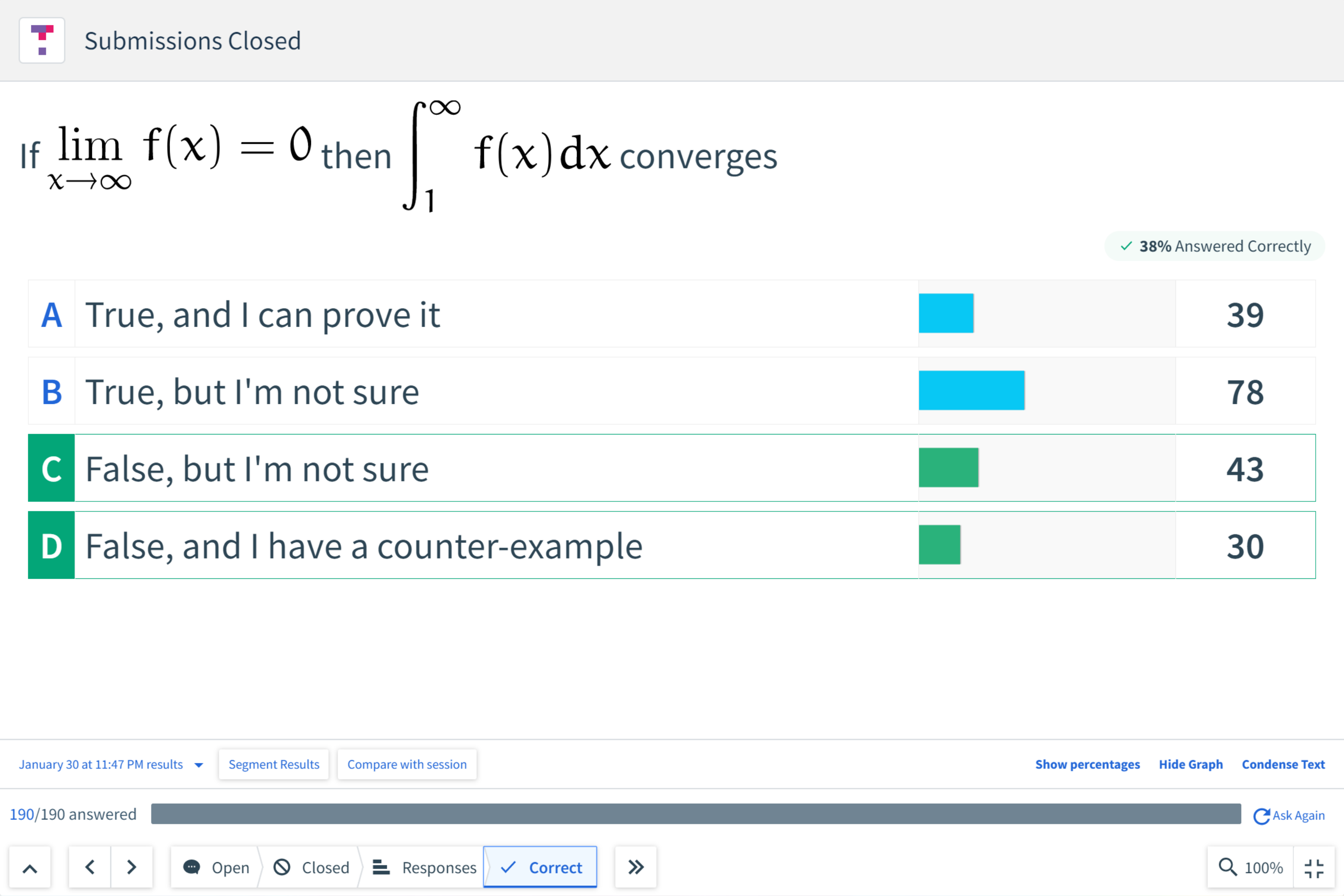
An Example – Taking the Limit

- Split the integral into two improper integrals
- Turn each integral into a limit
- Take the limit
- **Do the limits converge?**

What does this tell us about $\int_{-6}^{6} \frac{1}{x} dx$? Plug this in to WolframAlpha!

Takeaway

When evaluating improper integrals, you might need to split them up!



Submissions Closed

If $\lim_{x \longrightarrow \infty} f(x) = 0$ then $\int_1^\infty f(x)\,dx$ converges

✓ 38% Answered Correctly

A	True, and I can prove it	39
B	True, but I'm not sure	78
C	False, but I'm not sure	43
D	False, and I have a counter-example	30

190/190 answered

Criterion For Convergence

For which p does $\int_0^1 x^p dx$ converge?

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- Get into groups, and assign each group member a different number.

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2 min Check whether $\int_0^1 x^p dx$ converges where p is your number.

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2 min Compare answers with your neighbours, and form a conjecture.

Criterion For Convergence

For which p does $\int_0^1 x^p dx$ converge?

- Get into groups, and assign each group member a different number.

2 min Check whether $\int_0^1 x^p dx$ converges where p is your number.

2 min Compare answers with your neighbours, and form a conjecture.

2 min Check your conjecture by hand or on WolframAlpha

Takeaway

The integral $\int_0^1 x^p dx$ **converges when** $p > -1$

Roy and the Big Barn

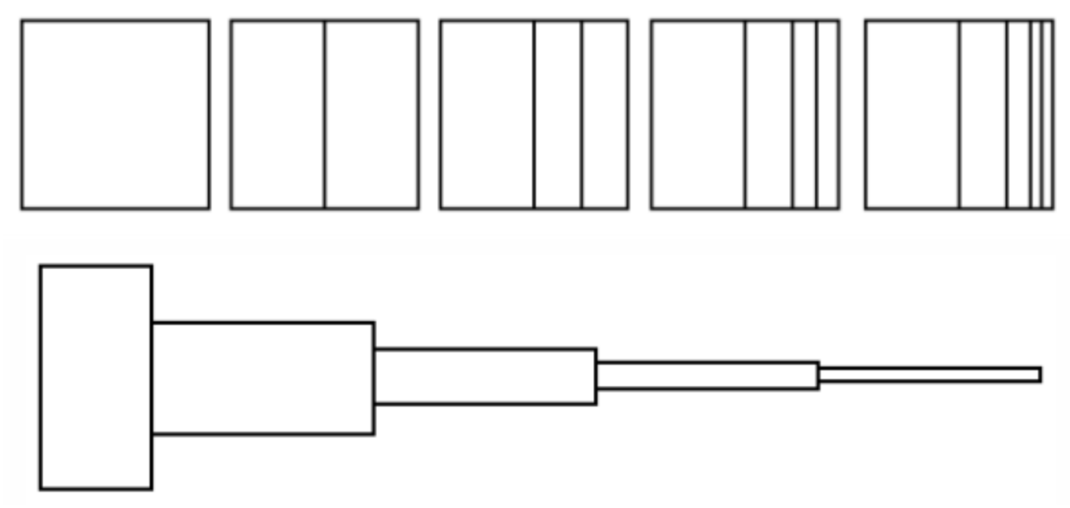
Roy the kitten is walking around the barn, and says the following:

> "I know this barn in and out, and I can confidently say that it has a finite area. I don't know its shape, but because it has finite area, I should be able to circumnavigate it in finite time."

Write a sentence explaining to Roy where he is wrong. Be sure to give an example.

Roy and the Big Barn

Here's a helpful picture:



Plans for the Future

For next time:

WeBWork 7.7 and read section 7.7

Welcome to MAT135 LEC0501 (Assaf)

Think of a hobby or a skill you have. Did you get a chance to do it this year?

S7.7 – Improper Integrals – Comparisons, Estimation, and Guessing

Assaf Bar-Natan

> “Laughing like children, living like lovers
> Rolling like thunder under the covers
> And I guess that’s why they call it the blues”
>
> –“I Guess That’s Why They Call it the Blues”, Elton John

Feb. 3, 2020

The CIQ

Things I noticed

- TopHat and working together got a lot of people engaged
- Things we dislike:
  - When people aren't participating
  - **When Assaf skips things**
  - **Unexplained answers**
- Things we like:
  - Explaining after TopHats
  - Other people helping us understand
- **Reading summary at the start of class before self-work**
- Things that surprised you:
  - How welcoming you were to each other
  - How many friends you made
  - The style of the class

How Do We Learn?

How do people learn?

How Do We Learn?

How do people learn?

What is something you are good at? How did you learn it?

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How do people learn?

What is something you are good at? How did you learn it?

Why do we ask you to read before class?

Good Reading Strategies

What are some good reading strategies for math?

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What are some good reading strategies for math?

Three-time rule:

- **Skim** – don't worry about understanding, just read! (10 mins)
- **Note** – take meticulous notes, and read carefully! (one hour)
- **Own** – Read things one last time to pick up pieces you've missed (10 mins)

Good Reading Strategies

What are some good reading strategies for math?

Three-time rule:

- **Skim** – don't worry about understanding, just read! (10 mins)
- **Note** – take meticulous notes, and read carefully! (one hour)
- **Own** – Read things one last time to pick up pieces you've missed (10 mins)

Other ideas:

- Ask friends for help
- **TAKE NOTES**
- Do the problems

Ice-Cream Sandwich

Spend a minute to think about:

- Something in the chapter that you've mastered.
- Something in the chapter that you've learned
- Something in the chapter that you've got to revisit

Ice-Cream Sandwich

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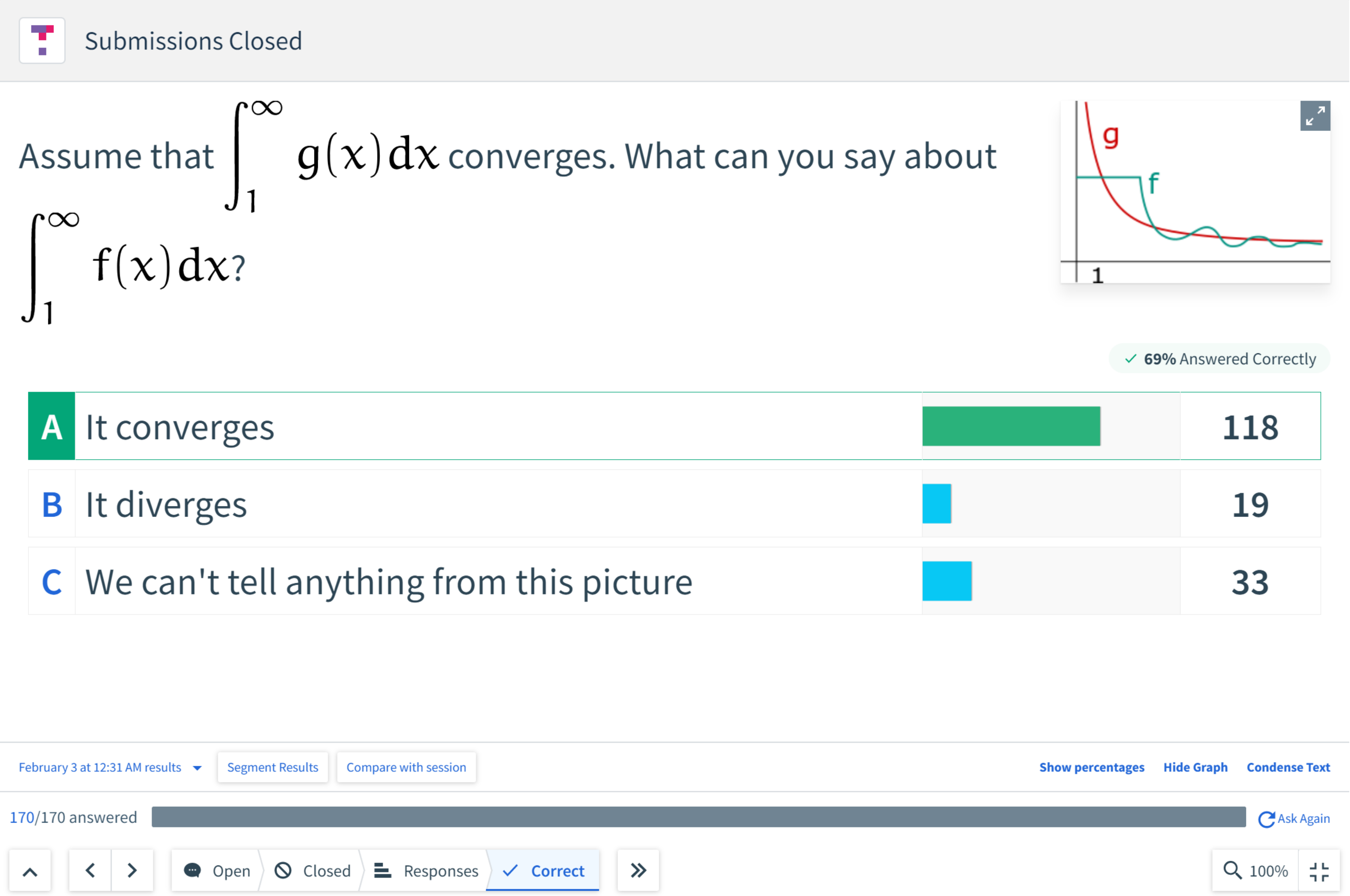
Share with your neighbours

The Idea of Comparisons

"If it looks like a cat, and meows like a cat, it converges like a cat"

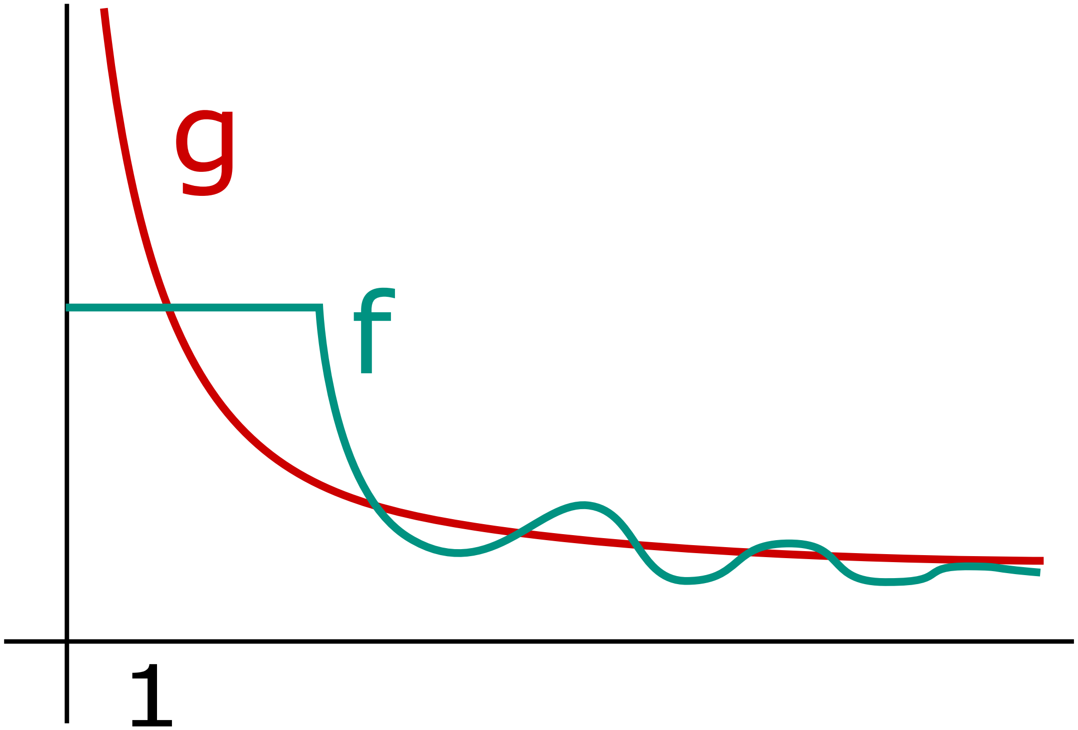
If $f \leq g$ then $\int_a^b f \leq \int_a^b g$, so if g converges, then f converges.



Assume that $\int_1^\infty g(x)\,dx$ converges. What can you say about $\int_1^\infty f(x)\,dx$?

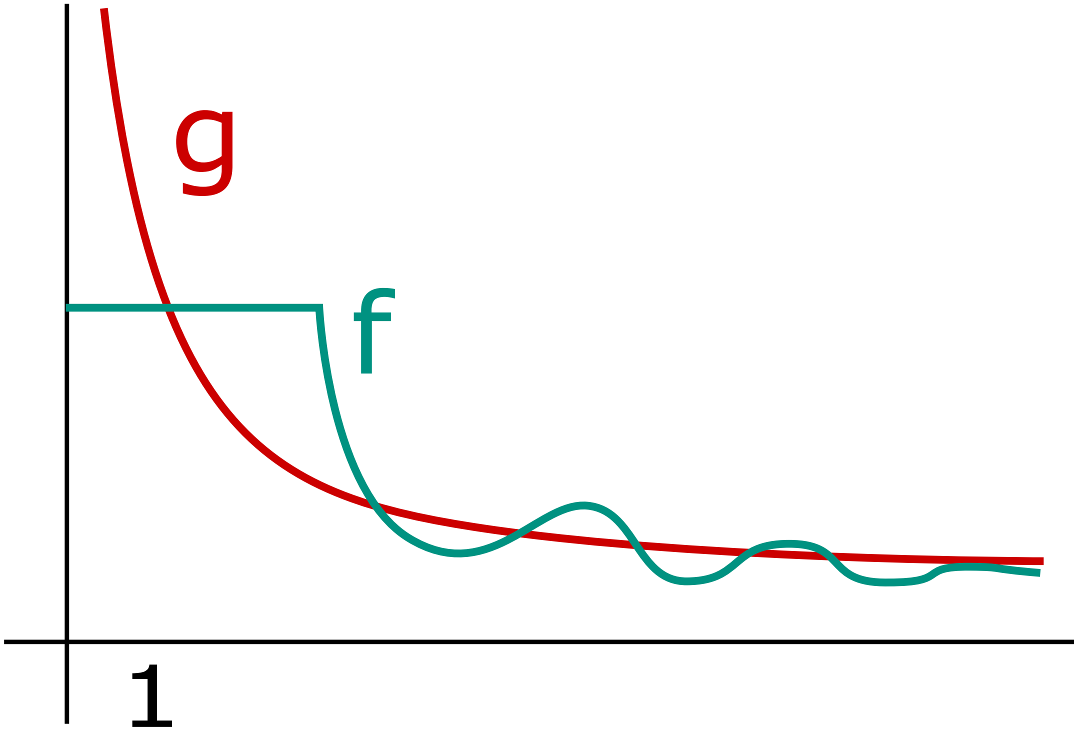
A	It converges	118
B	It diverges	19
C	We can't tell anything from this picture	33

A Graphical Example



If $\int_1^\infty g(x)dx$ converges, what can we say about $\int_1^\infty f(x)dx$?
It must converge, by the comparison test, since f looks like g.
If $\int_0^1 g(x)dx$ diverges, what can we say about $\int_0^1 f(x)dx$?

A Graphical Example



If $\int_1^\infty g(x)dx$ converges, what can we say about $\int_1^\infty f(x)dx$?
It must converge, by the comparison test, since f looks like g.
If $\int_0^1 g(x)dx$ diverges, what can we say about $\int_0^1 f(x)dx$?
$\int_0^1 f(x)dx$ is not an improper integral, so it converges.

Takeaway

When looking at what integrals to infinity do, we only care about the tail. If the tails look similar, then the functions converge and diverge together.

Spot the Error

Peek, the curious cat, is trying to compute:

$$\int_1^\infty \frac{-1}{x}dx$$

She writes:

> “I know that
>
> $$\int_1^\infty \frac{1}{x^2} = 1$$
>
> I also know that $\frac{-1}{x} \leq \frac{1}{x^2}$ for all $x \geq 1$
> So by the comparison test, I can conclude
> that $\int_1^\infty \frac{-1}{x}dx$ converges.”

What was her mistake? Write a takeaway from this example.

Takeaway

When dealing with negative integrands, we can't just bound things from one side.

Takeaway

When dealing with negative integrands, we can't just bound things from one side.

Aside: This should remind some of you of the squeeze theorem...

Review: Known Improper Integrals

Integral	Condition on parameter (p or a)	Converges/diverges
$\int_0^1 x^p$	$p > -1$	
$\int_0^1 x^p$	$p \leq -1$	
$\int_1^\infty x^p$	$p \geq -1$	
$\int_1^\infty x^p$	$p < -1$	
$\int_0^\infty e^{-ax}$	$a > 0$	
$\int_0^\infty e^{-ax}$	$a \leq 0$	

Review: Known Improper Integrals

Integral	Condition on parameter (p or a)	Converges/diverges
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$\int_1^\infty x^p$	$p \geq -1$	Diverges
$\int_1^\infty x^p$	$p < -1$	Converges
$\int_0^\infty e^{-ax}$	$a > 0$	Converges
$\int_0^\infty e^{-ax}$	$a \leq 0$	Diverges

An Algebraic Example

"If it looks like a cat, and meows like a cat, it converges like a cat"

What known improper integrals do the following integrals look like:

$$\int_6^\infty \frac{1}{(x-5)^2}\,dx$$

$$\int_0^5 \frac{1+\sin^2(x)}{\sqrt{x}}\,dx$$

$$\int_5^\infty \frac{1+\sin^2(x)}{\log(x)}\,dx$$

Meows Like a Cat

$$\int_6^\infty \frac{1}{(x-5)^2}\,dx$$

Key ideas:

- $x - 5 < x$ so $\frac{1}{(x-5)^2} \geq \frac{1}{x^2}$. This won't help.
- Substitute $u = x - 5$ to get $\int_1^\infty \frac{1}{u^2}du$
- When x is big, $\frac{1}{(x-5)^2} \approx \frac{1}{x^2}$

Meows Like a Cat

$$\int_6^\infty \frac{1}{(x-5)^2}dx$$

Key ideas:

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- Substitute $u=x-5$ to get $\int_1^\infty \frac{1}{u^2}du$
- When x is big, $\frac{1}{(x-5)^2} \approx \frac{1}{x^2}$

This integral converges

Meows Like a Cat

$$\int_0^5 \frac{1+\sin^2(x)}{\sqrt{x}}\,dx$$

Key ideas:

- $\frac{1}{\sqrt{x}} \le \frac{1+\sin^2(x)}{\sqrt{x}} \le \frac{2}{\sqrt{x}}$
- When x is small, integrand looks like $\frac{1}{\sqrt{x}}$

Meows Like a Cat

$$\int_0^5 \frac{1+\sin^2(x)}{\sqrt{x}}\,dx$$

Key ideas:

- $\frac{1}{\sqrt{x}} \leq \frac{1+\sin^2(x)}{\sqrt{x}} \leq \frac{2}{\sqrt{x}}$
- When x is small, integrand looks like $\frac{1}{\sqrt{x}}$

This integral converges

Meows Like a Cat

$$\int_5^\infty \frac{1+\sin^2(x)}{\log(x)}dx$$

Key ideas:

- The $1+\sin^2(x)$ term is a distraction that just oscillates a bit.
- Looks like $\int_5^\infty \frac{1}{\log(x)}$
- When x is big, $x > \log(x)$ so $\frac{1}{x} < \frac{1}{\log(x)}$

Meows Like a Cat

$$\int_5^\infty \frac{1+\sin^2(x)}{\log(x)}dx$$

Key ideas:

- The $1+\sin^2(x)$ term is a distraction that just oscillates a bit.
- Looks like $\int_5^\infty \frac{1}{\log(x)}$
- When x is big, $x > \log(x)$ so $\frac{1}{x} < \frac{1}{\log(x)}$

This integral diverges

Takeaway

When comparing integrals, be mindful of easy substitutions, but also watch for the bounds!

The Cat's Tail

Does the integral:

$$\int_a^\infty \frac{1}{x^2}dx$$

(where $a > 1$) converge?

The Cat’s Tail

Does the integral:

$$\int_a^\infty \frac{1}{x^2}dx$$

(where $a > 1$) converge? Yes!

$$\int_1^\infty \frac{1}{x^2}dx = \int_1^a \frac{1}{x^2}dx + \int_a^\infty \frac{1}{x^2}dx$$

So we get:

$$1 = 1 - \frac{1}{a} + \int_a^\infty \frac{1}{x^2}dx$$

and we can solve for the integral.

Plans for the Future

For next time:
WeBWork 11.1 and actively read section 11.1

Welcome to MAT135 LEC0501 (Assaf)

Last week, $u = \tan(x)$

$$\int \sqrt{\tan(x)}dx = \int \frac{\sqrt{u}}{u^2+1}du$$

Now, substitute $s = \sqrt{u}$:

$$\int \frac{\sqrt{u}}{u^2+1}du = 2\int \frac{s^2}{s^4+1}ds$$

Next week: a clever trick.

S11.1 – Differential Equations – Modeling the World

Assaf Bar-Natan

“You realize that life goes fast
It’s hard to make the good things last
You realize the sun doesn’t go down
It’s just an illusion caused by the world spinning round”

–“Do You Realize??”, The Flaming Lips

Feb. 5, 2020

What Is a Differential Equation?

A differential equation is an algebraic relation between functions and their derivatives. For example:

$$f'(t) = 4$$

$$f''(t) = f'(t) + 1$$

$$F = ma = m\frac{d^2 s}{dt^2}$$

Sometimes, these differential equations have solutions.

What Is a Differential Equation?

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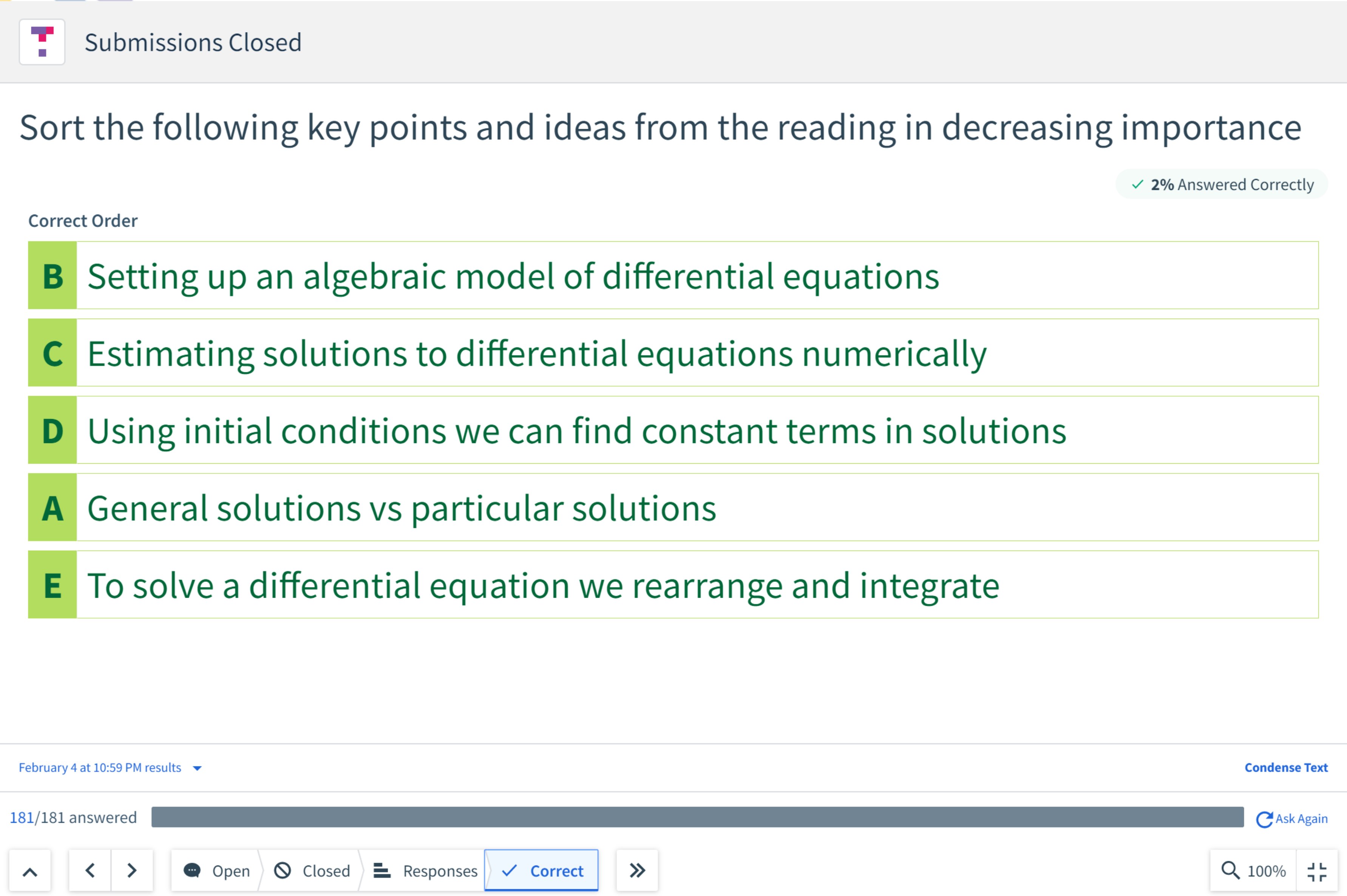
$$F = ma = m\frac{d^2s}{dt^2}$$

Sometimes, these differential equations have solutions.
For which values of k is $te^t - e^t + k$ a solution to the differential equation:

$$f'(t) = f(t) + e^t$$

Key Points from Reading

In groups of $3-4$, take turns listing a key point from the reading. Make sure to explain why you think these are key points.



Sort the following key points and ideas from the reading in decreasing importance

✓ **2%** Answered Correctly

Correct Order

- **B** Setting up an algebraic model of differential equations
- **C** Estimating solutions to differential equations numerically
- **D** Using initial conditions we can find constant terms in solutions
- **A** General solutions vs particular solutions
- **E** To solve a differential equation we rearrange and integrate

February 4 at 10:59 PM results

181/181 answered

The SI Model – Cat's Cold

The cats are sick with a cold. For now, we will make the following assumptions:

- A cat is either susceptible to the virus or infected
- A cat that is infected stays infected
- Let $S(t)$ be the number of susceptible cats after t days
- Let $I(t)$ be the number of infected cats after t days

The SI Model – Cat's Cold

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- A cat is either susceptible to the virus or infected
- A cat that is infected stays infected
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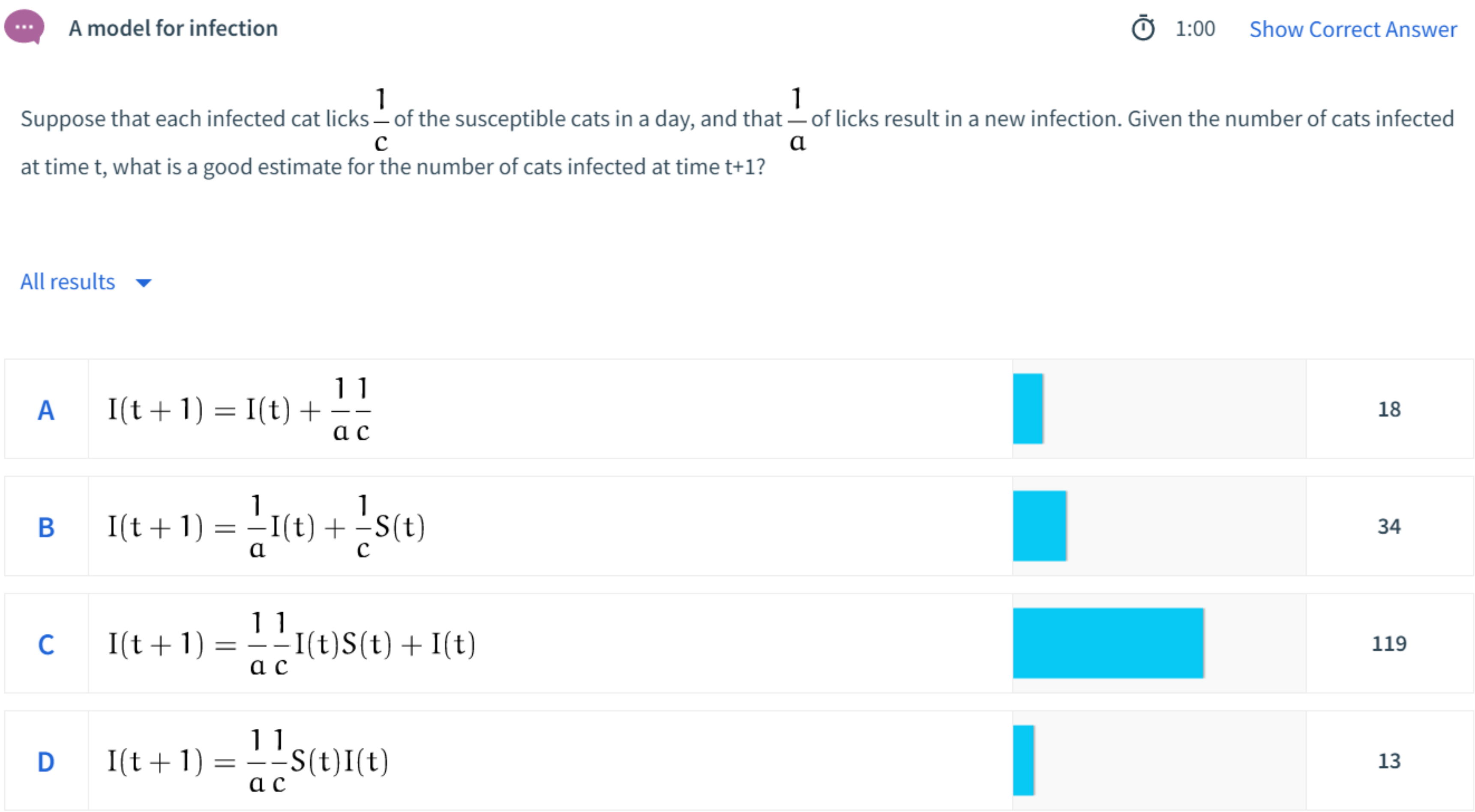
- Let $I(t)$ be the number of infected cats after t days

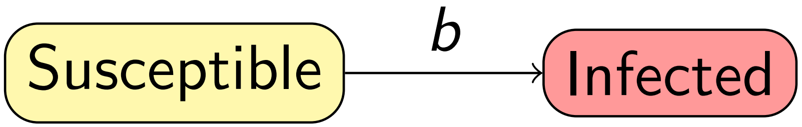
Question: Explain why $\frac{dI}{dt} = -\frac{dS}{dt}$



Suppose that each infected cat licks $\frac{1}{c}$ of the susceptible cats in a day, and that $\frac{1}{a}$ of licks result in a new infection. Given the number of cats infected at time t, what is a good estimate for the number of cats infected at time t+1?

A	$I(t+1) = I(t) + \frac{1}{a}\frac{1}{c}$	18
B	$I(t+1) = \frac{1}{a}I(t) + \frac{1}{c}S(t)$	34
C	$I(t+1) = \frac{1}{a}\frac{1}{c}I(t)S(t) + I(t)$	119
D	$I(t+1) = \frac{1}{a}\frac{1}{c}S(t)I(t)$	13

Deriving the SI Model – Cat's Cold

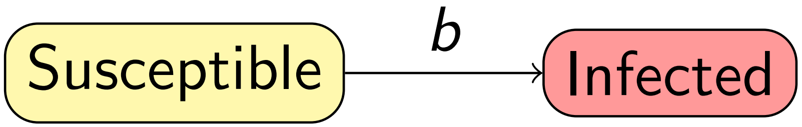


Susceptible $\xrightarrow{b}$ Infected

If we define $b = \frac{1}{ac}$, then we know that:

$$I(t+1) - I(t) = bI(t)S(t)$$

Deriving the SI Model – Cat's Cold

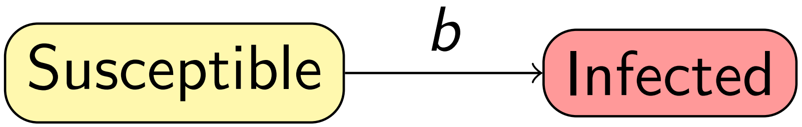


If we define $b = \frac{1}{ac}$, then we know that:

$$I(t+1) - I(t) = bI(t)S(t)$$

Question: What is the verbal interpretation of $I'(t)$?

Deriving the SI Model – Cat's Cold



Susceptible

b

Infected

If we define $b = \frac{1}{ac}$, then we know that:

$$I(t+1) - I(t) = bI(t)S(t)$$

Question: What is the verbal interpretation of $I'(t)$?
$I'(t) = A$ means that A new cats have been infected between t and $t+1$. In other words, $I(t+1) - I(t) = A$. So we can write:

$$I'(t) = bI(t)S(t)$$

SIR Model – The Cats’ Recovery

Eventually, all the cats are infected. Luckily, they start recovering.

Every day, a fraction $k < 1$ of the infected cats end up recovering.
Let $R(t)$ be the number of cats recovered at day t.
Question: Write an expression for $R(t+1) - R(t)$

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$$R(t+1) - R(t) = kI(t)$$

Question: What is the corresponding differential equation?

SIR Model – The Cats' Recovery

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Every day, a fraction $k < 1$ of the infected cats end up recovering.
Let $R(t)$ be the number of cats recovered at day t.
Question: Write an expression for $R(t+1) - R(t)$

$$R(t+1) - R(t) = kI(t)$$

Question: What is the corresponding differential equation?

$$R'(t) = kI(t)$$

SIR Model – The Cats' Recovery

We now add the following assumptions:

- Cats can be susceptible, infected, or recovered.
- When a cat recovers, they are immune to the virus.
- An infected cat can recover from the virus.

SIR Model – The Cats' Recovery

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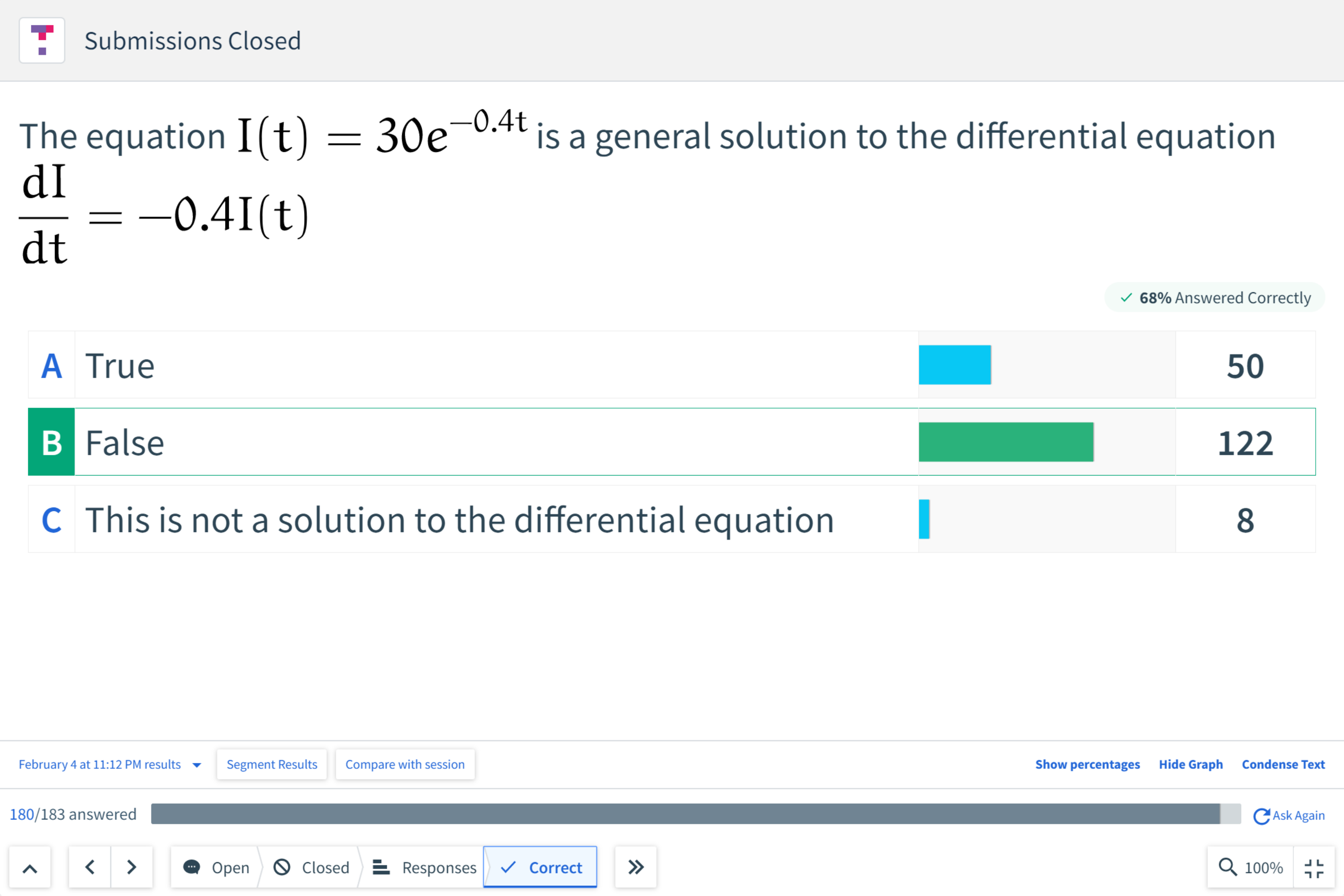
- Cats can be susceptible, infected, or recovered.
- When a cat recovers, they are immune to the virus.
- An infected cat can recover from the virus.

When all of the cats have been infected, and start recovering, we have $I + R = constant$, so

$$\frac{dI}{dt} = -\frac{dR}{dt} = -kI(t)$$

Assume that at $t = 0$, all 30 cats were infected, and none have recovered. If $k = 0.4$, how many cats will have recovered after 3 days? You might want to use the table below as a guide:

t	0	1	2	3
$I(t)$	30			
$I'(t) \approx$				



Submissions Closed

The equation $I(t) = 30e^{-0.4t}$ is a general solution to the differential equation $\frac{dI}{dt} = -0.4I(t)$

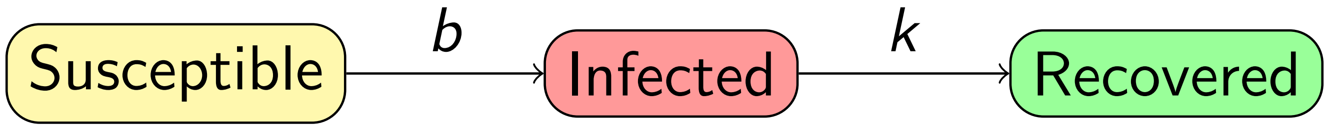
✓ 68% Answered Correctly

A	True	50
B	False	122
C	This is not a solution to the differential equation	8

180/183 answered

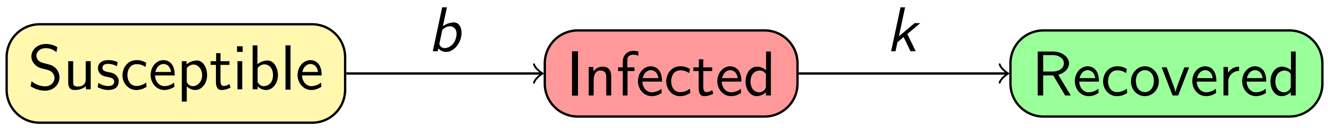
The SIR Model – Wrap-Up

The SIR Model – Wrap-Up


Susceptible $\xrightarrow{b}$ Infected $\xrightarrow{k}$ Recovered

Question: What two terms will contribute to a change in I? Use this to write a formula for $I'(t)$ *Hint: look at previous slides*

The SIR Model – Wrap-Up



Susceptible $\xrightarrow{b}$ Infected $\xrightarrow{k}$ Recovered

Question: What two terms will contribute to a change in I? Use this to write a formula for $I'(t)$ *Hint: look at previous slides*

$$\begin{aligned} S'(t) &= -bI(t)S(t) \\ I'(t) &= bI(t)S(t) - kI(t) \\ R'(t) &= kI(t) \end{aligned}$$

The SIR Model – Wrap-Up

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$$\begin{aligned} S'(t) &= -bI(t)S(t) \\ I'(t) &= bI(t)S(t) - kI(t) \\ R'(t) &= kI(t) \end{aligned}$$

What are some of the shortcomings of the SIR model?

Plans for the Future

For next time:

Review session

For Monday:

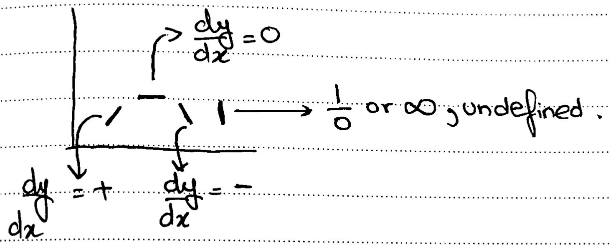
WeBWork 11.2 and actively read section 11.2

11.2 : Slope fields

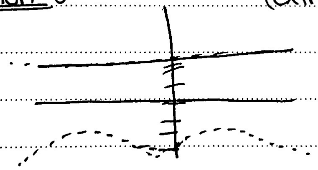
Webwork discussion

- you do not necessarily need to solve a differential equation to find its solution
- slope fields ⟶ way to qualitively measure the solution of a differential equation

- Shortcuts to solving slope field questions

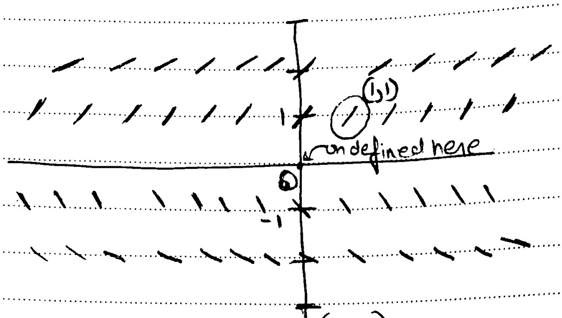


- Start from the given coordinate and trace out slope field.
- Webwork question 5 (extra notes)



When $y =$ any multiple of $\frac{\pi}{2}$, and $x = 0$, then always a horizontal line , therefore $\frac{dy}{dx} = 0$ and Solution $y =$ integer.

if $\frac{dy}{dx} = \frac{1}{y}$ (1,1)



Slope = 1 ~~isb~~ at (1,1)

increasing

steepness of slope at y = any number is the same for all x i.e

$\frac{dy}{dx}$ is not dependent on x.

slope less steep as y increases

Lecture Notes (continued) 11.2.

february 10th, 2020.

$\overset{(1,1)}{}$

$\frac{dy}{dx} = \frac{1}{y}$ Slope = 1 at (1,1)

Solve

$\int y\, dy = \int 1\, dx$ (numerical solution)

$\frac{y^2}{2} = x + C$

$y = \sqrt{2x + C}$ → general solutions

What is the solution of differential equation:

↳ a function that has the derivative equal to the differential equation.

$1 = \sqrt{2 + C}$ ⟶ general

$1 = 2 + C$ $C = -1$ ↳ particular solution

generally for asymptotic solutions

february 12th, 2020.

11.3 : Euler's method

numerically plotting points on a solution curve.

$\Delta y = (\text{Slope at } P_x)\,\Delta x$

y value at P_{x+1} = (y value at P_x)~~(Δy)~~ + Δy.

Error $\propto \frac{1}{n}$.

Error = Exact − Approximate ~~value~~ value.

Welcome to MAT135 LEC0501 (Assaf)

Last week, $u = \tan(x)$, then $s = \sqrt{u}$ gave us:

$$\int \sqrt{\tan(x)}dx = 2\int \frac{s^2}{s^4+1}ds$$

Here's the trick:

$$2\int \frac{s^2}{s^4+1}ds = \int \frac{1}{\sqrt{2}}\left(\frac{s}{s^2-\sqrt{2}s+1} - \frac{s}{s^2+\sqrt{2}s+1}\right)ds$$

Next week: we'll compute one of these terms.

S11.3 – Euler's Method – Stop, Point, Shoot, Repeat

Assaf Bar-Natan

“ Eat, sleep, rave, repeat
Eat, sleep, rave, repeat
Eat, sleep, rave, repeat
Eat, sleep, rave, repeat”

–“Eat, Sleep, Rave, Repeat”, Fatboy Slim

Feb. 12, 2020

What Is Euler's Method?

Euler's method is a bit like a biathelon.

Myriam Bédard, Canadian gold medalist in Biathelon, 1994 Winter Olympics

In a nutshell:

- Pick a starting point
- Use derivative to estimate change
- Move to next point
- Repeat

Example: What is Euler's Method?

What does this mean? Let's assume that:

$$y'(t) = f(y, t)$$

This differential equation has a family of solutions. If we specify that our solution passes through $(0, 0)$, then we know:

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$$y'(t) = f(y, t)$$

This differential equation has a family of solutions. If we specify that our solution passes through $(0, 0)$, then we know:

$$y'(0) = f(0, 0)$$

This lets us estimate $y(0.01) \approx 0.01y'(0)$, giving us a new point to start with.

Takeaway

To estimate the solution of a differential equation at a point, we can apply Euler's method

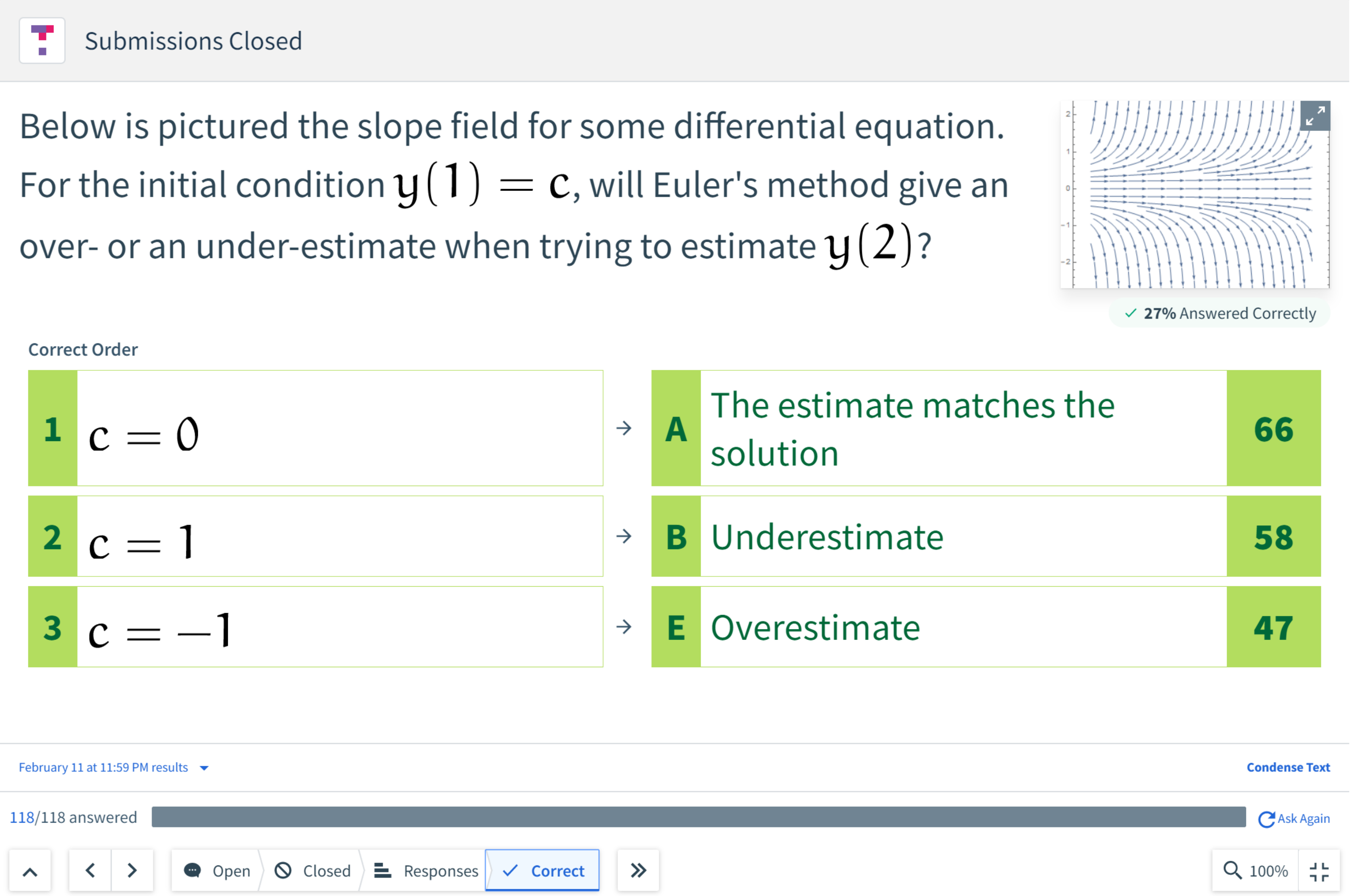
Round Robin: WeBWork

In your groups, share a question in the WeBWork that you either found challenging or a question that you found interesting.

Round Robin: WeBWork

In your groups, share a question in the WeBWork that you either found challenging or a question that you found interesting.

Why did we include this question in the WeBWork? What key point from the chapter does it relate to?



Submissions Closed

Below is pictured the slope field for some differential equation. For the initial condition $y(1) = c$, will Euler's method give an over- or an under-estimate when trying to estimate $y(2)$?

✓ **27%** Answered Correctly

Correct Order

1	$c = 0$	→	A	The estimate matches the solution	66
2	$c = 1$	→	B	Underestimate	58
3	$c = -1$	→	E	Overestimate	47

118/118 answered

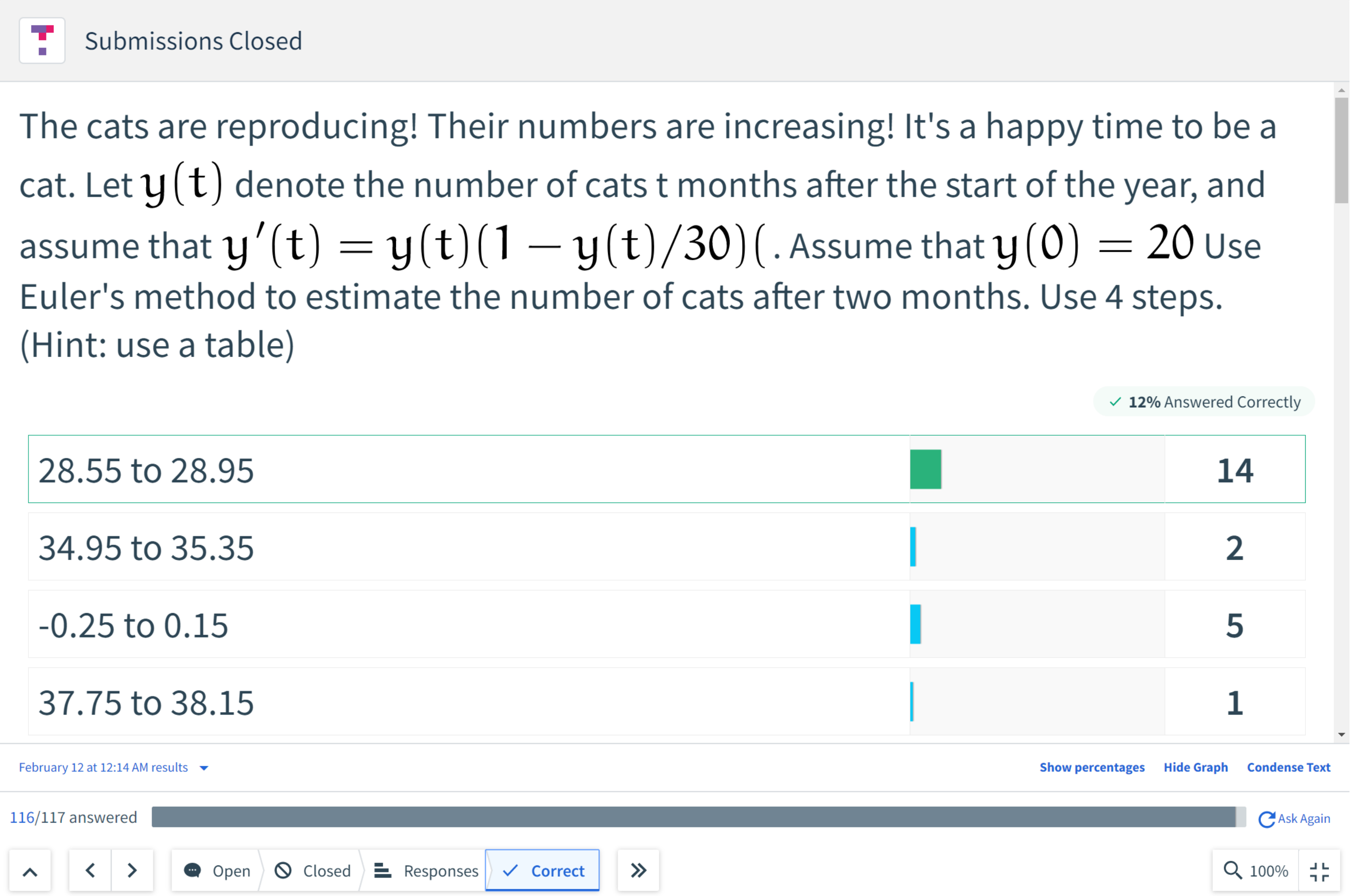
Writing Exercise

Write a short paragraph explaining to the students who aren't here why Euler's method works, and how we can make it more precise.

Writing Exercise

Write a short paragraph explaining to the students who aren't here why Euler's method works, and how we can make it more precise.

We can make Euler's method more precise by making the jumps smaller. That way, the estimate of the derivative is better.



The cats are reproducing! Their numbers are increasing! It's a happy time to be a cat. Let $y(t)$ denote the number of cats t months after the start of the year, and assume that $y'(t) = y(t)(1 - y(t)/30)($. Assume that $y(0) = 20$ Use Euler's method to estimate the number of cats after two months. Use 4 steps. (Hint: use a table)

✓ **12%** Answered Correctly

28.55 to 28.95	14
34.95 to 35.35	2
-0.25 to 0.15	5
37.75 to 38.15	1

February 12 at 12:14 AM results

116/117 answered

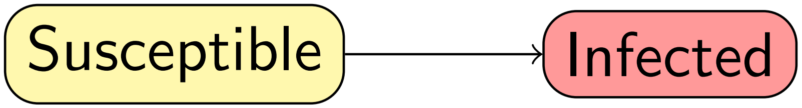
Bonus: Chaos, Fractals, Dynamics

An interesting video related to this:

`https://www.youtube.com/watch?v=ovJcsL7vyrk`

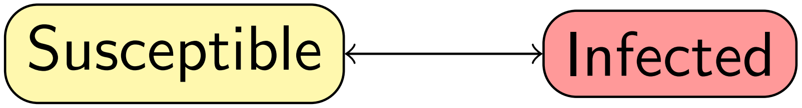
SIS? SI? SIR?

The SI model:



The SIR model:

The SIS model:

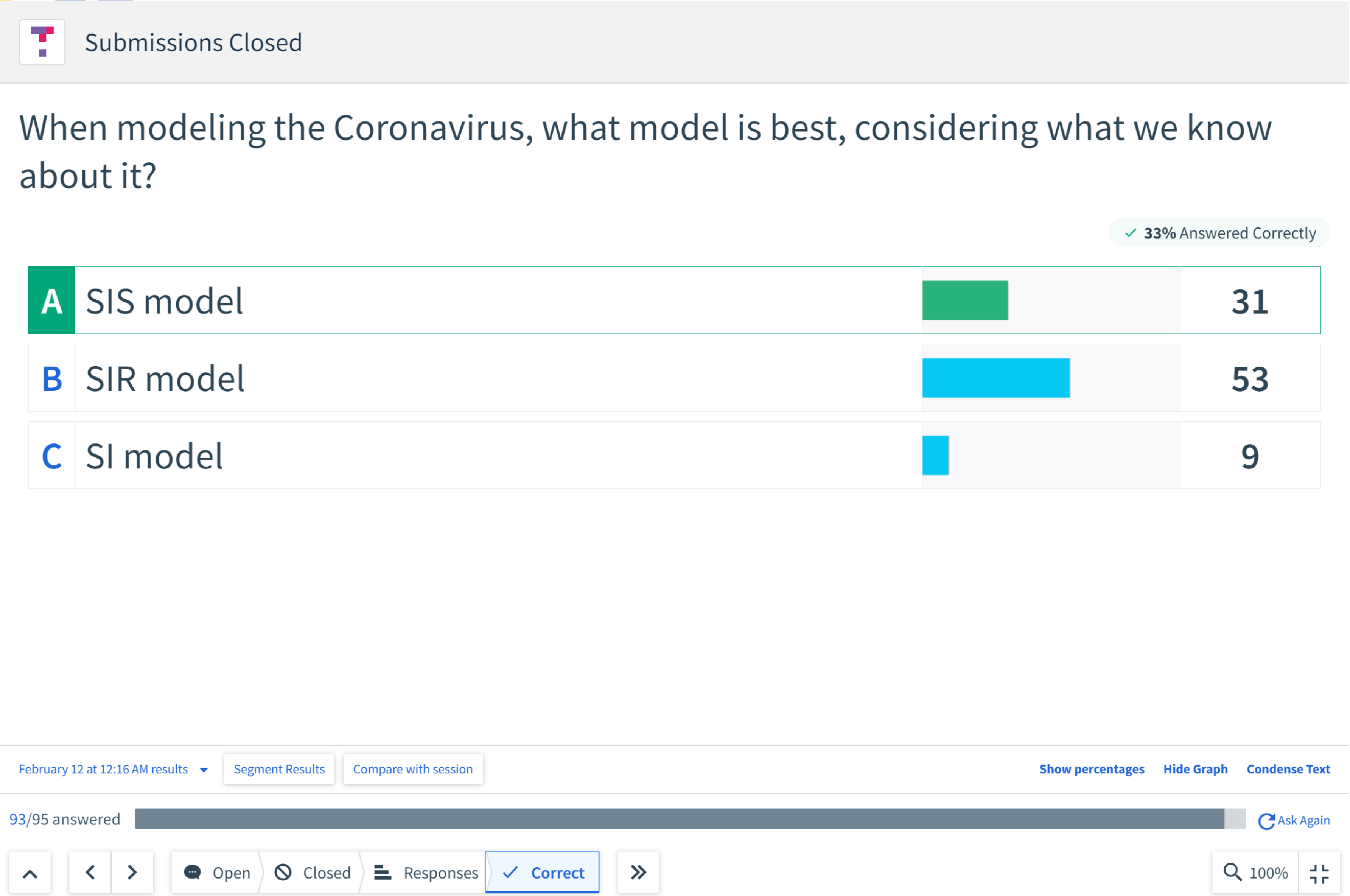


Coronavirus – SIS? SI? SIR?

Go online, and search for information about the current Coronavirus outbreak. Which model is best? What searches did you make?

Coronavirus – SIS? SI? SIR?

Go online, and search for information about the current Coronavirus outbreak. Which model is best? What searches did you make? After infection, what happens to surviving patients?



Submissions Closed

When modeling the Coronavirus, what model is best, considering what we know about it?

✓ **33%** Answered Correctly

A	SIS model	31
B	SIR model	53
C	SI model	9

93/95 answered

Plans for the Future

For next time:

WeBWork 11.4 and actively read section 11.4

Welcome to MAT136 LEC0501 (Assaf)

Was the midterm what you expected? What surprised you? What would you change next time?

S11.4 – Separation of Variables – $\frac{dy}{dx}$ is Still not a Fraction

Assaf Bar-Natan

“ How long, how long will I slide?
Separate my side, I don’t
I don’t believe it’s bad”

–“Otherside”, Red Hot Chili Peppers

Feb. 14, 2020

Ice Cream Sandwich

In your groups, share:

- A time you had a good success

Ice Cream Sandwich

In your groups, share:

- A time you had a good success
- A time you failed

Ice Cream Sandwich

In your groups, share:

- A time you had a good success
- A time you failed
- A time you recovered

What is Separation of Variables?

We wish to solve:

$$\frac{dy}{dx} = g(x)f(y)$$

Thinking of $\frac{dy}{dx}$ as a ratio (it's not), we get:

$$\int \frac{1}{f(y)}dy = \int g(x)dx$$

What is Separation of Variables?

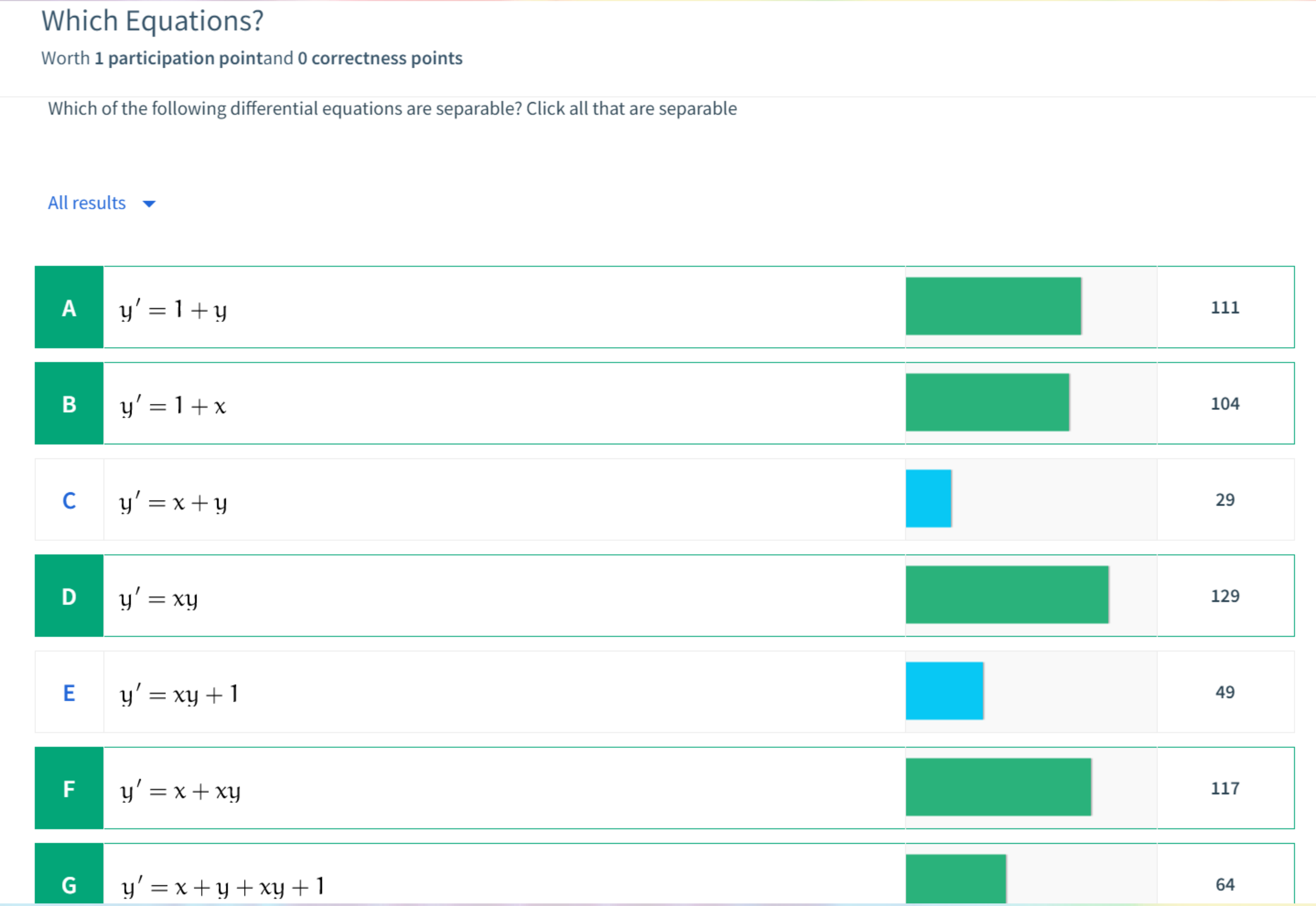
We wish to solve:

$$\frac{dy}{dx} = g(x)f(y)$$

Thinking of $\frac{dy}{dx}$ as a ratio (it's not), we get:

$$\int \frac{1}{f(y)}dy = \int g(x)dx$$

This gives us an relation between x and y, which is the solution to the differential equation



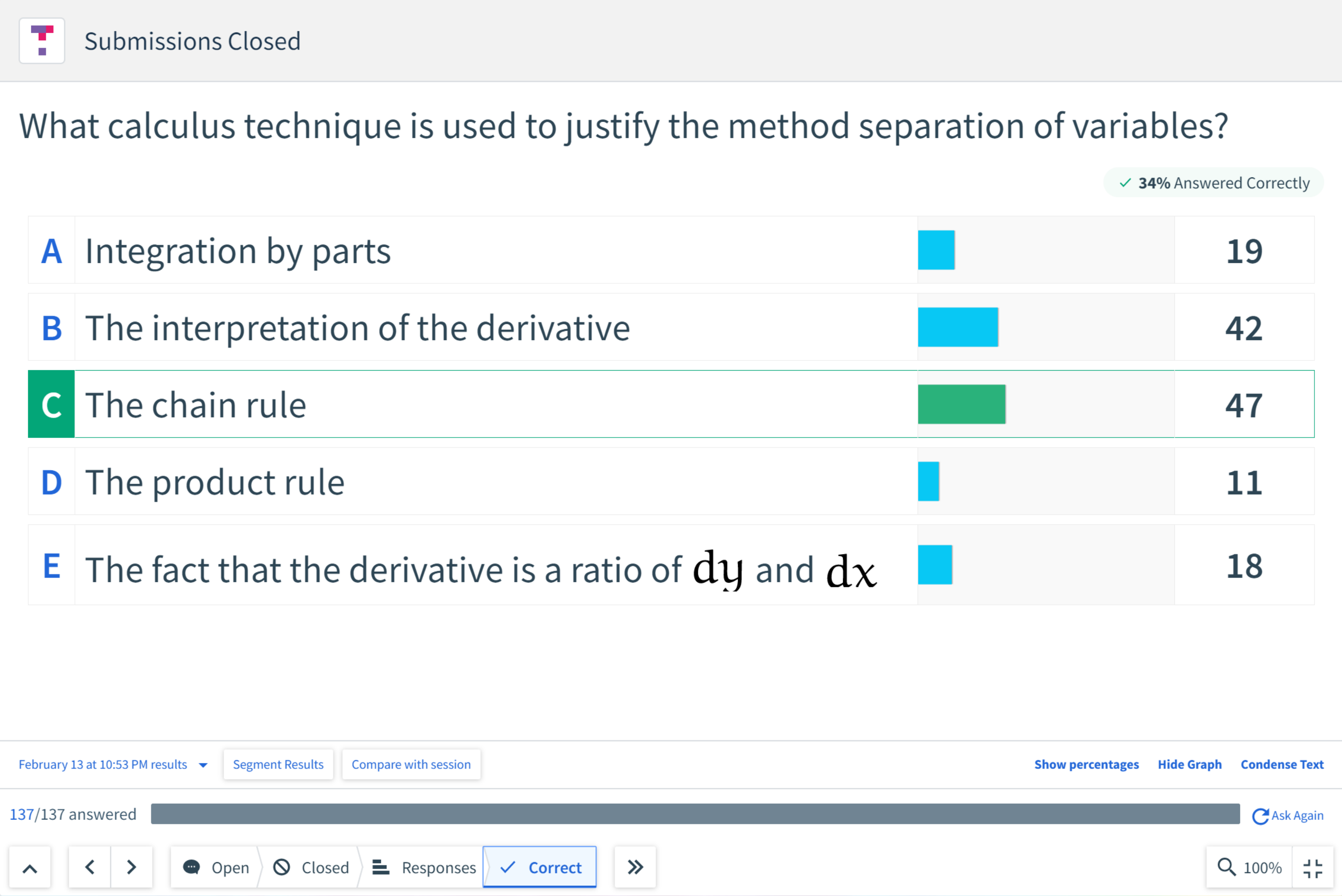
Which Equations?

Worth **1 participation point**and **0 correctness points**

Which of the following differential equations are separable? Click all that are separable

All results ▾

A	$y' = 1 + y$	111
B	$y' = 1 + x$	104
C	$y' = x + y$	29
D	$y' = xy$	129
E	$y' = xy + 1$	49
F	$y' = x + xy$	117
G	$y' = x + y + xy + 1$	64



Submissions Closed

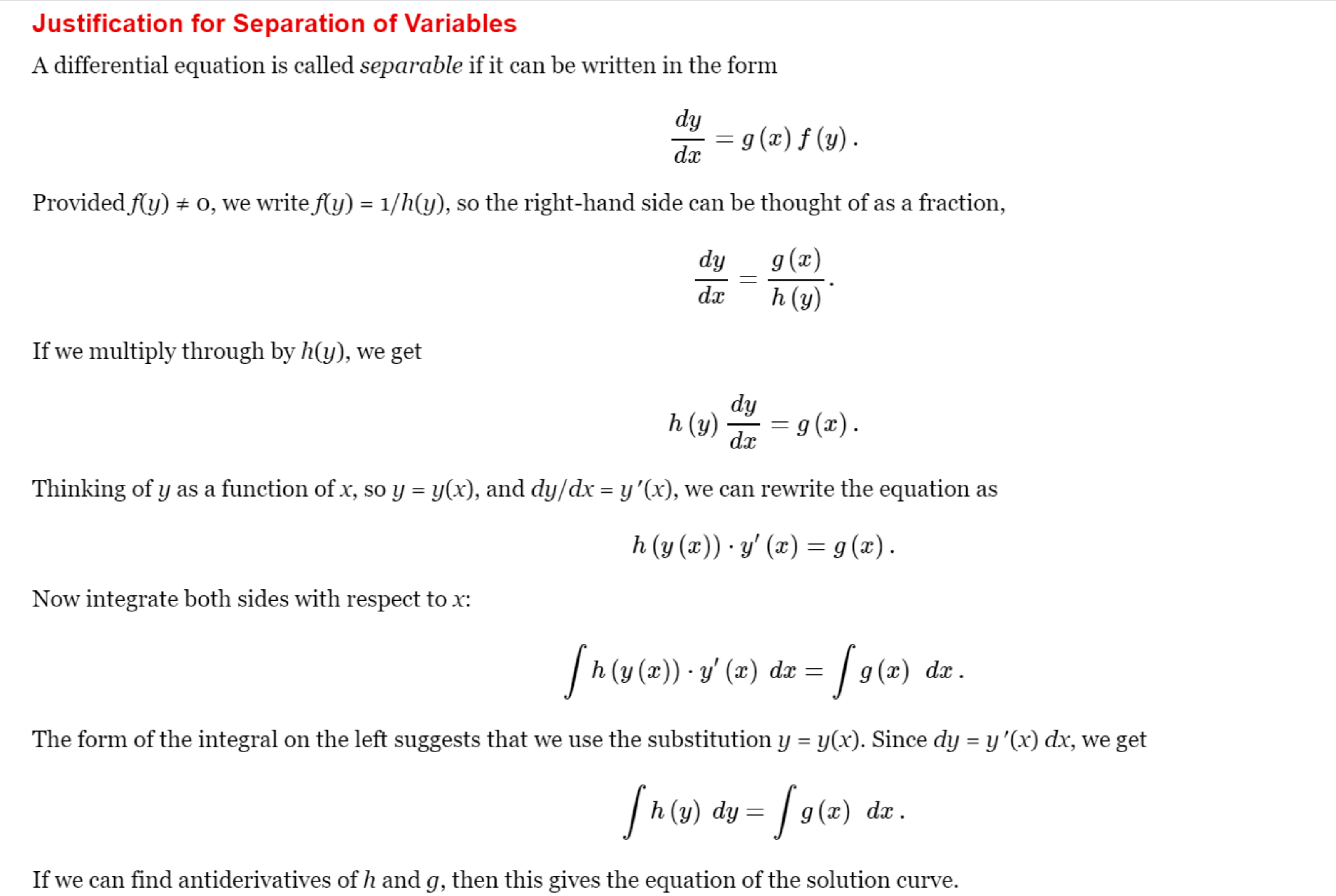
What calculus technique is used to justify the method separation of variables?

✓ **34%** Answered Correctly

A	Integration by parts	19
B	The interpretation of the derivative	42
C	The chain rule	47
D	The product rule	11
E	The fact that the derivative is a ratio of dy and dx	18

February 13 at 10:53 PM results

137/137 answered



Justification for Separation of Variables

A differential equation is called *separable* if it can be written in the form

$$\frac{dy}{dx} = g\left(x\right) f\left(y\right).$$

Provided $f(y) \neq 0$, we write $f(y) = 1/h(y)$, so the right-hand side can be thought of as a fraction,

$$\frac{dy}{dx} = \frac{g\left(x\right)}{h\left(y\right)}.$$

If we multiply through by $h(y)$, we get

$$h\left(y\right)\frac{dy}{dx} = g\left(x\right).$$

Thinking of y as a function of x, so $y = y(x)$, and $dy/dx = y\,'(x)$, we can rewrite the equation as

$$h\left(y\left(x\right)\right)\cdot y'\left(x\right) = g\left(x\right).$$

Now integrate both sides with respect to x:

$$\int h\left(y\left(x\right)\right)\cdot y'\left(x\right)\ dx = \int g\left(x\right)\ dx\,.$$

The form of the integral on the left suggests that we use the substitution $y = y(x)$. Since $dy = y\,'(x)\,dx$, we get

$$\int h\left(y\right)\ dy = \int g\left(x\right)\ dx\,.$$

If we can find antiderivatives of h and g, then this gives the equation of the solution curve.

Takeawy

While $\frac{dy}{dx}$ is not a fraction, it can be useful to think of it as one.

The textbook is a useful resource!!!!!

Punctuated Lecture: The Cat Population

Last time, we modeled the population of cats by:

$$\frac{dy}{dt} = y(1 - y/30)$$

Question: Use separation of variables to write this differential equation as an equality of integrals.

Punctuated Lecture: The Cat Population

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$$t = \int \frac{dy}{y - y^2/30}$$

Punctuated Lecture: The Cat Population

Last time, we modeled the population of cats by:

$$\frac{dy}{dt} = y(1 - y/30)$$

This is solved by:

$$t = \int \frac{dy}{y - y^2/30}$$

Question: Verify that

$$\log\left(\frac{y}{30 - y}\right)$$

is an antiderivative of $\frac{1}{y - y^2/30}$. (You may use a computer)

Punctuated Lecture: The Cat Population

Last time, we modeled the population of cats by:

$$\frac{dy}{dt} = y(1 - y/30)$$

This is solved by:

$$t = \log\left(\frac{y}{30 - y}\right)$$

Question: Write y as a function of t.

Punctuated Lecture: The Cat Population

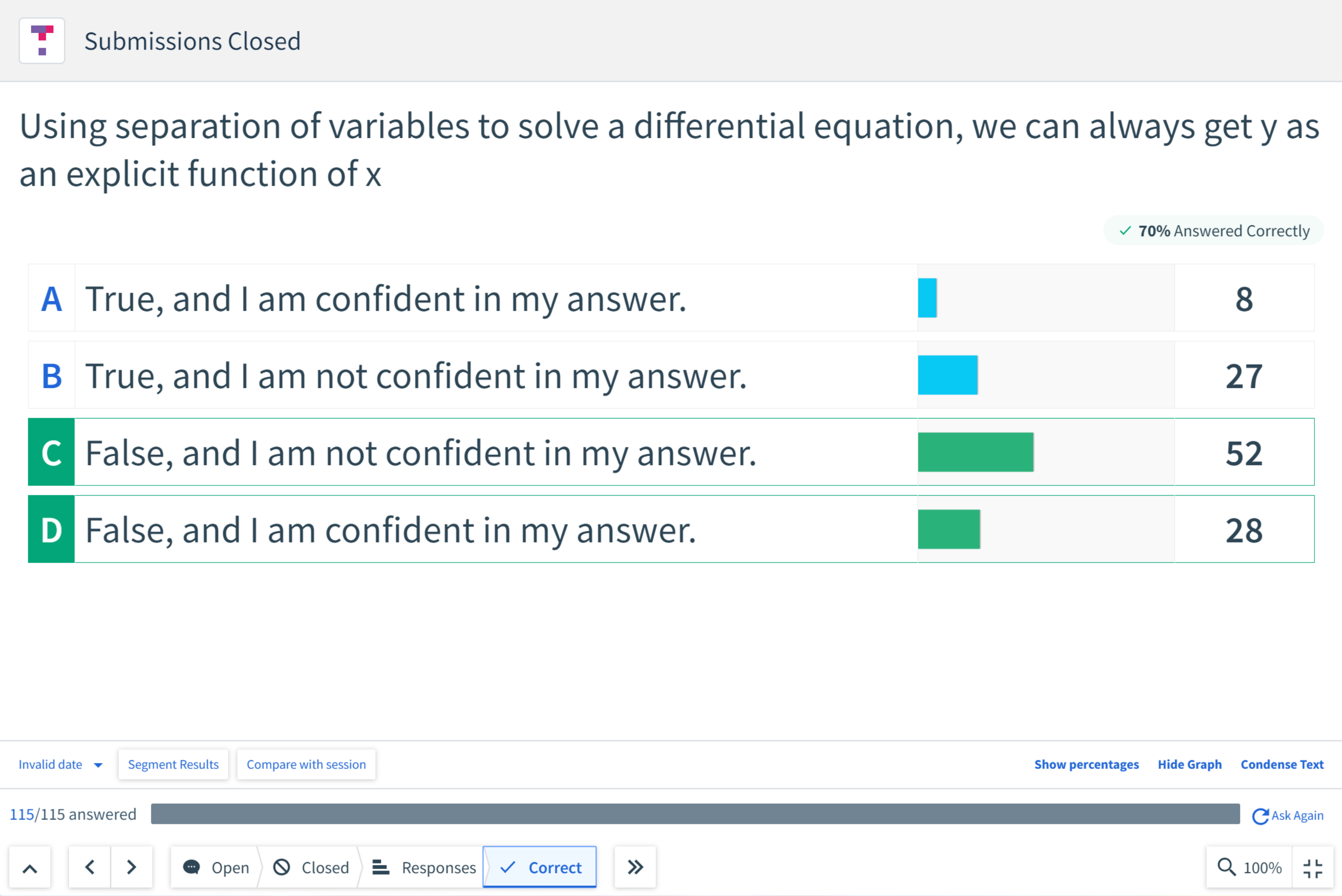
Last time, we modeled the population of cats by:

$$\frac{dy}{dt} = y(1 - y/30)$$

This is solved by:

$$y(t) = \frac{30e^t}{1 + e^t}$$

Question: We earlier said that the number of cats at $t = 0$ was 20, but plugging in $t = 0$ above does not yield 20. What happened?



Using separation of variables to solve a differential equation, we can always get y as an explicit function of x

✓ **70%** Answered Correctly

A	True, and I am confident in my answer.	8
B	True, and I am not confident in my answer.	27
C	False, and I am not confident in my answer.	52
D	False, and I am confident in my answer.	28

115/115 answered

Separation of Variables – Practice

Solve the following differential equation using separation of variables:

$$y' = \frac{1}{1+y^4}$$

Takeaway

Separation of variables gives an implicit solution to the differential equation, not an explicit one

Plans for the Future

For next time:
WeBWork 11.5 and actively read section 11.5

Welcome to MAT136 LEC0501 (Assaf)

Over reading week, did you do something:

- Fun?
- Hard?
- Rewarding?

S11.5 – Growth Models

Assaf Bar-Natan

> “ Now, for ten years we’ve been on our own
> And moss grows fat on a rolling stone
> But, that’s not how it used to be”
>
> –“American Pie”, Don McLean

Feb. 24, 2020

Game Plan

- Today: section 11.5
- Wednesday & Friday: section 11.8
- New WeBWork: Taylor polynomials review "136TaylorSolutions"

Key Points Round Robin

Get into groups of three or four

Key Points Round Robin

Get into groups of three or four

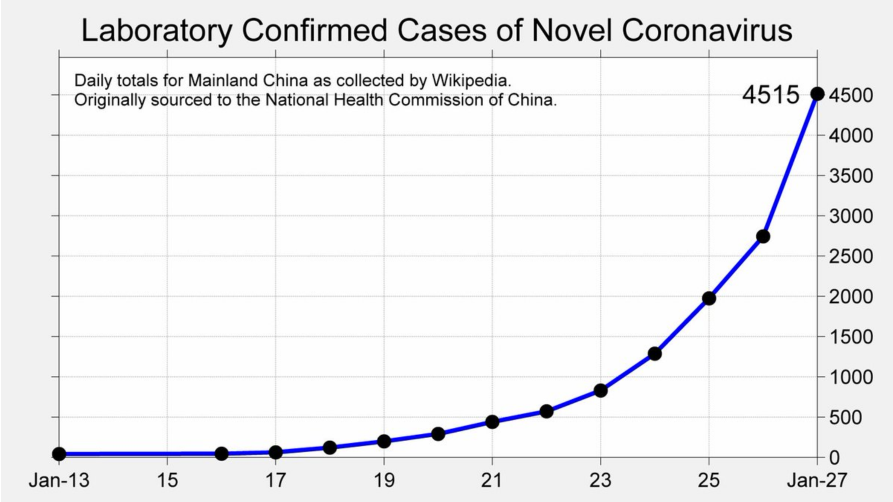
- As a group, come up with three big key ideas from this chapter.

Key Points Round Robin

Get into groups of three or four

- As a group, come up with three big key ideas from this chapter.
- Pick a WeBWork problem from section 11.5. What key ideas does it relate to?

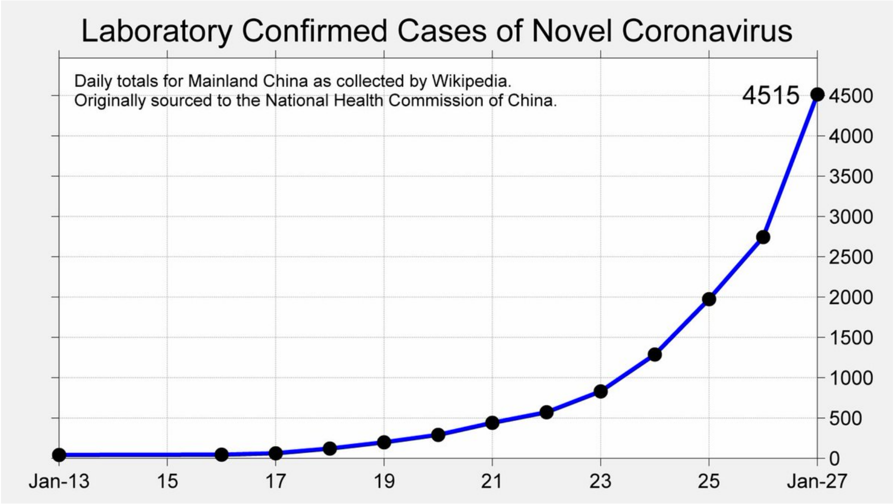
COVID-19 Growth


Laboratory Confirmed Cases of Novel Coronavirus

Daily totals for Mainland China as collected by Wikipedia.
Originally sourced to the National Health Commission of China.

What function could model this data?

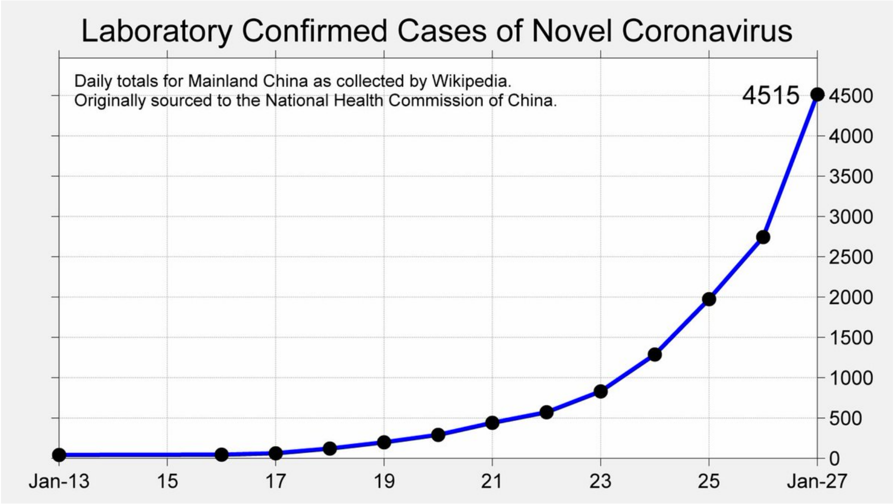
COVID-19 Growth



A reasonable guess:

$$I(t) = I_0 e^{kt}$$

COVID-19 Growth



A reasonable guess:

$$I(t) = I_0 e^{kt}$$

What value should we choose for k?

Possible Reasons for Discrepancy

- Data is imprecise
- S is approximately constant, so I' is approximately proportional to I
- The exponential model is not a good model to use in this case
- The data is not actually an exponential.

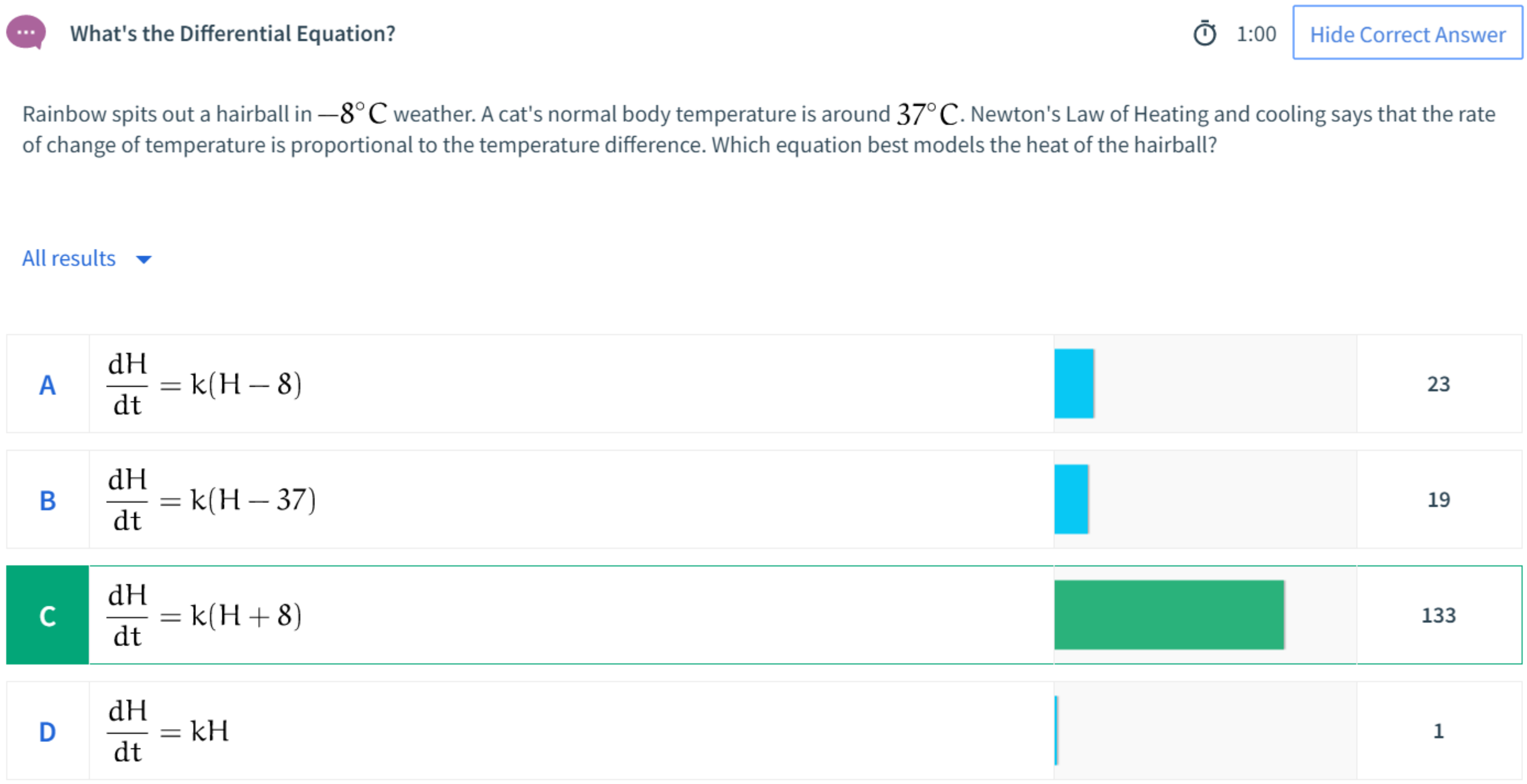
`https://www.worldometers.info/coronavirus/`

Takeaways

We can use a graph to track in real-time whether the SIS model is a good model

Punctuated Lecture: Rainbow’s Hairball

Rainbow spits out a hairball in $-8^\circ C$ weather. A cat’s normal body temperature is around $37^\circ C$. After one minute, the ball’s temperature was $20^\circ C$. We will try to model the hairball’s temperature as a function of time.



What's the Differential Equation?

Rainbow spits out a hairball in $-8^\circ C$ weather. A cat's normal body temperature is around $37^\circ C$. Newton's Law of Heating and cooling says that the rate of change of temperature is proportional to the temperature difference. Which equation best models the heat of the hairball?

A	$\frac{dH}{dt} = k(H-8)$	23
B	$\frac{dH}{dt} = k(H-37)$	19
C	$\frac{dH}{dt} = k(H+8)$	133
D	$\frac{dH}{dt} = kH$	1

Punctuated Lecture: Rainbow's Hairball

Rainbow spits out a hairball in $-8^\circ C$ weather. A cat's normal body temperature is around $37^\circ C$. After one minute, the ball's temperature was $20^\circ C$. We will try to model the hairball's temperature as a function of time.

$$\frac{dH}{dt} = k(H+8)$$

Punctuated Lecture: Rainbow's Hairball

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$$\frac{dH}{dt} = k(H+8)$$

Q: Should k be positive of negative?

Punctuated Lecture: Rainbow's Hairball

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$$\frac{dH}{dt} = k(H+8)$$

Q: Should k be positive of negative?

Q: Solve this differential equation.

Punctuated Lecture: Rainbow's Hairball

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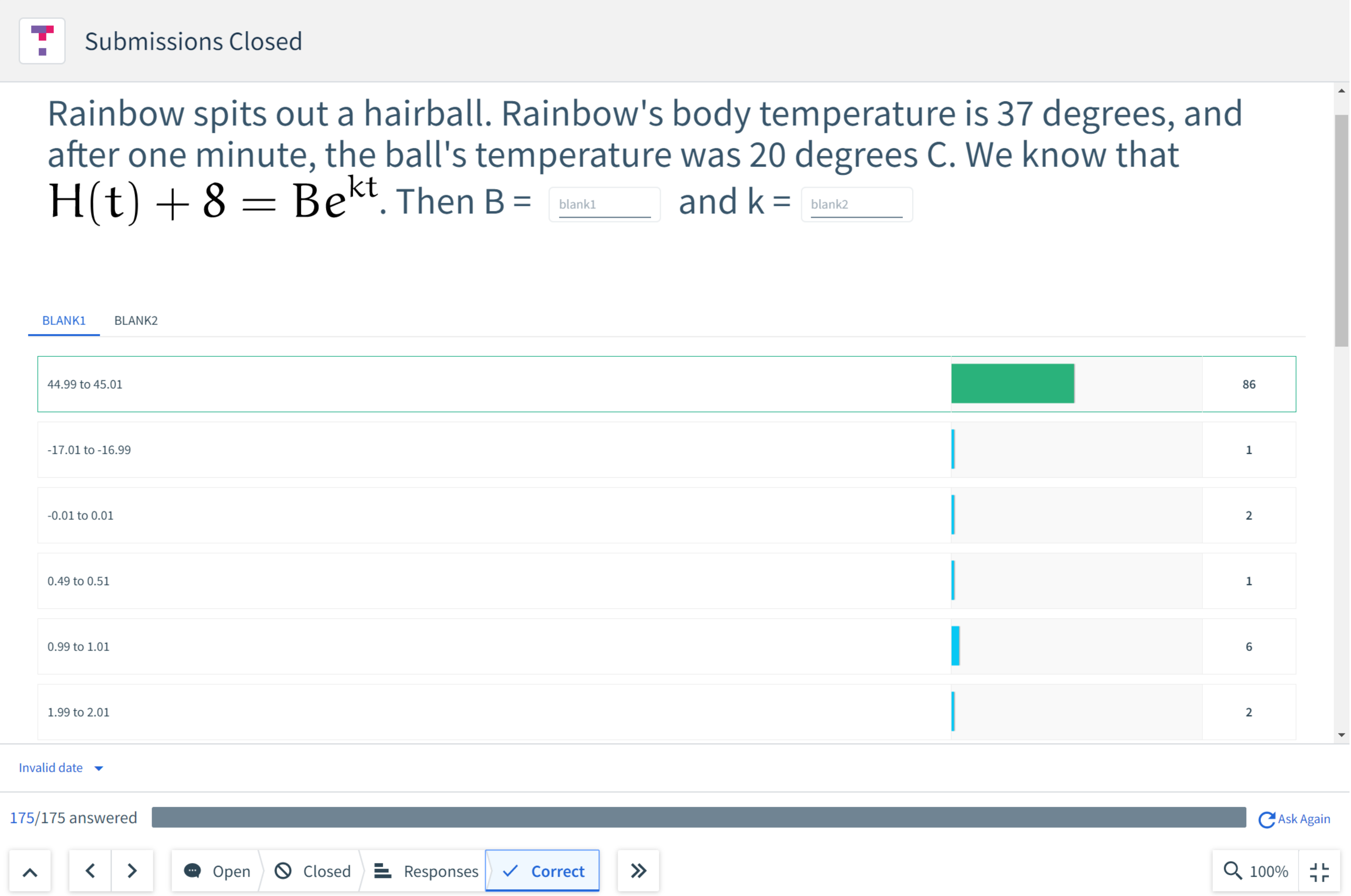
$$\frac{dH}{dt} = k(H+8)$$

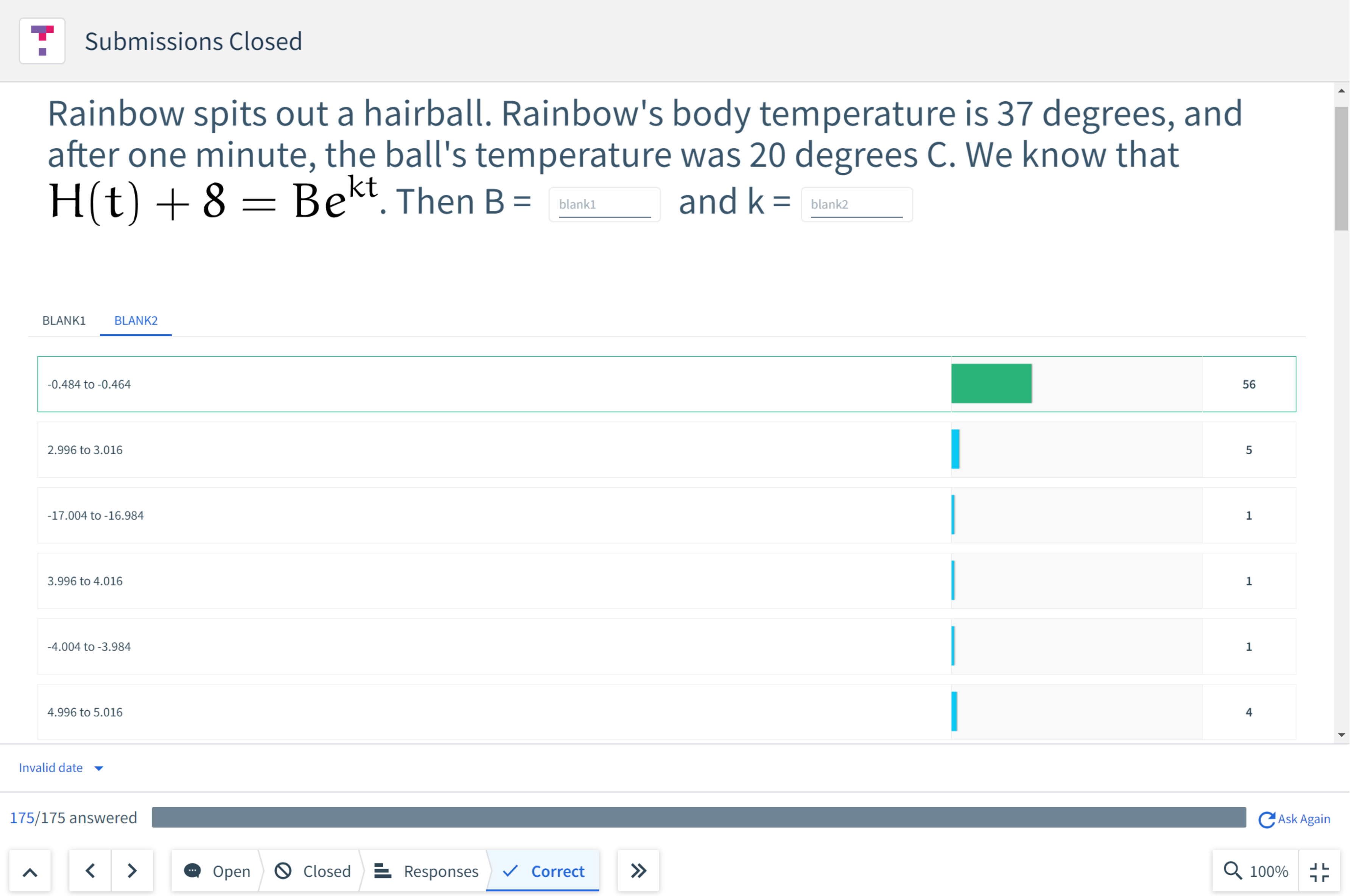
Q: Should k be positive of negative?
Q: Solve this differential equation.
A: Using separation of variables, $H(t) + 8 = Be^{kt}$.



Submissions Closed

Rainbow spits out a hairball. Rainbow's body temperature is 37 degrees, and after one minute, the ball's temperature was 20 degrees C. We know that $H(t) + 8 = Be^{kt}$. Then B = blank1 and k = blank2

44.99 to 45.01	86
-17.01 to -16.99	1
-0.01 to 0.01	2
0.49 to 0.51	1
0.99 to 1.01	6
1.99 to 2.01	2



Submissions Closed

Rainbow spits out a hairball. Rainbow's body temperature is 37 degrees, and after one minute, the ball's temperature was 20 degrees C. We know that $H(t)+8=Be^{kt}$. Then B = blank1 and k = blank2

BLANK1 BLANK2

-0.484 to -0.464	56
2.996 to 3.016	5
-17.004 to -16.984	1
3.996 to 4.016	1
-4.004 to -3.984	1
4.996 to 5.016	4

Invalid date

175/175 answered

Punctuated Lecture: Rainbow's Hairball

$$H(t) + 8 = Be^{kt}$$

We are given that $H(0) = 37$, so this means that $B = 45$.

Punctuated Lecture: Rainbow's Hairball

$$H(t) + 8 = Be^{kt}$$

We are given that $H(0) = 37$, so this means that $B = 45$. Moreover, we know $H(1) = 20$, so:

$$28 = 45e^{k}$$

giving us $k \approx -0.474$

Takeaway

We can use initial conditions and another point to find constants that give a particular solution to a heat-law-type problem

Equilibrium Points

Here's a totally wonky, and completely random differential equation:

$$\frac{dy}{dx} = (y-1)(y+1)$$

Q: What are its equilibrium solutions?

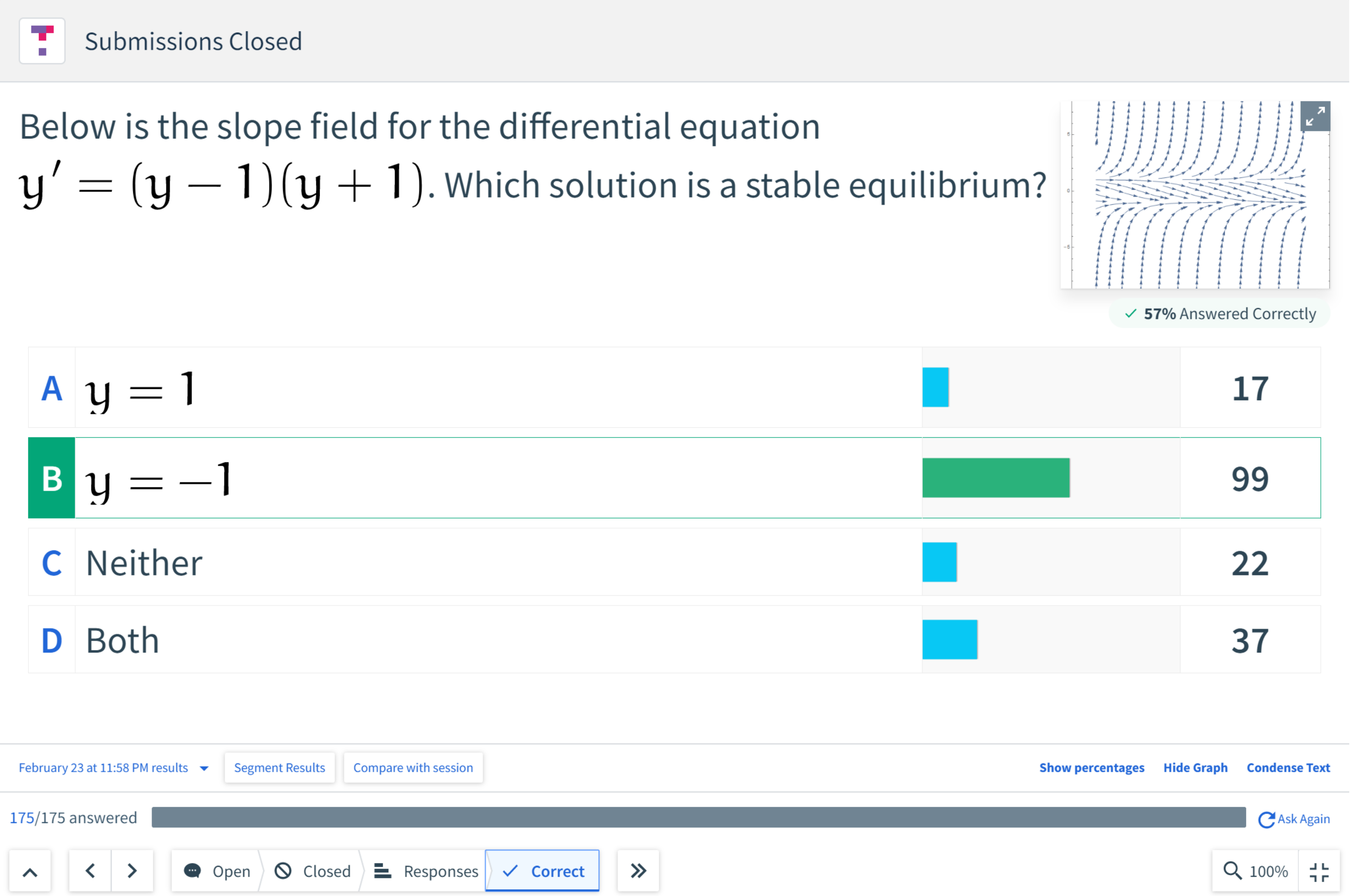
Equilibrium Points

Here's a totally wonky, and completely random differential equation:

$$\frac{dy}{dx} = (y-1)(y+1)$$

Q: What are its equilibrium solutions?

A: $y = 1$ and $y = -1$



Submissions Closed

Below is the slope field for the differential equation $y' = (y-1)(y+1)$. Which solution is a stable equilibrium?

✓ **57%** Answered Correctly

A	$y = 1$	17
B	$y = -1$	99
C	Neither	22
D	Both	37

175/175 answered

Takeaway

We can tell the difference between stable and unstable equilibria by looking at the slope fields.

Plans for the Future

For next time:

WeBWork 11.8 and actively read section 11.8

Welcome to MAT136 LEC0501 (Assaf)

Weather is finally nice! How've you been enjoying it?

S11.8 Systems of ODE's and The SIR Model (Part 1)

Assaf Bar-Natan

> “ For there's Basie, Miller, Satchmo
> And the king of all, Sir Duke
> And with a voice like Ella's ringing out
> There's no way the band can lose”
>
> –“SIR Duke”, Stevie Wonder

Feb. 26, 2020

The SIR Model

Reminder: The SIR model says:

$$\frac{dS}{dt} = -\alpha SI$$

$$\frac{dI}{dt} = \alpha SI - kI$$

$$\frac{dR}{dt} = kI$$

We used k, the textbook uses β.

The SIR Model

Reminder: The SIR model says:

$$\frac{dS}{dt} = -\alpha SI$$
$$\frac{dI}{dt} = \alpha SI - kI$$
$$\frac{dR}{dt} = kI$$

We used k, the textbook uses β.

Q: Remind yourself what S, I, and R mean in the SIR model.

Finding Values for β and α

Get into groups of three or four, and open up a spreadsheeting program.

- Title the first column: DATA

Finding Values for β and α

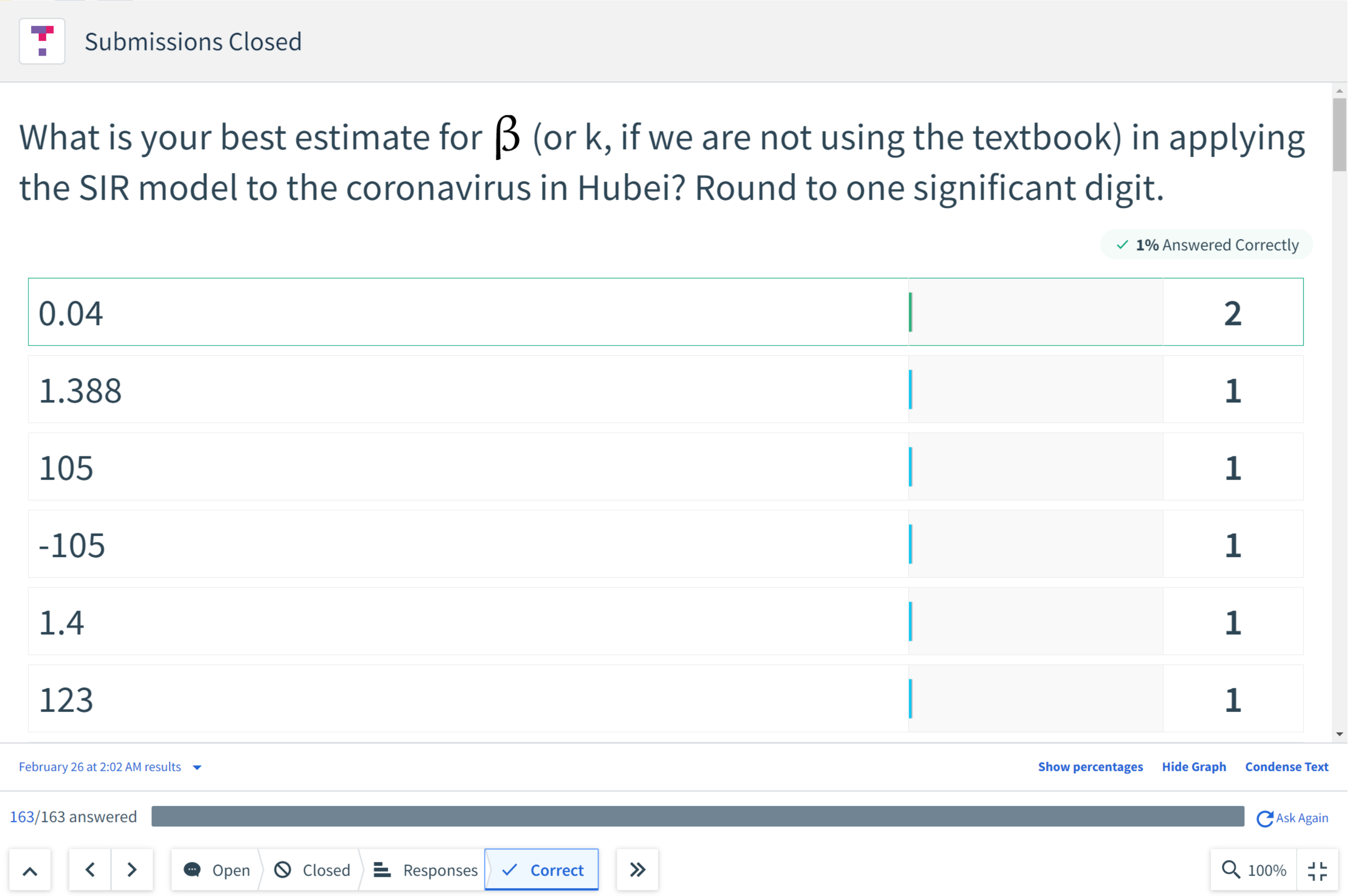
Get into groups of three or four, and open up a spreadsheeting program.

- Title the first column: DATA
- Navigate to: `https://covid2019.azurewebsites.net`, and explore the data on the bottom bar of the site

Finding Values for β and α

Get into groups of three or four, and open up a spreadsheeting program.

- Title the first column: DATA
- Navigate to: `https://covid2019.azurewebsites.net`, and explore the data on the bottom bar of the site
- For Hubei, copy down $I(t)$ into the first column of a spreadsheet (use only the data from the first 15-16 days)



Submissions Closed

What is your best estimate for β (or k, if we are not using the textbook) in applying the SIR model to the coronavirus in Hubei? Round to one significant digit.

✓ **1%** Answered Correctly

0.04	2
1.388	1
105	1
-105	1
1.4	1
123	1

February 26 at 2:02 AM results

163/163 answered

Takeaway

β is easily measured as the death and recovery rate

Making a Model

In your groups:

- Make a new column in the spreadsheet. Label it S
- Make a new column in the spreadsheed. Label it I
- Make a new column in the spreadsheet. Label it R
- What should $S(1)$ be? What should $R(1)$ be?

We will next use Euler's method to fill in the rest of the model.

Making a Model

In your groups:

- Make a new column in the spreadsheet. Label it S
- Make a new column in the spreadsheed. Label it I
- Make a new column in the spreadsheet. Label it R
- What should $S(1)$ be? What should $R(1)$ be? $S(1)$ is the population of Hubei, $R(1) = 0$

We will next use Euler's method to fill in the rest of the model.

Making a Model

- Write a formula for $I(2)$, $S(2)$, and $R(2)$ involving $S(1)$, $I(1)$, $R(1)$, the constant $\beta = 0.04$, and an unknown constant, α (maybe start by plugging in $\alpha = 0.000001$.)
- Extend the formula down (click and drag) to predict $I(t)$, $S(t)$, and $R(t)$. **Note: they will have to depend on each other!**
- Do your predictions match the data column? What parameter should you change?

Hint: $I(t+1) \approx I(t) + I'(t)$

Takeaway

We can use a spreadsheet and Euler's method to solve an ODE, and to make predictions

Interpreting the Constants

When we developed the SIR model:

- α represented the infection rate per sick person per day.
- $k = \beta$ represented the rate at which people recovered.

Go back in your notes, or to lecture 14, and remind yourself how we used these interpretations to derive the SIR model.

Interpreting the Constants

When we developed the SIR model:

- α represented the infection rate per sick person per day.
- $k = \beta$ represented the rate at which people recovered.

Now: $\frac{1}{k}$ can also be interpreted as the average amount of time a person is sick with the virus.

Interpreting the Constants

When we developed the SIR model:

- α represented the infection rate per sick person per day.
- $k = \beta$ represented the rate at which people recovered.

Now: $\frac{1}{k}$ can also be interpreted as the average amount of time a person is sick with the virus.

How can we use units to understand this interpretation? What are the units of k? What are the units of $\frac{1}{k}$?

Phase-Plane Introduction

We use the chain rule:

$$\frac{dI}{dS} = \frac{\frac{dI}{dt}}{\frac{dS}{dt}}$$

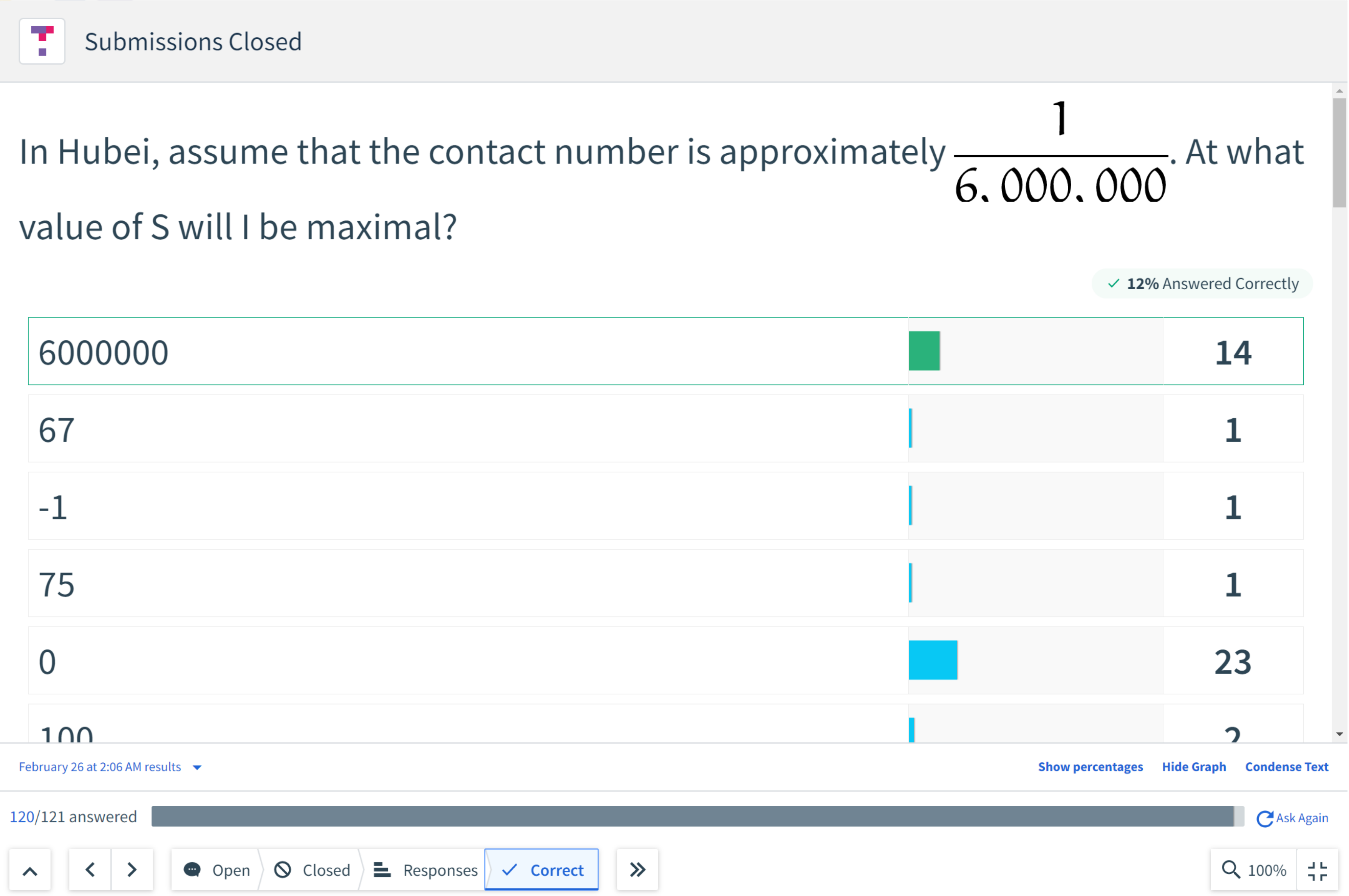
This, along with the SIR model equations, allows us to solve for I in terms of S.

Phase-Plane Introduction

We use the chain rule:

$$\frac{dI}{dS} = \frac{\frac{dI}{dt}}{\frac{dS}{dt}}$$

This, along with the SIR model equations, allows us to solve for I in terms of S.
In your groups, write $\frac{dI}{dS}$ exclusively in terms of S, α, and β (or k).



Submissions Closed

In Hubei, assume that the contact number is approximately $\frac{1}{6,000,000}$. At what value of S will I be maximal?

✓ **12%** Answered Correctly

6000000	14
67	1
-1	1
75	1
0	23
100	2

February 26 at 2:06 AM results

120/121 answered

Takeaways

$c = \frac{1}{6{,}000{,}000}$ means that on average, an infected person has close contact with about $\frac{1}{6{,}000{,}000}$th of the population of Hubei. This is around 10 people, which is quite reasonable.

The constant c, is called the contact number, and next time, we will see how it can be used to help prevent an epidemic.

Plans for the Future

For next time:
actively read section 11.8

Welcome to MAT136 LEC0501 (Assaf)

Last time, we had a trick: if $s = \sqrt{\tan(x)}$, then:

$$\int \sqrt{\tan(x)}dx = \int \frac{1}{\sqrt{2}}\left(\frac{s}{s^2-\sqrt{2}s+1} - \frac{s}{s^2+\sqrt{2}s+1}\right)ds$$

We work on the first term (the second is similar):

$$\int \frac{s}{s^2-\sqrt{2}s+1}ds = \int \frac{2s-\sqrt{2}}{2(s^2-\sqrt{2}s+1)} + \frac{1}{\sqrt{2}(s^2-\sqrt{2}s+1)}ds$$

You already know how to compute the first term here!

S11.8 Part 2 – The Perils of: Phase Diagrams, War, and Modeling

Assaf Bar-Natan

> “ I fought the war but the war won’t
> stop for the love of god.
> I fought the war but the war won”
>
> –“Monster Hospital”, Metric

Feb. 28, 2020

The SIR Model – Contact Number

$$\frac{dS}{dt} = -\alpha SI$$
$$\frac{dI}{dt} = \alpha SI - \beta I$$

So:

$$\frac{dI}{dS} = \frac{\alpha SI - \beta I}{-\alpha SI} = -1 + \frac{\beta}{\alpha}\frac{1}{S}$$
$$= -1 + \frac{1}{cS}$$

Where we define $c = \frac{\alpha}{\beta}$, the contact number.

Why Contact Numbers

Parameter	What does is Measure?	Units	Interpretation
α	Spreadability		Fraction of S who are infected, per sick person per day.
β	Removal rate		Percent of I that get better per day
$\frac{1}{\beta}$			
c			

Why Contact Numbers

Parameter	What does is Measure?	Units	Interpretation
α	Spreadability	$\frac{1}{t \times ppl}$	Fraction of S who are infected, per sick person per day.
β	Removal rate	$\frac{1}{t}$	Percent of I that get better per day
$\frac{1}{\beta}$		t	
c		$\frac{1}{ppl}$	

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Why Contact Numbers

Parameter	What does is Measure?	Units	Interpretation
α	Spreadability	$\frac{1}{t \times ppl}$	Fraction of S who are infected, per sick person per day.
β	Removal rate	$\frac{1}{t}$	Percent of I that get better per day
$\frac{1}{\beta}$		t	Average amount of time someone is sick
c		$\frac{1}{ppl}$	Fraction of S that are infected per sick person

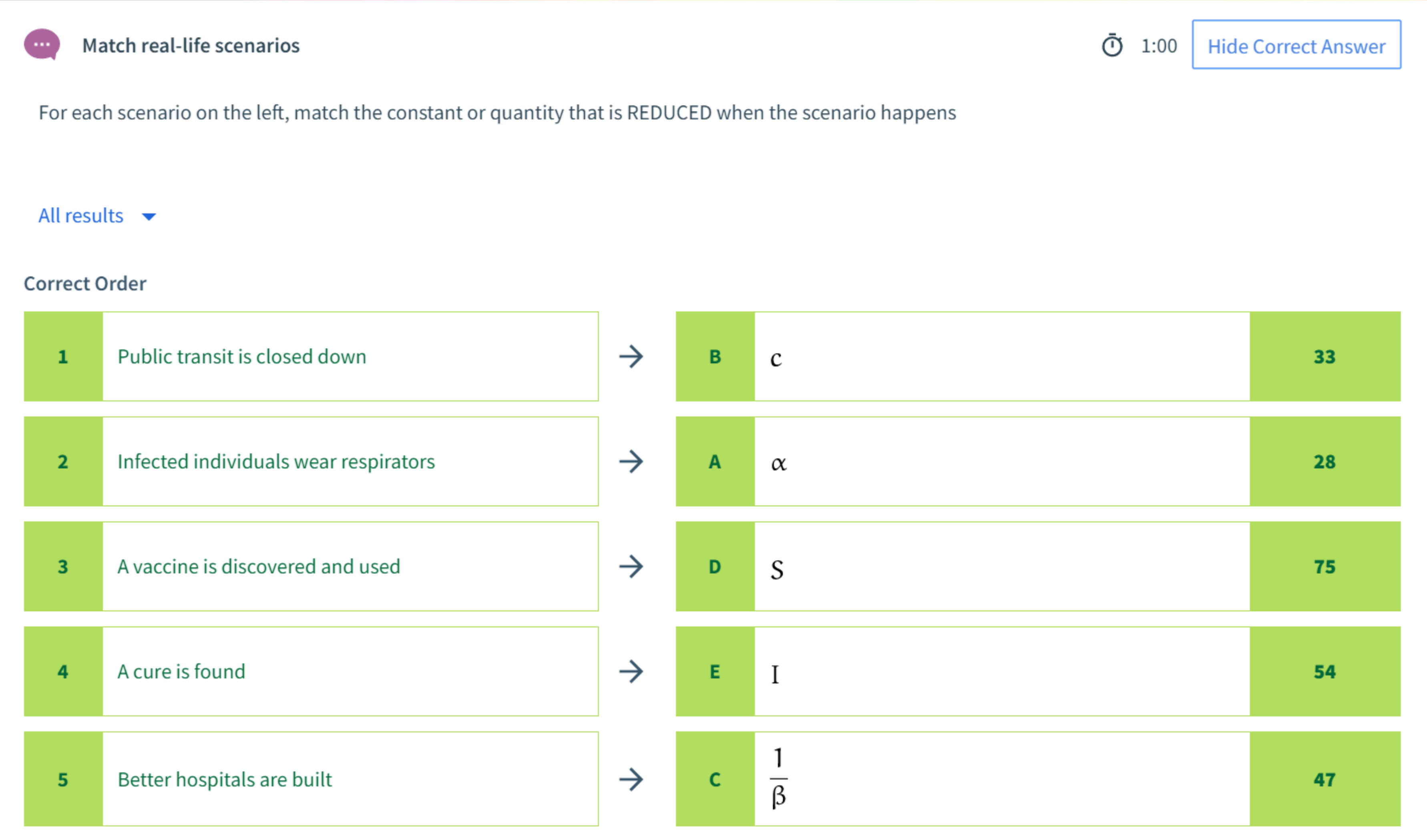
Takeaway

c is a measure of "contagion". It's a quantity that determines how many healthy people a sick person infects, all things considered.

Takeaway

c is a measure of "contagion". It's a quantity that determines how many healthy people a sick person infects, all things considered.

WARNING: some models use $s = \frac{S}{N}$, some models use different constants!



Match real-life scenarios

For each scenario on the left, match the constant or quantity that is REDUCED when the scenario happens

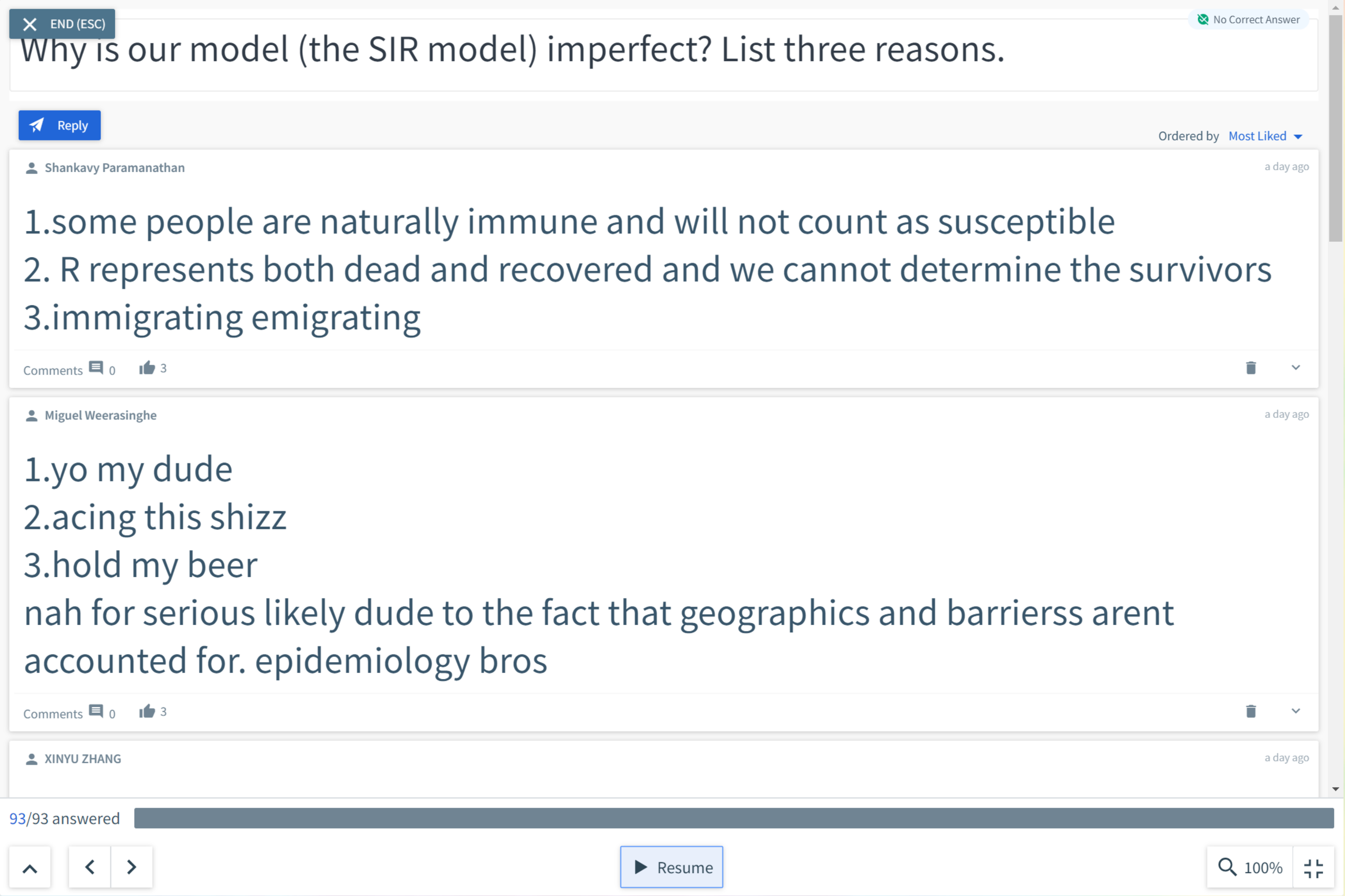
Correct Order

1	Public transit is closed down	→	B	c	33
2	Infected individuals wear respirators	→	A	α	28
3	A vaccine is discovered and used	→	D	S	75
4	A cure is found	→	E	I	54
5	Better hospitals are built	→	C	$\frac{1}{\beta}$	47

UofT Model

You, too, can play with the parameters of the SIR model:

`https://art-bd.shinyapps.io/nCov_control/`



Why is our model (the SIR model) imperfect? List three reasons.

Shankavy Paramanathan a day ago

1.some people are naturally immune and will not count as susceptible
2. R represents both dead and recovered and we cannot determine the survivors
3.immigrating emigrating

Miguel Weerasinghe a day ago

1.yo my dude
2.acing this shizz
3.hold my beer
nah for serious likely dude to the fact that geographics and barrierss arent
accounted for. epidemiology bros

XINYU ZHANG a day ago

93/93 answered

Discussion: Why is our model imperfect?

- Changes in policy
- Constants are not actually constant
- Demographics are different
- Vaccines, medications
- ...

Cat Fight!

Marzipan and Rainbow are having a fight, and they brought all the other cats into it. They muster up their armies, and fight at midnight. Let $R(t)$ be the number of cats remaining in Rainbow's army, t minutes after midnight. Define $M(t)$ similarly.

We apply Lanchester's model:

$$\frac{dR}{dt} = -0.5M(t)$$

$$\frac{dM}{dt} = -0.3R(t)$$

Cat Fight!

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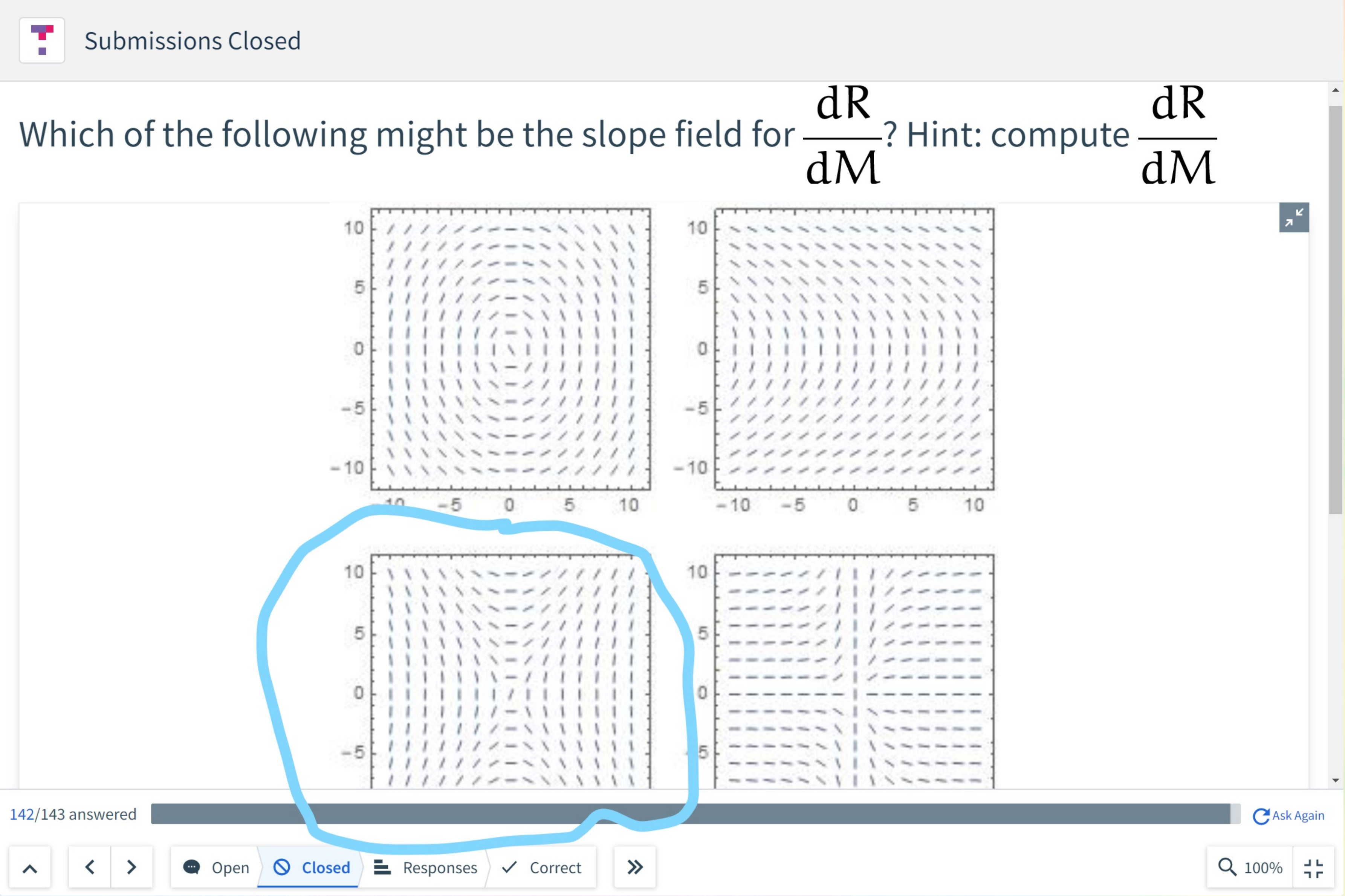
We apply Lanchester's model:

$$\frac{dR}{dt} = -0.5M(t)$$
$$\frac{dM}{dt} = -0.3R(t)$$

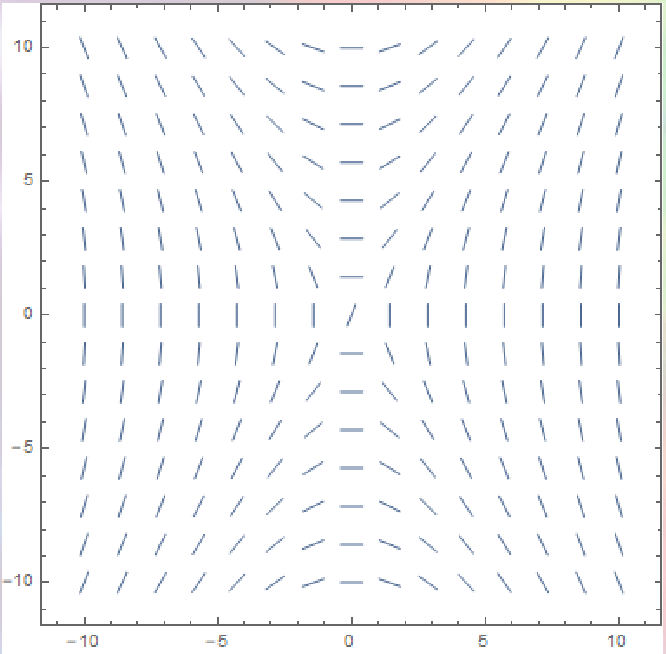
> "I don't care how long the battle takes, I just want to win."
>
> –Marzipan



Submissions Closed

Which of the following might be the slope field for $\frac{dR}{dM}$? Hint: compute $\frac{dR}{dM}$

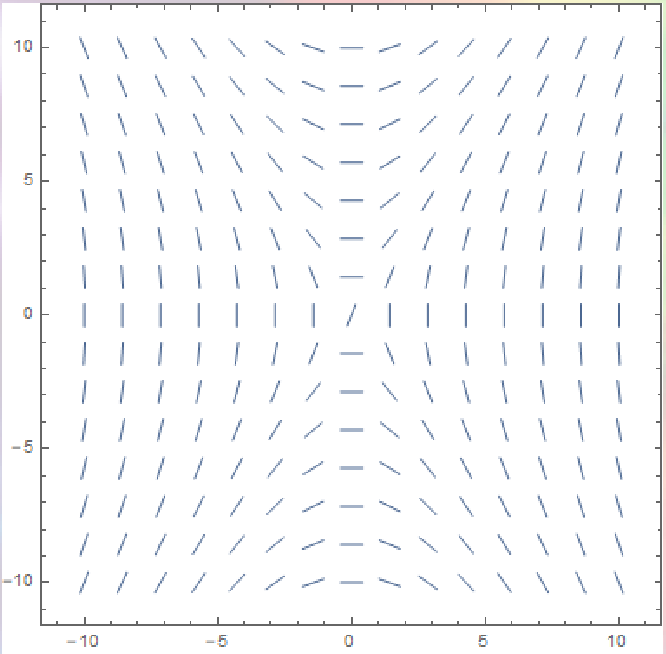
$$\frac{dR}{dM} = \frac{0.5}{0.3}\frac{M}{R}$$



An **Equilibrium point** is a point where:

$$\frac{dR}{dt} = 0$$
$$\frac{dM}{dt} = 0$$

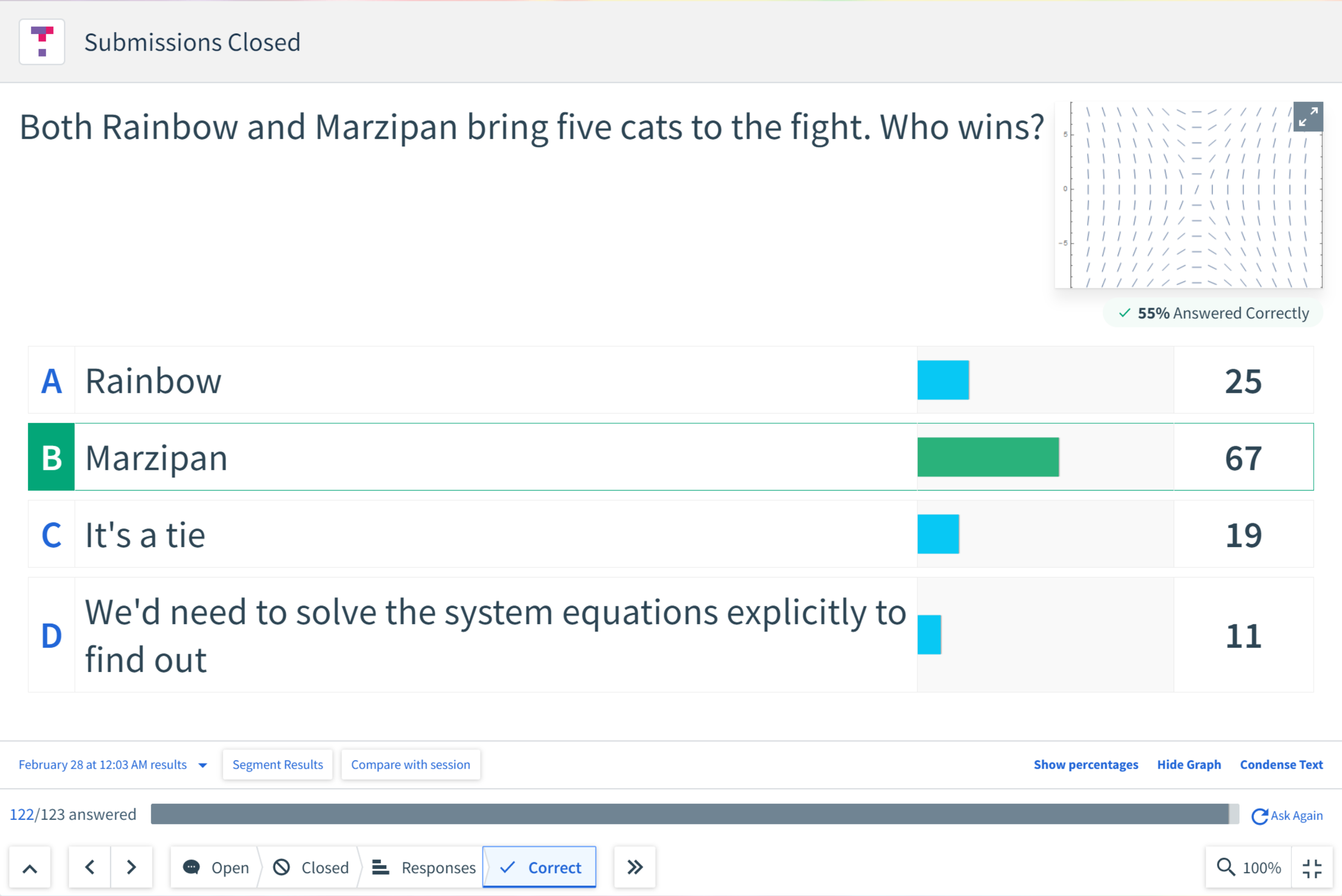
$$\frac{dR}{dM} = \frac{0.5}{0.3}\frac{M}{R}$$



An **Equilibrium point** is a point where:

$$\frac{dR}{dt} = 0$$
$$\frac{dM}{dt} = 0$$

Q: Does there exist an equilibrium point for this system of differential equations? Yes! At $R = 0, M = 0$



Submissions Closed

Both Rainbow and Marzipan bring five cats to the fight. Who wins?

✓ **55%** Answered Correctly

A	Rainbow	25
B	Marzipan	67
C	It's a tie	19
D	We'd need to solve the system equations explicitly to find out	11

122/123 answered

Takeaway

We don't need to solve the differential equation! The slope field can tell us quite a bit!

For more: see the SIR model example in the text.

Plans for the Future

For next time:
Review Taylor polynomials!

Welcome to MAT136 LEC0501 (Assaf)

`https://tinyurl.com/Unit2-3CIQ`

Taylor Expansions and ODEs

Assaf Bar-Natan

“The game has been disbanded
My mind has been expanded”

–“Rose Tint My World”, Susan Sarandon, et. al.

March 3, 2020

Critical Incident Questionnaire

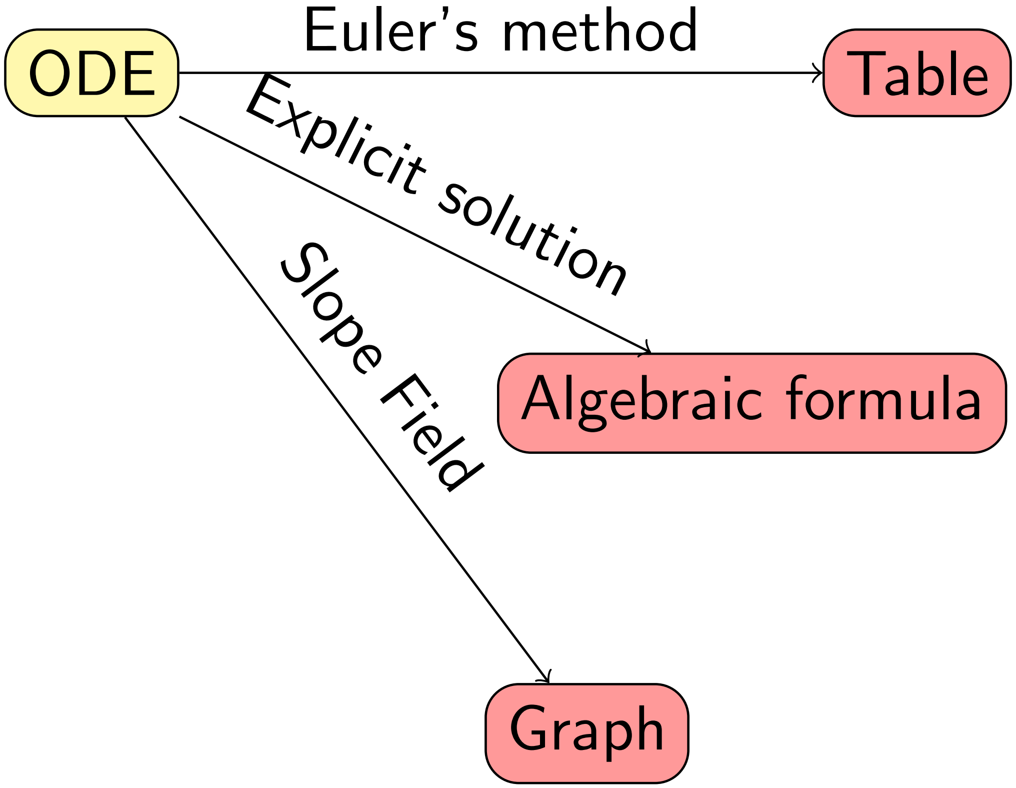
`https://tinyurl.com/Unit2-3CIQ`

What is a Solution?

How do we solve an ODE?

What is a Solution?

How do we solve an ODE?



ODE

Euler's method

Table

Explicit solution

Algebraic formula

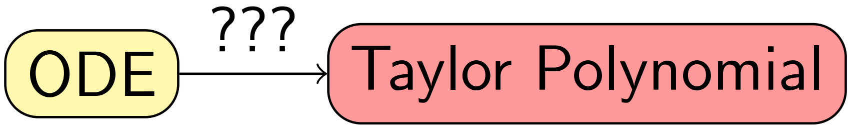
Slope Field

Graph

Takeaway

Depending on the ODE, and depending on how we want our solution presented, we use different solution strategies.

A New Way: Taylor Solutions



ODE

???

Taylor Polynomial

Key idea: Express a function as a Taylor polynomial, and solve for the coefficients.

A First Example

$$\frac{dy}{dx} = y$$
$$y(0) = 2$$

We compute the Taylor expansion of y around 0:

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

A First Example

$$\frac{dy}{dx} = y$$
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We compute the Taylor expansion of y around 0:

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Sanity check: What is the formula for a_2?

A First Example

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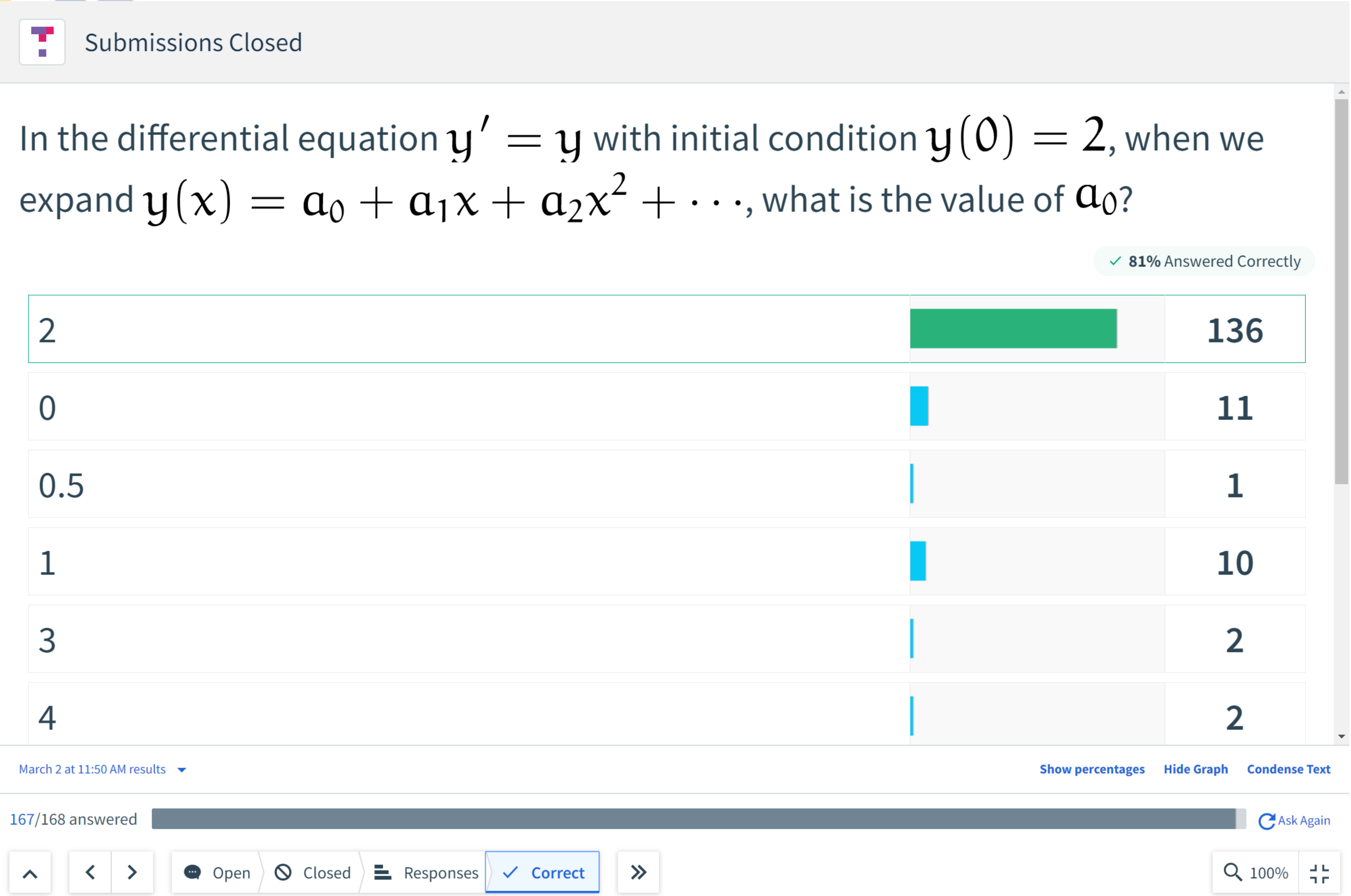
$$y(0) = 2$$

We compute the Taylor expansion of y around 0:

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Sanity check: What is the formula for a_2? $a_2 = \frac{y''(0)}{2}$
Now, we differentiate:

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$



Submissions Closed

In the differential equation $y' = y$ with initial condition $y(0) = 2$, when we expand $y(x) = a_0 + a_1x + a_2x^2 + \cdots$, what is the value of a_0?

✓ **81%** Answered Correctly

2	136
0	11
0.5	1
1	10
3	2
4	2

March 2 at 11:50 AM results

167/168 answered

A First Example

$$\frac{dy}{dx} = y \qquad y(0) = 2$$

We have:

$$\begin{aligned} y(x) &= 2 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\ y' &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots \end{aligned}$$

To solve the equation, we set them to be equal to each other!

$$\begin{aligned} &a_1 + 2a_2 x + 3a_3 x^2 + \cdots = y' = y \\ =&a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \end{aligned}$$

A First Example

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We have:

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Q: What is a_2 in terms of a_1?

A First Example

$$\frac{dy}{dx} = y \qquad y(0) = 2$$

We have:

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Q: What is a_2 in terms of a_1? $a_2 = \frac{a_1}{2}$

Sanity Check

We have that for the differential equation $y' = y$,

$$\frac{y''(0)}{2} = a_2 = \frac{a_1}{2}$$

but we also know: $a_1 = y'(0)$ (Taylor polynomial) and:
$y'(0) = y''(0)$ (y is a solution to the ODE).

This coincides with what we have!

A First Example

$$\frac{dy}{dx} = y$$
$$y(0) = 2$$

We have:

$$y(x) = 0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$
$$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots$$

Check that $a_n = \frac{2}{n!}$
Q: What is $y(x)$?

A First Example

$$\frac{dy}{dx} = y$$
$$y(0) = 2$$

We have:

$$y(x) = 0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Check that $a_n = \frac{2}{n!}$

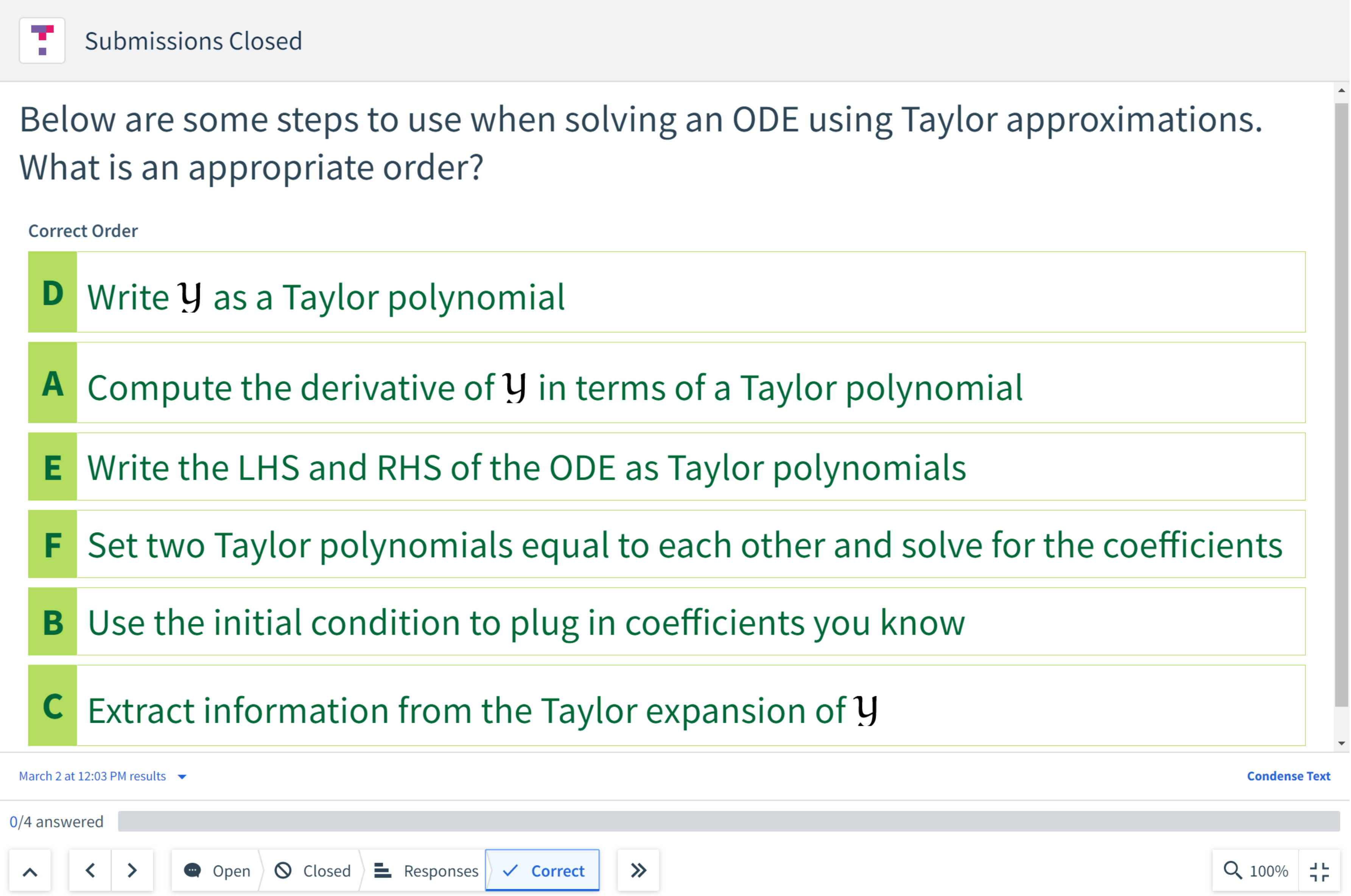
Q: What is $y(x)$?

$$y(x) = \sum_{n=0}^{\infty} \frac{2}{n!} x^n = 2e^x$$

Takeaway

Using an initial condition and the differential equation, we can write the solution as a Taylor polynomial, and solve for the coefficients

This is an entirely new way to solve ODEs!


Submissions Closed

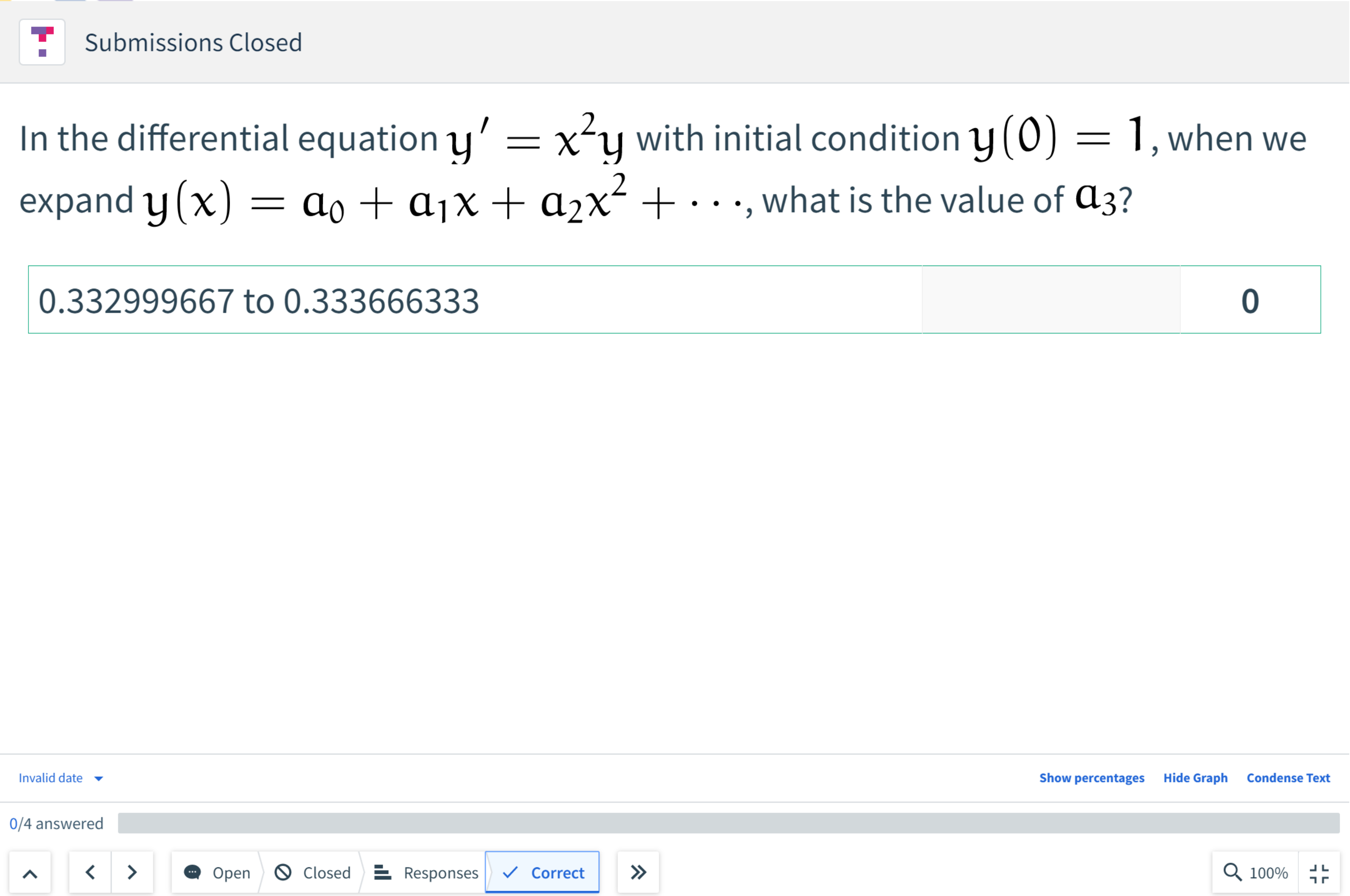
Below are some steps to use when solving an ODE using Taylor approximations. What is an appropriate order?

Correct Order

- **D** Write y as a Taylor polynomial
- **A** Compute the derivative of y in terms of a Taylor polynomial
- **E** Write the LHS and RHS of the ODE as Taylor polynomials
- **F** Set two Taylor polynomials equal to each other and solve for the coefficients
- **B** Use the initial condition to plug in coefficients you know
- **C** Extract information from the Taylor expansion of y

March 2 at 12:03 PM results

0/4 answered



In the differential equation $y' = x^2 y$ with initial condition $y(0) = 1$, when we expand $y(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, what is the value of a_3?

0.332999667 to 0.333666333	0

A Hard Differential Equation

$$\begin{aligned} y' &= x^2 y \\ y(0) &= 1 \end{aligned}$$

Write:

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\ x^2 y(x) &= \qquad\qquad\qquad\quad a_0 x^2 + a_1 x^3 + a_2 x^4 + a_3 x^5 \\ y'(x) &= a_1 + 2a_2 x + 3a_3 x^2 \end{aligned}$$

We know:

$$\begin{aligned} a_0 &= 1 & a_1 &= 0 \\ a_2 &= 0 & a_3 &= \frac{1}{3} a_0 = \frac{1}{3} \end{aligned}$$

Using Taylor Solutions to Estimate

Assume that:

$$y' = x^2 y$$
$$y(0) = 1$$

Use Taylor approximations to estimate $y(0.5)$.

Using Taylor Solutions to Estimate

Assume that:

$$y' = x^2 y$$
$$y(0) = 1$$

Use Taylor approximations to estimate $y(0.5)$.
We know that:

$$y(x) = 1 + 0x + 0x^2 + \frac{1}{3}x^3$$

at $x = 0.5$, we get: $y(0.5) \approx 1.04$.

Using Taylor Solutions to Estimate

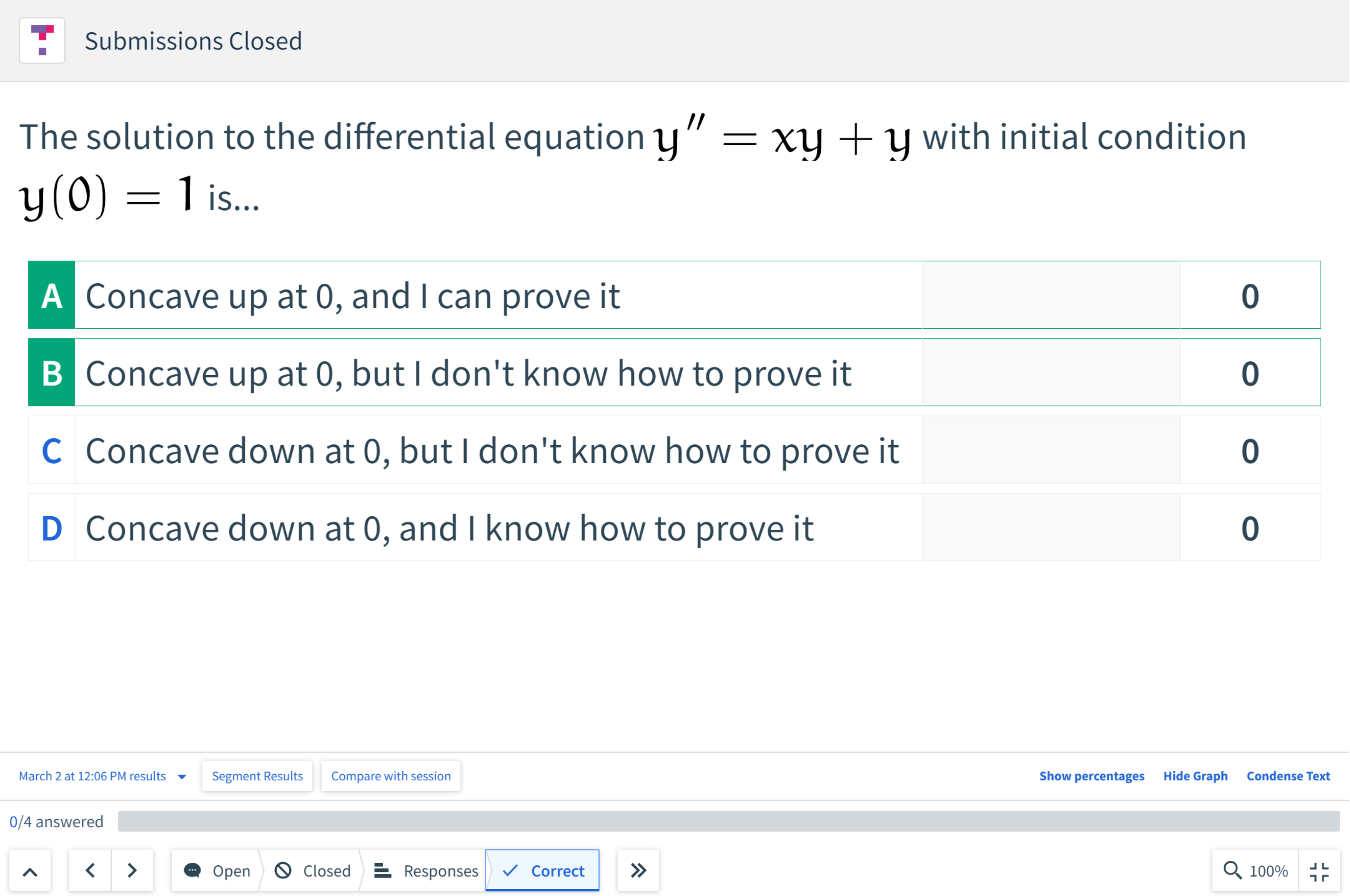
Assume that:

$$y' = x^2 y$$
$$y(0) = 1$$

Use Taylor approximations to estimate $y(0.5)$.
We know that:

$$y(x) = 1 + 0x + 0x^2 + \frac{1}{3}x^3$$

at $x = 0.5$, we get: $y(0.5) \approx 1.04$.
The actual solution to this ODE (it's separable) is $y(x) = e^{x^3}$. How close is our estimate?



Submissions Closed

The solution to the differential equation $y'' = xy + y$ with initial condition $y(0) = 1$ is...

A	Concave up at 0, and I can prove it	0
B	Concave up at 0, but I don't know how to prove it	0
C	Concave down at 0, but I don't know how to prove it	0
D	Concave down at 0, and I know how to prove it	0

Takeaway

We can get information about convexity or other properties of the function by looking at the coefficients of its solution. For example, increasing, decreasing, concavity,...

Plans for the Future

For next time:

WeBWork 8.1 and actively read section 8.1

Welcome to MAT136 LEC0501 (Assaf)

We continue to solve $\int \sqrt{\tan(x)}$. We substitute $w = s^2 - \sqrt{2}s + 1$ to get:

$$
\begin{aligned}
\int \frac{s}{s^2 - \sqrt{2}s + 1} ds &= \int \frac{2s - \sqrt{2}}{2(s^2 - \sqrt{2}s + 1)} + \frac{1}{\sqrt{2}(s^2 - \sqrt{2}s + 1)} ds \\
&= \int \frac{1}{2w} dw + \int \frac{1}{\sqrt{2}(s^2 - \sqrt{2}s + 1)} ds \\
&= \frac{\log(w)}{2} + \frac{1}{\sqrt{2}} \int \frac{1}{(s - \frac{1}{\sqrt{2}})^2 + \frac{1}{2}}
\end{aligned}
$$

The last integral is computed using an inverse trig substitution. This is covered in chapter 7.4

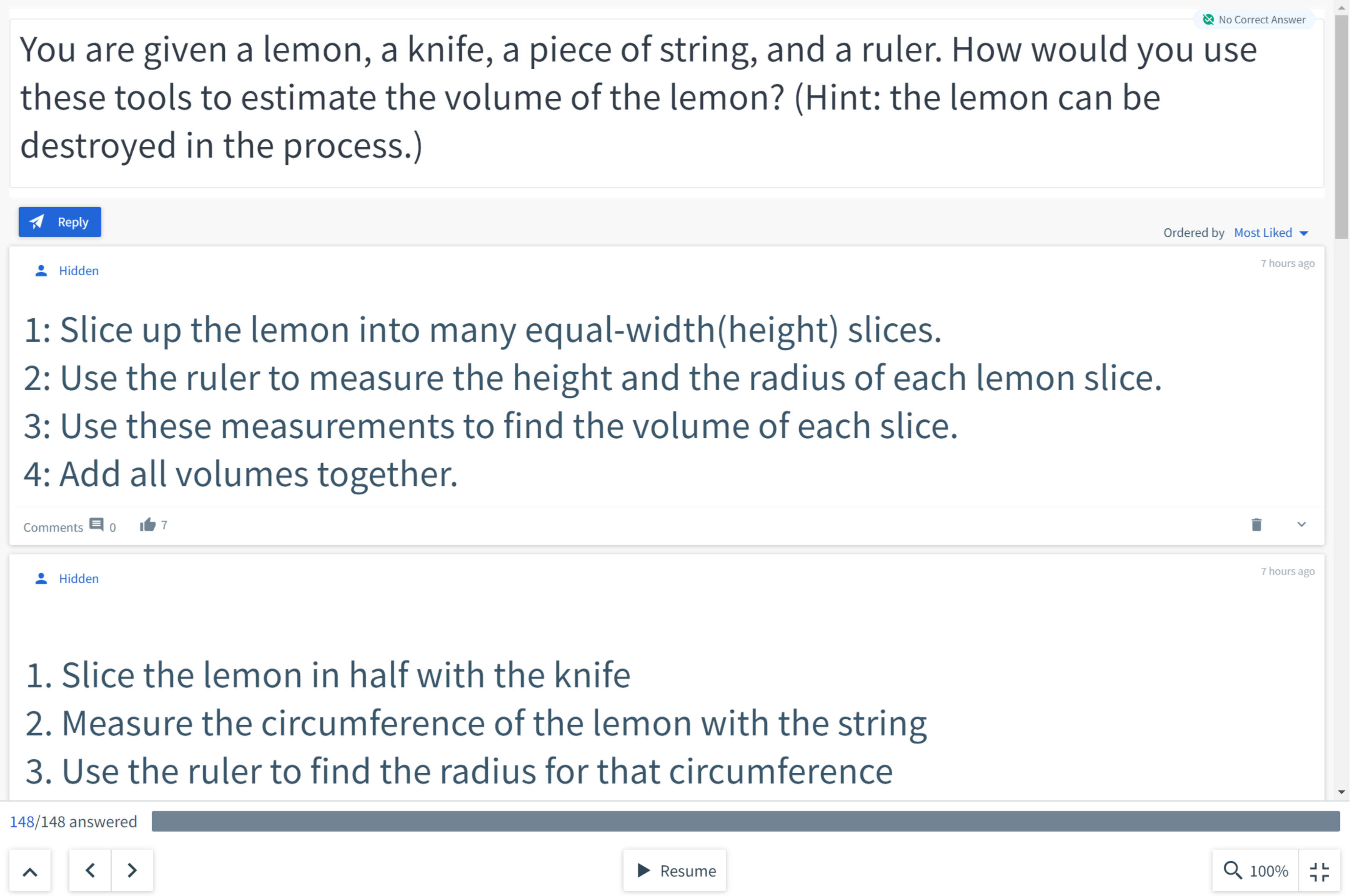
Areas and Volumes – Slice ’em, Dice ’em, Integrate ’em

Assaf Bar-Natan

“Where trouble melts like lemon drops
High above the chimney top
That’s where you’ll find me”

–“Somewhere Over The Rainbow”, Israel Kamakawiwo’ole

March 4, 2020



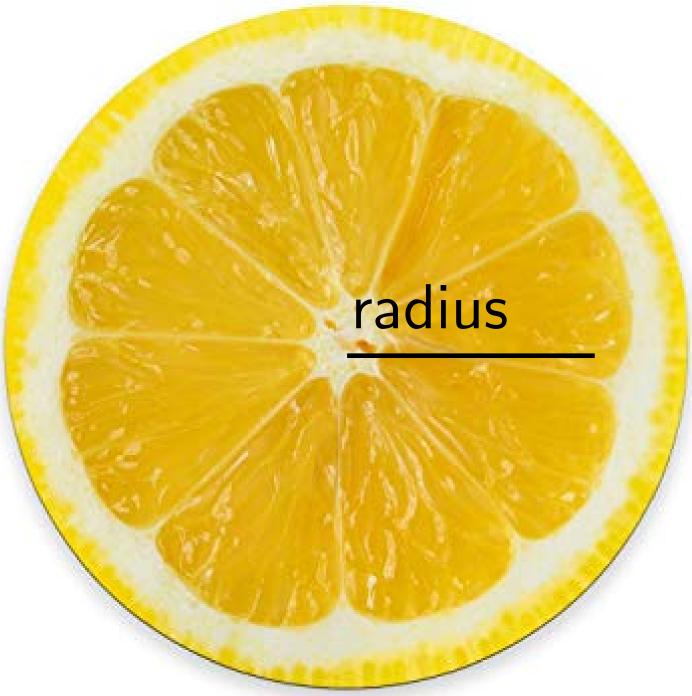
You are given a lemon, a knife, a piece of string, and a ruler. How would you use these tools to estimate the volume of the lemon? (Hint: the lemon can be destroyed in the process.)

1: Slice up the lemon into many equal-width(height) slices.
2: Use the ruler to measure the height and the radius of each lemon slice.
3: Use these measurements to find the volume of each slice.
4: Add all volumes together.

1. Slice the lemon in half with the knife
2. Measure the circumference of the lemon with the string
3. Use the ruler to find the radius for that circumference

Cutting a Lemon

“When life gives you lemons, cut them up, and compute their volume”



We can estimate the volume by slicing the lemon into 1*cm*-thick slices. Then:

$$\text{Vol} = \sum_{\text{slices}} \text{Area}(\text{slice}) \times 1cm$$

$$\text{Area}(\text{slice}) = \pi \times \text{radius}^2$$

The smaller the slices, the better the approximation.

A Stack of Post-Its

Sometimes, shapes that look irregular, have nice-looking cross-sections:

Q: A $3\mathrm{in}$ tall stack of post-its is twisted like in the picture. What is the volume of the stack?

A Stack of Post-Its

Sometimes, shapes that look irregular, have nice-looking cross-sections:

Q: A $3\mathrm{in}$ tall stack of post-its is twisted like in the picture. What is the volume of the stack?

A: Separate the stack into n individual post-its, each having a height of $\frac{3}{n}\mathrm{in}$. The total volume is:

$$\sum_{i=1}^{n} l \times w \times \frac{3}{n}$$

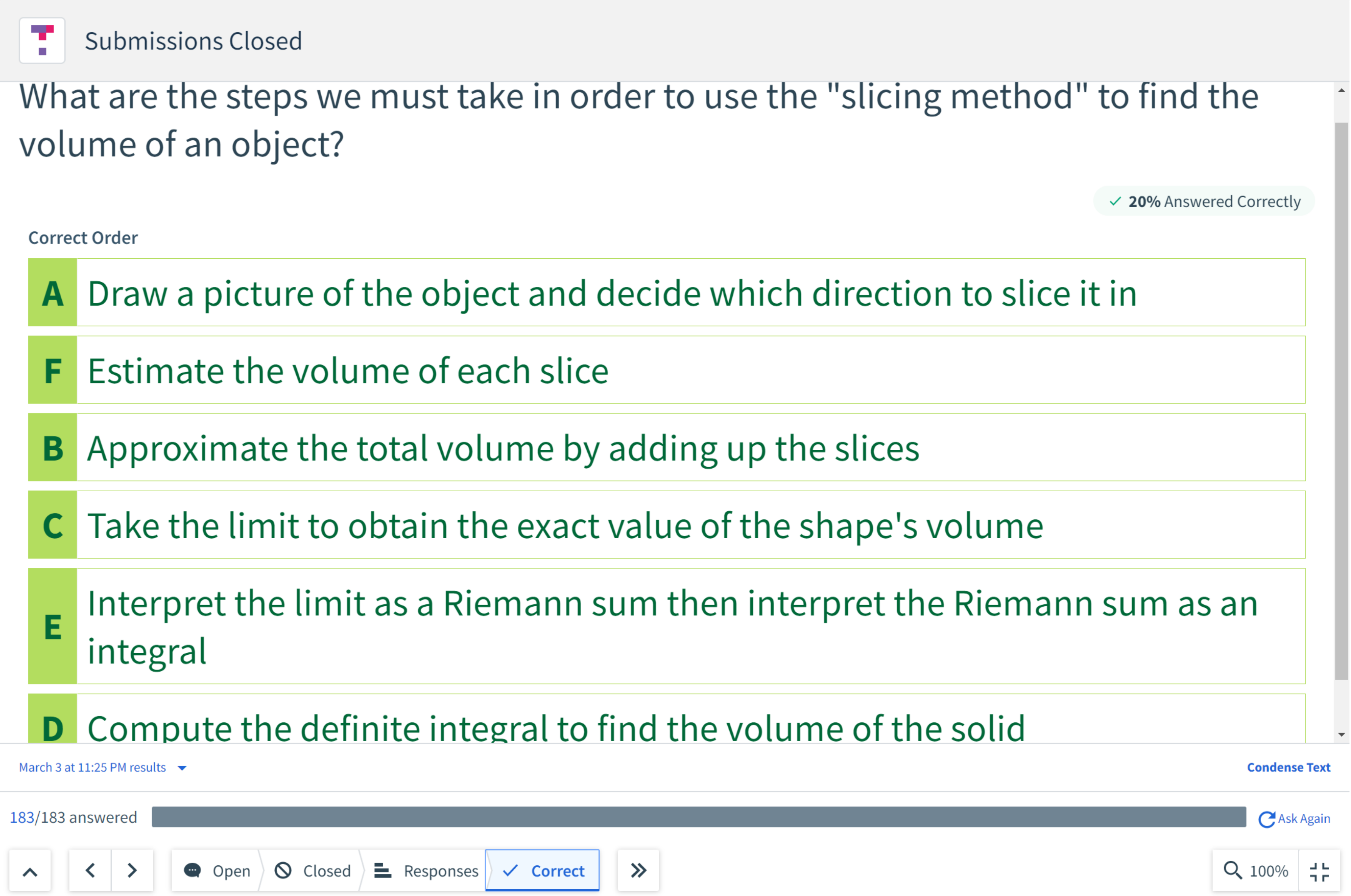
Takeaway

In the same way that we sliced regions into small rectangles to compute areas using integrals, we can slice solids into thin cross-sections to compute volumes using integrals.

The Cat's Nest

The cats are burrowing into the top of a square hay-bale. The hay bale has a width of 18", a length of 36", and a height of 14". The cats burrow a cavity from the top whose radius is changing with the height above the ground. The radius of the cavity is $\frac{\sqrt{h}}{3}$ feet, where h is measured in feet above the ground.

How much hay is in the bale?



Submissions Closed

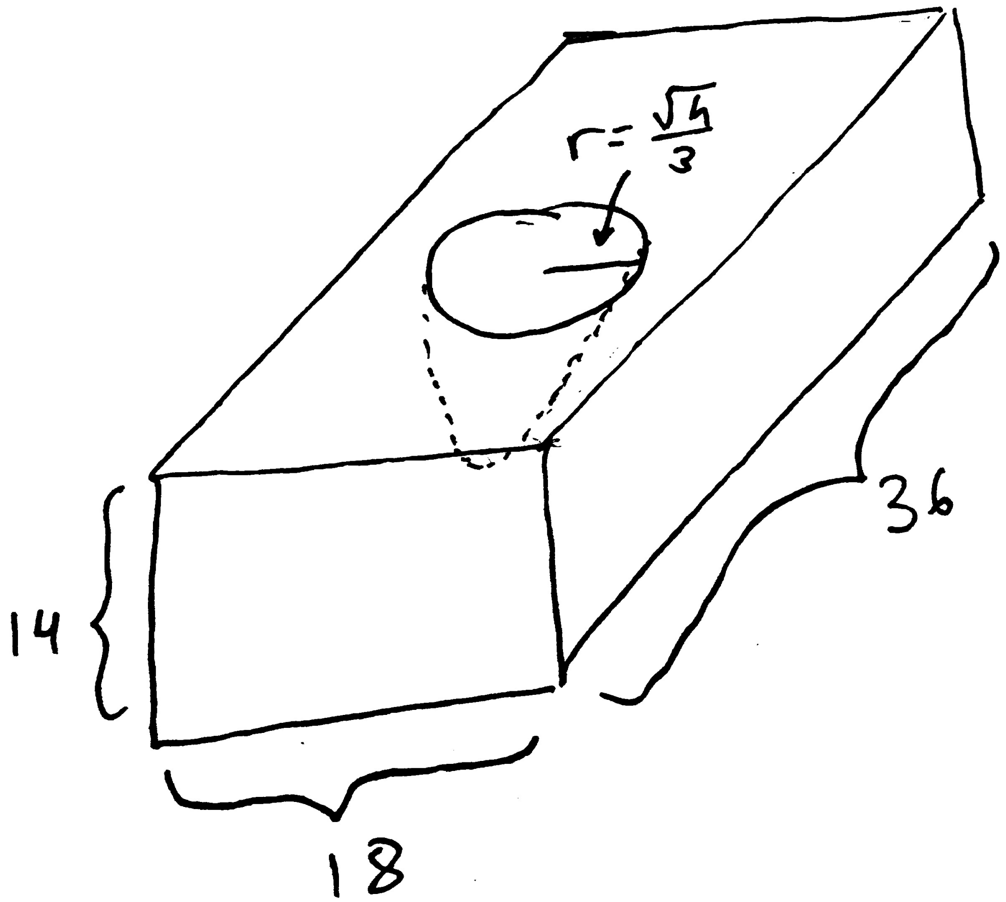
What are the steps we must take in order to use the "slicing method" to find the volume of an object?

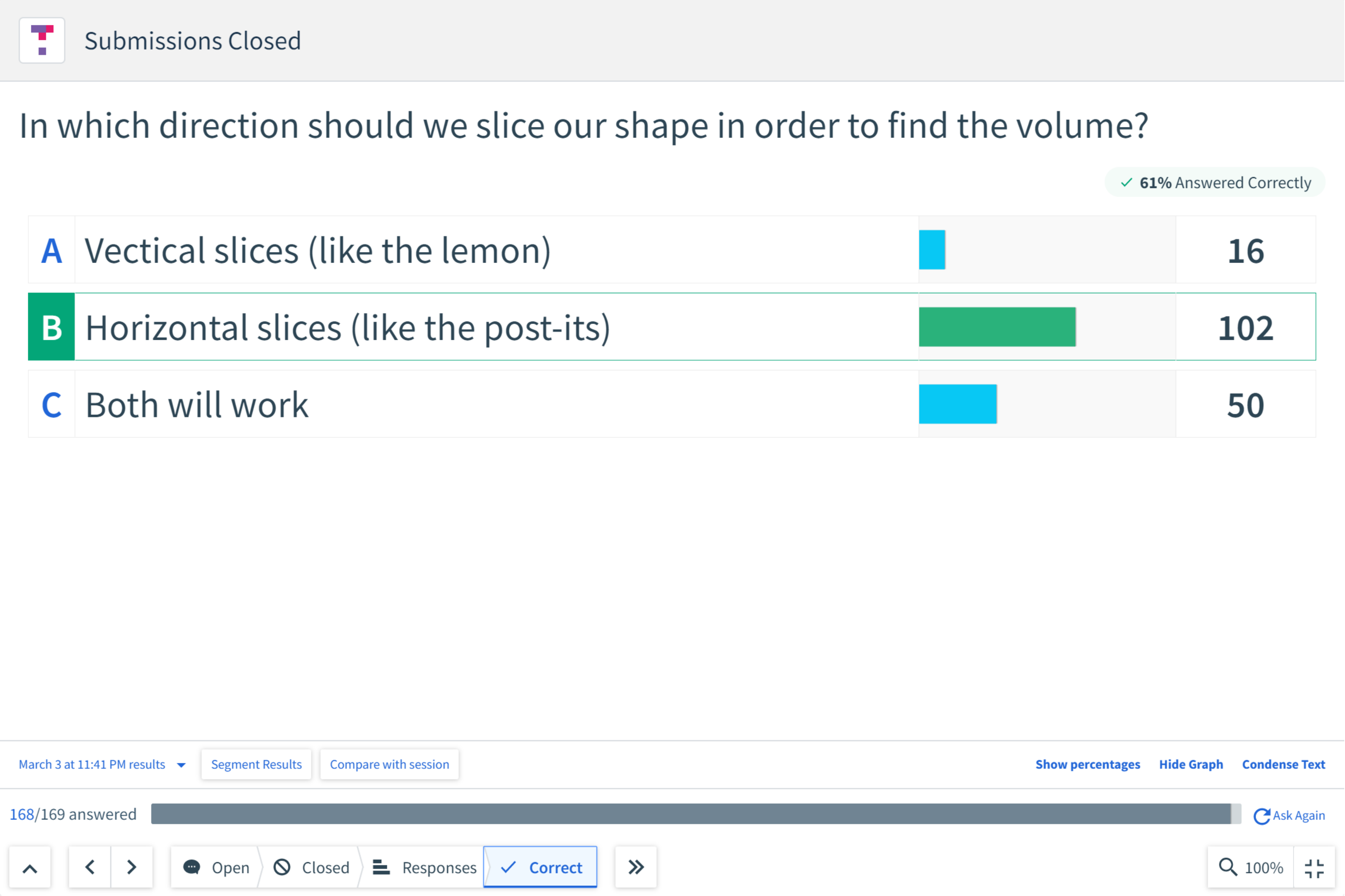
✓ **20%** Answered Correctly

Correct Order

- **A** Draw a picture of the object and decide which direction to slice it in
- **F** Estimate the volume of each slice
- **B** Approximate the total volume by adding up the slices
- **C** Take the limit to obtain the exact value of the shape's volume
- **E** Interpret the limit as a Riemann sum then interpret the Riemann sum as an integral
- **D** Compute the definite integral to find the volume of the solid

Draw a Picture





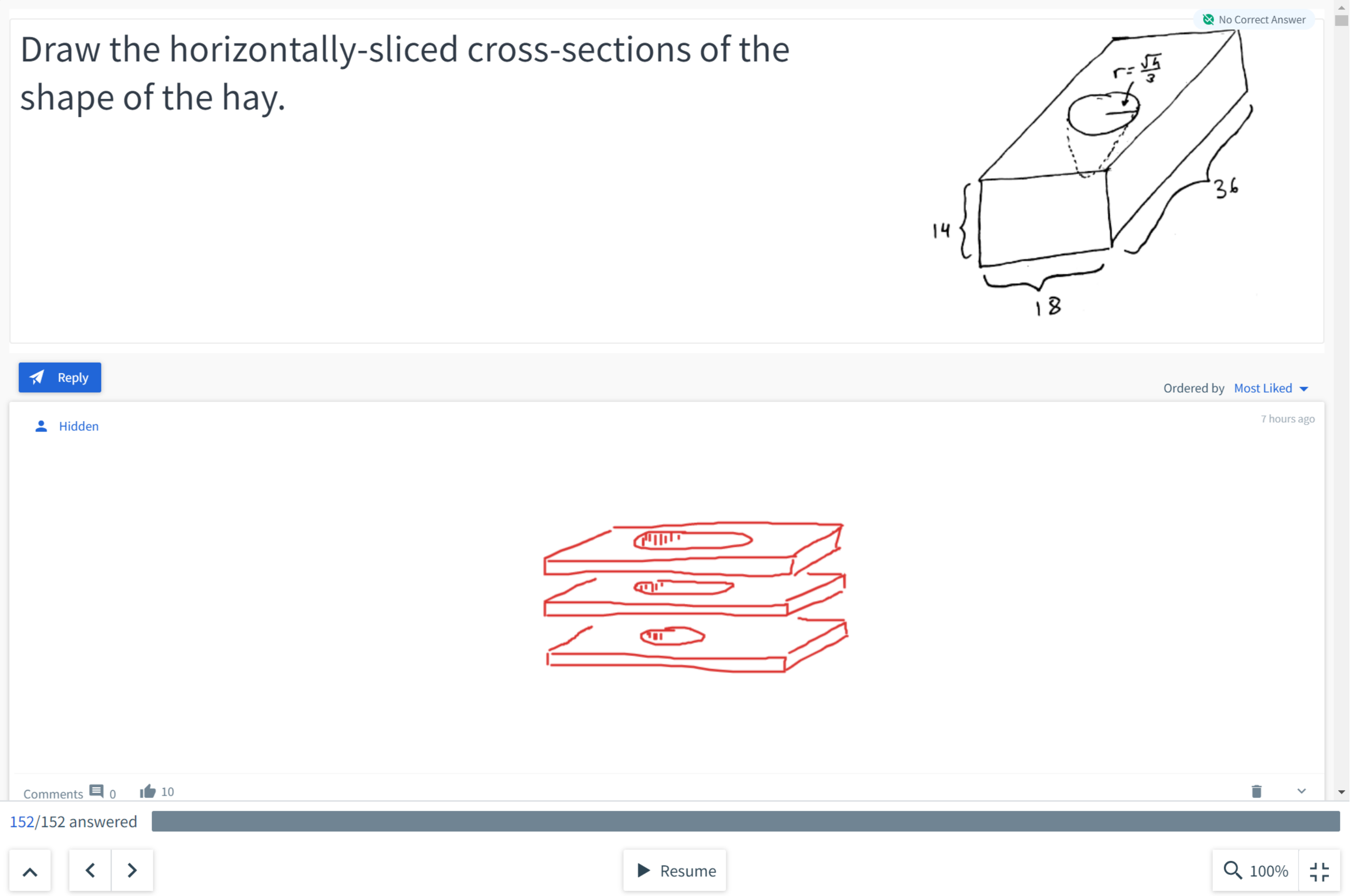
Submissions Closed

In which direction should we slice our shape in order to find the volume?

✓ **61%** Answered Correctly

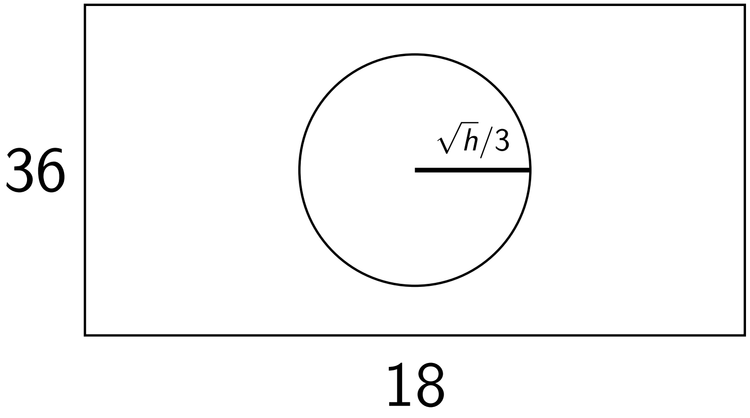
A	Vectical slices (like the lemon)	16
B	Horizontal slices (like the post-its)	102
C	Both will work	50

168/169 answered



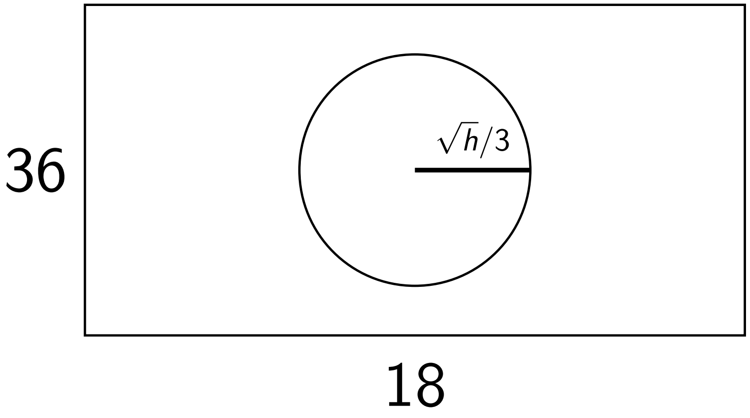
Draw the horizontally-sliced cross-sections of the shape of the hay.

Approximating the Volume



If the height of each cross-section is Δh, then the volume of the cross section at height h is:

Approximating the Volume



If the height of each cross-section is Δh, then the volume of the cross section at height h is:

$$V(h) = (18 \times 36 - \pi r^2)\Delta h$$
$$= 648\Delta h - \frac{\pi h}{9}\Delta h$$

Approximating the Total Volume

If the height of each cross-section is Δh, then the volume of the cross section at height h is:

$$\begin{aligned}\text{Cross-Sec. Vol} &= (18 \times 36 - \pi r^2)\Delta h \\ &= 648\Delta h - \frac{\pi h}{9}\Delta h\end{aligned}$$

Q: Write a Riemann sum that estimates the total volume (take $\Delta h = \frac{14}{n}$).

Approximating the Total Volume

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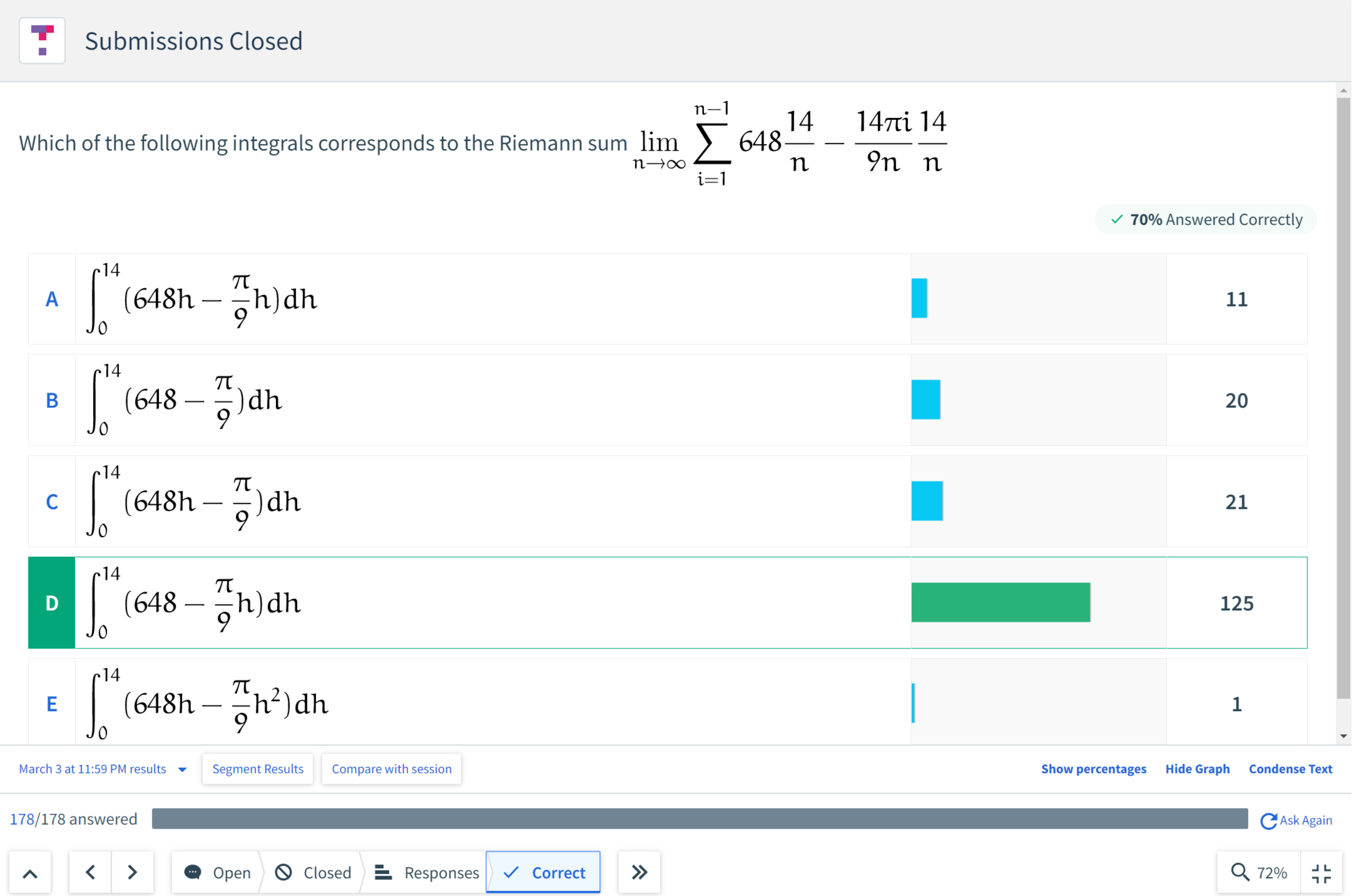
Q: Write a Riemann sum that estimates the total volume (take $\Delta h = \frac{14}{n}$).

$$\sum_{i=1}^{n} 648\frac{14}{n} - \frac{14\pi i}{9n}\frac{14}{n}$$

Taking the Limit

We've replaced h with $h_i = \frac{14}{n}i$, and $\Delta h = \frac{14}{n}$, then added it up. All that's left (in cubic inches) is to take the limit:

$$Vol = \lim_{n\to\infty} \sum_{i=1}^{n} 648\frac{14}{n} - \frac{14\pi i}{9n}\frac{14}{n}$$



Which of the following integrals corresponds to the Riemann sum $\lim_{n\to\infty}\sum_{i=1}^{n-1} 648\frac{14}{n} - \frac{14\pi i}{9n}\frac{14}{n}$

✓ **70%** Answered Correctly

A	$\int_0^{14}(648h - \frac{\pi}{9}h)dh$	11
B	$\int_0^{14}(648 - \frac{\pi}{9})dh$	20
C	$\int_0^{14}(648h - \frac{\pi}{9})dh$	21
D	$\int_0^{14}(648 - \frac{\pi}{9}h)dh$	125
E	$\int_0^{14}(648h - \frac{\pi}{9}h^2)dh$	1

178/178 answered

Evaluating the Integral

When all is said and done, the volume of hay left is:

$$\int_0^{14} \left(648 - \frac{\pi}{9}h\right) dh$$

Which is around $9000\mathrm{in}^3$, or, around 5.2 cubic feet.

Plans for the Future

For next time:

WeBWork 8.2 and actively read section 8.2

Ban cars on campus

Welcome to MAT136 LEC0501 (Assaf)

We continue to solve $\int \sqrt{\tan(x)}$. After computing the last integral, and subbing in everything...

$$\begin{aligned}\int \sqrt{\tan(x)}dx = & \frac{1}{2\sqrt{2}} \log(\tan(x) - \sqrt{2\tan(x)} + 1) \\ & - \frac{1}{2\sqrt{2}} \log(\tan(x) + \sqrt{2\tan(x)} + 1) \\ & + \frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2\tan(x)} + 1) \\ & - \frac{1}{\sqrt{2}} \tan^{-1}(1 - \sqrt{2\tan(x)})\end{aligned}$$

Easy, right?

Areas and Volumes – Rotating, Spinning, and a Bit of Slicing

Assaf Bar-Natan

"My head is in a spin,
My feet don't touch the ground.
Because you're near to me
My head goes round and round."

–"Feels Like I'm in Love", Kelly Marie

March 6, 2020

CIQ summary

- You were most engaged when using TopHat. Specifically, discussing with groups, and discussing things together.
- You were engaged during the lectures about COVID-19, SIR model, and the Excel spreadsheet activity
- You were sad that people leave before class is over, or talk over me.
- Some of you were distanced when doing the SIR model stuff. A lot of you were confused at slope fields.
- At times, the lecture was moving fast, and you disliked skipped questions.
- You did not gain a lot from classes when you did not do the reading.

CIQ summary – Cont.

Things you liked:

- Taking up WeBWork and TopHat questions (through discussions, or on slides)
- The amount of interaction and group discussion
- Traditional lecturing
- Each other

Things that confused you:

CIQ summary – Cont.

Things you liked:

- Taking up WeBWork and TopHat questions (through discussions, or on slides)
- The amount of interaction and group discussion
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- Each other

Things that confused you:

- idk (this annoyed a lot of of you, too)

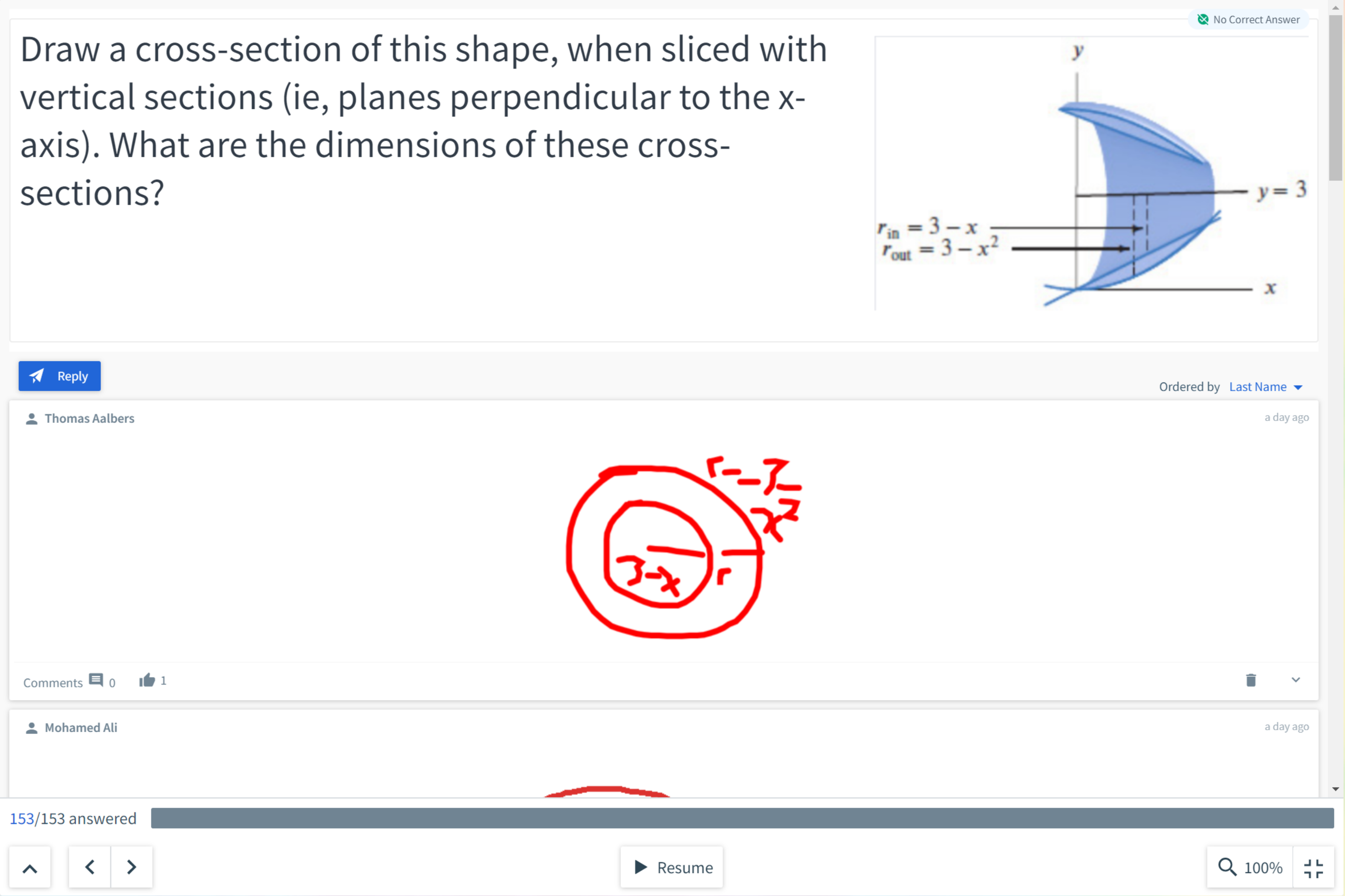
CIQ summary – Cont.

Things you liked:

- Taking up WeBWork and TopHat questions (through discussions, or on slides)
- The amount of interaction and group discussion
- Traditional lecturing
- Each other

Things that confused you:

- idk (this annoyed a lot of of you, too)
- Going too fast



Draw a cross-section of this shape, when sliced with vertical sections (ie, planes perpendicular to the x-axis). What are the dimensions of these cross-sections?

Thomas Aalbers

a day ago

Comments 0

1

Mohamed Ali

a day ago

153/153 answered

CIQ summary – Cont.

Things that surprised you:

CIQ summary – Cont.

Things that surprised you:

- $0/1000000000$

CIQ summary – Cont.

Things that surprised you:

- 0/1000000000
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CIQ summary – Cont.

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CIQ summary – Cont.

Things that surprised you:

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- “The friends I made in this class”
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- Difference in marks between those who attended the class on the day of the mid term and those who did not

CIQ summary – Cont.

Things that surprised you:

- 0/1000000000
- “The friends I made in this class”
- SIR/Coronavirus modeling
- Difference in marks between those who attended the class on the day of the mid term and those who did not
- Cats?

A First Application of Slices

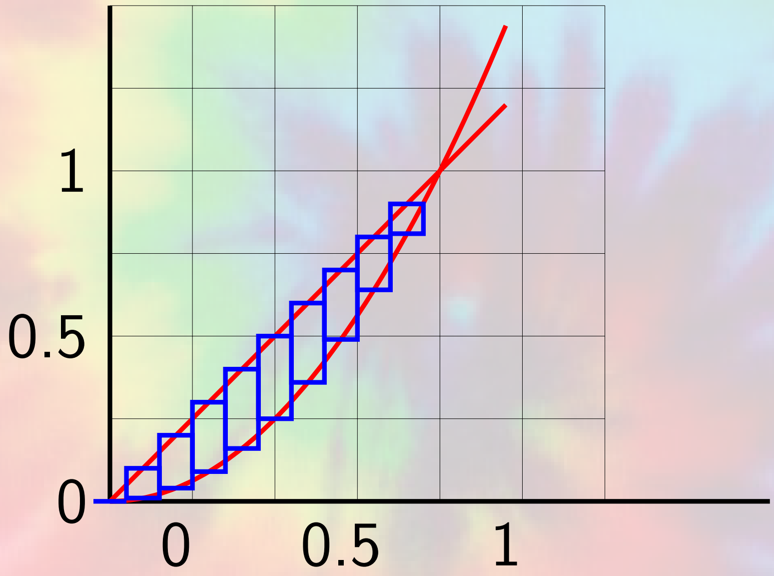
To find the arc-length of a function:

$$\begin{aligned} \Delta \text{Arc length} &= \sqrt{\Delta x^2 + \Delta y^2} \\ &= \sqrt{\Delta x^2 + (f'(x)^2)\Delta x^2} \\ &= \sqrt{1 + (f')^2}\Delta x \end{aligned}$$

Integrate to get $\mathit{Arclength} = \int \sqrt{1 + (f')^2} dx$

Slices – Areas and Volumes

Previously, on MAT136:



Taking the slices to be really small...

$$\text{Area} = \int_a^b (f(x) - g(x))dx$$

Slices – Areas and Volumes

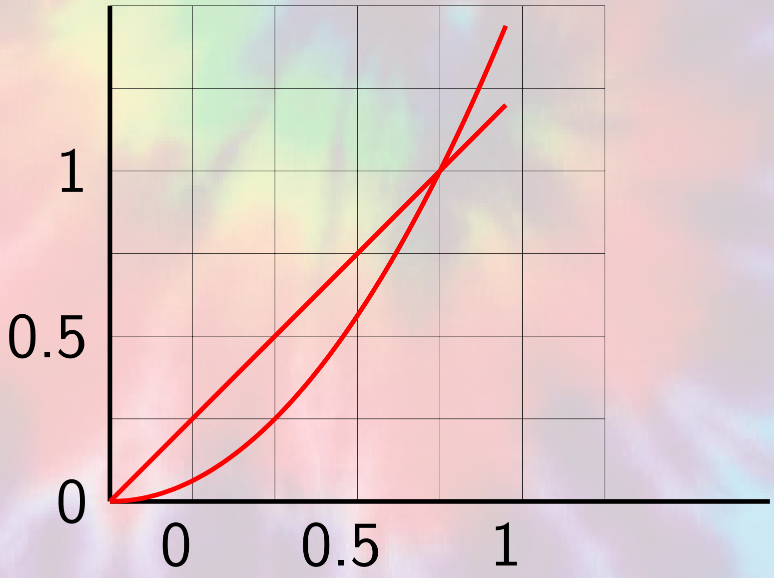
Now, on MAT136: **Find the volume of the solid obtained by rotating the region in the first quadrant bounded by and** $y = x^2$**,** $y = x$ **around the** $y = 3$ **axis.**

Q: What does the base region look like?

Slices – Areas and Volumes

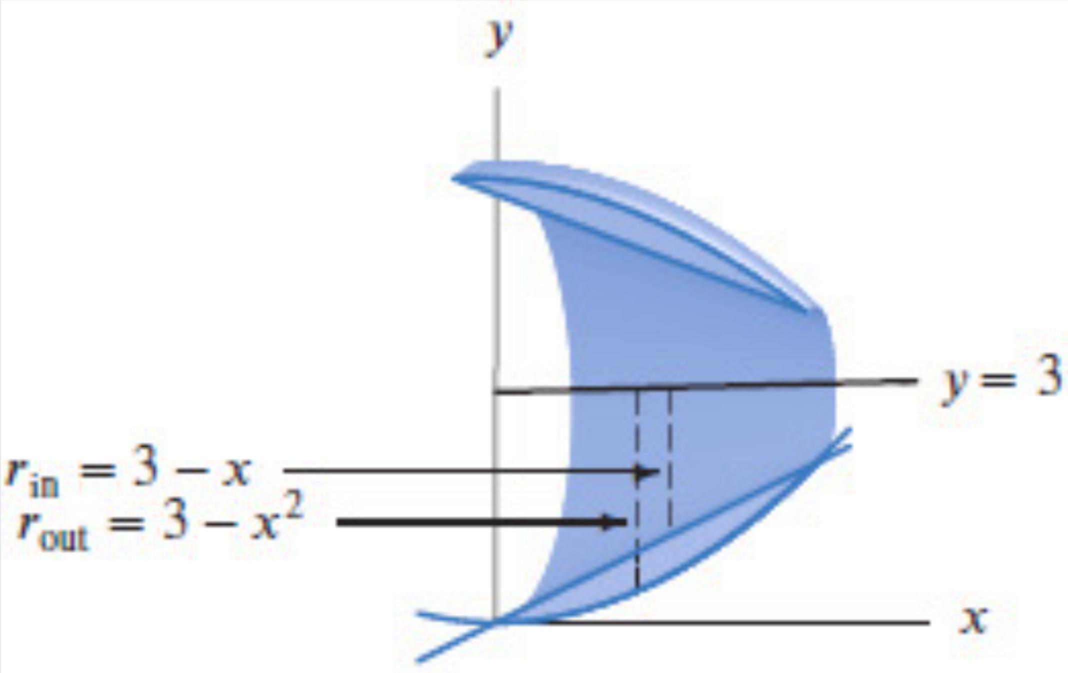
Now, on MAT136: **Find the volume of the solid obtained by rotating the region in the first quadrant bounded by and** $y = x^2$**,** $y = x$ **around the** $y = 3$ **axis.**

Q: What does the base region look like?



Slices – Areas and Volumes

rotate the region between the curves around the line $y = 3$



Use slices here, and make them really small...

Finding the Volume

Get into groups.

- In your groups, draw the cross-section of the slices

Finding the Volume

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- In your groups, draw the cross-section of the slices
- Write an expression for the volume of each slice.

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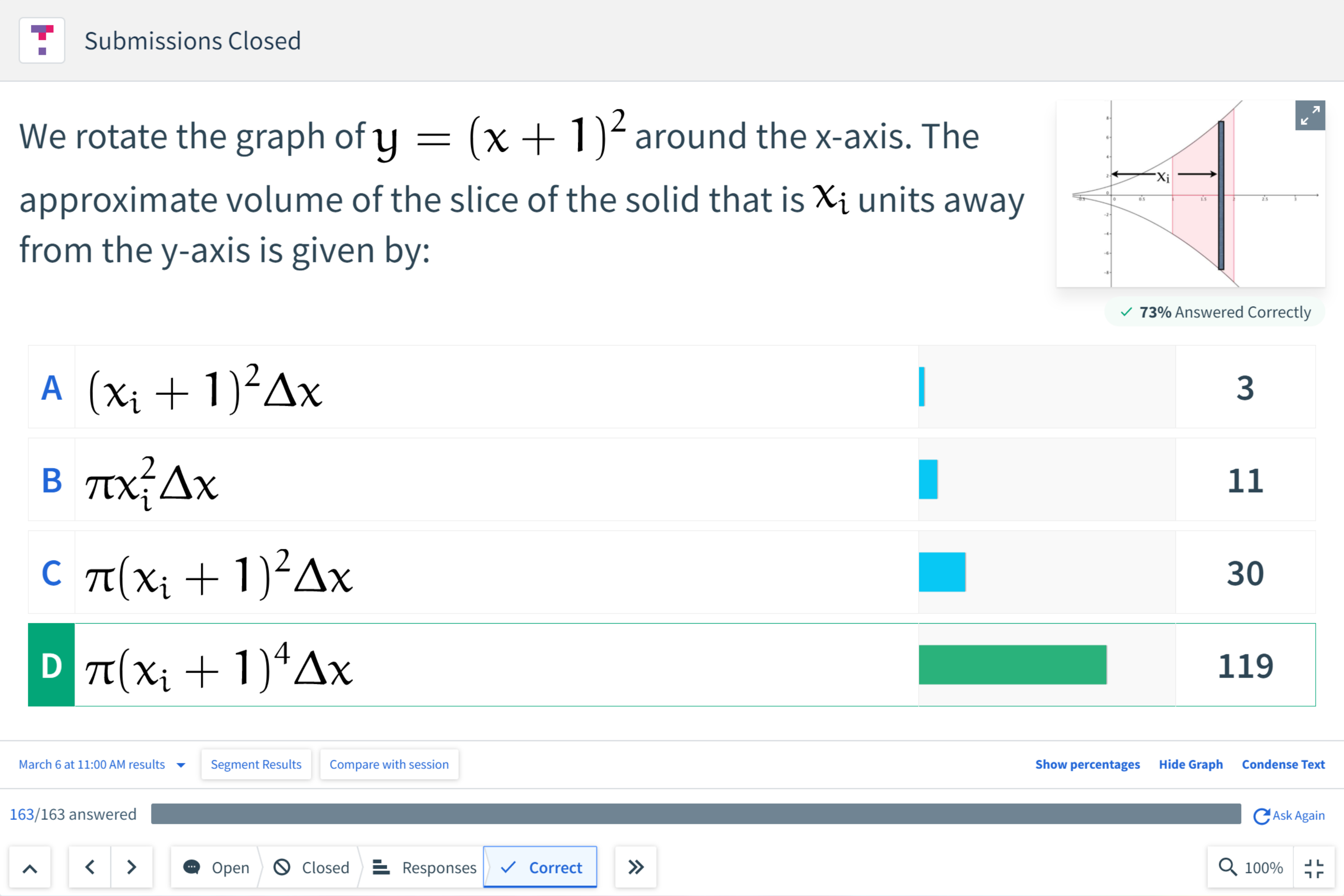
- Write an expression for the volume of each slice.
  $V = \Delta x\pi((3 - x^2)^2 - (3 - x)^2)$
- Write an integral that computes the total volume

Finding the Volume

Get into groups.

- In your groups, draw the cross-section of the slices
- Write an expression for the volume of each slice.
  $V = \Delta x\pi((3-x^2)^2-(3-x)^2)$
- Write an integral that computes the total volume

$$\int_0^1 \pi((3-x^2)^2-(3-x)^2)dx$$



We rotate the graph of $y = (x+1)^2$ around the x-axis. The approximate volume of the slice of the solid that is x_i units away from the y-axis is given by:

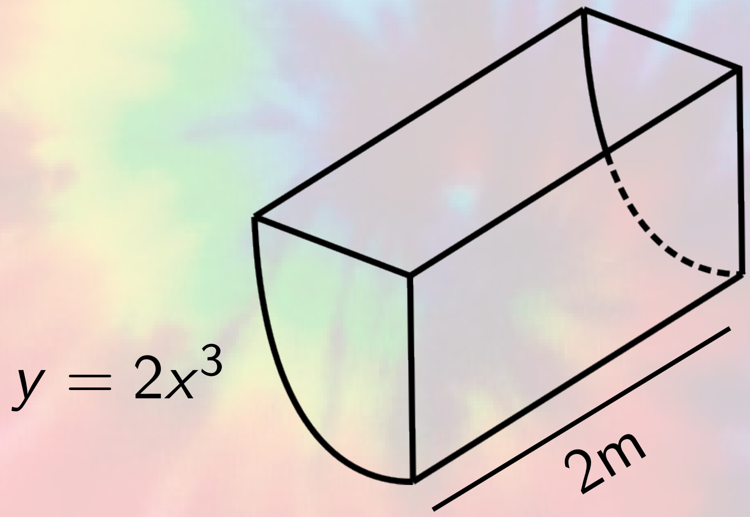
✓ **73%** Answered Correctly

A	$(x_i+1)^2\Delta x$	3
B	$\pi x_i^2 \Delta x$	11
C	$\pi(x_i+1)^2\Delta x$	30
D	$\pi(x_i+1)^4\Delta x$	119

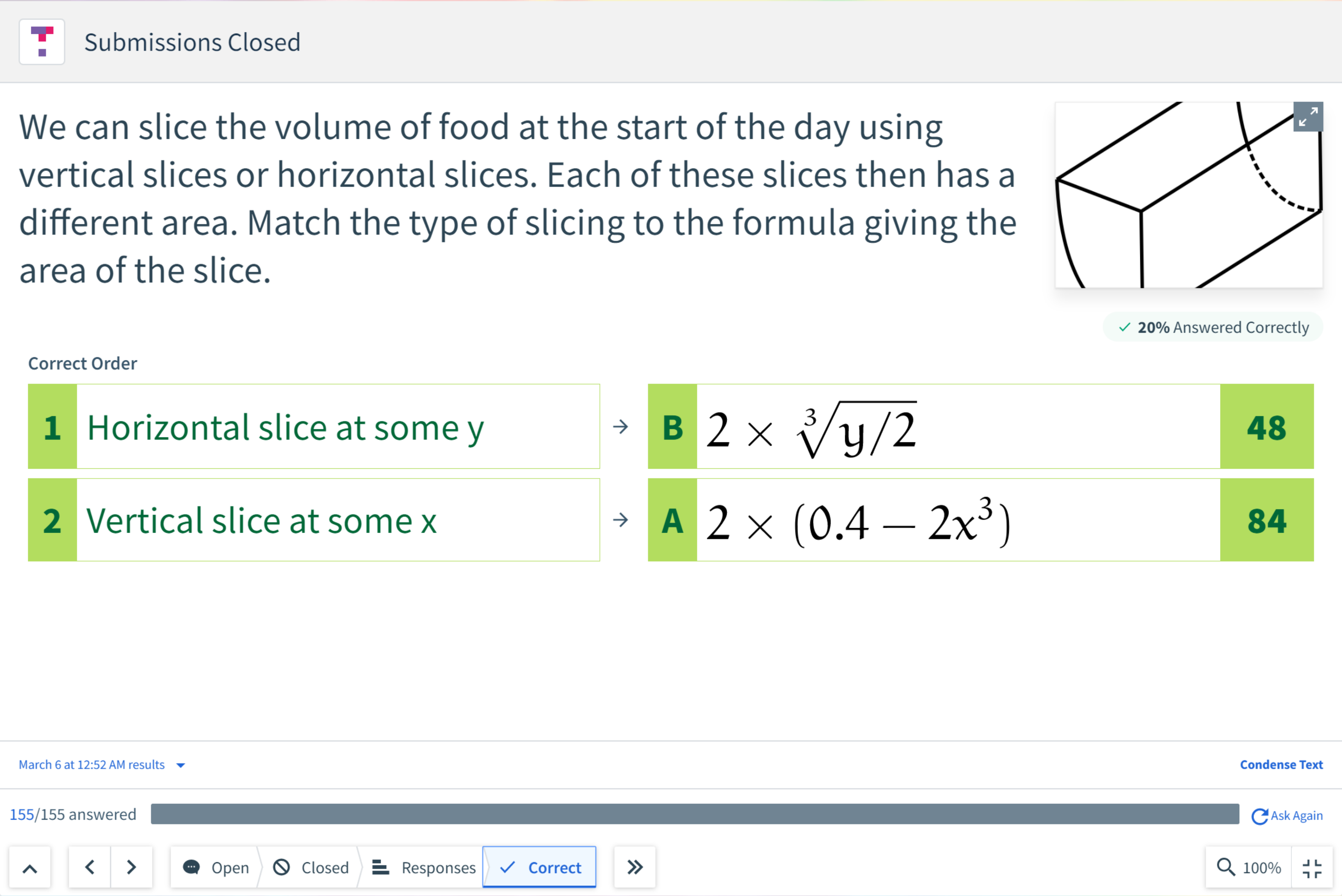
163/163 answered

Cats and Troughs

The cats are stealing food from the sheep's trough (pictured below schematically):



Assume that the trough has a cubic cross-section $y = 2x^3$, and that its length is $2m$. At the start of the day, the food in the trough is $0.4m$ high in the trough. At the end of the day, it's $0.39m$ high. How much food did the cats eat?



Submissions Closed

We can slice the volume of food at the start of the day using vertical slices or horizontal slices. Each of these slices then has a different area. Match the type of slicing to the formula giving the area of the slice.

✓ **20%** Answered Correctly

Correct Order

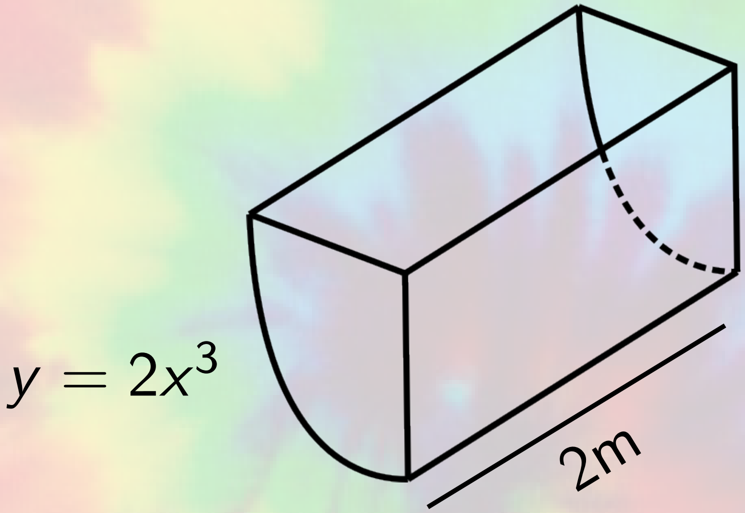
1	Horizontal slice at some y	→	B	$2 \times \sqrt[3]{y/2}$	48
2	Vertical slice at some x	→	A	$2 \times (0.4 - 2x^3)$	84

March 6 at 12:52 AM results

Condense Text

155/155 answered

Cats and Troughs

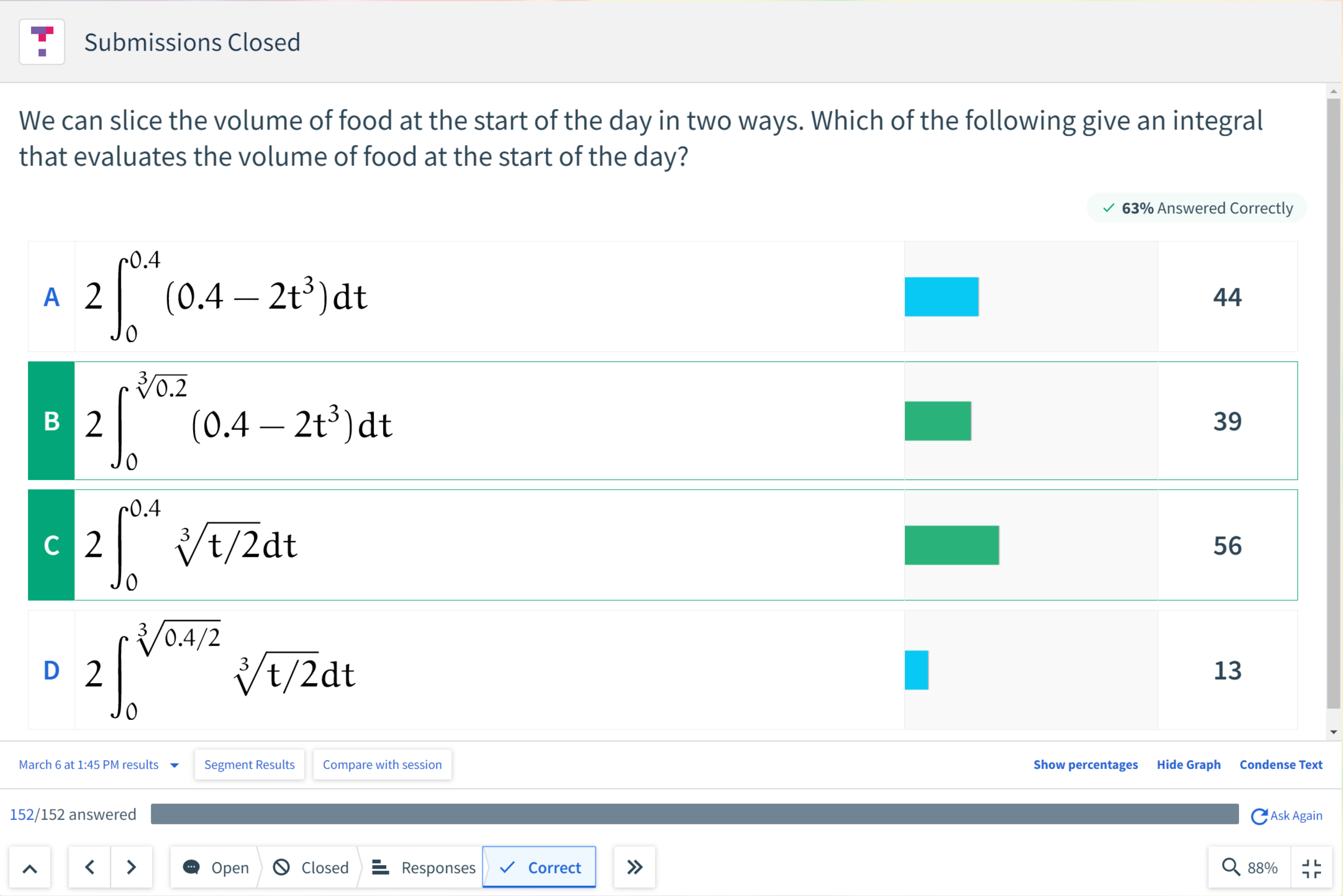


Vertical Slices

$$\Delta\text{V(start of day)}$$
$$= 2 \times (0.4 - 2x^3) \times \Delta x$$

Horizontal Slices

$$\Delta\text{V(start of day)}$$
$$= 2 \times \sqrt[3]{y/2} \times \Delta y$$



Submissions Closed

We can slice the volume of food at the start of the day in two ways. Which of the following give an integral that evaluates the volume of food at the start of the day?

✓ **63%** Answered Correctly

A	$2\int_0^{0.4}(0.4-2t^3)\,dt$	44
B	$2\int_0^{\sqrt[3]{0.2}}(0.4-2t^3)\,dt$	39
C	$2\int_0^{0.4}\sqrt[3]{t/2}\,dt$	56
D	$2\int_0^{\sqrt[3]{0.4/2}}\sqrt[3]{t/2}\,dt$	13

152/152 answered

Cats and Troughs

At the start of the day...

Vertical Slices	**Horizontal Slices**
$\Delta\mathrm{V} = 2 \times (0.4 - 2x^3) \times \Delta x$	$\Delta\mathrm{V} = 2 \times \sqrt[3]{y/2} \times \Delta y$
$\mathrm{V} = 2\int_0^{\sqrt[3]{0.2}} (0.4 - 2x^3)dx$	$\mathrm{V} = 2\int_0^{0.4} \sqrt[3]{y/2}dy$

Both are equal to ≈ 0.3508

Cats and Troughs

So how much did the cats eat?

Cats and Troughs

So how much did the cats eat?

$$\begin{aligned} \text{V(start of day)} - \text{V(end of day)} \\ &= 2\int_0^{0.4} \sqrt[3]{y/2}dy - 2\int_0^{0.39} \sqrt[3]{y/2}dy \\ &= 2\int_{0.39}^{0.4} \sqrt[3]{y/2}dy \approx 0.011m^2 = 11L \end{aligned}$$

Plans for the Future

For next time:

WeBWork 8.4 and actively read section 8.4

Welcome to MAT136 LEC0501 (Assaf)

`https://www.youtube.com/watch?v=Kas0tIxDvrg`

An interesting video on COVID-19 modeling and exponential growth.

Q: What model (SI, SIR, or SIS) is this video using?

S8.4 – Density and Slicing

Assaf Bar-Natan

“ Come gather ’round people
Wherever you roam
And admit that the waters
Around you have grown”

–“The Times They Are ’a Changin’”, Simon and Garfunkel

March 9, 2020

WeBWork Round Robin

In your groups, go in a circle, and:

- Say a problem from the WeBWork you struggled with.

WeBWork Round Robin

In your groups, go in a circle, and:

- Say a problem from the WeBWork you struggled with.
- Discuss the solution to each problem that the group mentioned.

WeBWork Round Robin

In your groups, go in a circle, and:

- Say a problem from the WeBWork you struggled with.
- Discuss the solution to each problem that the group mentioned.
- Write a hint for a student struggling with the problem.

Takeaway

In life, and on the exam, you will be asked to communicate your math using complete sentences.

The writing exercises we do in class are for your practice!

What is Density?

Flood has a long tail, and the fur-density is given by a function, $h(l)\frac{\text{hairs}}{\text{m}}$, where l is the length along her tail. If Flood's tail is 30cm long, how many hairs does Flood have?

What is Density?

Flood has a long tail, and the fur-density is given by a function, $h(l)\frac{\mathrm{hairs}}{\mathrm{m}}$, where l is the length along her tail. If Flood's tail is $30\mathrm{cm}$ long, how many hairs does Flood have?

$$\mathrm{Hairs} \approx \sum h(l)\Delta l = \int_a^b h(l)dl$$

Q: What are *a* and *b*? (Hint: units!)

What is Density?

Flood has a long tail, and the fur-density is given by a function, $h(l)\frac{\mathrm{hairs}}{\mathrm{m}}$, where l is the length along her tail. If Flood's tail is $30\mathrm{cm}$ long, how many hairs does Flood have?

$$\mathrm{Hairs} \approx \sum h(l)\Delta l = \int_a^b h(l)dl$$

Q: What are a and b? (Hint: units!) $a = 0$ and $b = 0.3\mathrm{m} = 30\mathrm{cm}$.

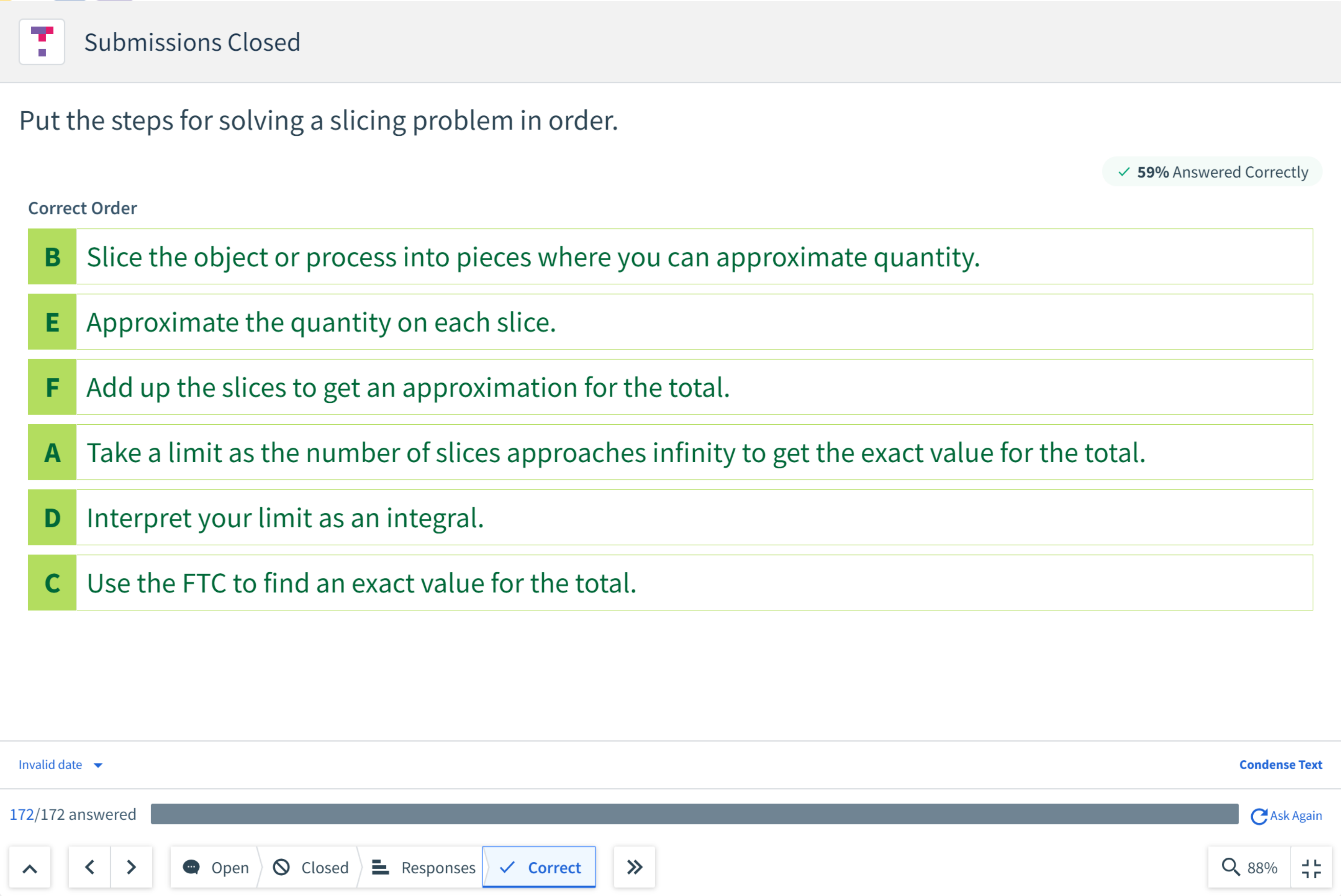
Takeaway

Always make sure that the units work out!

Torontopolis

The fictional city of Torontopolis radially has a population density of $4000e^{-0.02r^2}$ people per km^2, where r is the radius (in km) from the CM-tower.

We are interested in finding the total population living within a certain radius of the CM-tower.



Submissions Closed

Put the steps for solving a slicing problem in order.

✓ **59%** Answered Correctly

Correct Order

- **B** Slice the object or process into pieces where you can approximate quantity.
- **E** Approximate the quantity on each slice.
- **F** Add up the slices to get an approximation for the total.
- **A** Take a limit as the number of slices approaches infinity to get the exact value for the total.
- **D** Interpret your limit as an integral.
- **C** Use the FTC to find an exact value for the total.

Invalid date

172/172 answered

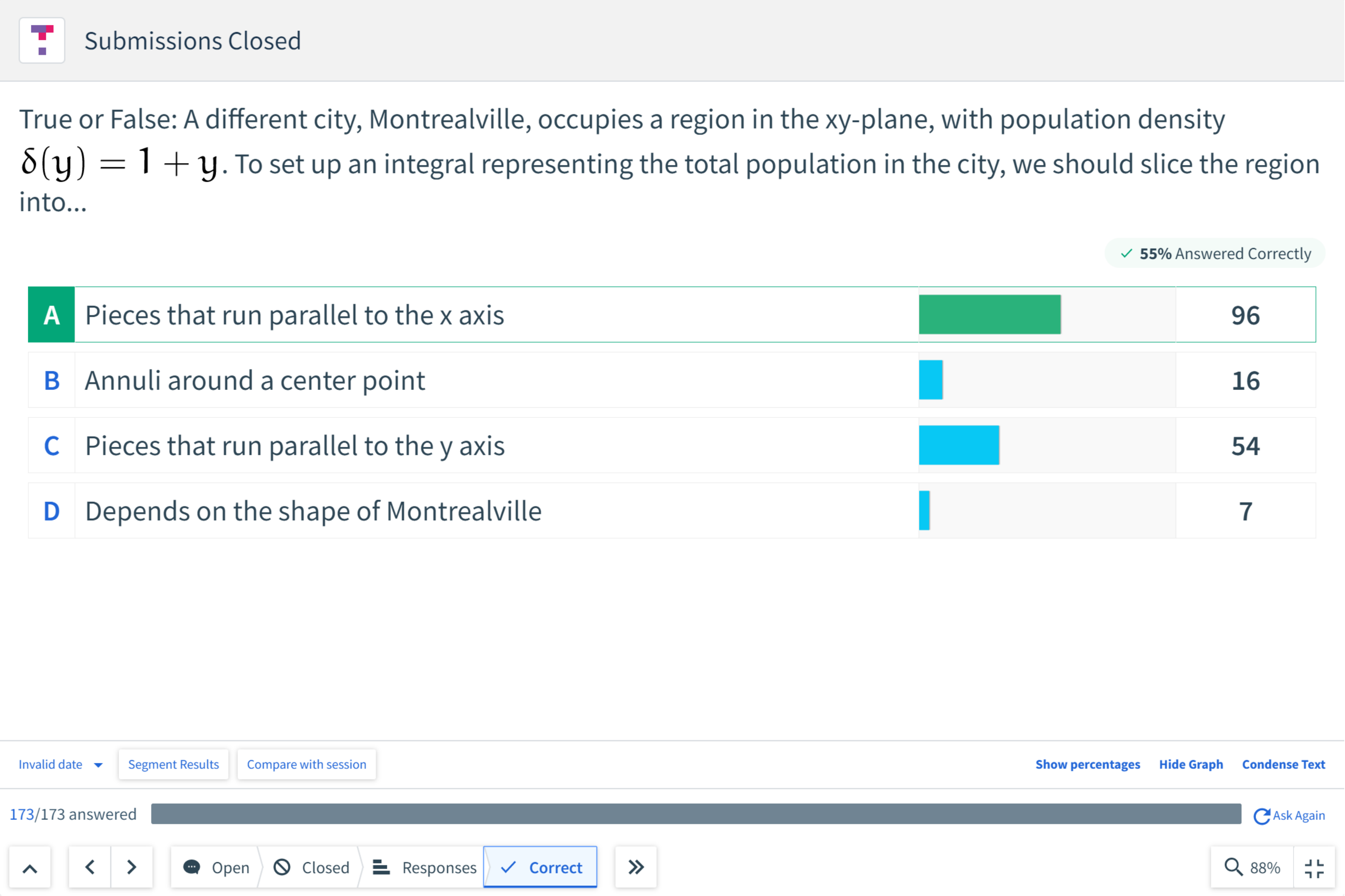
Slice object where density is constant

Discussion: Along what "slices" of Torontopolis is the population density approximately constant?

Slice object where density is constant

Discussion: Along what "slices" of Torontopolis is the population density approximately constant?

A: Annuli of small thickness centered at the CM-tower.



Submissions Closed

True or False: A different city, Montrealville, occupies a region in the xy-plane, with population density $\delta(y) = 1 + y$. To set up an integral representing the total population in the city, we should slice the region into...

✓ **55%** Answered Correctly

A	Pieces that run parallel to the x axis	96
B	Annuli around a center point	16
C	Pieces that run parallel to the y axis	54
D	Depends on the shape of Montrealville	7

173/173 answered

Add up slices

Discussion: What is the total population living on an annulus of radius r_i and of width Δr?

Add up slices

Discussion: What is the total population living on an annulus of radius r_i and of width Δr?

A: $4000e^{-0.02r^2} \times 2\pi r \times \Delta r$

Interpret as Riemann sum

Discussion: What is the total number of people who live within a $3km$ radius of the CM-tower? Write your answer as a Riemann sum.

Interpret as Riemann sum

Discussion: What is the total number of people who live within a $3km$ radius of the CM-tower? Write your answer as a Riemann sum.

A: We partition the interval $[0, 3]$ into n pieces. So $\Delta r = \frac{3}{n}$.

What is r_i?

Interpret as Riemann sum

Discussion: What is the total number of people who live within a $3km$ radius of the CM-tower? Write your answer as a Riemann sum.

A: We partition the interval $[0, 3]$ into n pieces. So $\Delta r = \frac{3}{n}$.

What is r_i?

A: $r_i = \frac{3i}{n}$, so the sum becomes:

Interpret as Riemann sum

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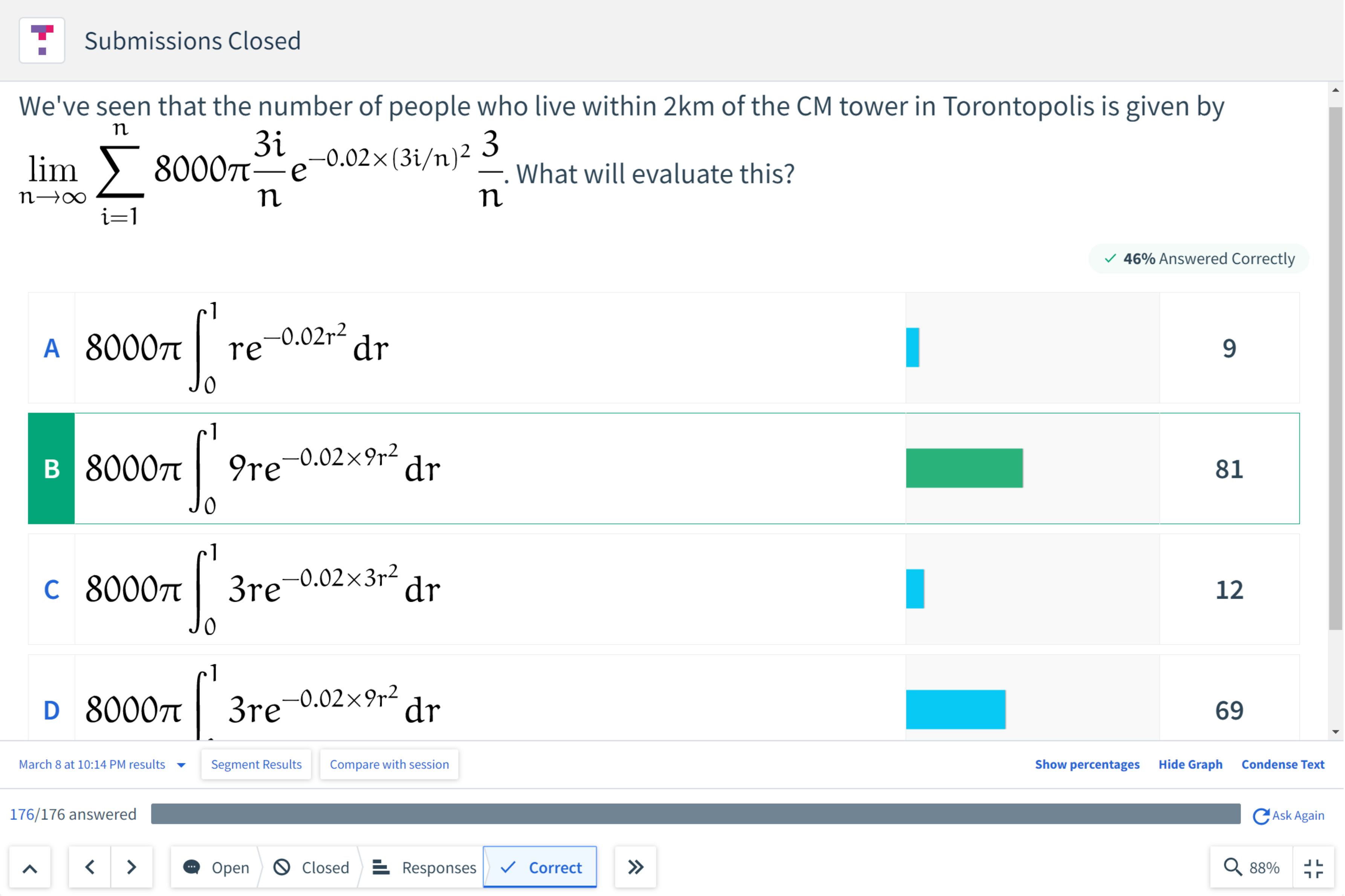
What is r_i?

A: $r_i = \frac{3i}{n}$, so the sum becomes:

$$\sum_{i=1}^{n} 2\pi r_i \times 4000 e^{-0.02 r_i^2} \times \frac{3}{n}$$

To get the true quantity, take the limit.



Submissions Closed

We've seen that the number of people who live within 2km of the CM tower in Torontopolis is given by $\lim_{n\to\infty}\sum_{i=1}^{n} 8000\pi\frac{3i}{n}e^{-0.02\times(3i/n)^2}\frac{3}{n}$. What will evaluate this?

✓ **46%** Answered Correctly

A	$8000\pi\int_0^1 re^{-0.02r^2}\,dr$	9
B	$8000\pi\int_0^1 9re^{-0.02\times9r^2}\,dr$	81
C	$8000\pi\int_0^1 3re^{-0.02\times3r^2}\,dr$	12
D	$8000\pi\int^1 3re^{-0.02\times9r^2}\,dr$	69

176/176 answered

Compute the Integral

The total number of people who live within a 3km radius of the CM-tower is:

$$8000\pi \int_0^1 9re^{-0.02\times 9r^2}\,dr = 8000\pi \int_0^3 re^{-0.02r^2}\,dr$$

Compute the Integral

The total number of people who live within a 3km radius of the CM-tower is:

$$8000\pi \int_0^1 9re^{-0.02\times 9r^2}dr = 8000\pi \int_0^3 re^{-0.02r^2}dr \approx 103,000$$

Takeaway

Reminder: for ALL slicing problems, you need to show all the steps on the exam!

Plans for the Future

For next time:
Go over WeBWork 8.4 and section 8.4

Ban cars on campus

Welcome to MAT136 LEC0501 (Assaf)

Final exam is in three weeks – Do you have a study plan?

Applications for Slicing

Assaf Bar-Natan

"Money. It's a crime
Share it fairly, but don't take a slice of my pie
Money. So they say
Is the root of all evil today. "

–"Money", Pink Floyd

March 11, 2020

Today's Plan

Today: practice for the short answer problems on the final

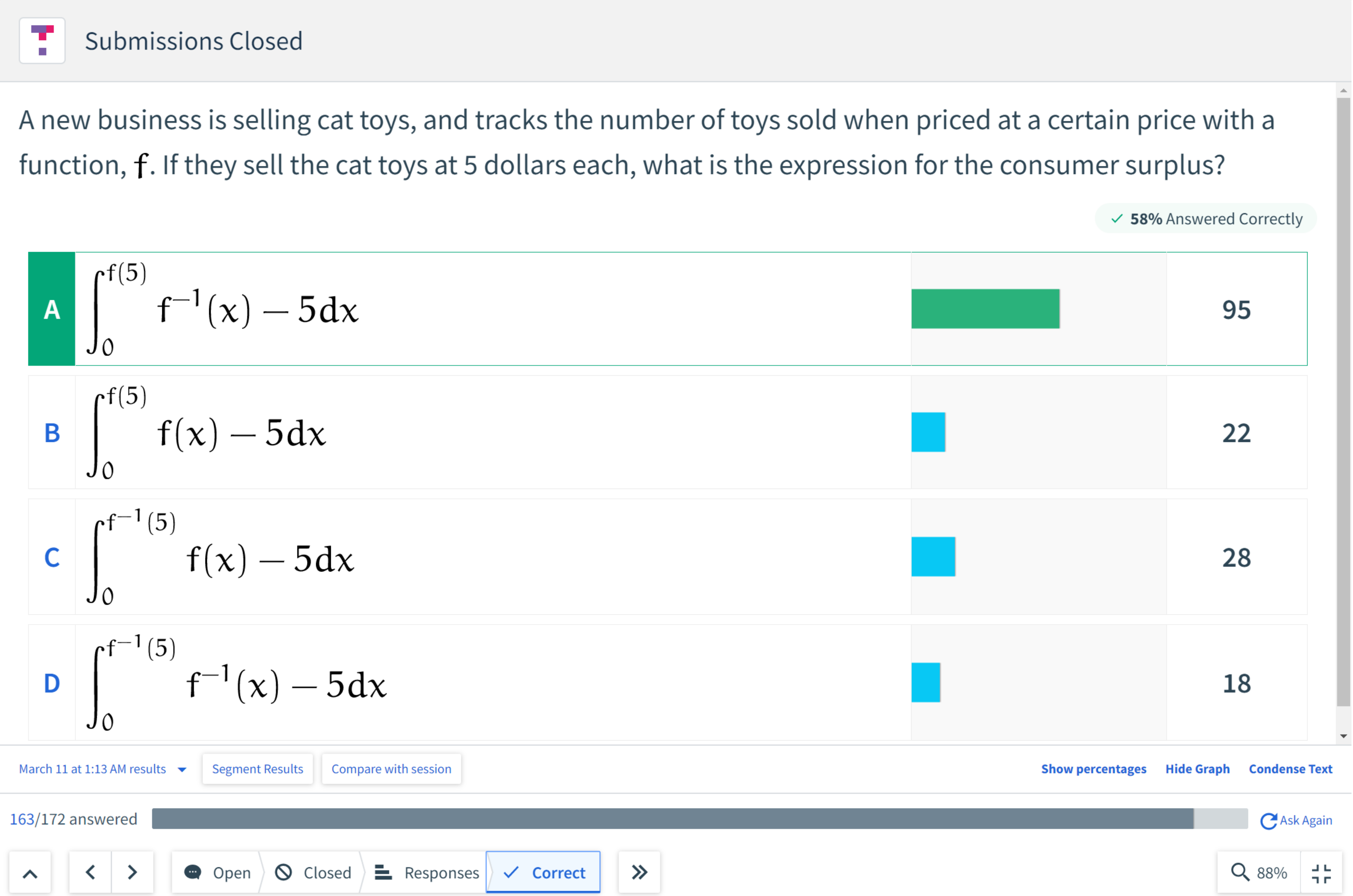
- Read a text on slicing problems
- Summarize the text
- TopHat

Please open the text (Week 9 on Quercus)

Consumer Surplus

Main points from the reading:

- The **demand curve** plots the price of a product as a function of how many will sell at that price.
- The difference between what a consumer pays and what they are willing to pay is called the **consumer surplus**.
- Adding up (savings per unit)x(number of units) $= \sum (p(x_i) - P)\Delta x$ gives the total amount of money saved by everyone.
- The above is called the commodity **consumer surplus**
- We can compute the the commodity consumer surplus using an integral



Submissions Closed

A new business is selling cat toys, and tracks the number of toys sold when priced at a certain price with a function, f. If they sell the cat toys at 5 dollars each, what is the expression for the consumer surplus?

✓ **58%** Answered Correctly

A	$\int_0^{f(5)} f^{-1}(x) - 5dx$	95
B	$\int_0^{f(5)} f(x) - 5dx$	22
C	$\int_0^{f^{-1}(5)} f(x) - 5dx$	28
D	$\int_0^{f^{-1}(5)} f^{-1}(x) - 5dx$	18

163/172 answered

Laminar Flow

Main points from the reading:

- The laminar flow rate depends on the radius of the tube
- The total flow is the integral of the flow rate – use rings
- This reminds me of a WeBWork problem.... **Q:** which one?

- The **flux** is the amount of blood that passes through a section of the tube per unit time.
- **Poiseuille's Law** says that the flux is given by:

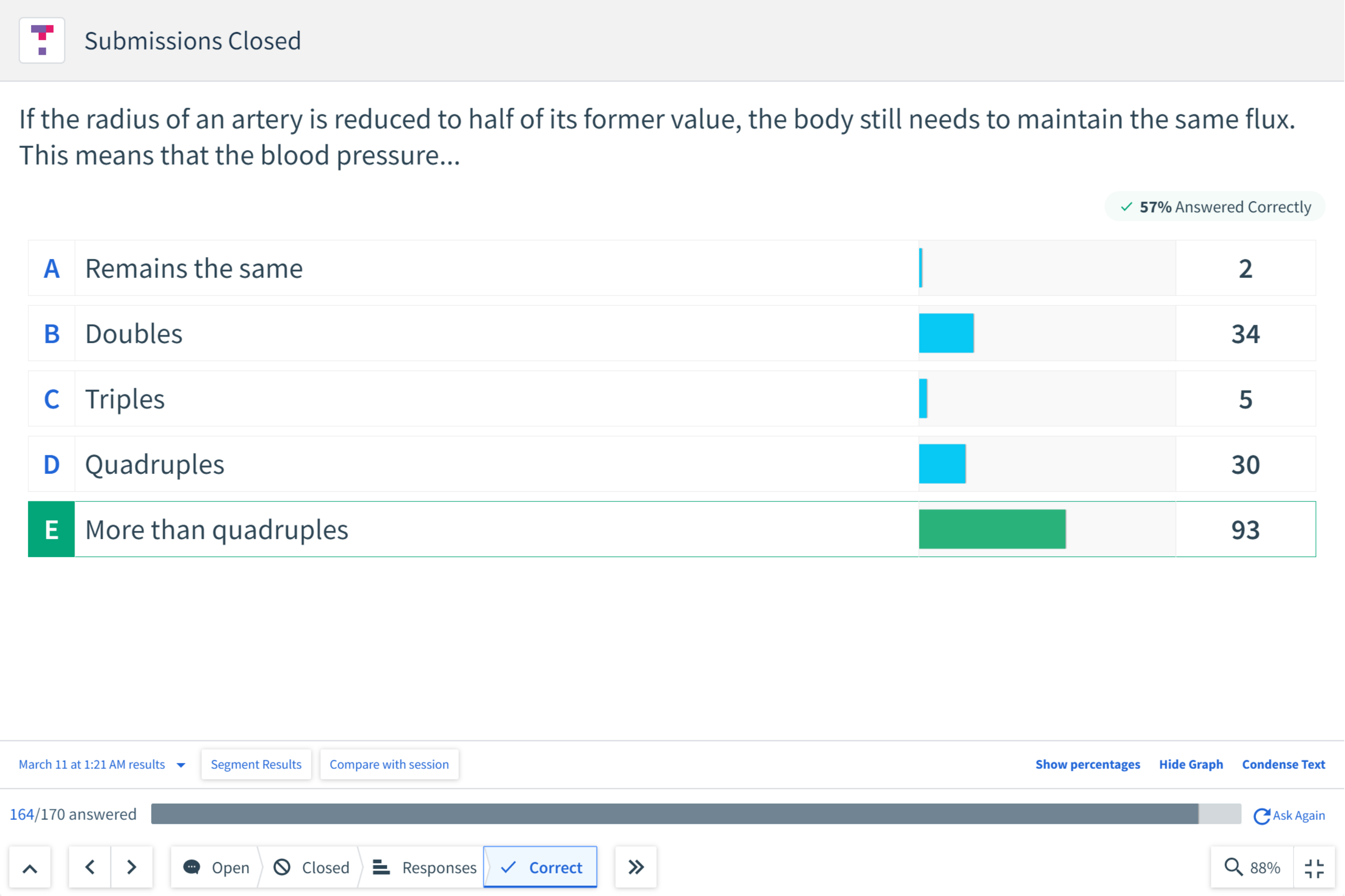
$$F = \frac{\pi P R^4}{8 \eta l}$$

Laminar Flow

Main points from the reading:

- The laminar flow rate depends on the radius of the tube
- The total flow is the integral of the flow rate – use rings
- This reminds me of a WeBWork problem.... **Q:** which one? WW8.4, number 5
- The **flux** is the amount of blood that passes through a section of the tube per unit time.
- **Poiseuille's Law** says that the flux is given by:

$$F = \frac{\pi P R^4}{8 \eta l}$$



Submissions Closed

If the radius of an artery is reduced to half of its former value, the body still needs to maintain the same flux. This means that the blood pressure...

✓ **57%** Answered Correctly

A	Remains the same	2
B	Doubles	34
C	Triples	5
D	Quadruples	30
E	More than quadruples	93

164/170 answered

Plans for the Future

For next time:

Go over WeBWork 9.1 and section 9.1

Welcome to MAT136 LEC0501 (Assaf)

Next week – We're going digital!
I don't care what the university says.

S9.1 – Sequences (AKA infinite lists)

Assaf Bar-Natan

> “Yeah yeah ’cause it goes on and on and on
> And it goes on and on and on yeah
> I throw my hands up in the air sometimes
> Saying ayeoh, gotta let go”
>
> –“Dynamite”, Taio Cruz

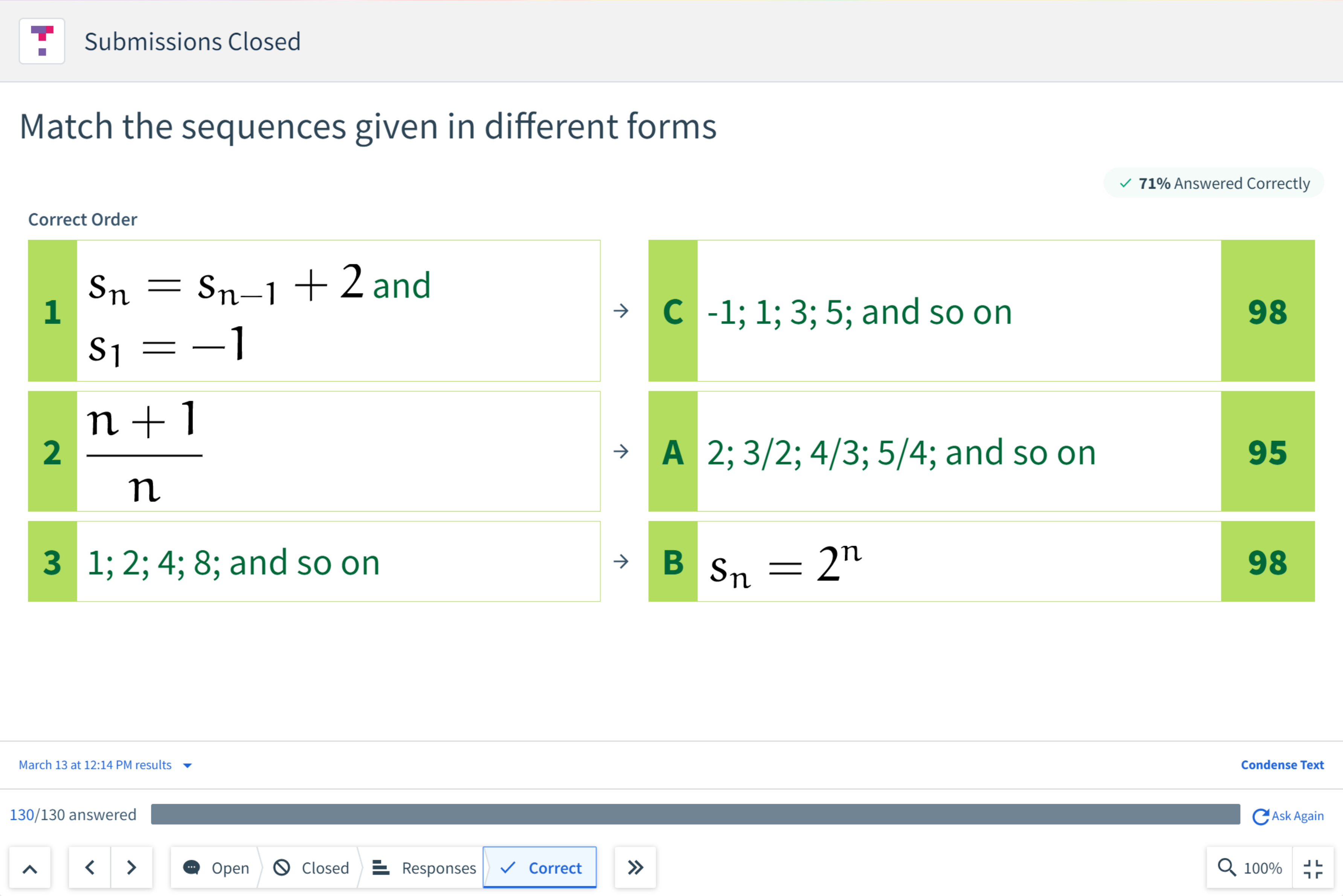
March 13, 2020

What is a sequence?

A sequence is an ordered list of numbers

We can give a sequence in a few ways:

- Explicity: $1, 4, 9, \ldots$ (like a table of values $f(n) = n^2$)
- Closed form: $c_n = \frac{1+2n}{3n-2}$ (like Taylor coefficients $c_n = \frac{1}{n!}\frac{d^n f}{dx^n}$)
- Recursive: $s_{n+1} = s_n + 1/n$ (like Euler's method)



Submissions Closed

Match the sequences given in different forms

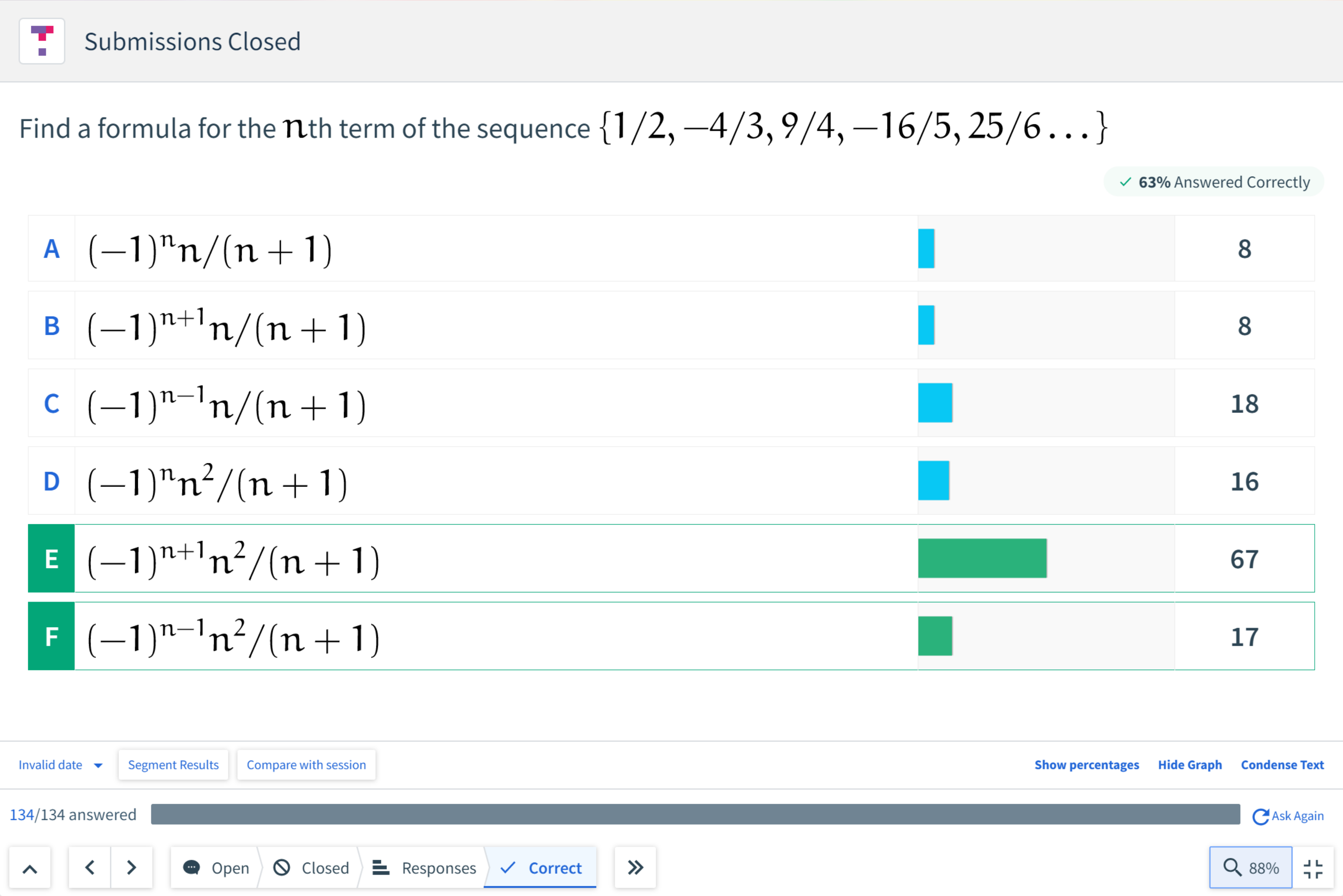
✓ **71%** Answered Correctly

Correct Order

1	$s_n = s_{n-1} + 2$ and $s_1 = -1$	→	C	-1; 1; 3; 5; and so on	98
2	$\frac{n+1}{n}$	→	A	2; 3/2; 4/3; 5/4; and so on	95
3	1; 2; 4; 8; and so on	→	B	$s_n = 2^n$	98

March 13 at 12:14 PM results

130/130 answered



Submissions Closed

Find a formula for the nth term of the sequence $\{1/2, -4/3, 9/4, -16/5, 25/6\ldots\}$

✓ 63% Answered Correctly

A	$(-1)^n n/(n+1)$	8
B	$(-1)^{n+1} n/(n+1)$	8
C	$(-1)^{n-1} n/(n+1)$	18
D	$(-1)^n n^2/(n+1)$	16
E	$(-1)^{n+1} n^2/(n+1)$	67
F	$(-1)^{n-1} n^2/(n+1)$	17

134/134 answered

Takeaway

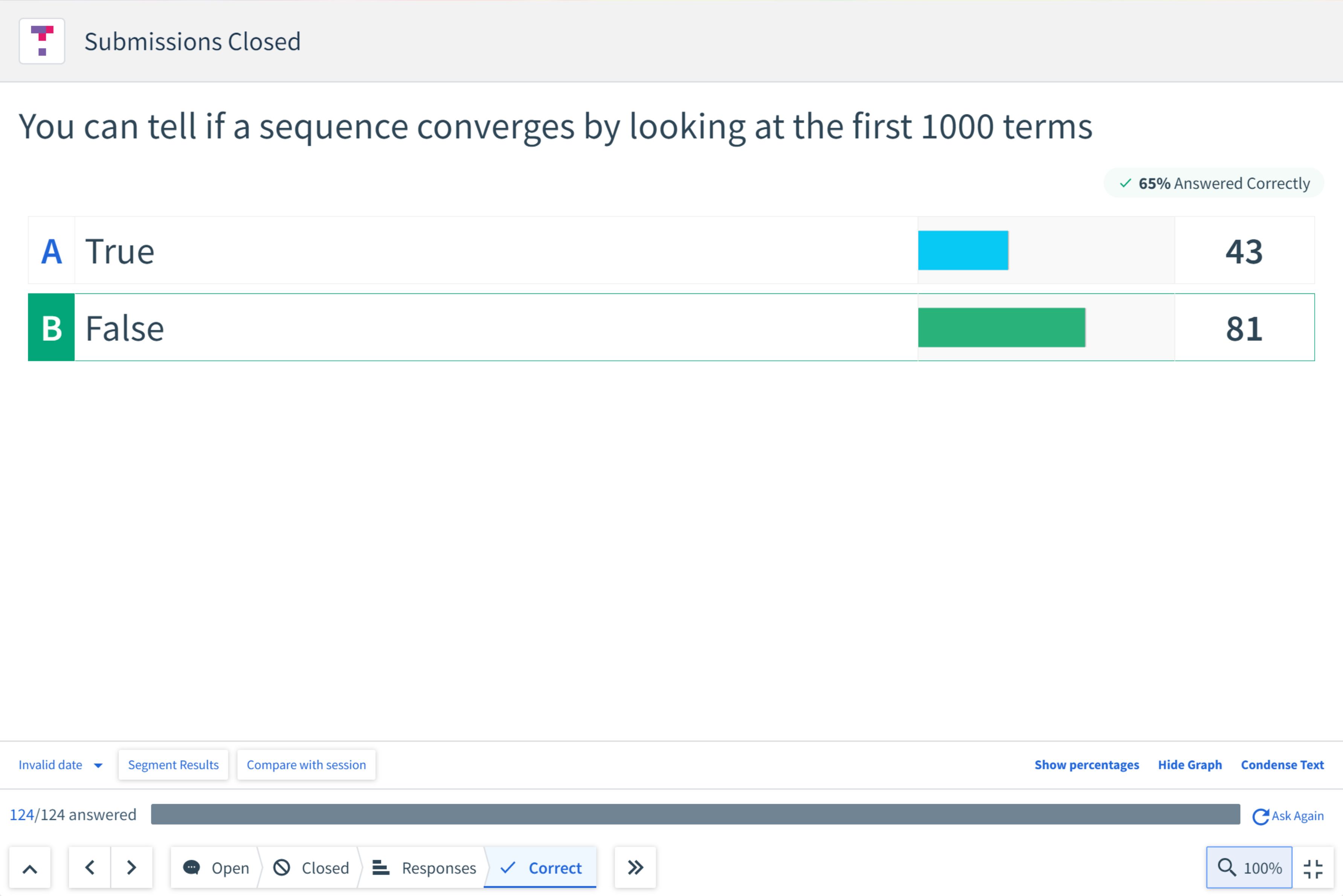
We can move back and forth between representations of sequences!

Fill in the Blanks

- If a sequence is m______ and b______, it converges.
- A sequence s_n converges to L if s_n is as close to ______ as we please if ______ is ______.
- A sequence is an ______ list of numbers.
- For a positive integer n, n! $=$ ______.
- A sequence is ______ defined if the equation for a general term depends on previous terms.

Fill in the Blanks

- If a sequence is **monotonic** and **bounded**, it converges.
- A sequence s_n converges to L if s_n is as close to **L** as we please if **n** is **large**.
- A sequence is an **ordered** list of numbers.
- For a positive integer n, n! $=$ $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$.
- A sequence is **recursively** defined if the equation for a general term depends on previous terms.



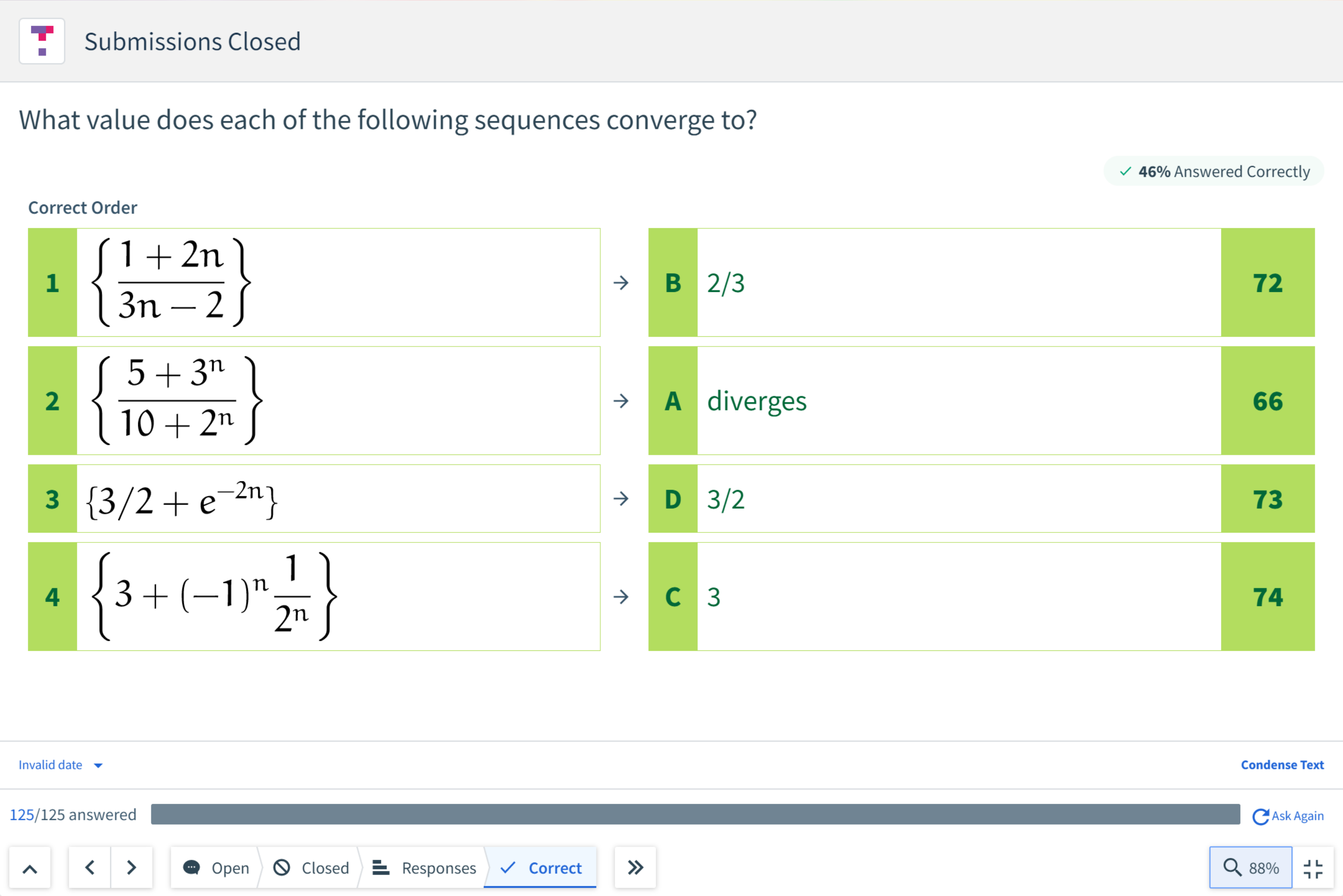
Submissions Closed

You can tell if a sequence converges by looking at the first 1000 terms

✓ **65%** Answered Correctly

A	True	43
B	False	81

124/124 answered



Submissions Closed

What value does each of the following sequences converge to?

✓ **46%** Answered Correctly

Correct Order

1	$\left\{\frac{1+2n}{3n-2}\right\}$	→	**B**	2/3	**72**
2	$\left\{\frac{5+3^n}{10+2^n}\right\}$	→	**A**	diverges	**66**
3	$\{3/2+e^{-2n}\}$	→	**D**	3/2	**73**
4	$\left\{3+(-1)^n\frac{1}{2^n}\right\}$	→	**C**	3	**74**

125/125 answered

Takeaway

We have a few ways to check if a sequence converges. One way is to look at the closed form and plug in big numbers

Champernowne constant

Consider the sequence:

- $C_1 = 0.1$
- $C_2 = 0.12$
- $C_3 = 0.123$

Q: Does this sequence converge? How do you know this?

A: This sequence converges because it is monotonic and bounded.

Champernowne constant

The limit of the sequence $0.1, 0.12, 0.123, \ldots$ **is called Champernowne constant, and its decimal expansion contains every number. Even your phone number!**

And now, we meet our friends...

The gang

Inspiration for cat opening mouth question

Kittens in hay

Cats looking

Cuddles

Bulking up for winter

Sunset

Plans for the Future

For next time:
Go over WeBWork 9.2 and section 9.2

Welcome to MAT136 LEC0501 (Assaf)

Administrative Announcements

- us Class will “meet” at 2:10pm MWF on BB Collaborate
- us Classes will all be recorded
- me My office hour times are now after every class, and will be held on BB Collaborate
- me In-Class TopHat is ungraded, and is replaced by assigned TopHat questions
- you Pre-reading, WeBWork stay the same
- you Watch the videos on the main site!

S9.2 – Geometric Series

Assaf Bar-Natan

"If I could only reach you
If I could make you smile,
If I could only reach you,
That would really be a breakthrough."

–"Breakthru", Queen

March 16, 2020

Welcome to MAT136 LEC0501 (Assaf)

Administrative Announcements

us	Class will "meet" at 2:10pm MWF on BB Collaborate
us	Classes will all be recorded
me	My office hour times are now after every class, and will be held on BB Collaborate
me	In-Class TopHat is ungraded, and is replaced by assigned TopHat questions (due at the end of the class day)
you	Pre-reading, WeBWork stay the same
you	Watch the videos on the main site!

Series

We've seen sequences:

$$a_1, a_2, a_3, \ldots$$

Now, we're going to add them up:

$$a_1$$

$$a_1 + a_2$$

$$a_1 + a_2 + \cdots + a_n + \cdots$$

Such a sum is called a **series**.

Takeaway

Q: What is the difference between a **sum** and a **series**?

Takeaway

Q: What is the difference between a **sum** and a **series**?
A sum only adds up finitely many elements, but a series adds up ininitely many elements.

Takeaway

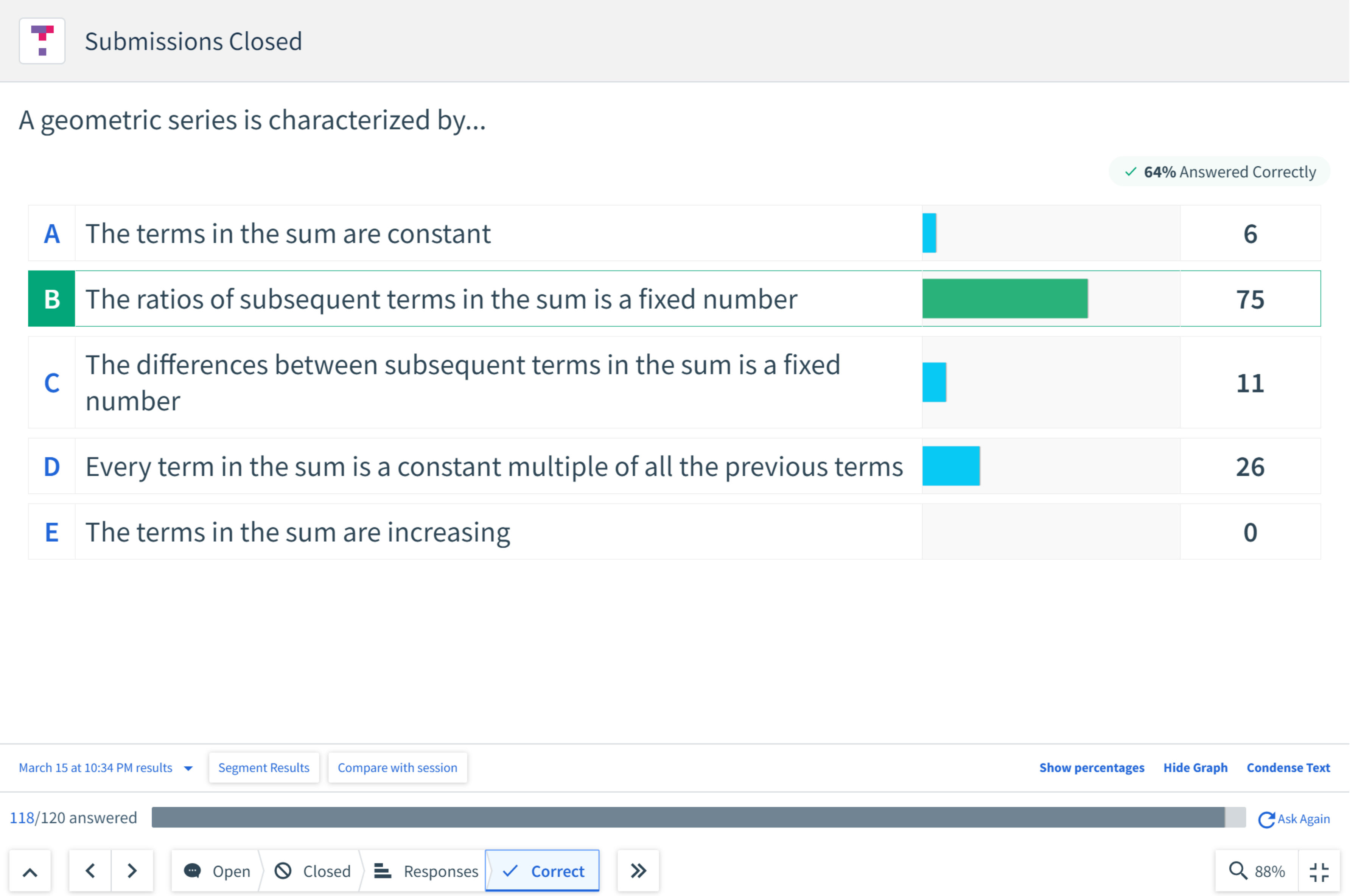
Q: What is the difference between a **sum** and a **series**?
A sum only adds up finitely many elements, but a series adds up ininitely many elements.

$$\sum_{i=0}^{n} f(x_i)\Delta x$$

is a sum.

$$\sum_{n=0}^{\infty} \frac{1}{2^n}$$

is a series (see Zeno's Paradox video)



Submissions Closed

A geometric series is characterized by...

✓ **64%** Answered Correctly

A	The terms in the sum are constant	6
B	The ratios of subsequent terms in the sum is a fixed number	75
C	The differences between subsequent terms in the sum is a fixed number	11
D	Every term in the sum is a constant multiple of all the previous terms	26
E	The terms in the sum are increasing	0

March 15 at 10:34 PM results

118/120 answered

Takeaway

A geometric series is a special kind of series, where the ratio between subsequent terms is constant.

Marzipan's Problem

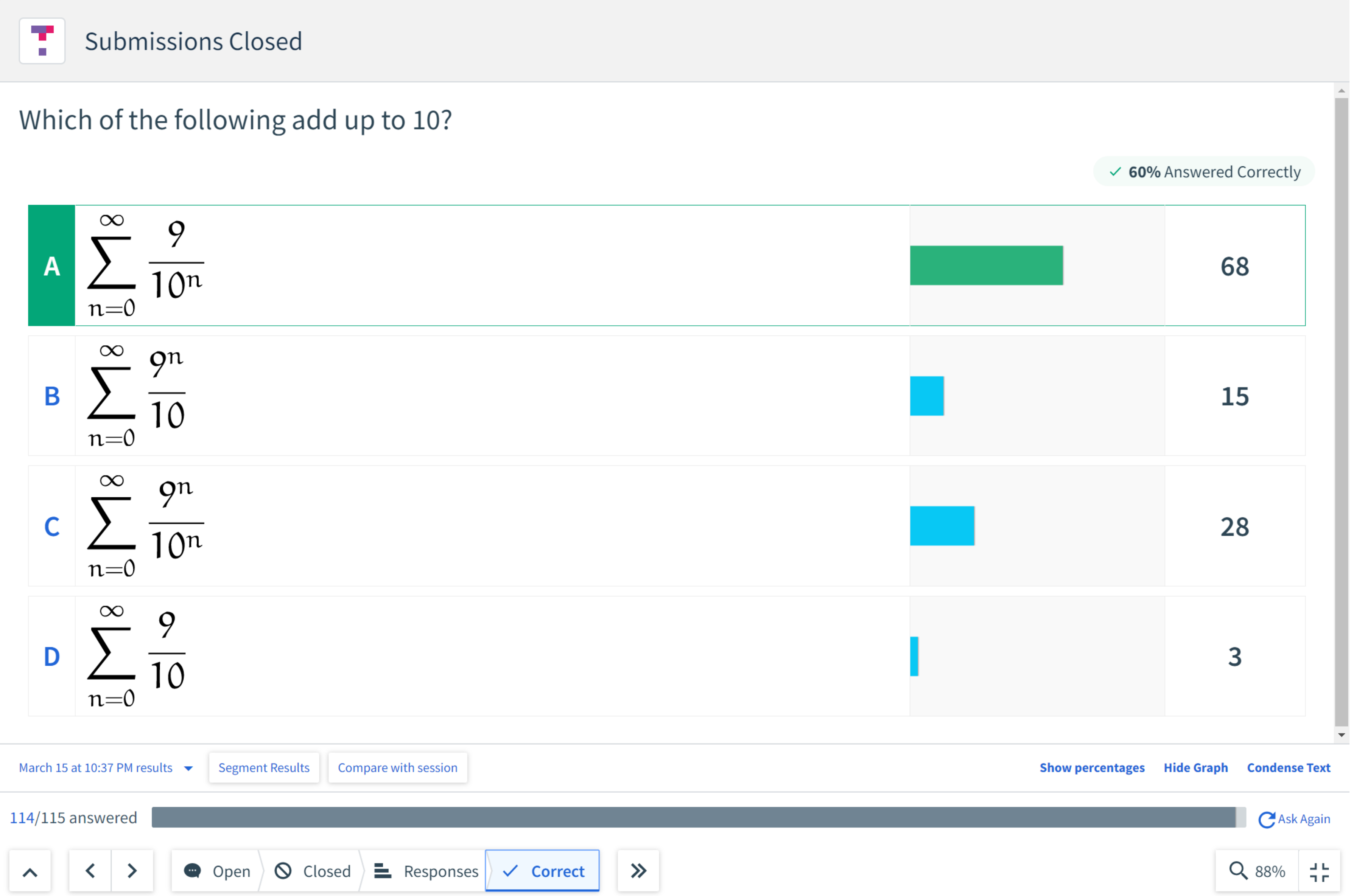
Marzipan is modeling the mouse population in the barn. She finds three mice in the barn, and measures that the number of mice is multiplied by a factor of 1.3 every week. She writes:

"I want to know how many mice will be in the barn by summertime. If summer many many weeks away, I'll approximate using the formula for the inifinite geometric series to get:

$$\text{number of mice} = 3 + 3(1.3) + 3(1.3)^2 + \cdots = \frac{3}{1-1.3} = -10$$

So there will be -10 mice over the summer."

Can you help Marzipan interpret her answer?



Submissions Closed

Which of the following add up to 10?

✓ 60% Answered Correctly

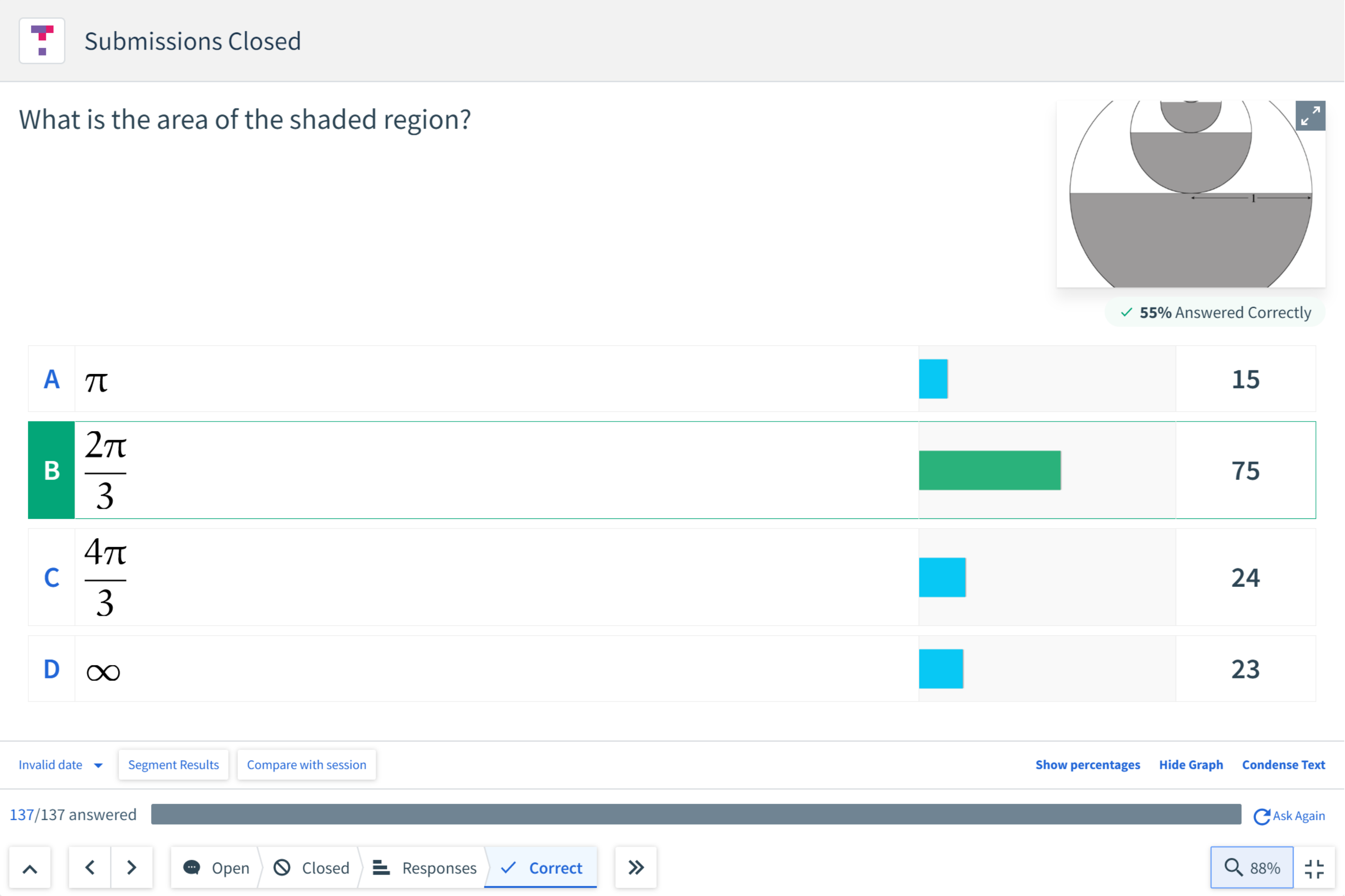
A	$\sum_{n=0}^{\infty} \frac{9}{10^n}$	68
B	$\sum_{n=0}^{\infty} \frac{9^n}{10}$	15
C	$\sum_{n=0}^{\infty} \frac{9^n}{10^n}$	28
D	$\sum_{n=0}^{\infty} \frac{9}{10}$	3

114/115 answered

Plans for the Future

For next time:

Do WeBWork 9.3 and actively read section 9.3



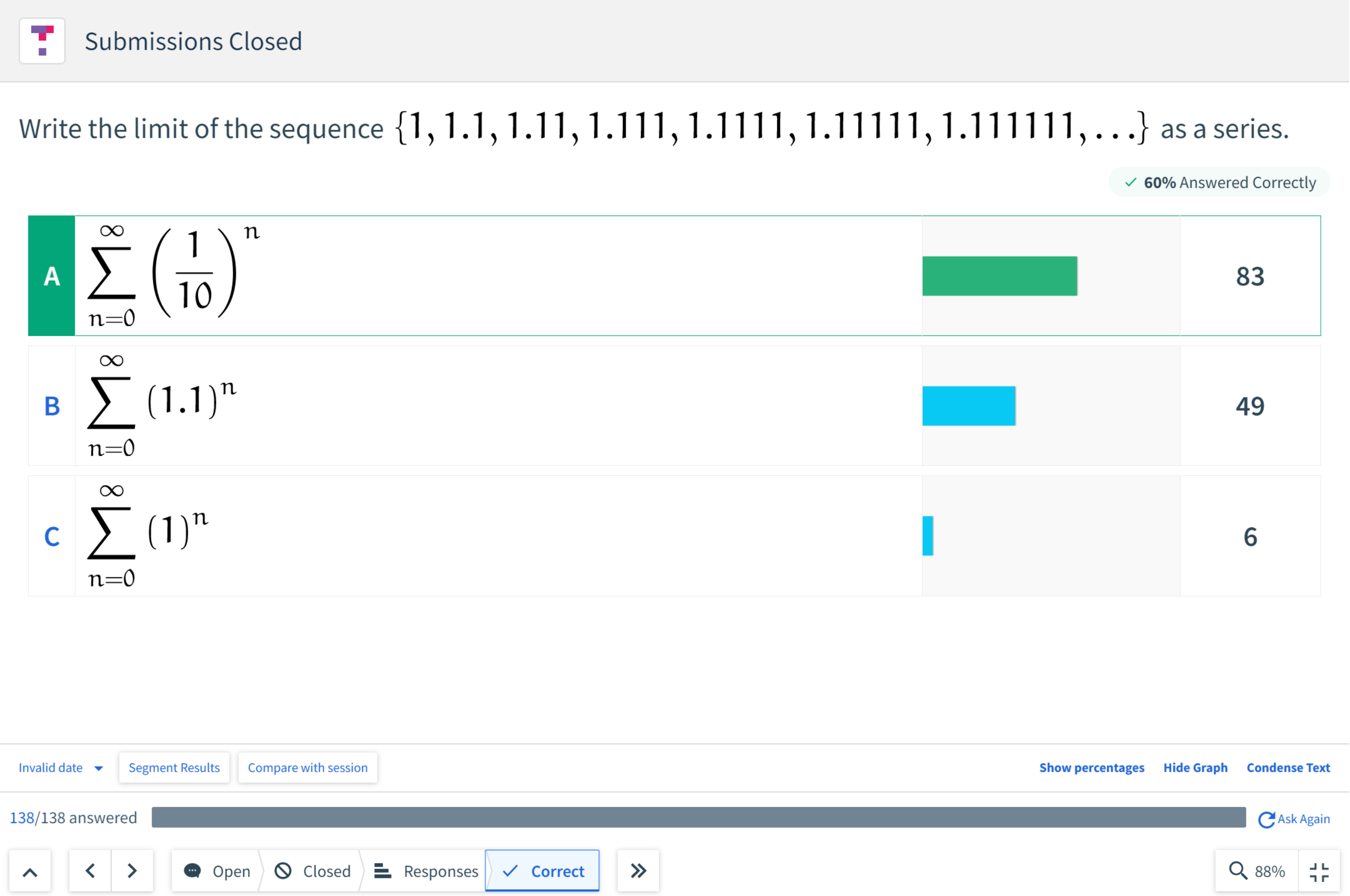
Submissions Closed

What is the area of the shaded region?

✓ **55%** Answered Correctly

A	π	15
B	$\frac{2\pi}{3}$	75
C	$\frac{4\pi}{3}$	24
D	∞	23

137/137 answered



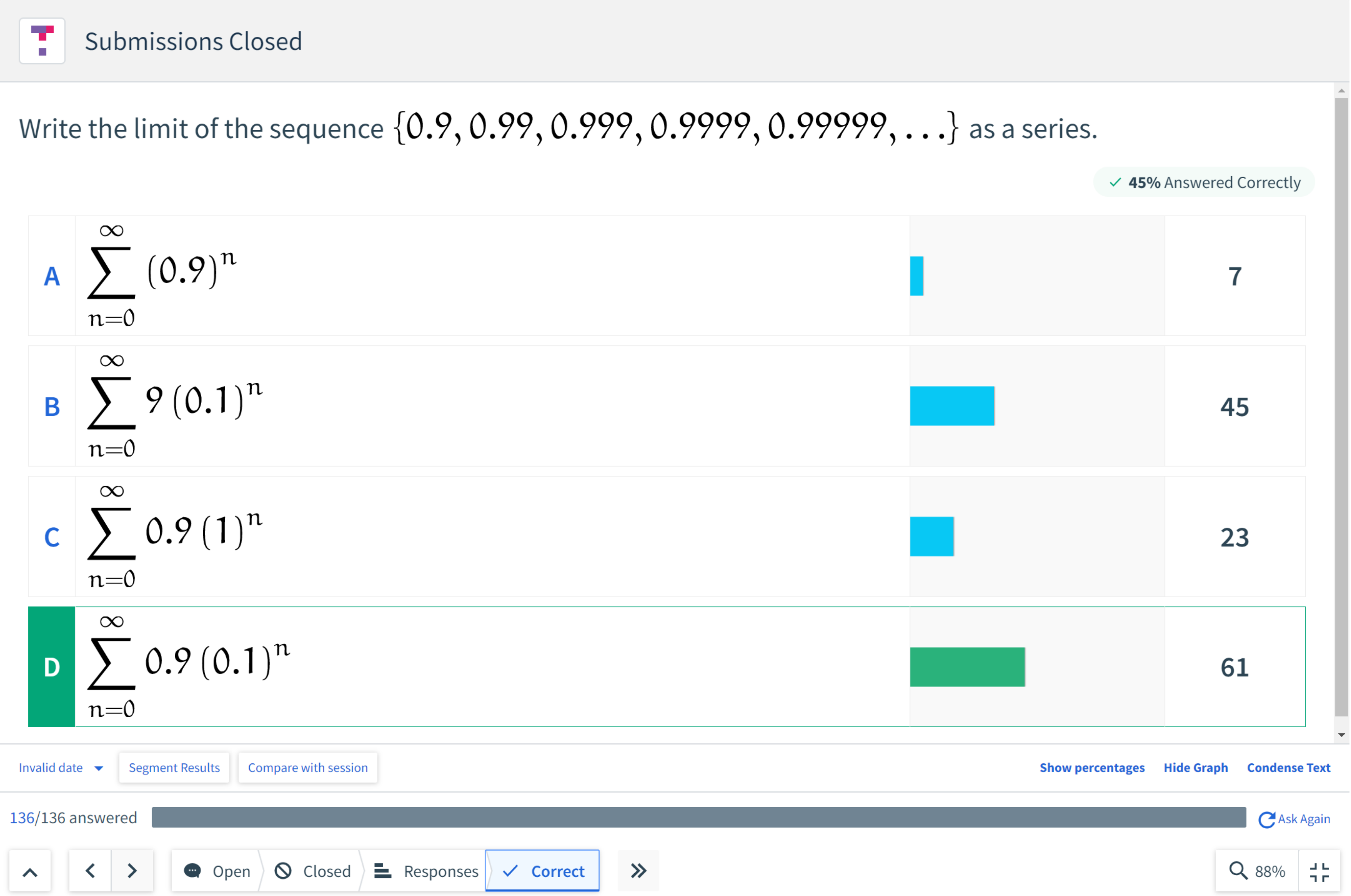
Submissions Closed

Write the limit of the sequence $\{1, 1.1, 1.11, 1.111, 1.1111, 1.11111, 1.111111, \ldots\}$ as a series.

✓ **60%** Answered Correctly

A	$\sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n$	83
B	$\sum_{n=0}^{\infty} (1.1)^n$	49
C	$\sum_{n=0}^{\infty} (1)^n$	6

138/138 answered



Submissions Closed

Write the limit of the sequence $\{0.9, 0.99, 0.999, 0.9999, 0.99999, \ldots\}$ as a series.

✓ **45%** Answered Correctly

A	$\sum_{n=0}^{\infty} (0.9)^n$	7
B	$\sum_{n=0}^{\infty} 9\,(0.1)^n$	45
C	$\sum_{n=0}^{\infty} 0.9\,(1)^n$	23
D	$\sum_{n=0}^{\infty} 0.9\,(0.1)^n$	61

136/136 answered

Welcome to MAT136 LEC0501 (Assaf)

How similar are other online classes to this one? What's different?

Answer in the chat.

S9.3 – Series & Convergence

Assaf Bar-Natan

> “One thing I can tell you is
> You got to be free
> Come together, right now
> Over me”
>
> –“Come Together”, The Beatles

March 18, 2020

Fill in the Blanks

- We say that a series $\sum_{k=1}^{\infty} a_k$ c______ if the p______ s______, $\sum_{k=1}^{n} a_k$ converge
- We define the value of a series as the ______ of the partial sums.
- The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ______ if $p \leq 1$, by the ______-test

Partial Sums and Convergence

When we write:

$$\sum_{k=1}^{\infty} a_k$$

what we really mean is:

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

Partial Sums and Convergence

When we write:

$$\sum_{k=1}^{\infty} a_k$$

what we really mean is:

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k$$

If we write $S_n = \sum_{k=1}^{n} a_k$, and call it the **partial sum**, then the series $\sum_{k=1}^{\infty} a_k$ converges when $\lim_{n\to\infty} S_n$ converges.

Partial Sums – Geometric Series

Consider the series:

$$1 + 0.2 + (0.2)^2 + (0.2)^3 + \cdots$$

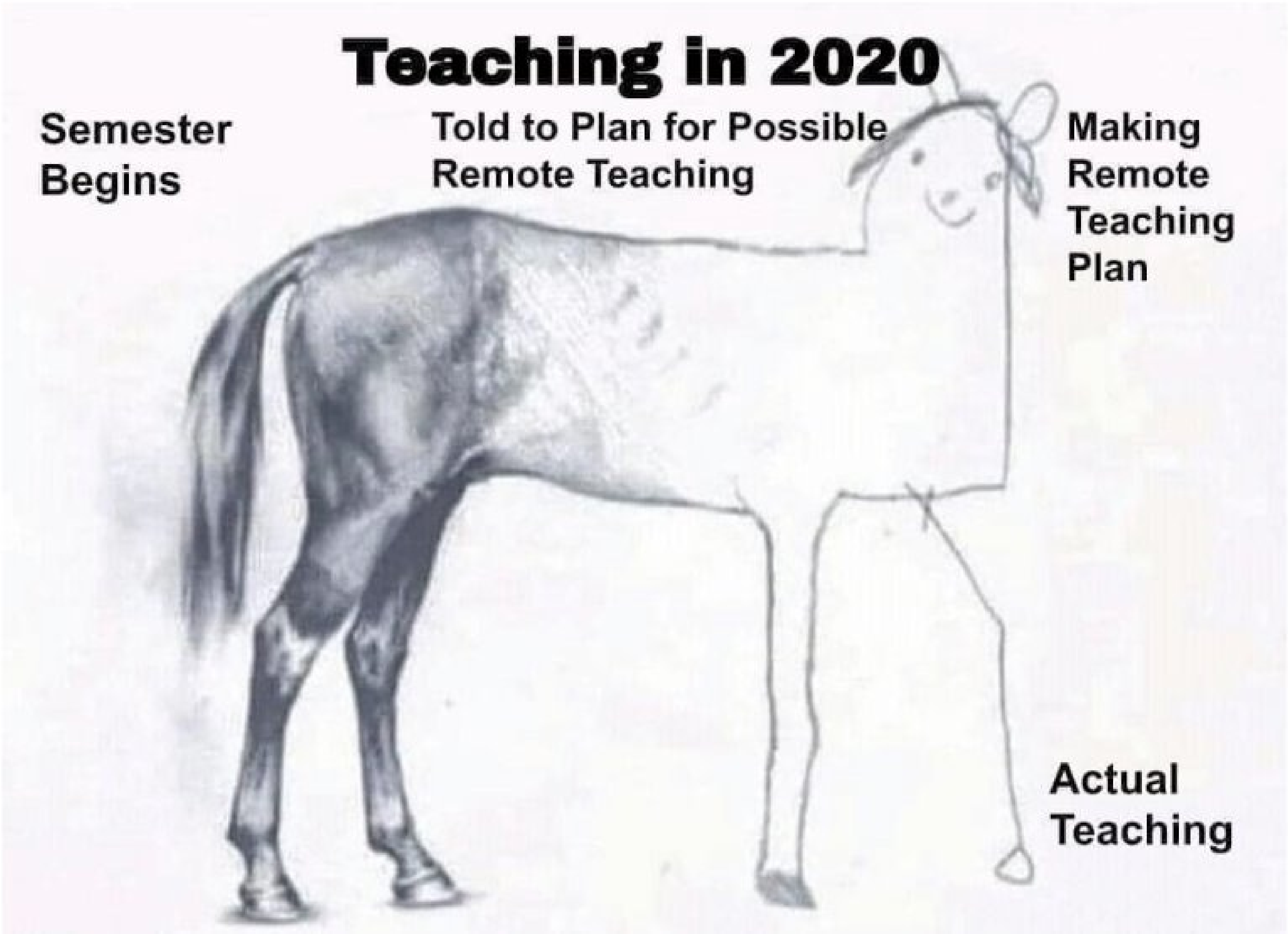
- What is a_k?
- What is S_n?
- What is $\lim_{n\to\infty} S_n$?
- What integral do we use in the integral test?

Partial Sums – Geometric Series

Consider the series:

$$1 + 0.2 + (0.2)^2 + (0.2)^3 + \cdots$$

- What is a_k?$a_k = (0.2)^k$
- What is S_n?$S_n = \frac{1-(0.2)^{n+1}}{0.8}$
- What is $\lim_{n\to\infty} S_n$?$\frac{1}{0.8}$
- What integral do we use in the integral test? **We use the integrand** $(0.2)^x$


Teaching in 2020
Semester Begins
Told to Plan for Possible Remote Teaching
Making Remote Teaching Plan
Actual Teaching

The Integral Idea

Suppose $a_n = f(n)$, where $f(x)$ is decreasing and positive.

- If $\int_1^\infty f(x)dx$ diverges, then $\sum a_n$ diverges.
- If $\int_1^\infty f(x)dx$ converges, then $\sum a_n$ converges.

Q: Does the series:

$$e^4 - 0.2 + \pi + 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

converge?

The Integral Idea

Suppose $a_n = f(n)$, where $f(x)$ is decreasing and positive.

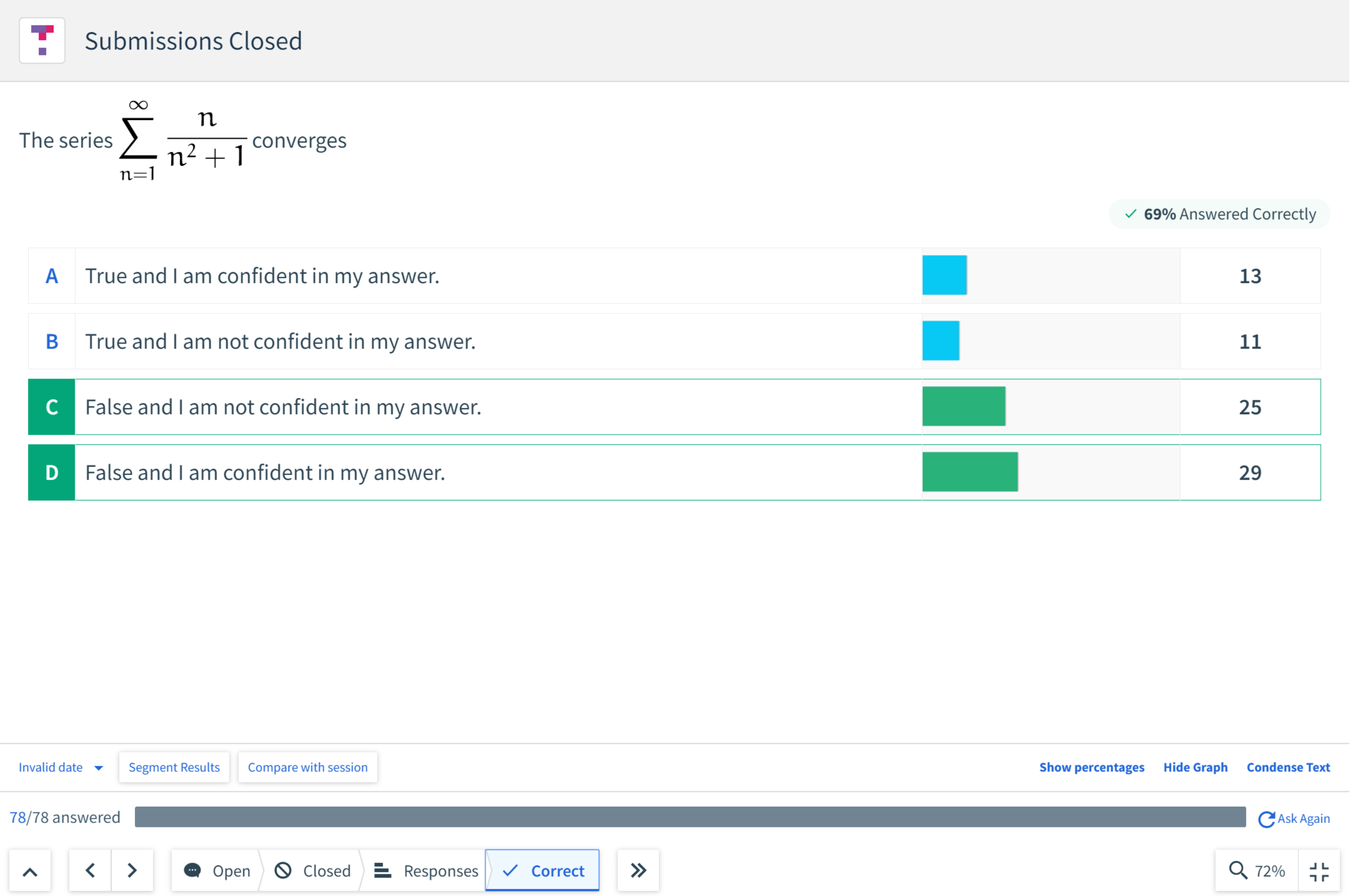
- If $\int_1^\infty f(x)dx$ diverges, then $\sum a_n$ diverges.
- If $\int_1^\infty f(x)dx$ converges, then $\sum a_n$ converges.

Q: Does the series:

$$e^4 - 0.2 + \pi + 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

converge?

A: Yes! We only care about the tail of the series, which converges by the integral test.



Submissions Closed

The series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ converges

✓ **69%** Answered Correctly

A	True and I am confident in my answer.	13
B	True and I am not confident in my answer.	11
C	False and I am not confident in my answer.	25
D	False and I am confident in my answer.	29

78/78 answered

Lexi's Series

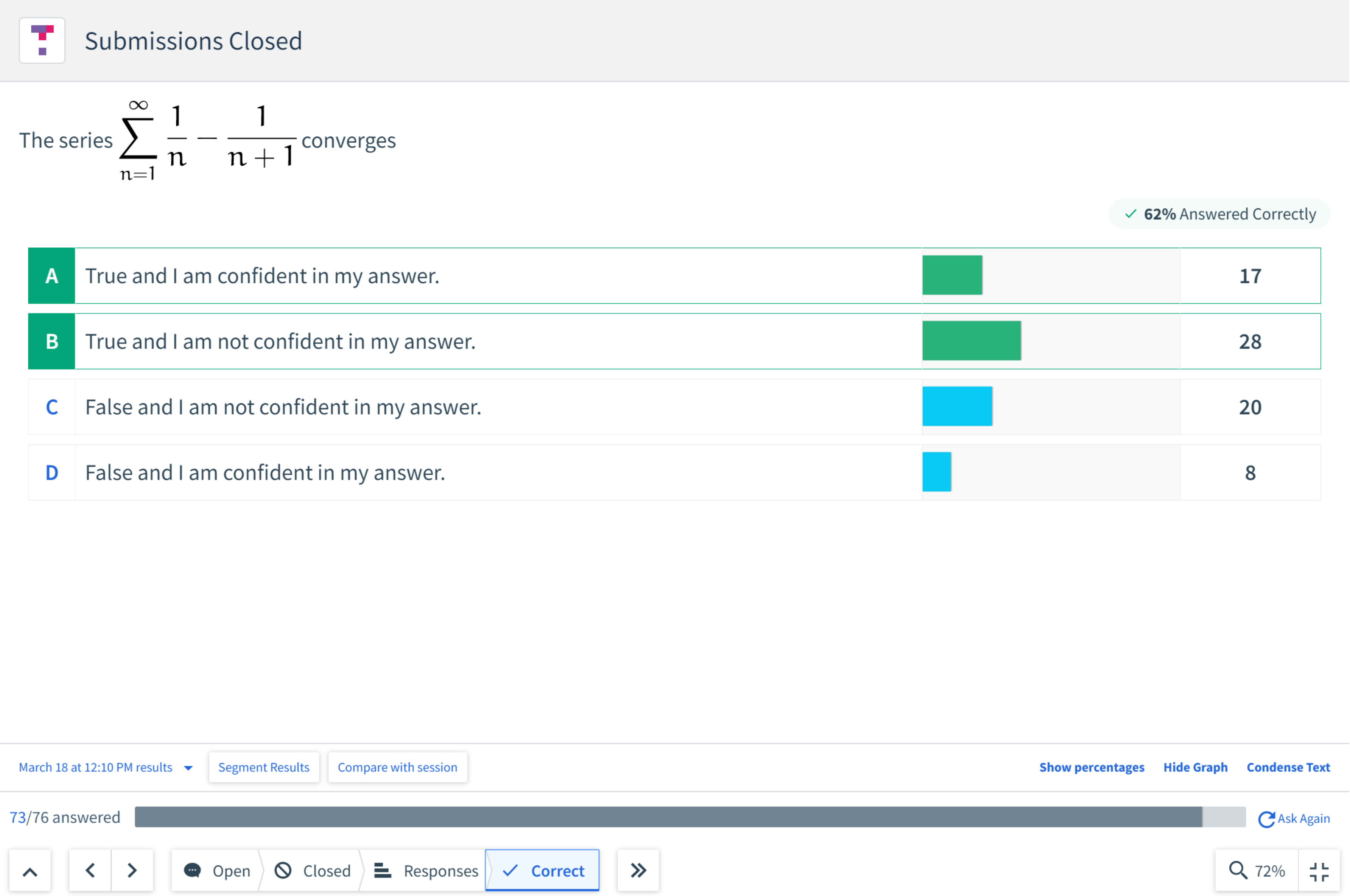
Lexi, the tail-less cat (she was born that way) is practicing her convergence properties. She writes:

" I want to see if the series $\sum(\frac{1}{n} - \frac{1}{n+1})$ converges. I'll split it up to get:

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+1}$$

The series on the right is the Harmonic series, which diverges, so the whole thing diverges."

Is Lexi's reasoning correct?



Submissions Closed

The series $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$ converges

✓ **62%** Answered Correctly

A	True and I am confident in my answer.	17
B	True and I am not confident in my answer.	28
C	False and I am not confident in my answer.	20
D	False and I am confident in my answer.	8

73/76 answered

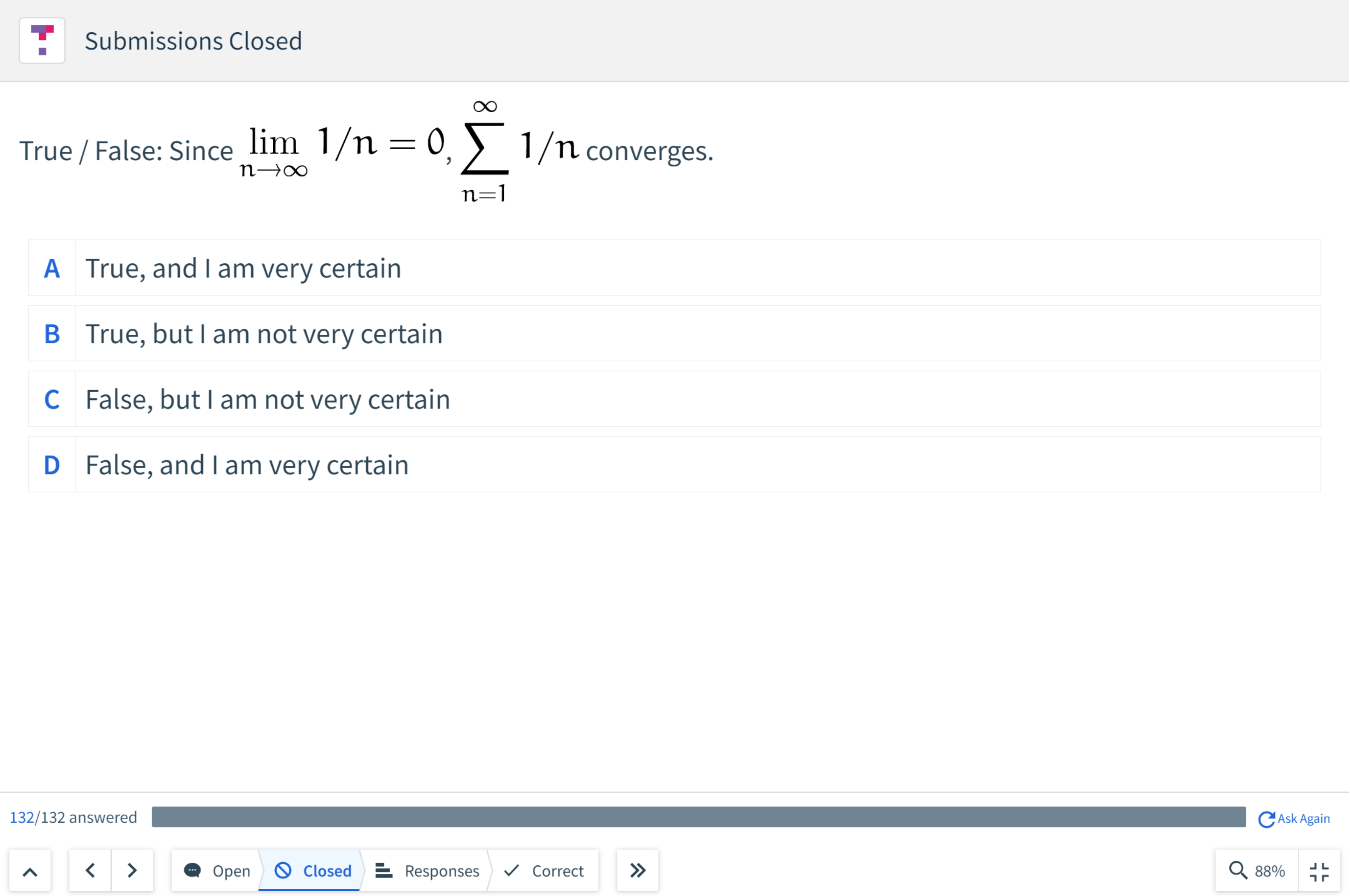
Takeaway

When all else fails, look at the partial sums!

Plans for the Future

For next time:

Do WeBWork 9.3 and actively read section 9.3



Submissions Closed

True / False: Since $\lim_{n\to\infty} 1/n = 0$, $\sum_{n=1}^{\infty} 1/n$ converges.

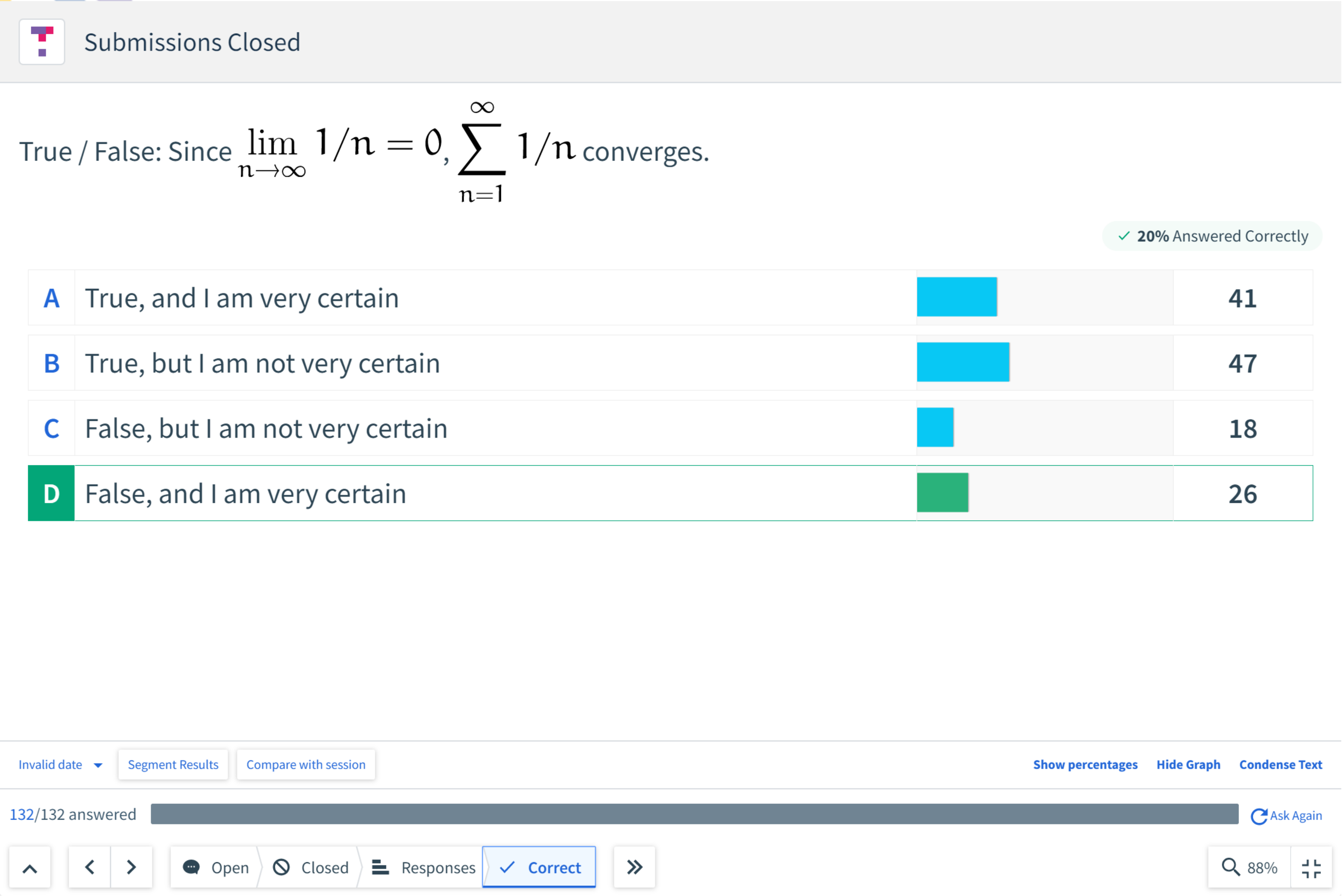
A True, and I am very certain

B True, but I am not very certain

C False, but I am not very certain

D False, and I am very certain

132/132 answered



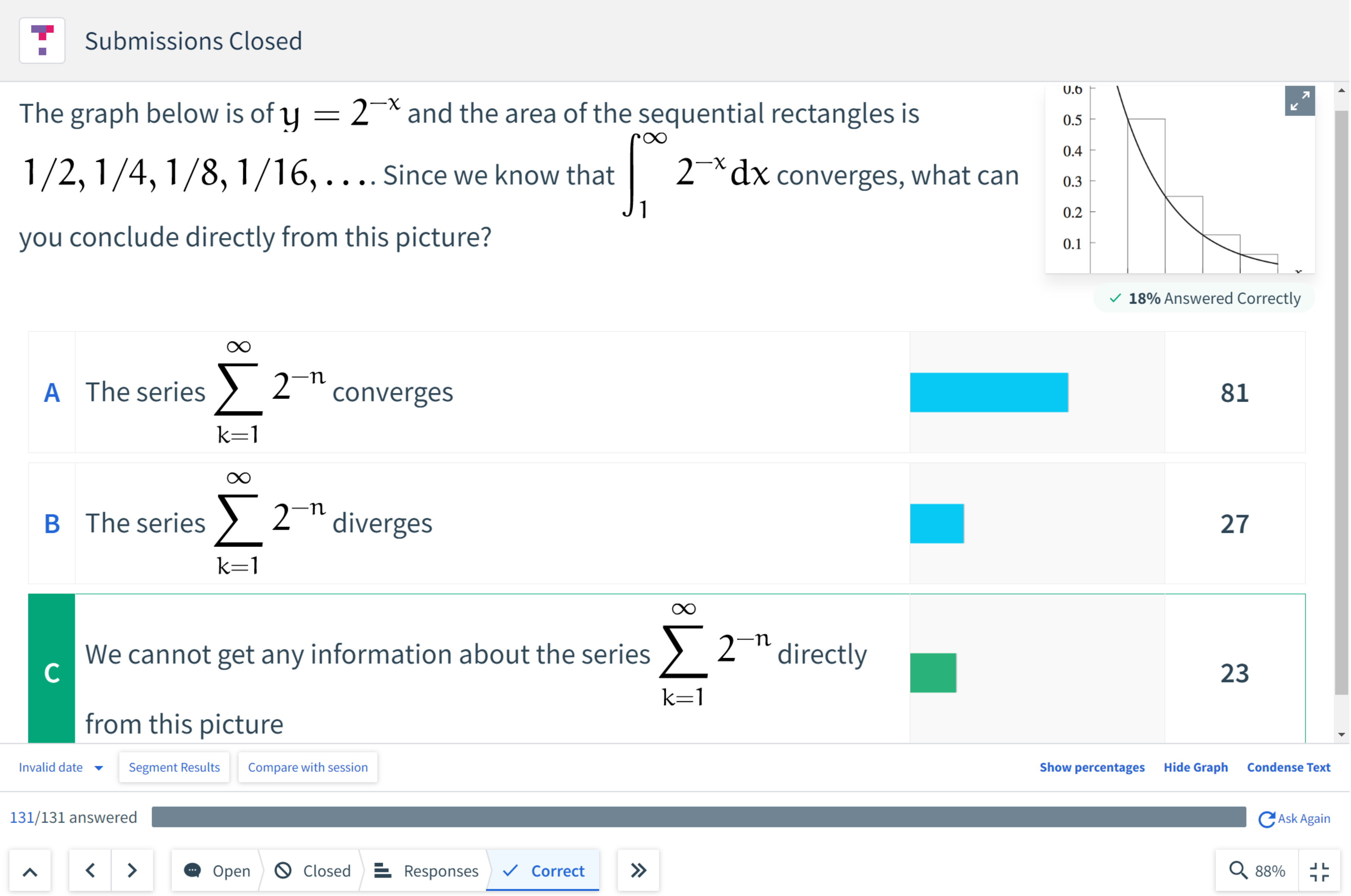
Submissions Closed

True / False: Since $\lim_{n\to\infty} 1/n = 0$, $\sum_{n=1}^{\infty} 1/n$ converges.

✓ **20%** Answered Correctly

A	True, and I am very certain	41
B	True, but I am not very certain	47
C	False, but I am not very certain	18
D	False, and I am very certain	26

132/132 answered



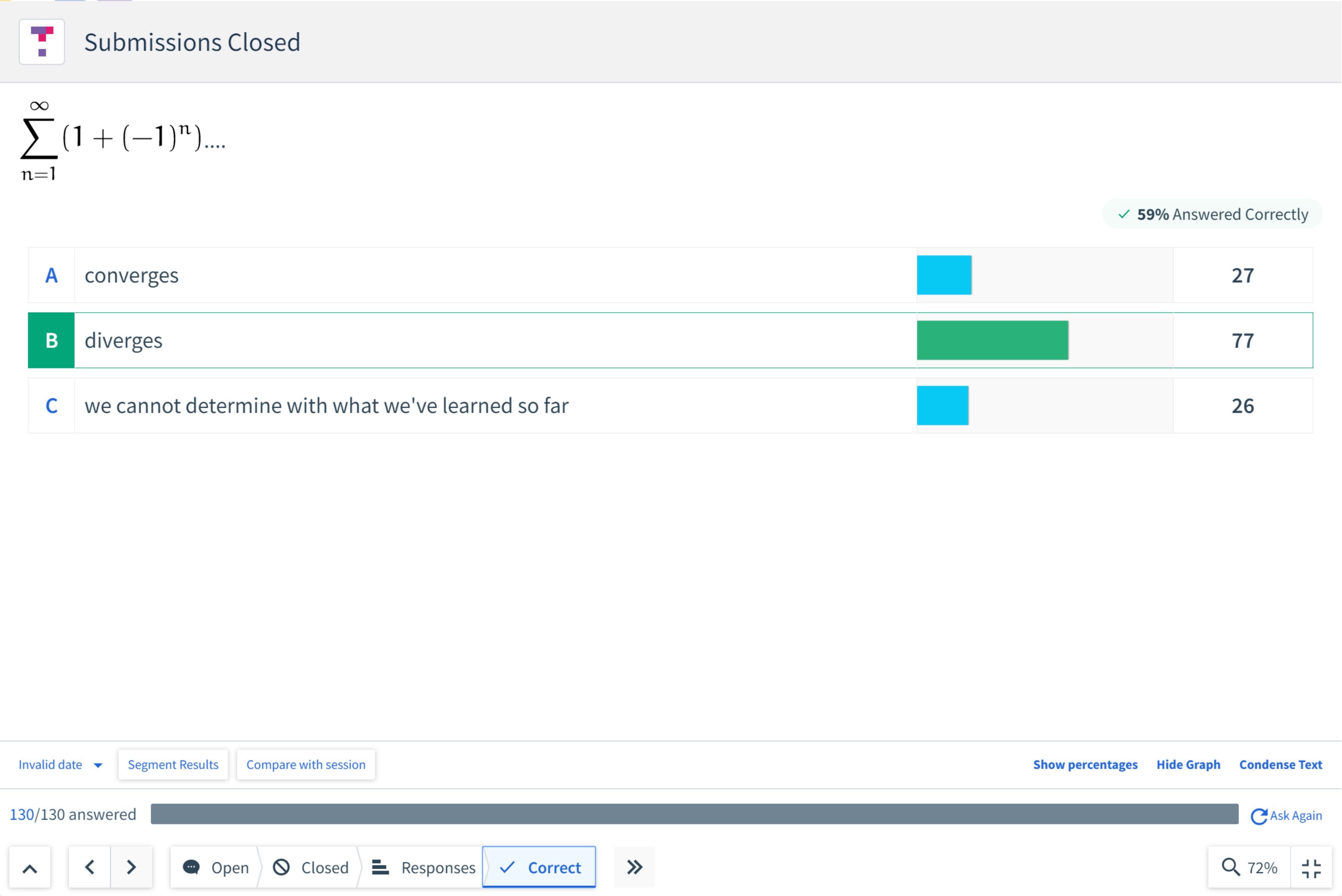
Submissions Closed

The graph below is of $y = 2^{-x}$ and the area of the sequential rectangles is $1/2, 1/4, 1/8, 1/16, \ldots$. Since we know that $\int_1^\infty 2^{-x}\,dx$ converges, what can you conclude directly from this picture?

✓ **18%** Answered Correctly

A	The series $\sum_{k=1}^{\infty} 2^{-n}$ converges	81
B	The series $\sum_{k=1}^{\infty} 2^{-n}$ diverges	27
C	We cannot get any information about the series $\sum_{k=1}^{\infty} 2^{-n}$ directly from this picture	23

131/131 answered



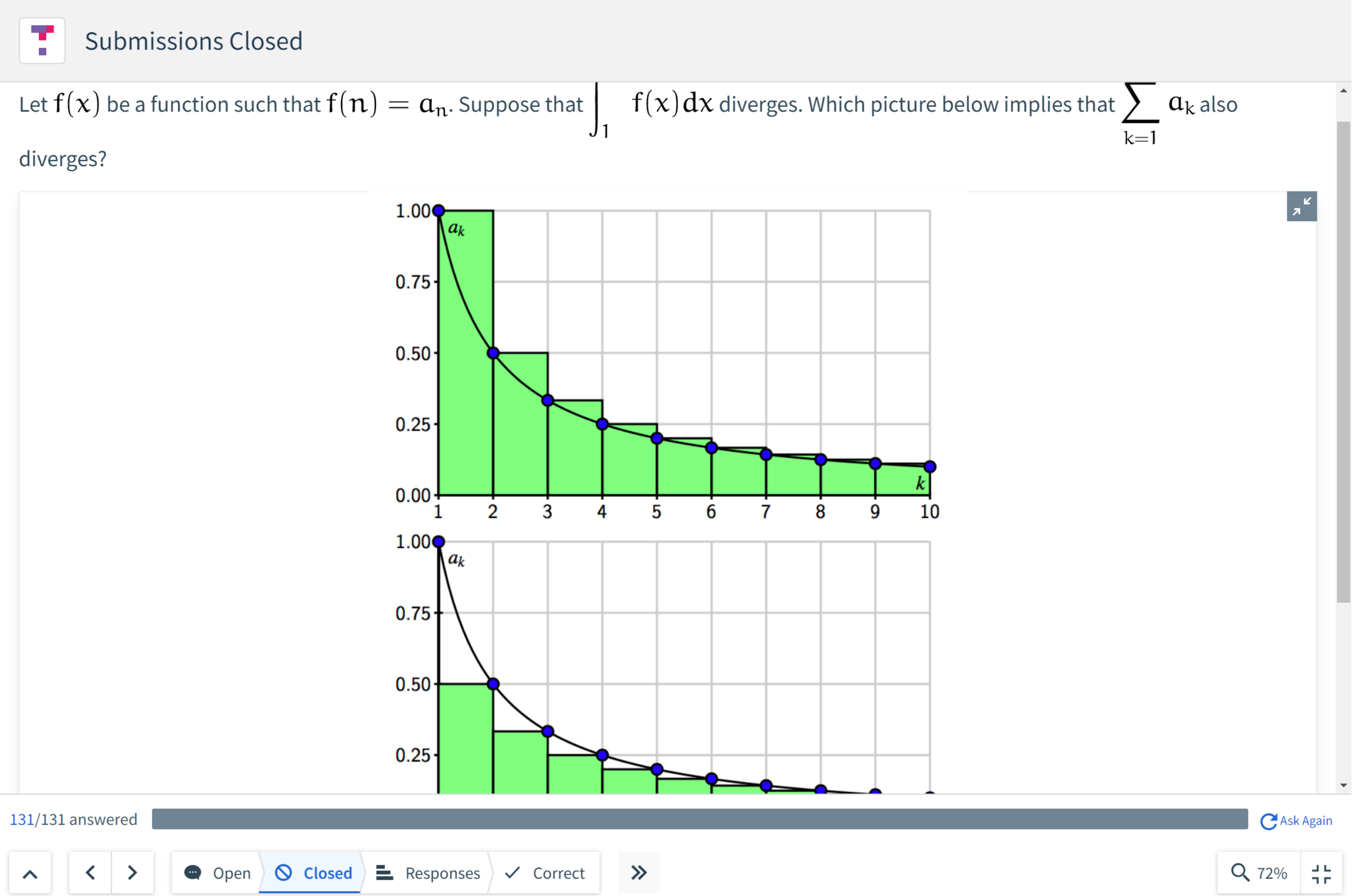
Submissions Closed

$$\sum_{n=1}^{\infty}(1+(-1)^n)....$$

✓ **59%** Answered Correctly

A	converges	27
B	diverges	77
C	we cannot determine with what we've learned so far	26

130/130 answered



Submissions Closed

Let $f(x)$ be a function such that $f(n) = a_n$. Suppose that $\int_1^{\infty} f(x)\,dx$ diverges. Which picture below implies that $\sum_{k=1}^{\infty} a_k$ also diverges?

Welcome to MAT136 LEC0501 (Assaf)

Online final exam is happening April 9 between 1:30pm and 5:30pm EST. See main course page for details (none posted yet).

S9.3 – Series & The Ratio Test

Assaf Bar-Natan

“Life is a series of hellos and goodbyes
I’m afraid it’s time for goodbye again
Say goodbye to Hollywood
Say goodbye my baby”

–“Say Goodbye to Hollywood”, Billy Joel

March 20, 2020

Fill in the Blanks

We have a series, $\sum a_n$.

If the ratios $\frac{a_{n+1}}{a_n}$ approach L, and $L < 1$, then the series $\sum a_n$ grows ______ a geometric series with factor ______, which is ___ $(<, >, =)$ 1. Hence, the series ______.

Fill in the Blanks

We have a series, $\sum a_n$.

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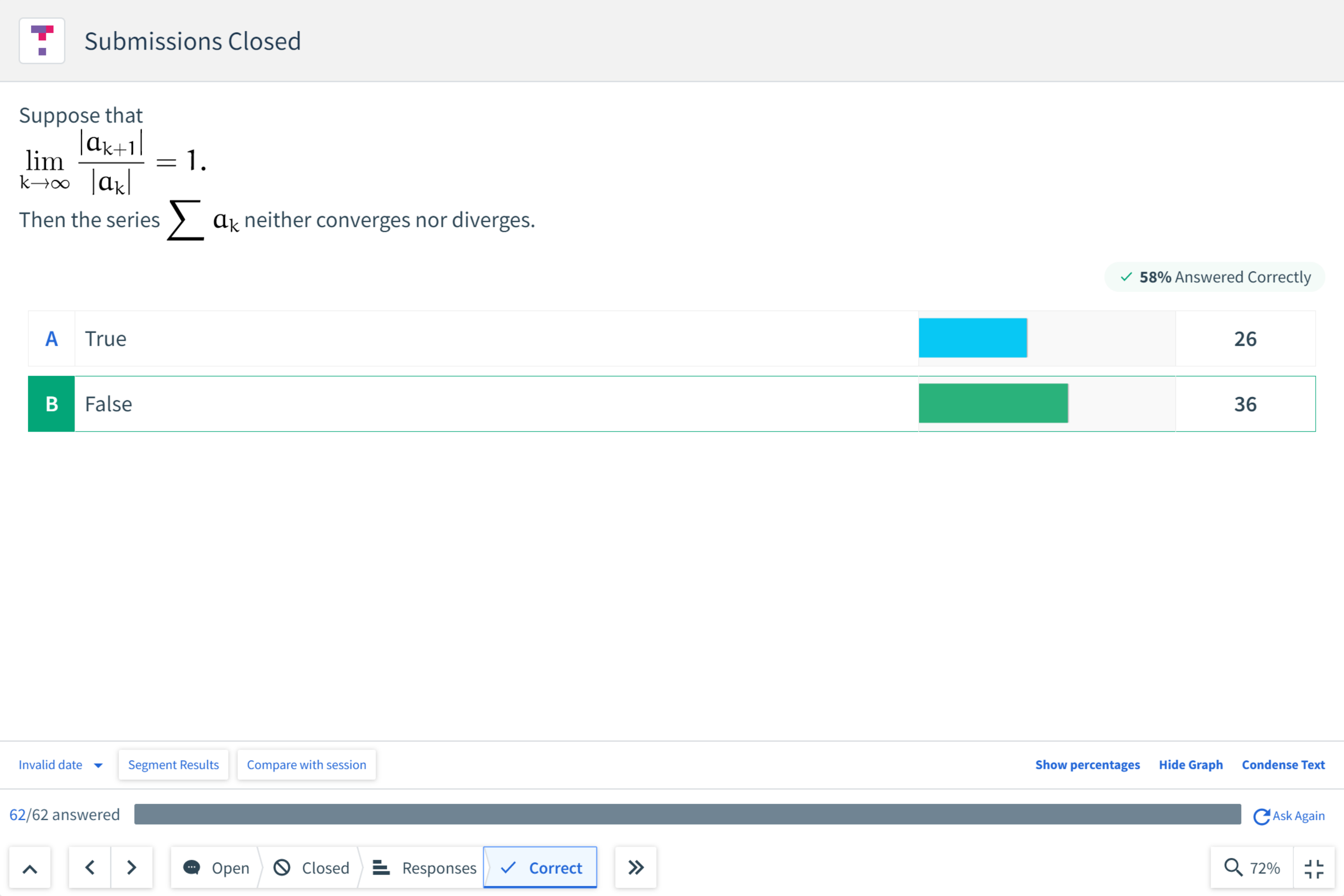

Submissions Closed

Suppose that

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = 1.$$

Then the series $\sum a_k$ neither converges nor diverges.

✓ **58%** Answered Correctly

A	True	26
B	False	36

62/62 answered

Why the Ratio Test

Let's assume that for any sufficiently large n, $\frac{a_{n+1}}{a_n} \approx L$. Then:

$$a_{k+1} \approx La_k$$
$$a_{k+2} \approx La_{k+1} \approx L^2 a_k$$

Continuing in this manner, we get:

Why the Ratio Test

Let's assume that for any sufficiently large n, $\frac{a_{n+1}}{a_n} \approx L$. Then:

$$\begin{aligned} a_{k+1} &\approx La_k \\ a_{k+2} &\approx La_{k+1} \approx L^2 a_k \end{aligned}$$

Continuing in this manner, we get:

$$a_k + a_{k+1} + a_{k+2} + \cdots \approx a_k \left(1 + L + L^2 + L^3 + \cdots\right)$$

If $L < 1$, then the right hand side is a geometric series, which converges!

Why the Ratio Test

Let's assume that for any sufficiently large n, $\frac{a_{n+1}}{a_n} \approx L$. Then:

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Continuing in this manner, we get:

$$a_k + a_{k+1} + a_{k+2} + \cdots \approx a_k \left(1 + L + L^2 + L^3 + \cdots\right)$$

If $L < 1$, then the right hand side is a geometric series, which converges!

If $L = 0$, replace all $\approx$ with $<$, and replace L with $\frac{1}{2}$

Takeaway

The ratio test measures how much a series looks like a geometric series. If the limit of the ratio $\frac{a_{n+1}}{a_n}$ is < 1, the series converges, and if it is > 1, it diverges. Just like a geometric series!

Obie and Limits

Obie (the bully cat) says:

"In examining the series:

$$\sum_{n=0}^{\infty} \frac{n}{(1.05)^n} = 0.95 + 1.181 + 2.59 + 3.29 + \cdots$$

I notice that the terms are getting larger, so $L > 1$. Thus, by the ratio test, this series diverges."

Is Obie correct?

Obie and Limits

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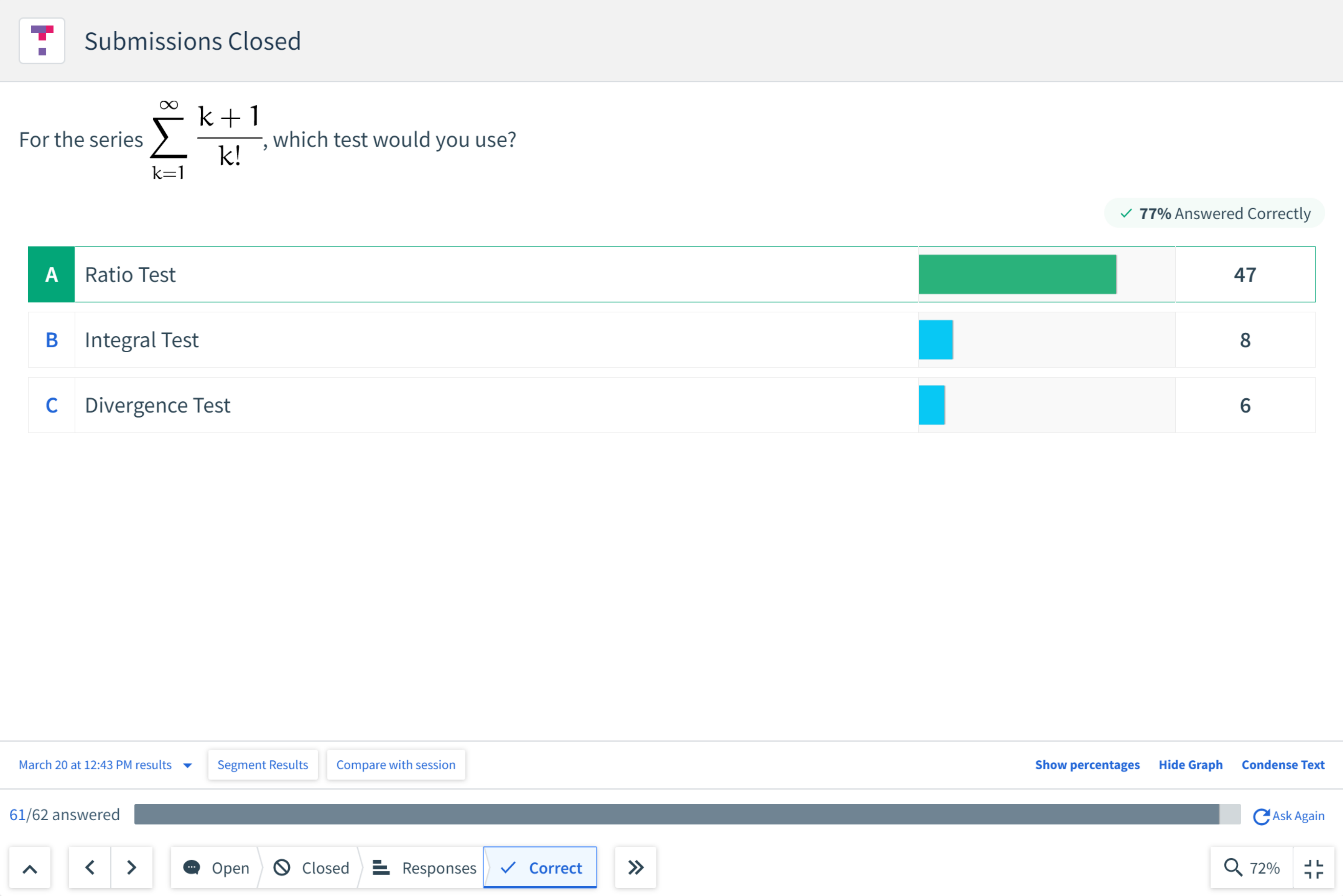
I notice that the terms are getting larger, so $L > 1$. Thus, by the ratio test, this series diverges.”

Is Obie correct?
If this still confuses you, write a star in your notebook to go over this later

Takeaway

We really do need to take a limit in the ratio test. We don't care what the series' terms do early. The limit captures this "end behaviour"

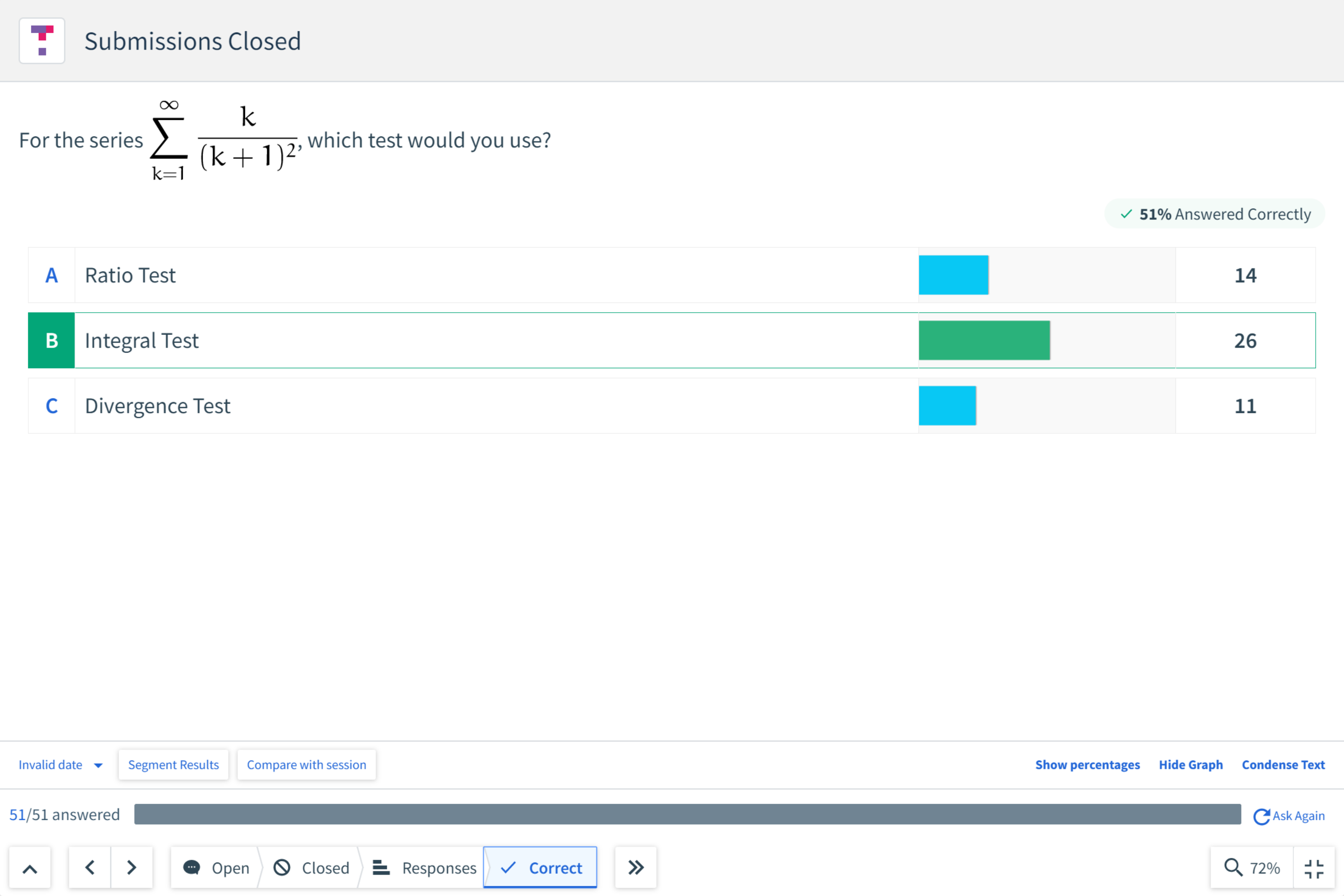


Submissions Closed

For the series $\sum_{k=1}^{\infty} \frac{k+1}{k!}$, which test would you use?

✓ 77% Answered Correctly

A	Ratio Test	47
B	Integral Test	8
C	Divergence Test	6



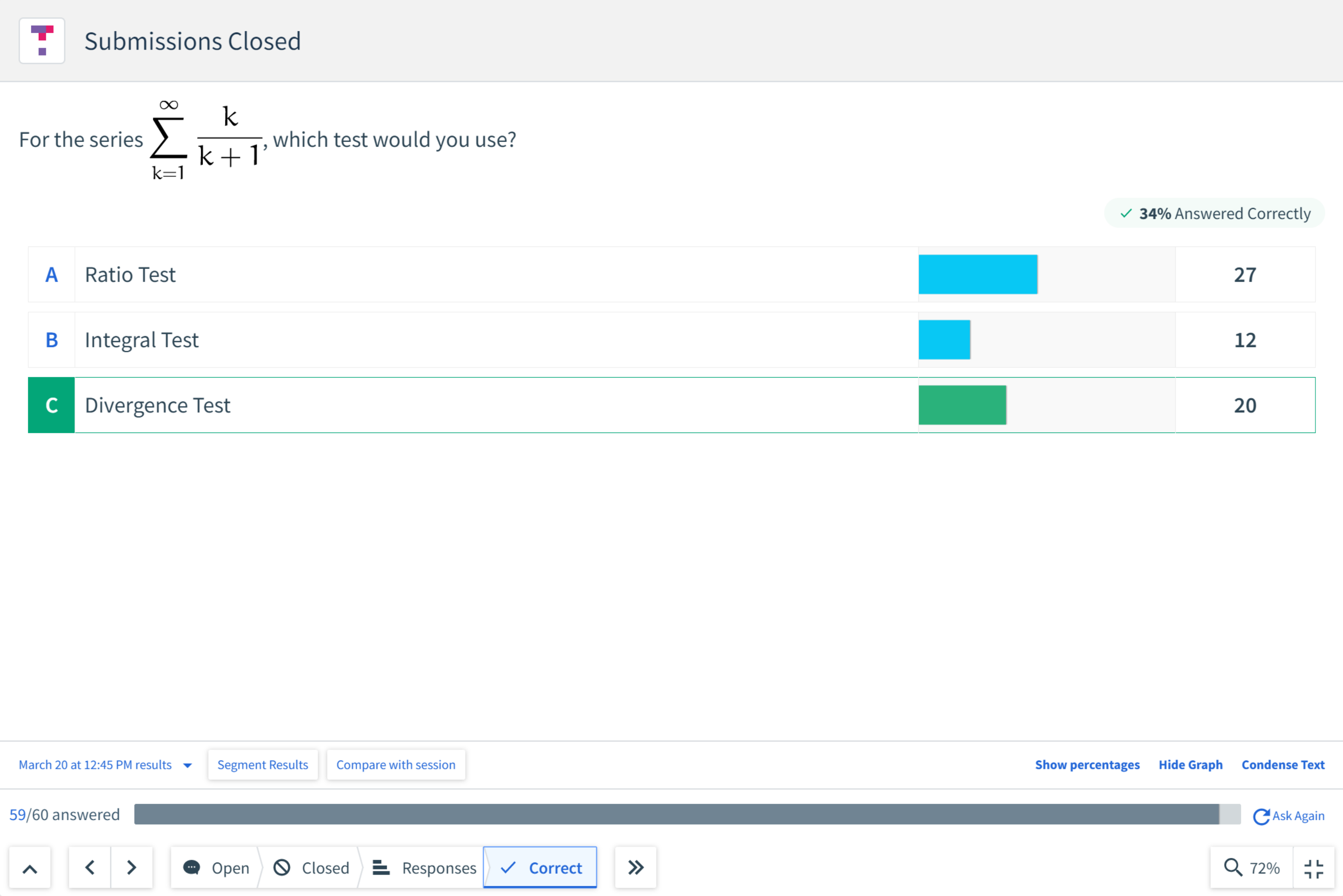
Submissions Closed

For the series $\sum_{k=1}^{\infty} \frac{k}{(k+1)^2}$, which test would you use?

✓ **51%** Answered Correctly

A	Ratio Test	14
B	Integral Test	26
C	Divergence Test	11

51/51 answered



Submissions Closed

For the series $\sum_{k=1}^{\infty} \frac{k}{k+1}$, which test would you use?

✓ **34%** Answered Correctly

A	Ratio Test	27
B	Integral Test	12
C	Divergence Test	20

March 20 at 12:45 PM results

59/60 answered

Inconclusive Test Results

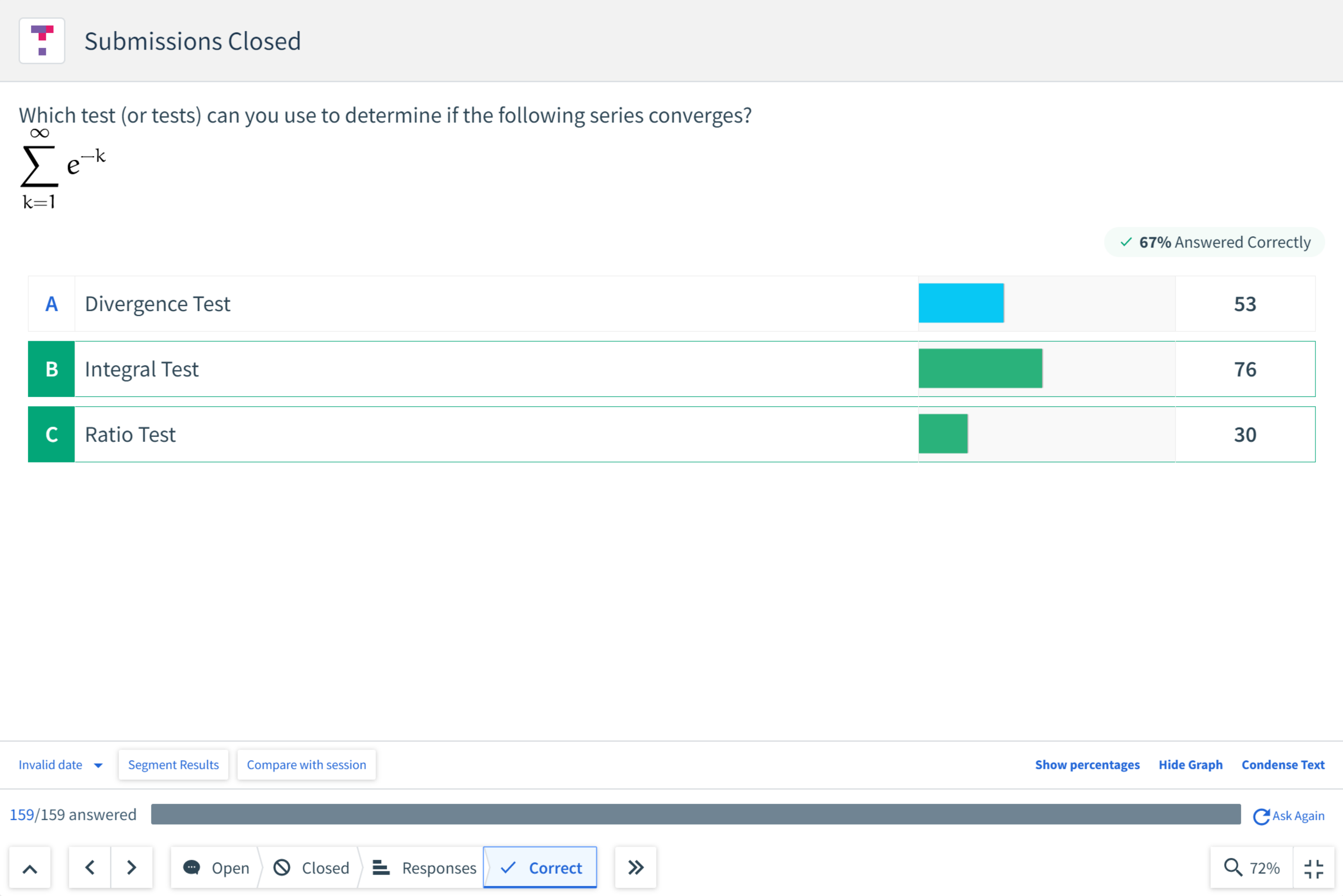
Write a series which diverges, but for which the ratio test gives a limit of 1.

Challenge: write a series which converges, but for which the ratio test gives a limit of 1.

Plans for the Future

For next time:

Do WeBWork 9.5, actively read section 9.5, and watch the videos!



Submissions Closed

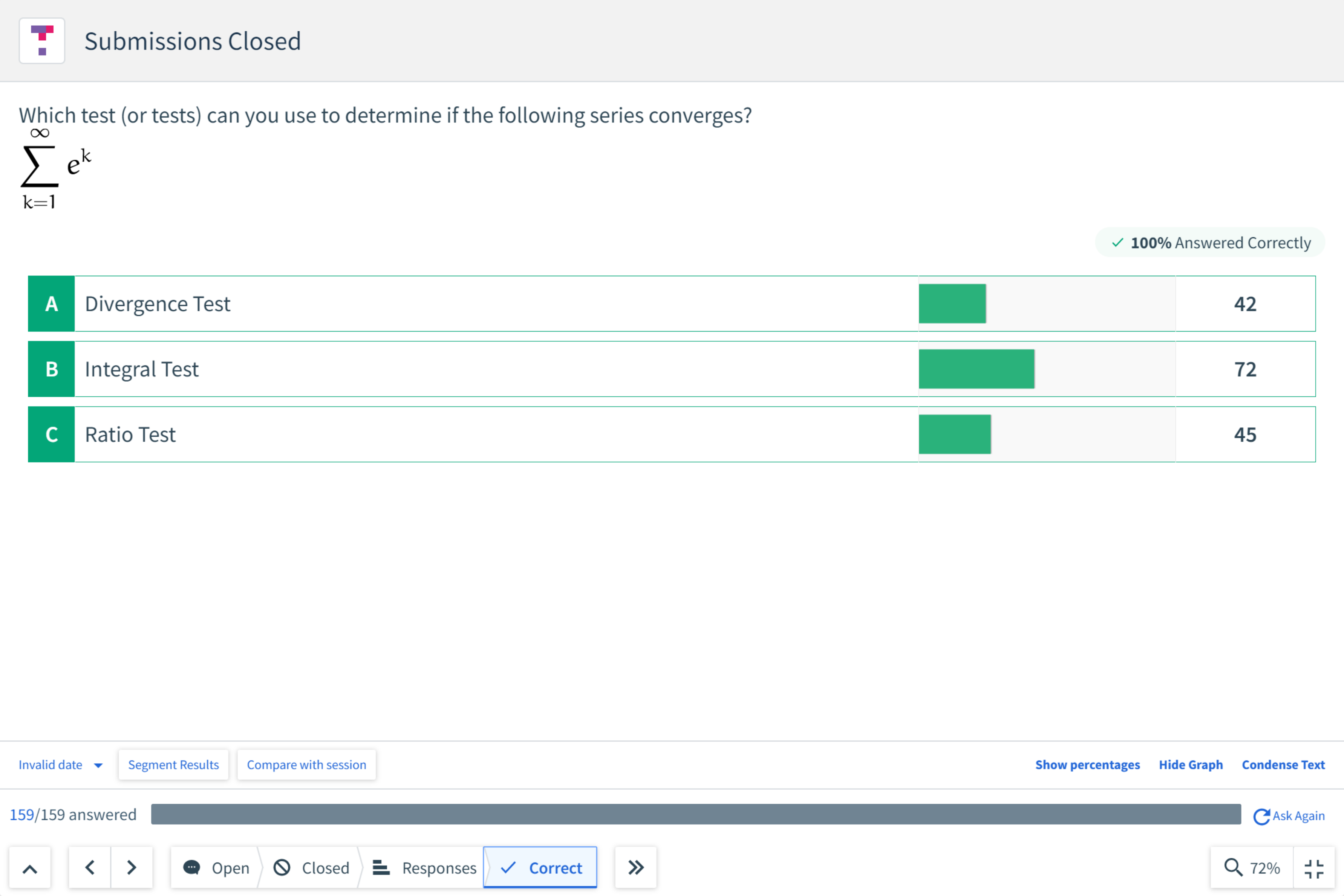
Which test (or tests) can you use to determine if the following series converges?

$$\sum_{k=1}^{\infty} e^{-k}$$

✓ **67%** Answered Correctly

A	Divergence Test	53
B	Integral Test	76
C	Ratio Test	30

159/159 answered



Submissions Closed

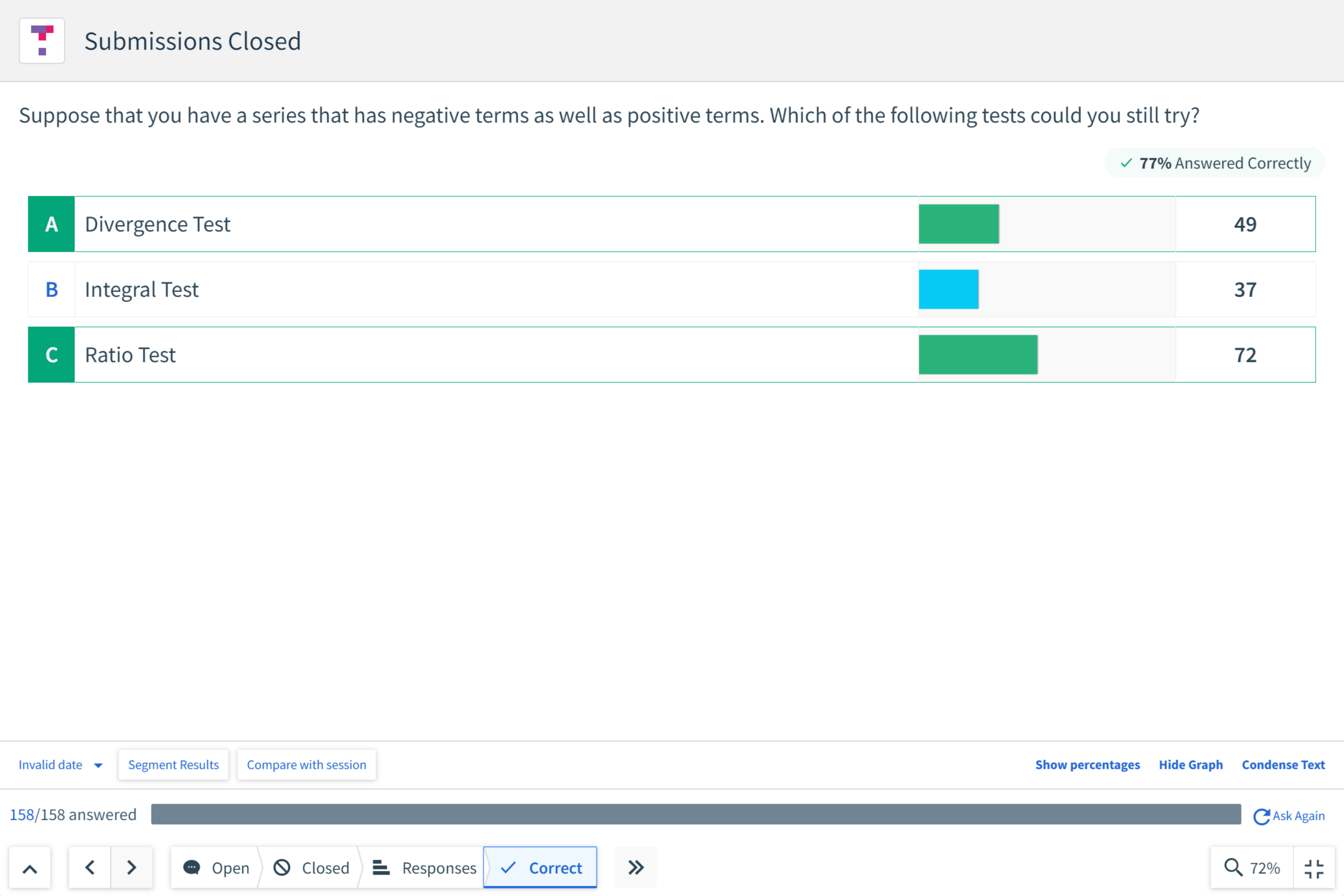
Which test (or tests) can you use to determine if the following series converges?

$$\sum_{k=1}^{\infty} e^k$$

✓ **100%** Answered Correctly

A	Divergence Test	42
B	Integral Test	72
C	Ratio Test	45

159/159 answered



Submissions Closed

Suppose that you have a series that has negative terms as well as positive terms. Which of the following tests could you still try?

✓ **77%** Answered Correctly

A	Divergence Test	49
B	Integral Test	37
C	Ratio Test	72

158/158 answered

Welcome to MAT136 LEC0501 (Assaf)

No more in-class TopHats. The software isn't working and I'm tired of fighting it.

S9.5 – Power Series & Convergence Interval

Assaf Bar-Natan

> "You and me got staying power yeah
> You and me we got staying power
> Staying power (I got it I got it)"
>
> –"Staying Power", Queen

March 23, 2020

Using the Ratio Test on a Power Series

We are given the power series:

$$\sum_{n=1}^{\infty} C_n(x-a)^n$$

To check for convergence, apply the **ratio test**:

$$\lim_{n\to\infty}\left|\frac{C_{n+1}(x-a)^{n+1}}{C_n(x-a)^n}\right| = \lim_{n\to\infty}|x-a|\left|\frac{C_{n+1}}{C_n}\right| = |x-a|\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right|$$

The series $\sum C_n(x-a)^n$ converges when the above is less than 1.

Using the Ratio Test on a Power Series

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$$\sum_{n=1}^{\infty} C_n(x-a)^n$$

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The series $\sum C_n(x-a)^n$ converges when the above is less than 1.
Q: If $\lim_{n\to\infty}|C_{n+1}/C_n| = 3$, what is the radius of convergence?

Using the Ratio Test on a Power Series

We are given the power series:

$$\sum_{n=1}^{\infty} C_n(x-a)^n$$

To check for convergence, apply the **ratio test**:

$$\lim_{n\to\infty}\left|\frac{C_{n+1}(x-a)^{n+1}}{C_n(x-a)^n}\right| = \lim_{n\to\infty}|x-a|\left|\frac{C_{n+1}}{C_n}\right| = |x-a|\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right|$$

The series $\sum C_n(x-a)^n$ converges when the above is less than 1.
Q: If $\lim_{n\to\infty}|C_{n+1}/C_n| = 3$, what is the radius of convergence?
A: We want $3|x-a| < 1$, so $|x-a| < \frac{1}{3}$, and this is the radius of convergence.

Variables, Indices, and Parameters

Consider the following power series:

$$\sum_{n=1}^{\infty} \frac{c^{n/2}}{n}(x-a)^n$$

Variables, Indices, and Parameters

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$$\sum_{n=1}^{\infty} \frac{c^{n/2}}{n}(x-a)^n$$

- What are the **variables**?
- What are the **parameters**?
- What plays the role of the **index**?

Variables, Indices, and Parameters

Consider the following power series:

$$\sum_{n=1}^{\infty} \frac{c^{n/2}}{n}(x-a)^n$$

- What are the **variables**?x
- What are the **parameters**?c and a
- What plays the role of the **index**?n
- What is the radius of convergence of this power series?

Variables, Indices, and Parameters

Consider the following power series:

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- What are the **variables**?x
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- What plays the role of the **index**?n
- What is the radius of convergence of this power series?
- What is the interval of convergence of this power series?

Variables, Indices, and Parameters

Consider the following power series:

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What is the radius of convergence of this power series?

Variables, Indices, and Parameters

Consider the following power series:

$$\sum_{n=1}^{\infty} \frac{c^{n/2}}{n}(x-a)^n$$

What is the radius of convergence of this power series?

We compute:

$$\lim_{n\to\infty} \frac{c^{(n+1)/2}}{c^{n/2}} \frac{n}{n+1} = \lim_{n\to\infty} \frac{n}{n+1} c^{1/2} = \sqrt{c}$$

So the radius of convergence is $\frac{1}{\sqrt{c}}$.

What is the interval of convergence of this power series?

Variables, Indices, and Parameters

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So the radius of convergence is $\frac{1}{\sqrt{c}}$.

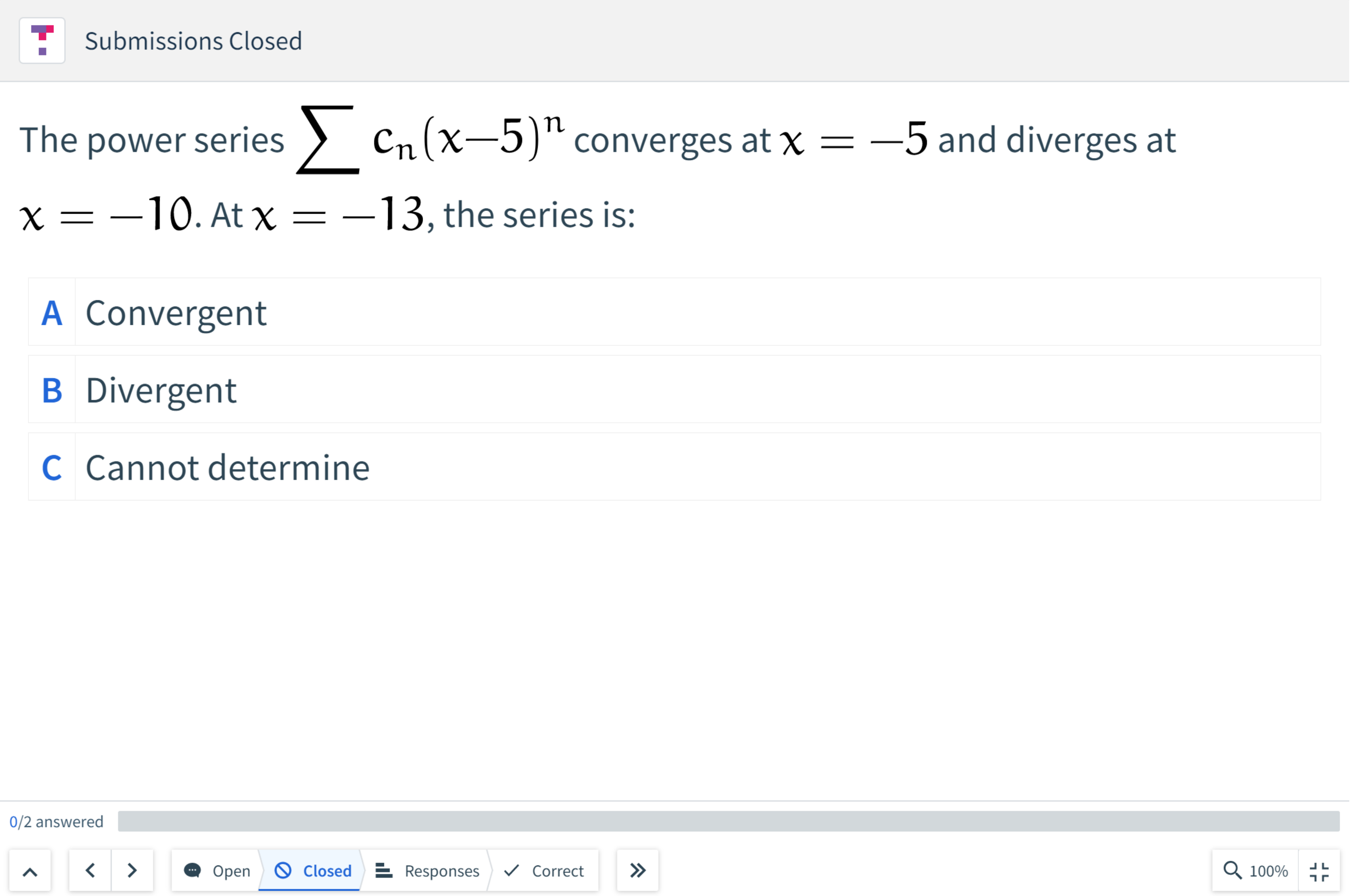
What is the interval of convergence of this power series?

The power series is centered at $x = a$, so it will converge for

$$a - \frac{1}{\sqrt{c}} < x < a + \frac{1}{\sqrt{c}}$$

Takeaway

In general, for $\sum c_n(x-a)^n$, the interval of convergence is centered at a.

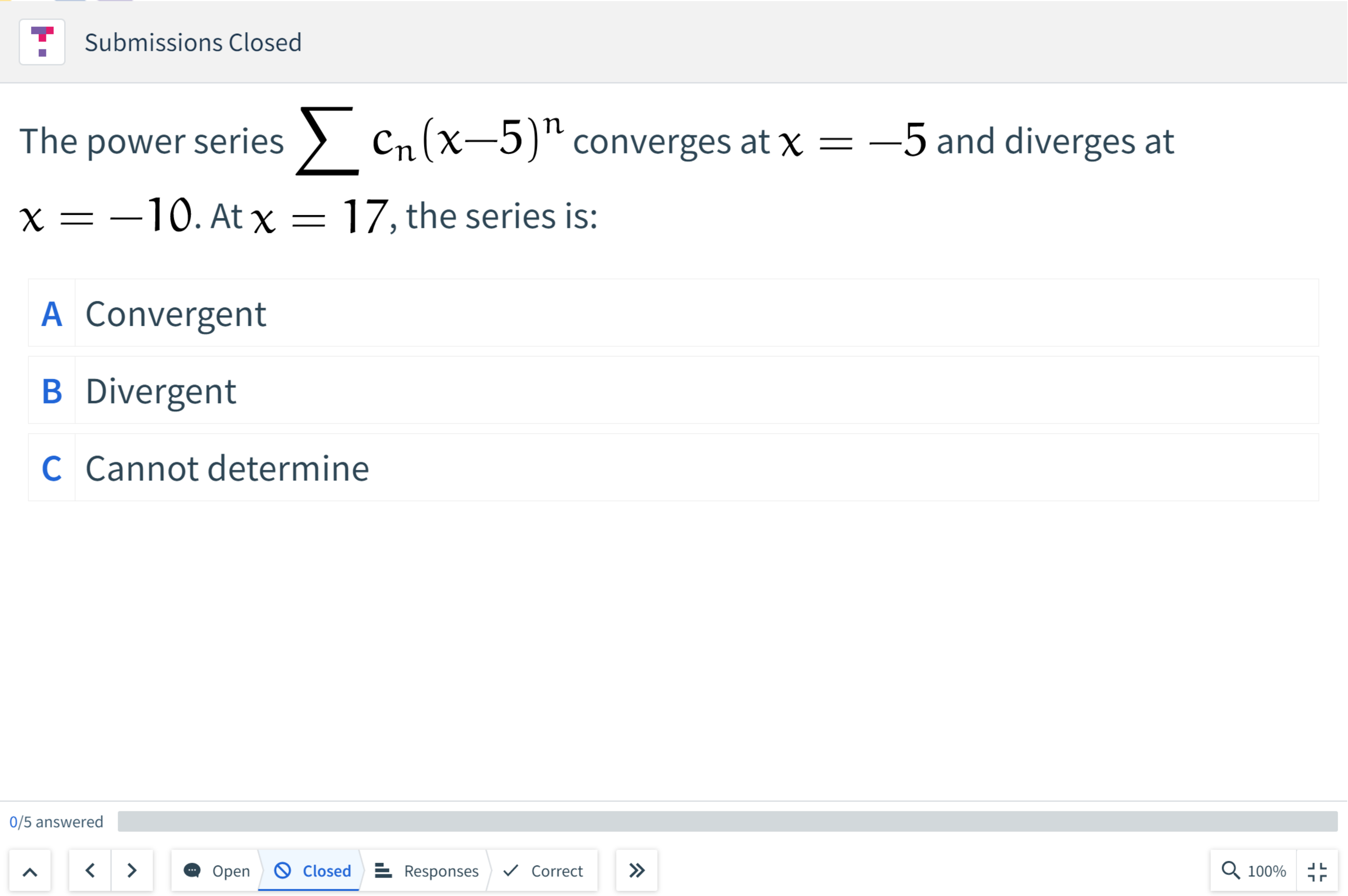


The power series $\sum c_n(x-5)^n$ converges at $x = -5$ and diverges at $x = -10$. At $x = -13$, the series is:

A Convergent

B Divergent

C Cannot determine



Submissions Closed

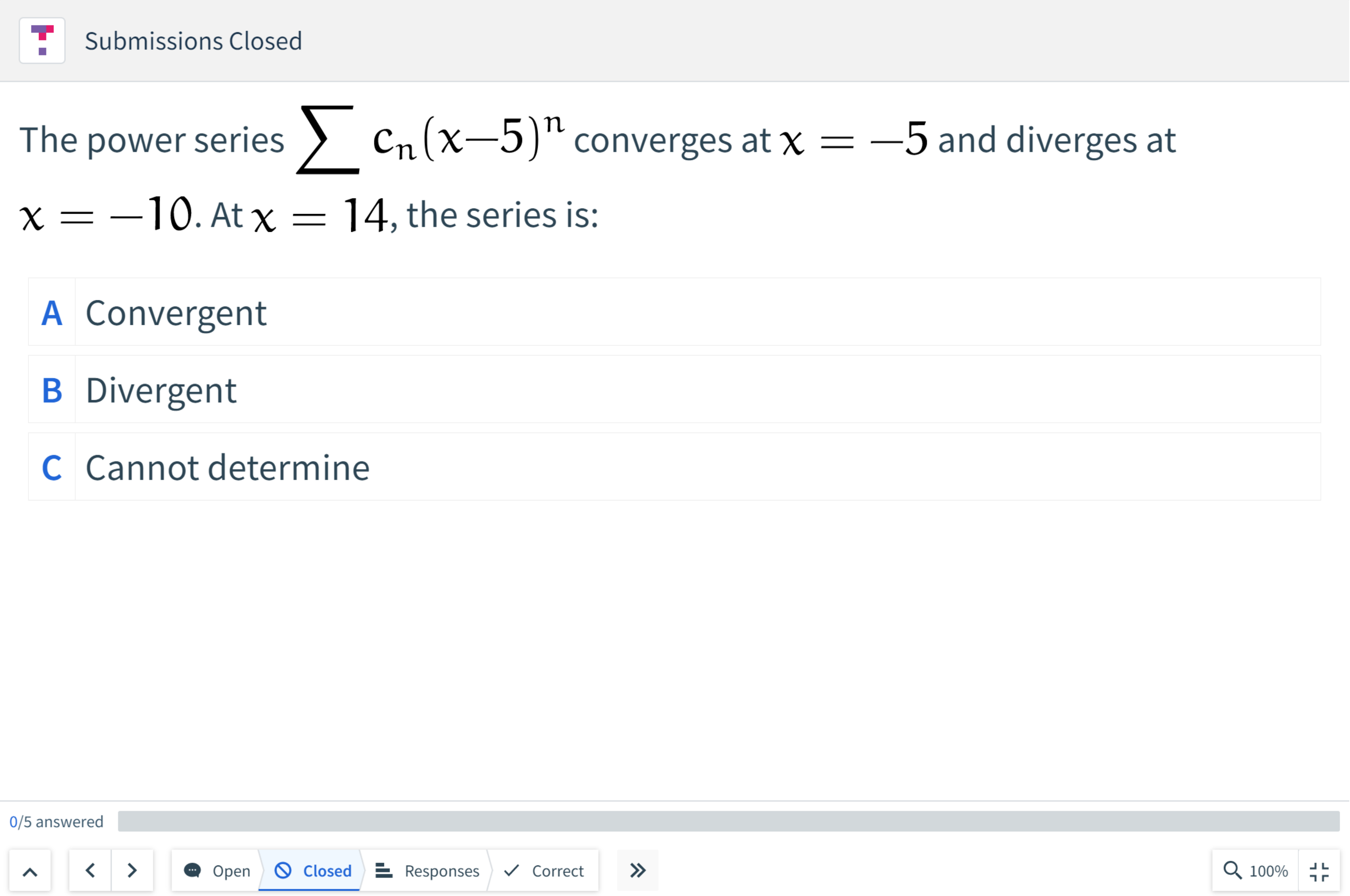
The power series $\sum c_n(x-5)^n$ converges at $x=-5$ and diverges at $x=-10$. At $x=17$, the series is:

A Convergent

B Divergent

C Cannot determine

0/5 answered



Submissions Closed

The power series $\sum c_n(x-5)^n$ converges at $x = -5$ and diverges at $x = -10$. At $x = 14$, the series is:

A Convergent

B Divergent

C Cannot determine

0/5 answered

What Possible Interval?

Draw a possible interval of convergence for $\sum c_n(x-5)^n$**, given that the series converges at** $x=-5$ **and diverges at** $x=-10$**.**

What Possible Interval?

Draw a possible interval of convergence for $\sum c_n(x-5)^n$, given that the series converges at $x=-5$ and diverges at $x=-10$.

We know that the interval needs to be centered at 5. Since the series converges at -5, this means that the radius of convergence is at least 10. Since the series diverges at $x=-10$, this means that the radius of convergence is less than 15. A possible interval of convergence is:

$$|x-5| < 11$$

$$-6 < x < 16$$

What Possible Interval?

Draw a possible interval of convergence for $\sum c_n(x-5)^n$, given that the series converges at $x=-5$ and diverges at $x=-10$.

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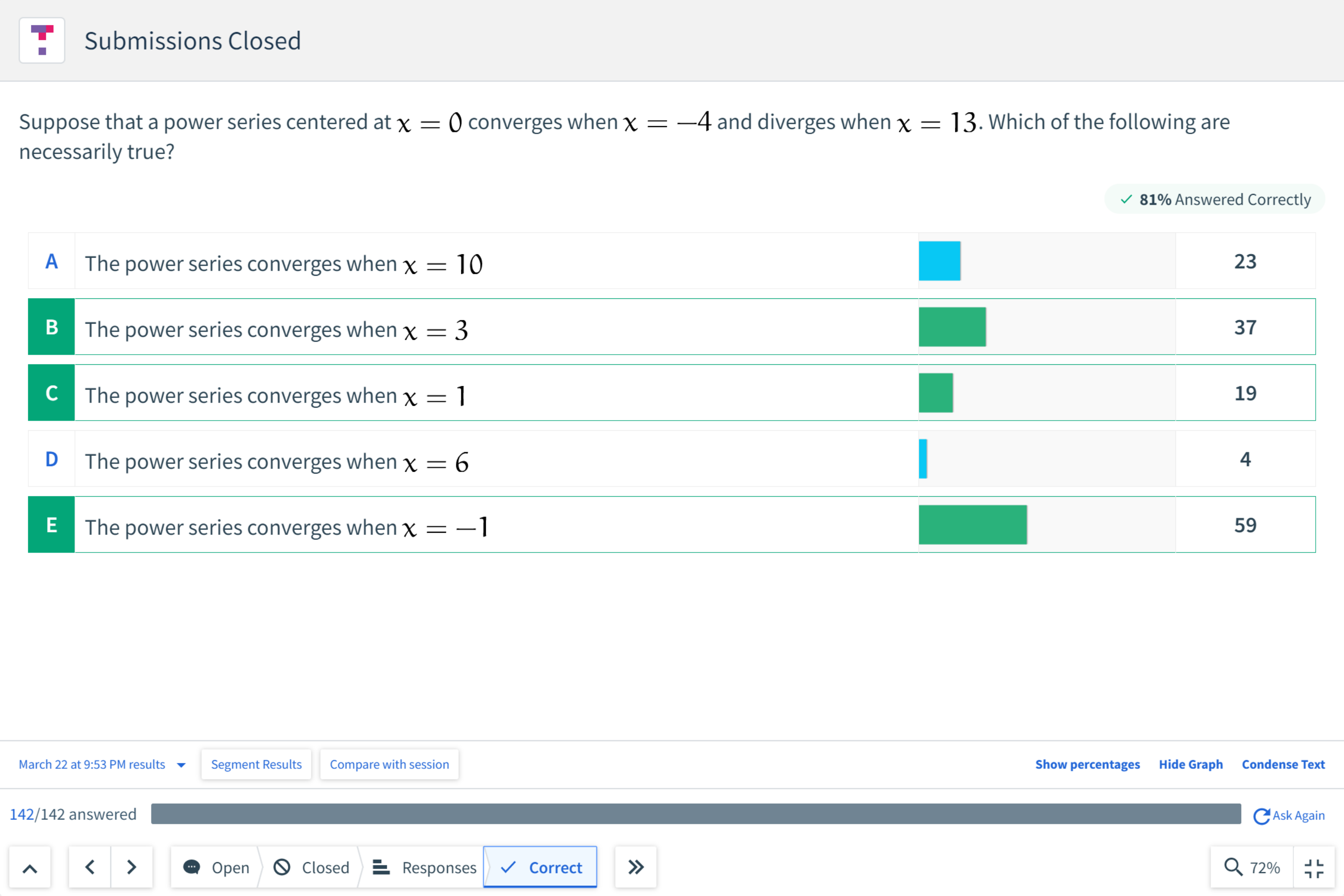
$$|x-5| < 11$$
$$-6 < x < 16$$

Note that the interval $|x-5| < 14$ (ie $-9 < x < 19$) is also possible

Plans for the Future

For next time:

Watch the week 11 videos, do WeBWork 10.2, actively read section 10.2



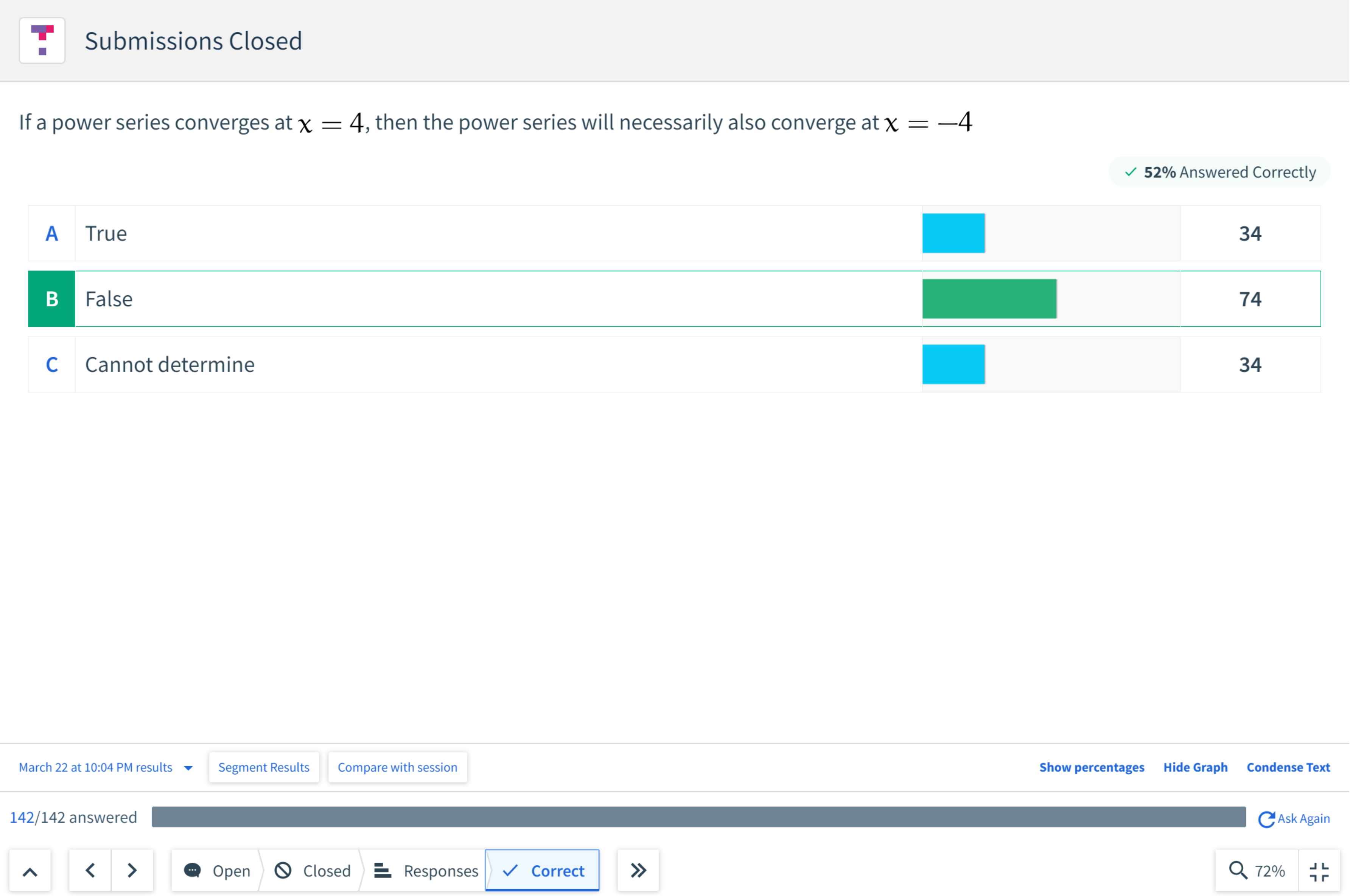
Submissions Closed

Suppose that a power series centered at $x = 0$ converges when $x = -4$ and diverges when $x = 13$. Which of the following are necessarily true?

✓ **81%** Answered Correctly

A	The power series converges when $x = 10$	23
B	The power series converges when $x = 3$	37
C	The power series converges when $x = 1$	19
D	The power series converges when $x = 6$	4
E	The power series converges when $x = -1$	59

142/142 answered



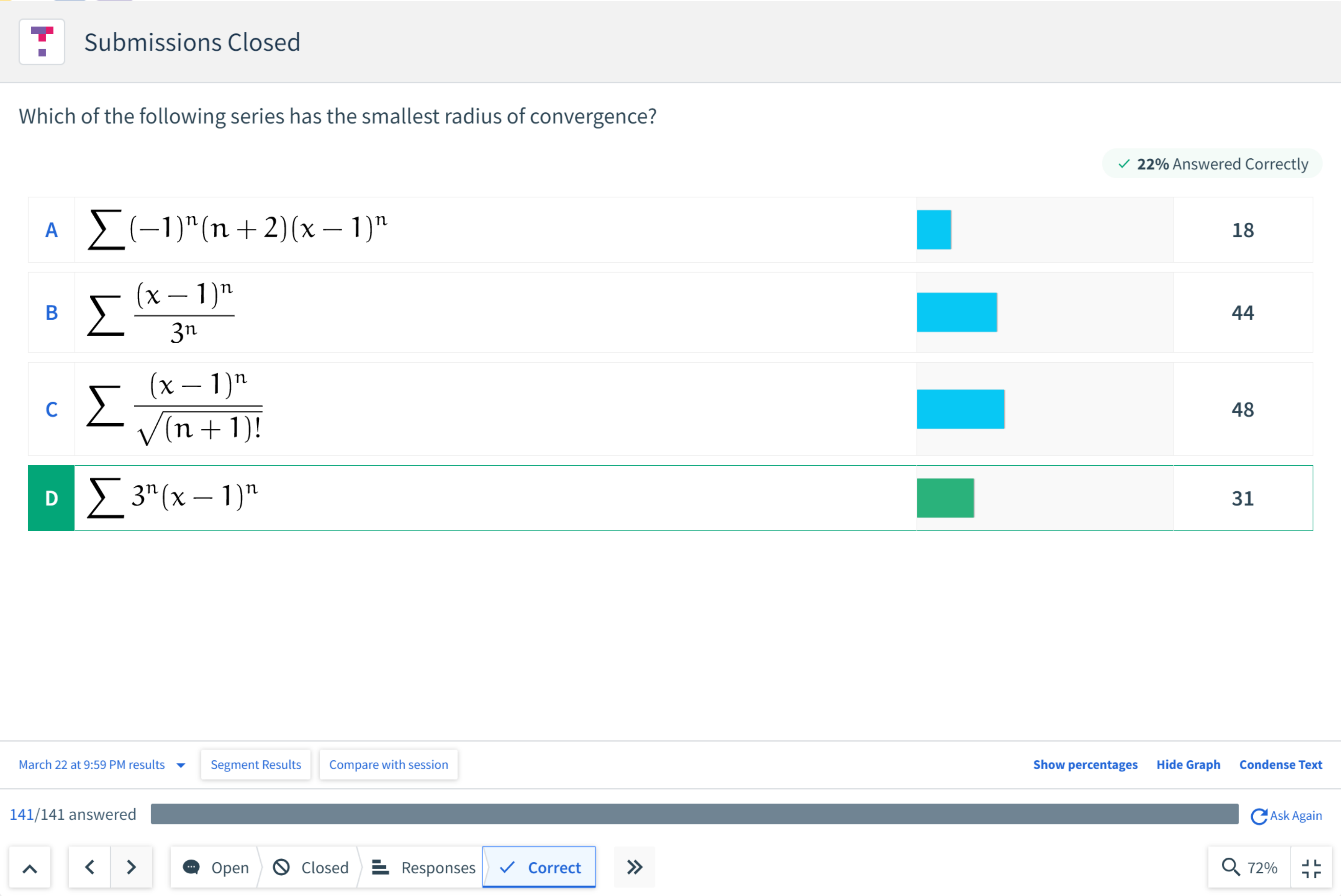
Submissions Closed

If a power series converges at $x = 4$, then the power series will necessarily also converge at $x = -4$

✓ **52%** Answered Correctly

A	True	34
B	False	74
C	Cannot determine	34

142/142 answered



Submissions Closed

Which of the following series has the smallest radius of convergence?

✓ **22%** Answered Correctly

A	$\sum (-1)^n (n+2)(x-1)^n$	18
B	$\sum \frac{(x-1)^n}{3^n}$	44
C	$\sum \frac{(x-1)^n}{\sqrt{(n+1)!}}$	48
D	$\sum 3^n (x-1)^n$	31

141/141 answered

Welcome to MAT136 LEC0501 (Assaf)

COURSE EVALUATIONS ARE OPEN! PLEASE PLEASE PLEASE SUBMIT THEM!!!

Especially important: feedback about online learning, the transition process, the overall course structure given the situation, and your suggestions for next semester if we still need to do things online by then.

S10.2 – Taylor Series – Back Again

Assaf Bar-Natan

“I never knew I’d love this world they’ve let me into
And the memories were lost long ago
So I’ll dance with these beautiful ghosts”

–“Beautiful Ghosts (Cats movie)”, Taylor Swift

March 25, 2020

Review: Taylor Polynomials

Recall that if f is some function, we can approximate f around a using a Taylor polynomial:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

where x is close to a.

Review: Taylor Polynomials

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where x is close to a.
Use the following Geogebra applet to investigate what happens when n gets big:

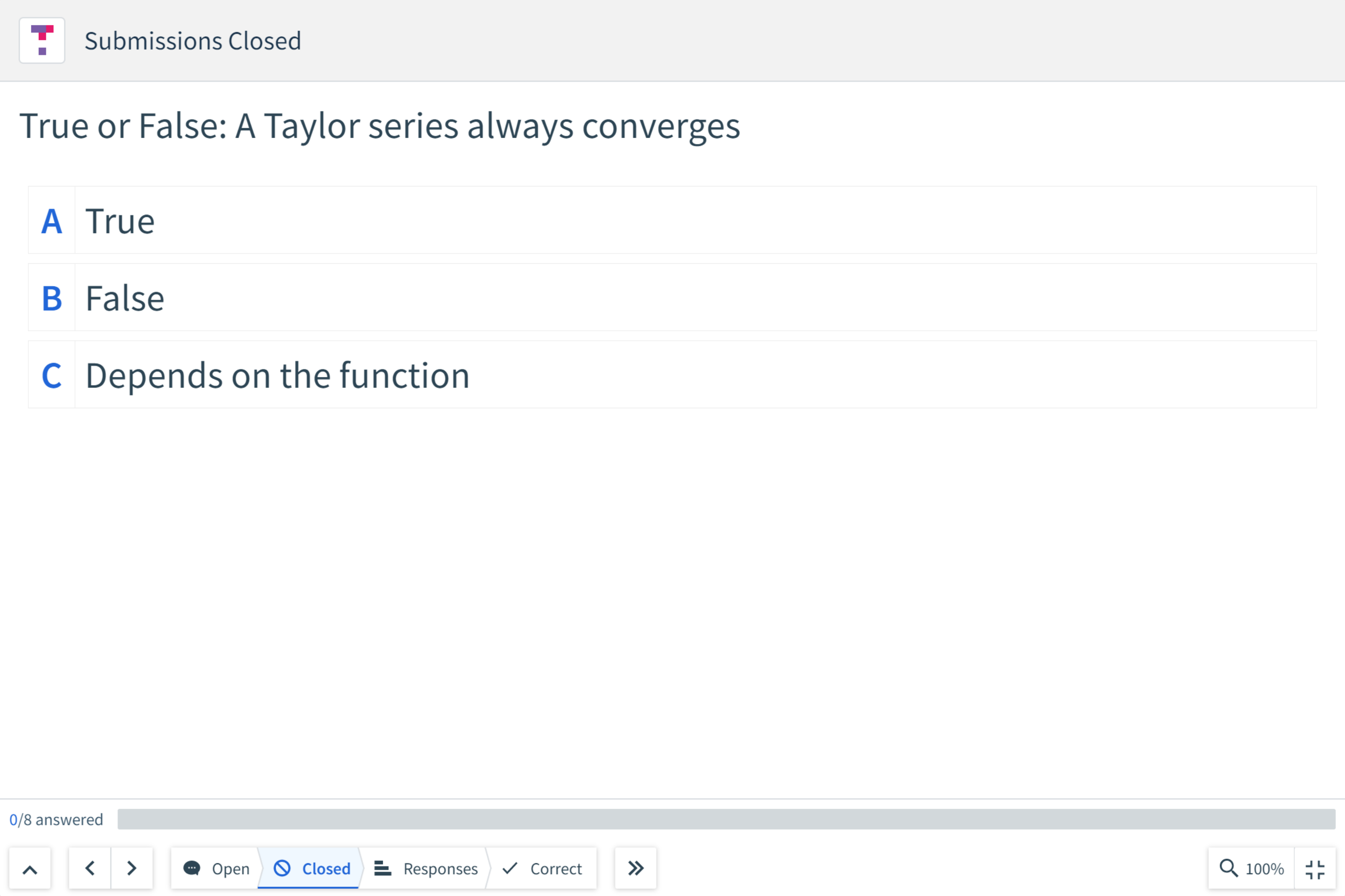
`https://www.geogebra.org/m/s9SkCsvC`

The Taylor Series

Take a Taylor polynomial to the extreme, and use a **power series** to approximate f:

$$f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots$$

This is called the **Taylor Series of** f **at** $x=a$



Submissions Closed

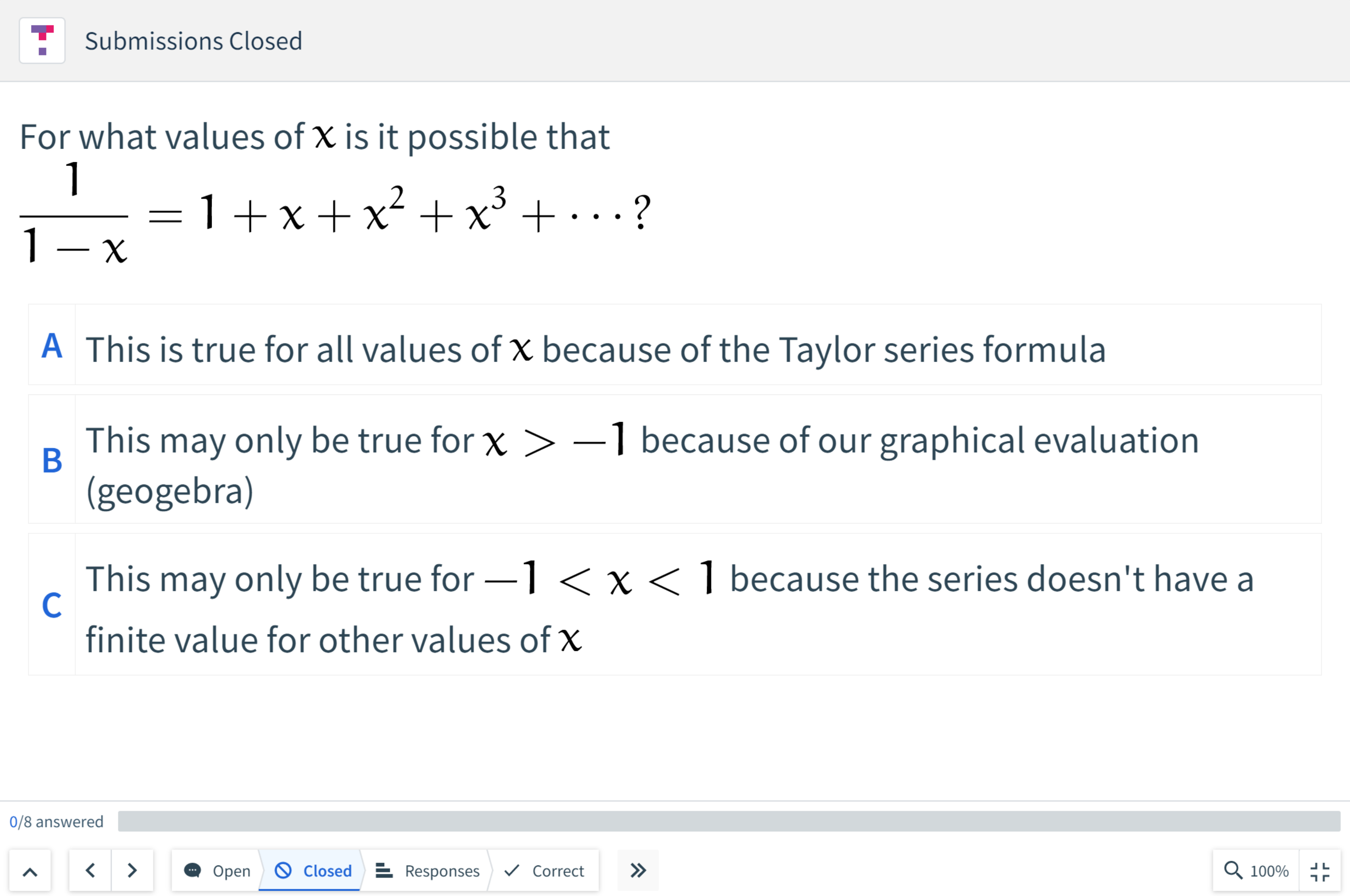
True or False: A Taylor series always converges

- **A** True
- **B** False
- **C** Depends on the function

0/8 answered

Takeaway

The Taylor series is a power series, so, just like any power series, it might converge for some values of x and diverge for other values of x.



For what values of x is it possible that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots ?$$

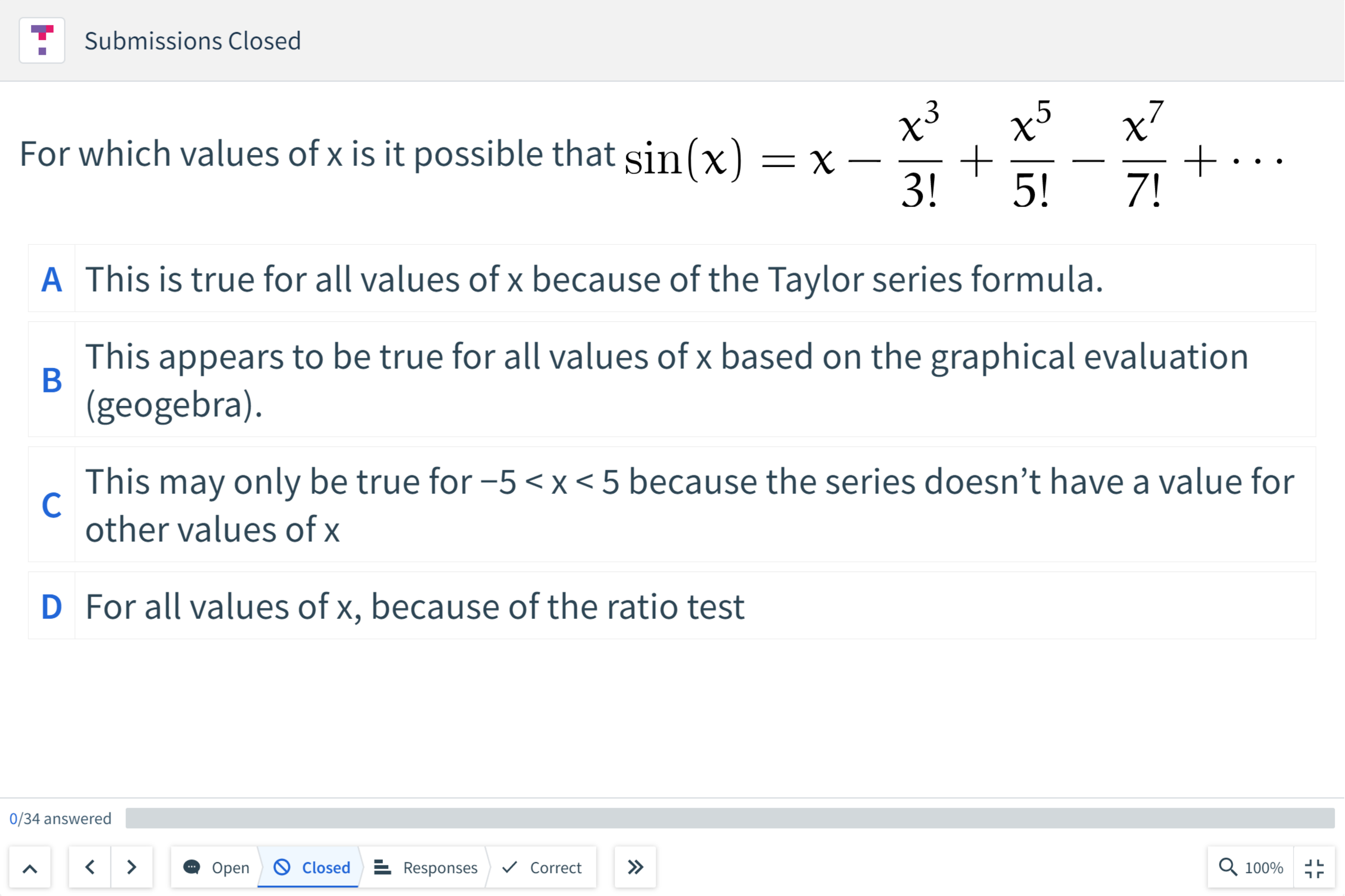
A This is true for all values of x because of the Taylor series formula

B This may only be true for $x > -1$ because of our graphical evaluation (geogebra)

C This may only be true for $-1 < x < 1$ because the series doesn't have a finite value for other values of x

Takeaway

We use the ratio test to check when a Taylor series converges



Submissions Closed

For which values of x is it possible that $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

A This is true for all values of x because of the Taylor series formula.

B This appears to be true for all values of x based on the graphical evaluation (geogebra).

C This may only be true for $-5 < x < 5$ because the series doesn’t have a value for other values of x

D For all values of x, because of the ratio test

The Miracle of Taylor Series

I'd like to use my Taylor series to approximate sin(1000000). I know:

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

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Let me try again....

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My first approximation will be $\sin(1000000) \approx 1000000$. But this is garbage! I know that $\sin(1000000) < 1$.
Let me try again....

$$\sin(1000000) \approx 1000000 - \frac{(1000000)^3}{6} \approx -1.6 \times 10^{17}$$

Wow! That's even worse than the first time... Let me try again...

The Miracle of Taylor Series

I'd like to use my Taylor series to approximate sin(1000000). I know:

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Let me try again....

$$\sin(1000000) \approx 1000000 - \frac{(1000000)^3}{6} \approx -1.6 \times 10^{17}$$

Wow! That's even worse than the first time... Let me try again...

$$\sin(1000000) \approx 1000000 - \frac{(1000000)^3}{6} + \frac{(1000000)^5}{5!} \approx 8.3 \times 10^{27}$$

The Miracle of Taylor Series

My approximations on the previous slide were trash.
$-1 < \sin(1000000) < 1$, but I kept getting absurdly high numbers.
Q: What is something I can do to get good approximations of $\sin(1000000)$?

The Miracle of Taylor Series

My approximations on the previous slide were trash.
$-1 < \sin(1000000) < 1$, but I kept getting absurdly high numbers.
Q: What is something I can do to get good approximations of $\sin(1000000)$?

- I could choose to expand at a point close to 1000000
- I could keep going, and take many many many more terms in the series
- I could use the fact that sin is periodic, and get that $\sin(1000000) = \sin(x)$, where $-\pi < x < \pi$. Then I could approximate.

Takeaway

Some functions have Taylor series that have an infinite radius of convergence (eg: $\sin$, $\cos$, e^x). For these functions, the Taylor series always converges, but it might converge very slowly!

Check that $\sin$, $\cos$, and e^x indeed have this property:
`https://www.geogebra.org/m/s9SkCsvC`

Euler’s Identity

We know:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

These look related

Euler's Identity

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$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

These look related

Let $i = \sqrt{-1}$ be an imaginary number (don't worry about it, just pretend that all algebra works the same, but $i^2 = -1$).
Q: Write the Taylor series for e^{ix}.

Euler's Identity

We know:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

These look related

Let $i = \sqrt{-1}$ be an imaginary number (don't worry about it, just pretend that all algebra works the same, but $i^2 = -1$).

Q: Write the Taylor series for e^{ix}.

$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

Euler's Identity

We know:

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots$$

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$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Q: Relate e^{ix}, $\cos(x)$, and $\sin(x)$ using the above.

Euler's Identity

We know:

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots$$

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$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Q: Relate e^{ix}, $\cos(x)$, and $\sin(x)$ using the above. Hint: multiply $\sin(x)$ by i...

$$e^{ix} = \cos(x) + i\sin(x)$$

Q: Compute $e^{i\pi}$.

$e^{i\pi} = -1$

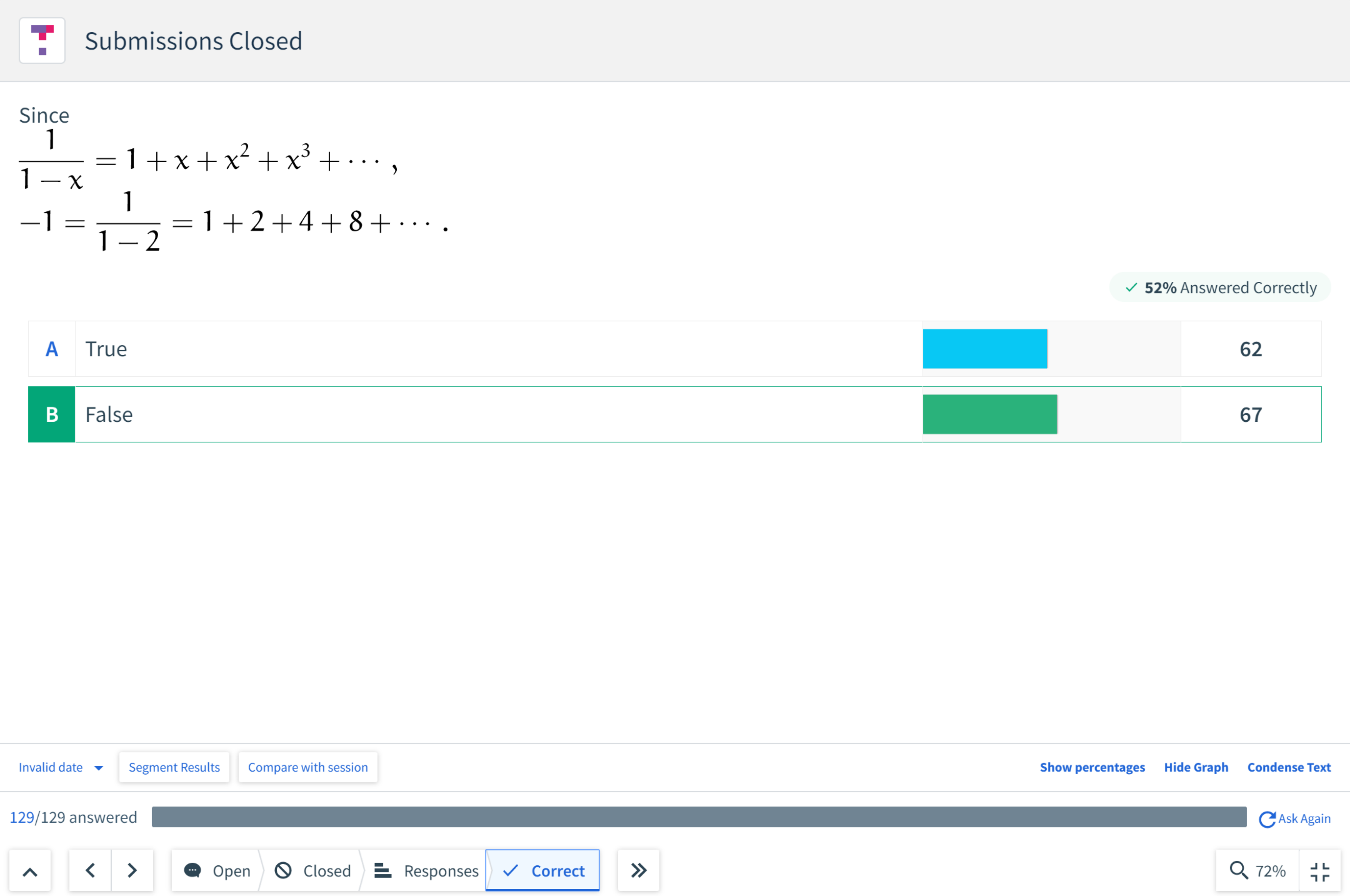
$$e^{i\pi} = -1$$

A good explanation of this:
`https://www.youtube.com/watch?v=v0YEaeIClKY`

Plans for the Future

For next time:

Watch the week 11 videos, do WeBWork 10.3, actively read section 10.3



Submissions Closed

Since

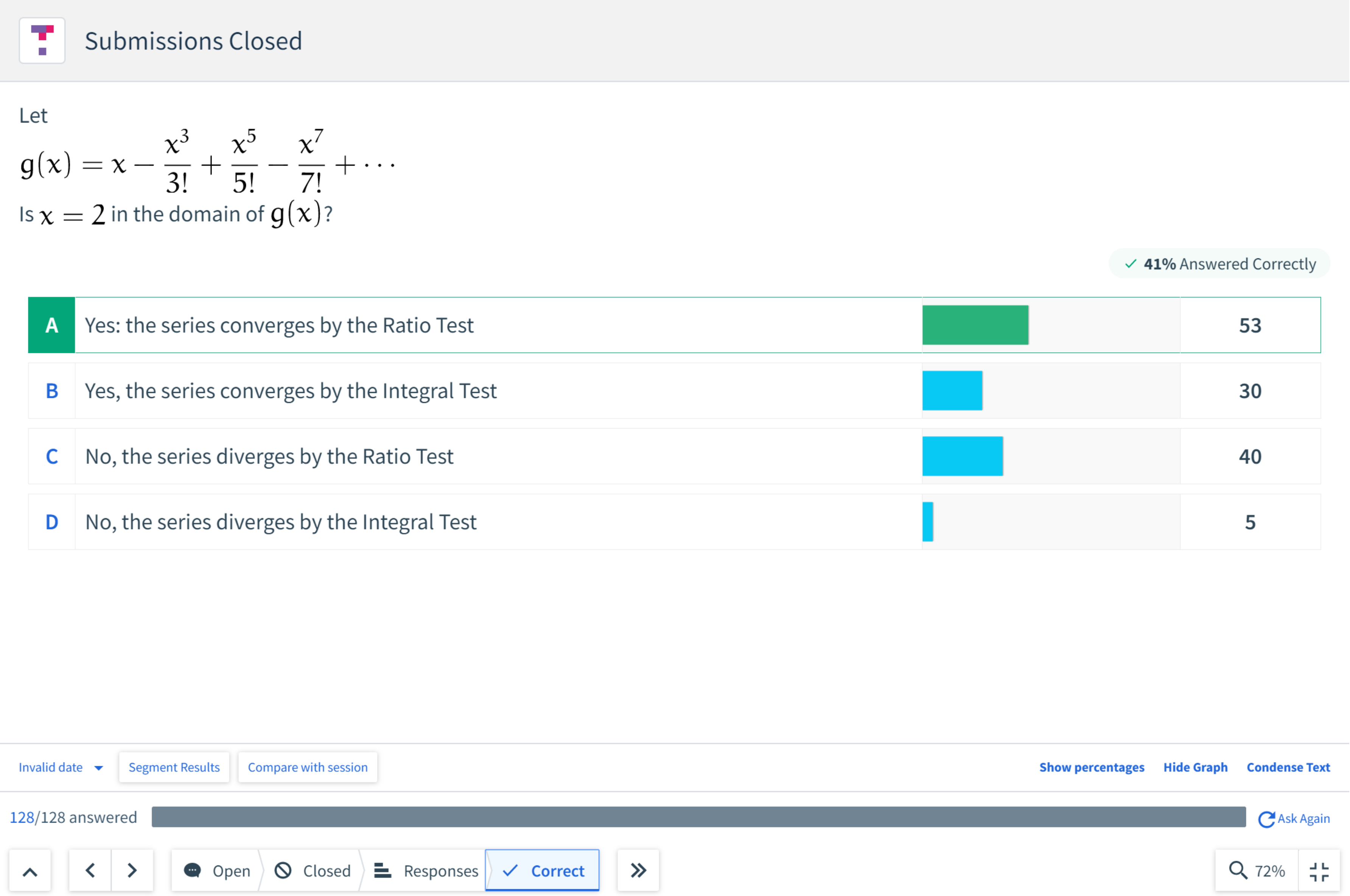
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots,$$

$$-1 = \frac{1}{1-2} = 1 + 2 + 4 + 8 + \cdots.$$

✓ 52% Answered Correctly

A	True	62
B	False	67

129/129 answered



Let

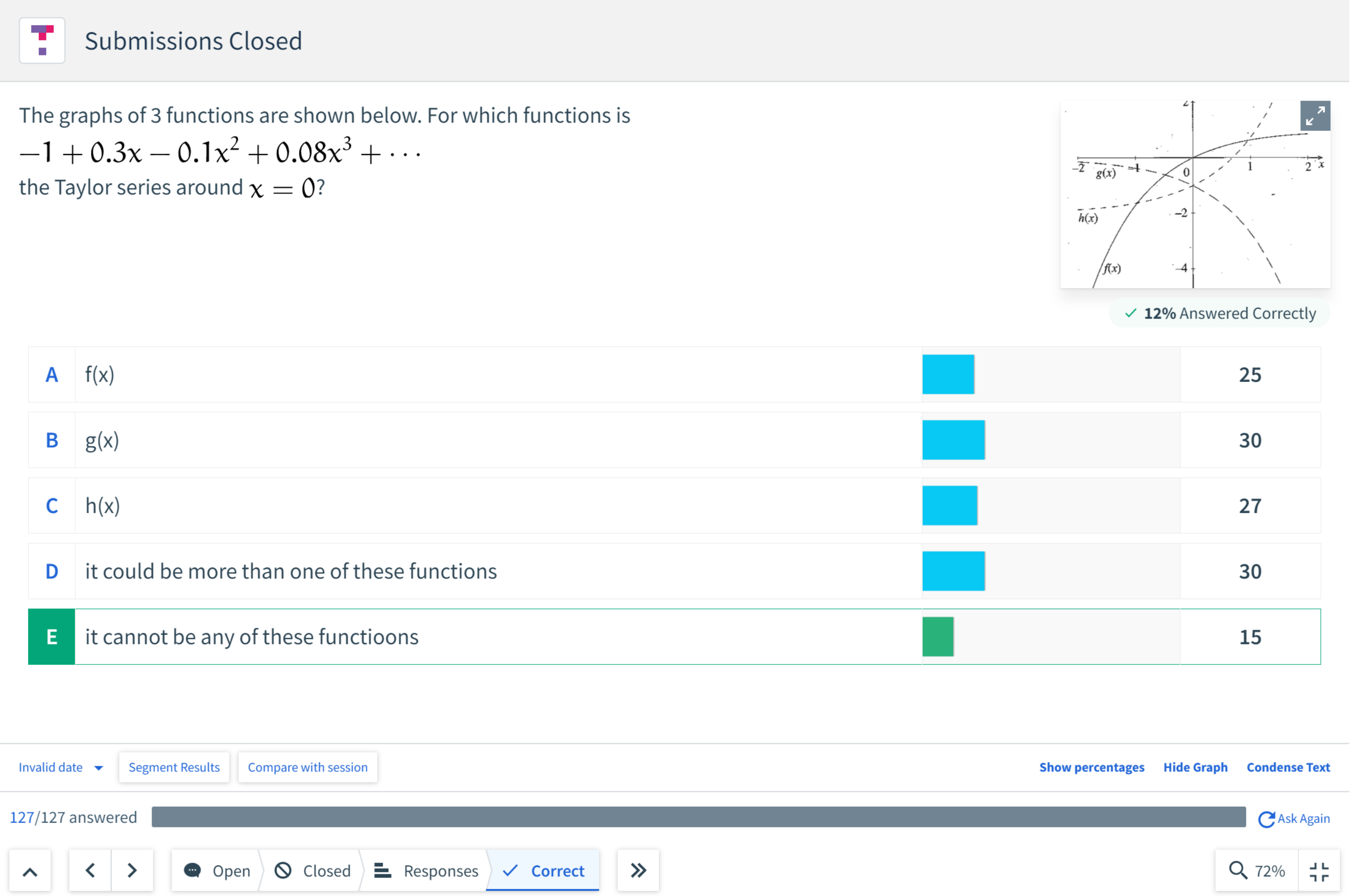
$$g(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Is $x = 2$ in the domain of $g(x)$?

✓ **41%** Answered Correctly

A	Yes: the series converges by the Ratio Test	53
B	Yes, the series converges by the Integral Test	30
C	No, the series diverges by the Ratio Test	40
D	No, the series diverges by the Integral Test	5

128/128 answered



The graphs of 3 functions are shown below. For which functions is

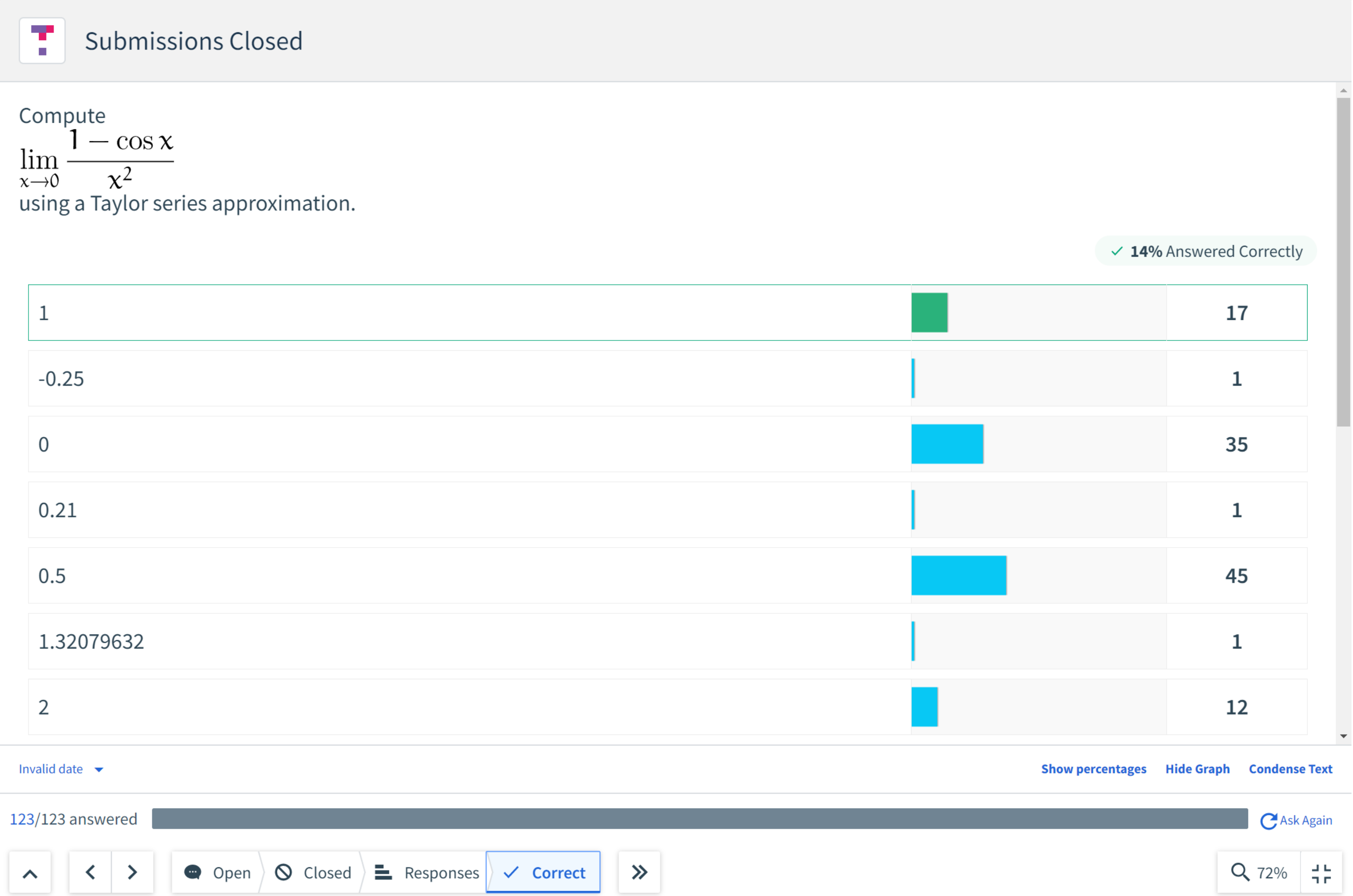
$-1 + 0.3x - 0.1x^2 + 0.08x^3 + \cdots$

the Taylor series around $x = 0$?

✓ **12%** Answered Correctly

A	f(x)	25
B	g(x)	30
C	h(x)	27
D	it could be more than one of these functions	30
E	it cannot be any of these functioons	15

127/127 answered



Submissions Closed

Compute

$$\lim_{x \to 0} \frac{1-\cos x}{x^2}$$

using a Taylor series approximation.

✓ **14%** Answered Correctly

1	17
-0.25	1
0	35
0.21	1
0.5	45
1.32079632	1
2	12

123/123 answered

Welcome to MAT136 LEC0501 (Assaf)

Final exam information is on the main course website, under Test & Exam

S10.3 – Taylor Series – Applications

Assaf Bar-Natan

“They’ll tell you I’m insane
But I’ve got a blank space baby
And I’ll write your name”

–“Blank Space”, Taylor Swift

March 27, 2020

Taylor Series and Substitution

Key observation: the equation

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

holds for **any** x.

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So, we can write:

$$f(3x^2) = \sum_{n=0}^{\infty} c_n (3x^2-a)^n$$

Taylor Series and Substitution

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Q: If the radius of convergence of $\sum_{n=0}^{\infty} c_n x^n$ is 3, what is the radius of convergence of $\sum_{n=0}^{\infty} c_n (3x^2)^n$?

Taylor Series and Substitution

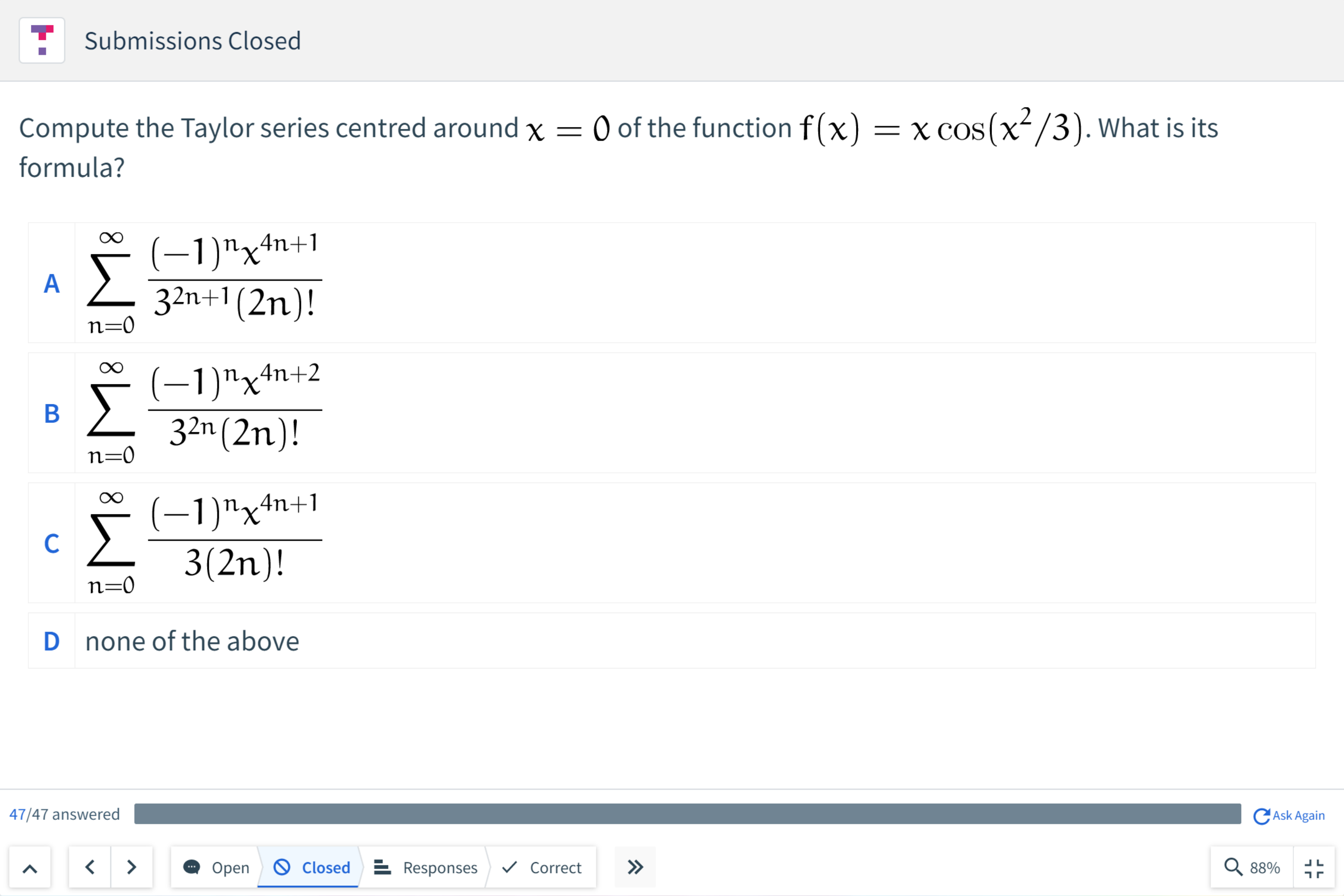
Key observation: the equation

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

holds for **any** x.
So, we can write:

$$f(3x^2) = \sum_{n=0}^{\infty} c_n (3x^2-a)^n$$

Q: If the radius of convergence of $\sum_{n=0}^{\infty} c_n x^n$ is 3, what is the radius of convergence of $\sum_{n=0}^{\infty} c_n (3x^2)^n$?
A: We need $-3 < 3x^2 < 3$, so this means that $-1 < x < 1$, and the radius of convergence is 1.



Submissions Closed

Compute the Taylor series centred around $x = 0$ of the function $f(x) = x\cos(x^2/3)$. What is its formula?

A $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{3^{2n+1}(2n)!}$

B $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{3^{2n}(2n)!}$

C $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{3(2n)!}$

D none of the above

47/47 answered

Radius of Convergence

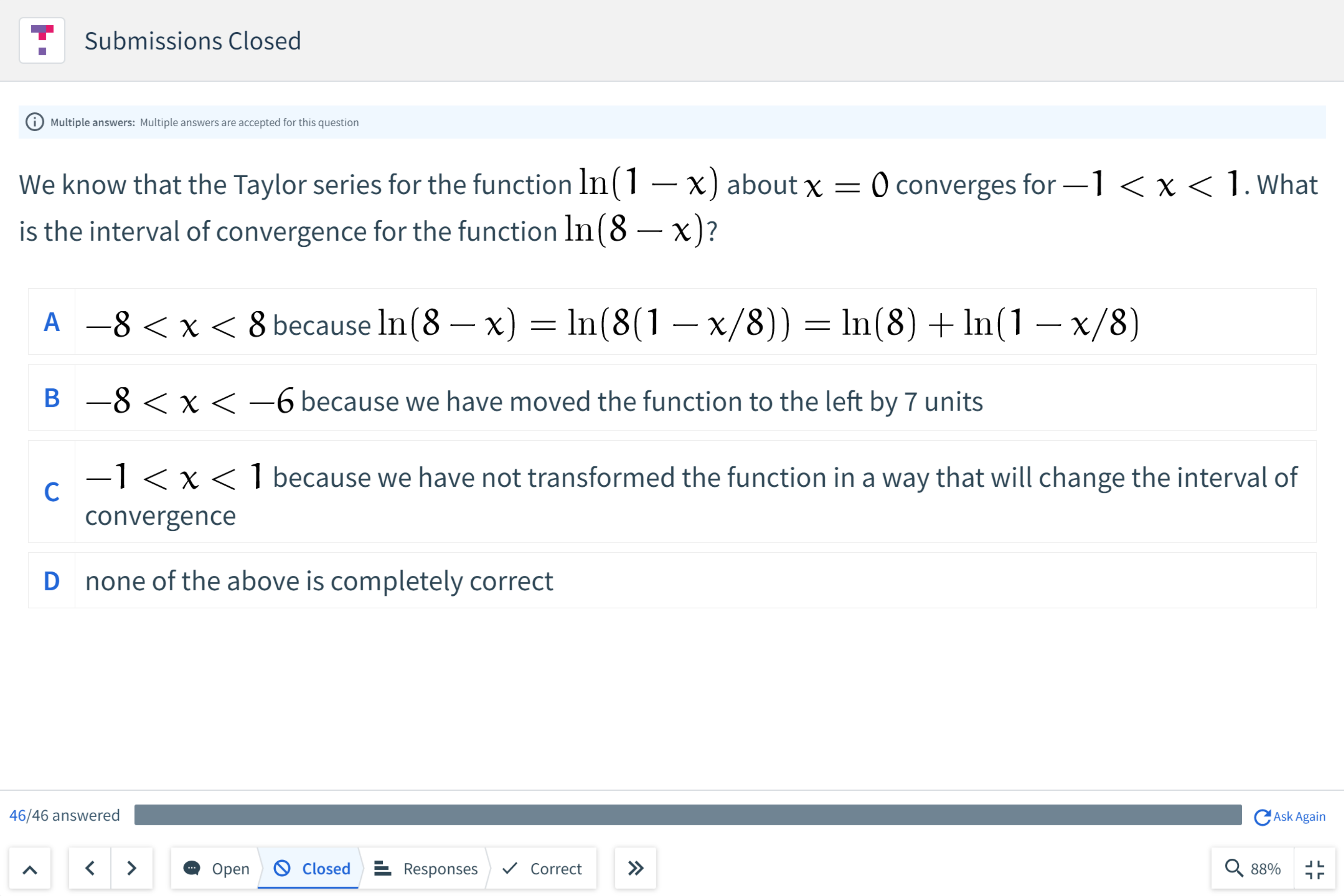
Compute the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{3^{2n}(2n)!}$$

Takeaway

To compute the interval of convergence of a substituted series, use the original interval of convergence and transform it

This should remind you of integration by substitution, and changing the bounds



Submissions Closed

Multiple answers: Multiple answers are accepted for this question

We know that the Taylor series for the function $\ln(1-x)$ about $x=0$ converges for $-1<x<1$. What is the interval of convergence for the function $\ln(8-x)$?

A $-8<x<8$ because $\ln(8-x)=\ln(8(1-x/8))=\ln(8)+\ln(1-x/8)$

B $-8<x<-6$ because we have moved the function to the left by 7 units

C $-1<x<1$ because we have not transformed the function in a way that will change the interval of convergence

D none of the above is completely correct

Fill in the Blanks

If a ______ series for $f(x)$ at $x = a$ converges to f for $|x - a| < R$, then the series found by term-by-term differentiation is the Taylor series for ______, and converges on the interval ______.

erf and Taylor Series Integration

Let's revisit our friend,

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

To estimate $erf(x)$ for small x, we will write it as a Taylor series.
Q: Write down three steps to computing the Taylor series of $erf(x)$ around $x = 0$.

erf and Taylor Series Integration

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$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

To estimate $erf(x)$ for small x, we will write it as a Taylor series.
Q: Write down three steps to computing the Taylor series of $erf(x)$ around $x = 0$.

- Write the Taylor series for e^x
- Plug in $x = -t^2$
- Integrate term-by-term

erf and Taylor Series Integration

- Write the Taylor series for e^x
- Plug in $x = -t^2$
- Integrate term-by-term

Let's do it:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

so

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

Q: What is the Taylor series for $erf(x)$?

erf and Taylor Series Integration

- Write the Taylor series for e^x
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Let's do it:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

so

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!}$$

Q: What is the Taylor series for $erf(x)$?

$$erf(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(n!)}$$

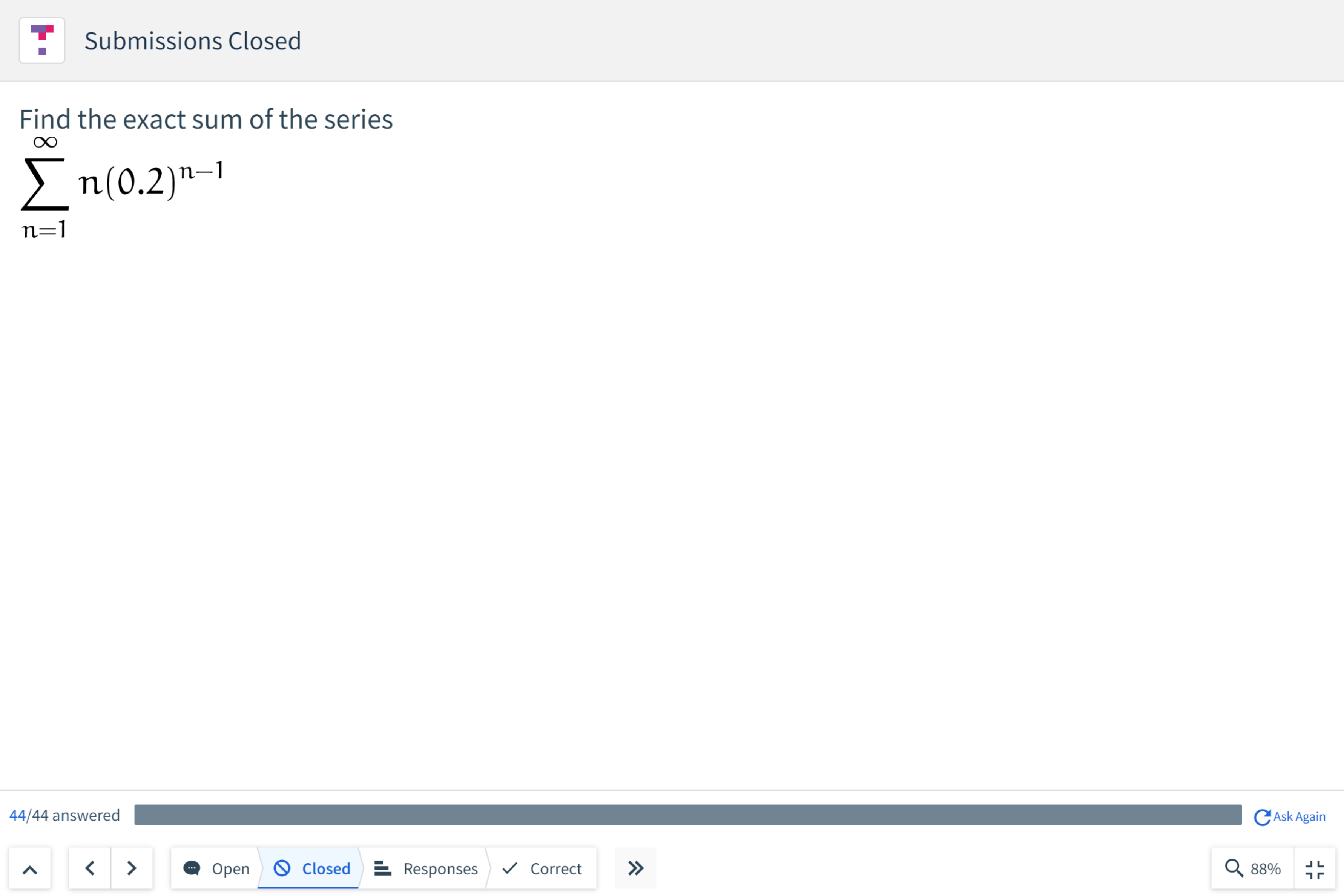
Computing Series Using Taylor Polynomials

WolframAlpha says:

$$erf(1) = 0.842\cdots$$

We can use this to compute:

$$0.746 \approx erf(1)\frac{\sqrt{\pi}}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n 1^{2n+1}}{(2n+1)(n!)}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(n!)}$$



Submissions Closed

Find the exact sum of the series

$$\sum_{n=1}^{\infty} n(0.2)^{n-1}$$

Takeaway

When we have a series, we can plug in a variable, x, then interpret it as a derivative or an integral of series that we know

Plans for the Future

For next time:

Watch the week 12 videos, and review section 10.3

Welcome to MAT136 LEC0501 (Assaf)

Critical Incident Questionnaire 3:
`https://tinyurl.com/March2020CIQ`

S10.3 – Taylor Series – Applications (Part 2)

Assaf Bar-Natan

"Everything will be alright, if
We just keep dancing like we're twenty-two..."

–"22", Taylor Swift

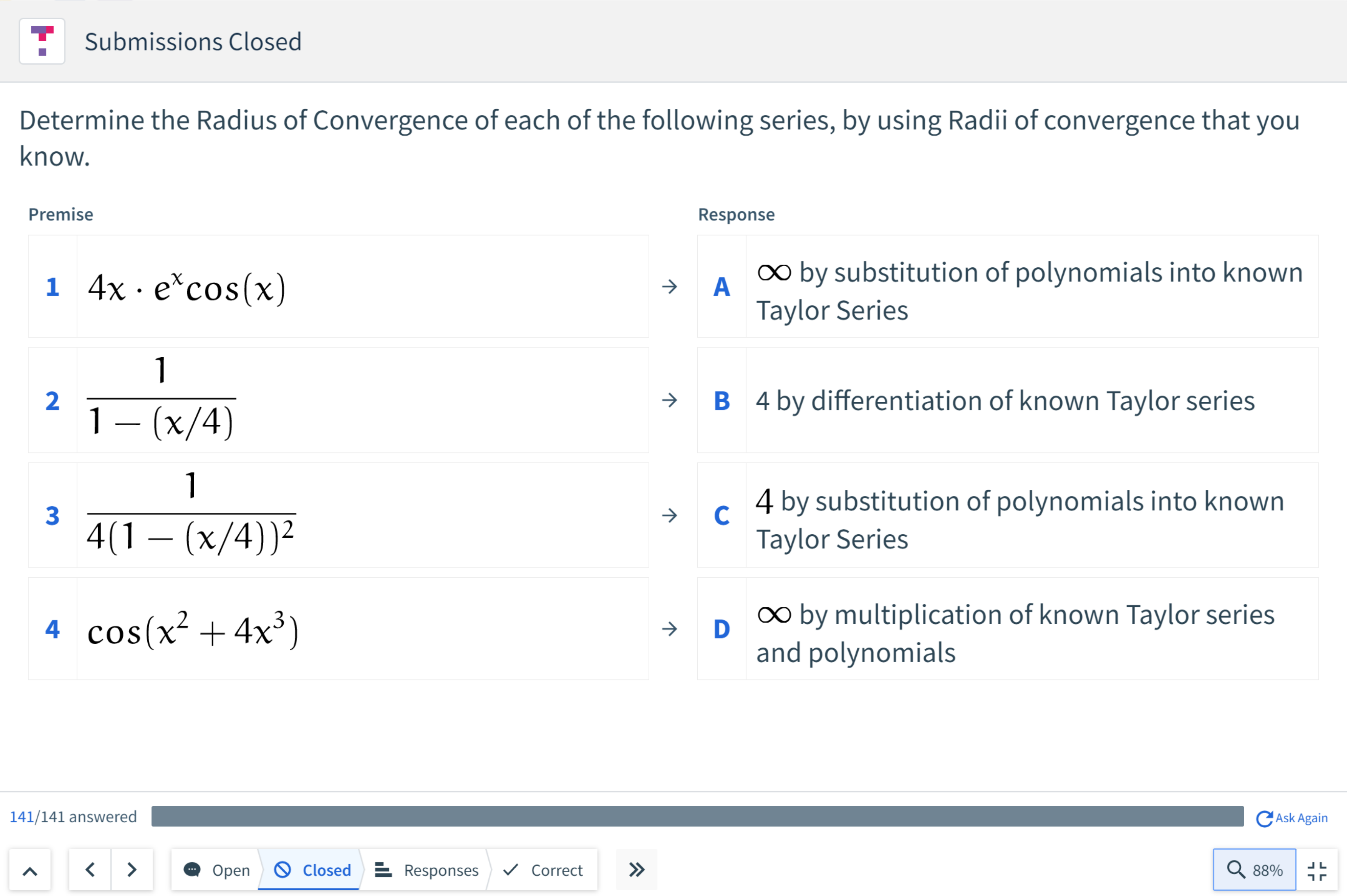
March 30, 2020

Taylor Series and Substitution

Recall:

If a Taylor series for $f(x)$ converges for x on some interval, then the Taylor series for $f(g(x))$ converges whenever $g(x)$ is in that interval

If a Taylor series for $f(x)$ converges for x on some interval, then the Taylor series for $f'(x)$ converges on the same interval



Submissions Closed

Determine the Radius of Convergence of each of the following series, by using Radii of convergence that you know.

Premise			Response	
1	$4x \cdot e^x \cos(x)$	→	A	∞ by substitution of polynomials into known Taylor Series
2	$\dfrac{1}{1-(x/4)}$	→	B	4 by differentiation of known Taylor series
3	$\dfrac{1}{4(1-(x/4))^2}$	→	C	4 by substitution of polynomials into known Taylor Series
4	$\cos(x^2+4x^3)$	→	D	∞ by multiplication of known Taylor series and polynomials

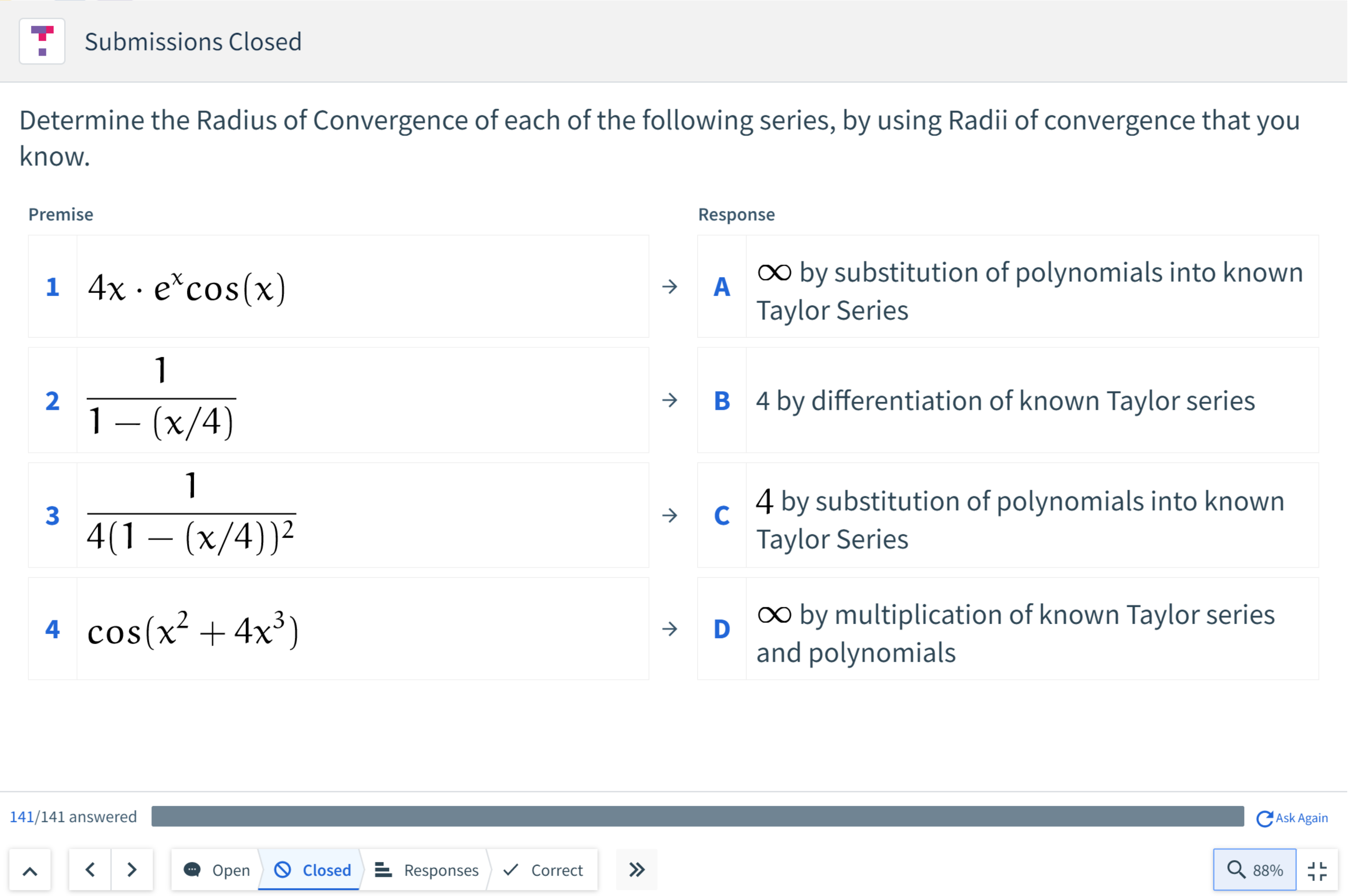
141/141 answered

Radius of Convergence

Consider $\frac{1}{1-(x/4)}$. The Taylor series around 0 is:

$$1 + y + y^2 + \cdots$$

Where $y = x/4$.
This converges when $-1 < y < 1$, ie, when $-4 < x < 4$, so the Taylor series converges on this interval by subtituting $x/4$ into a known Taylor series.



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141/141 answered

Radius of Convergence

Consider $\frac{1}{4(1-(x/4))^2}$. Can we interpret this function as a derivative of something?

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$$\frac{d}{dx}\left(\frac{1}{1-(x/4)}\right) = \frac{1}{4(1-(x/4))^2}$$

Radius of Convergence

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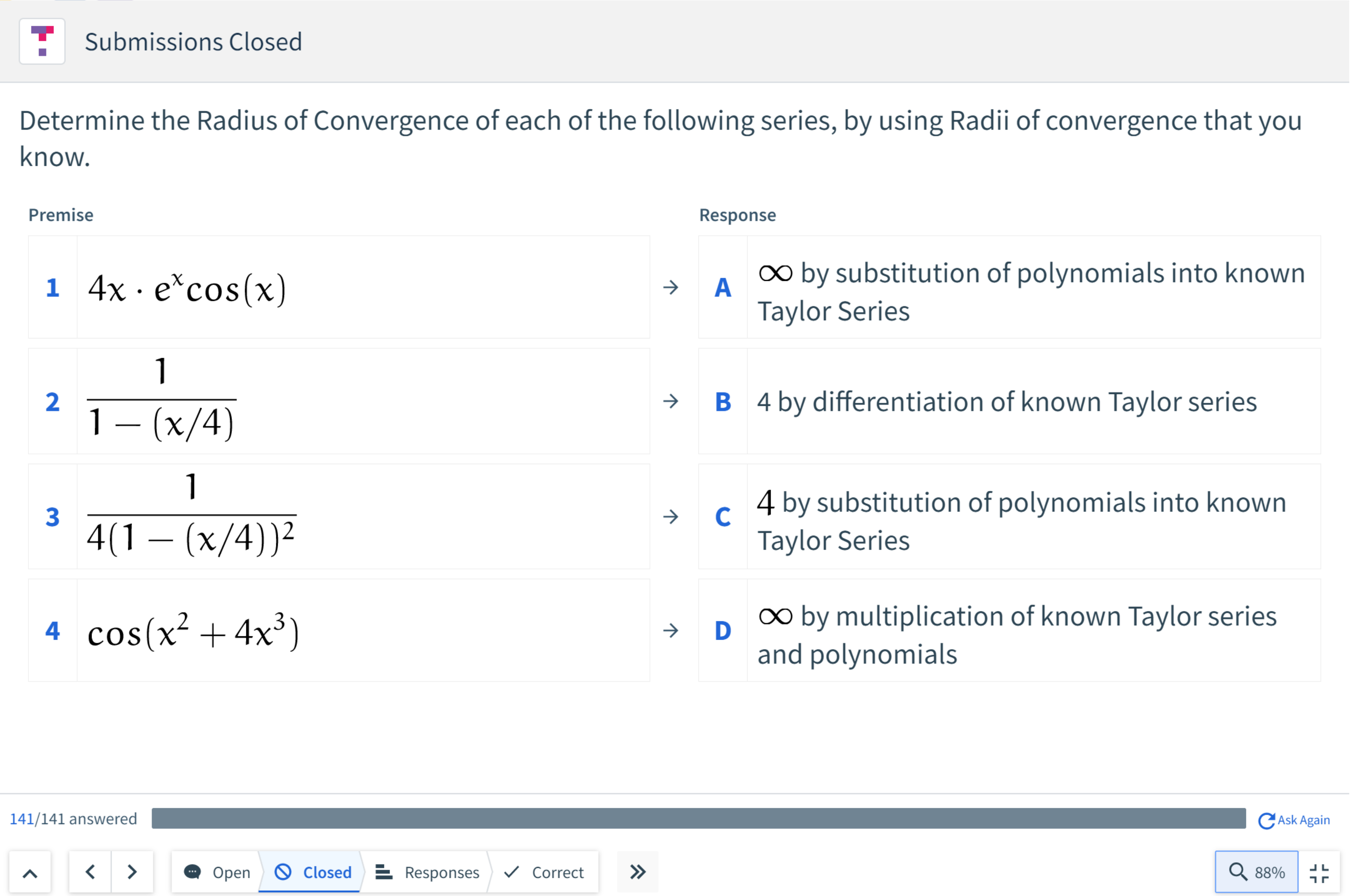
$$\frac{d}{dx}\left(\frac{1}{1-(x/4)}\right) = \frac{1}{4(1-(x/4))^2}$$

We know that converges when $-4 < x < 4$, because it's the derivative of a Taylor series that converges on that interval.



Submissions Closed

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Radius of Convergence

The Taylor series for $\cos(x)$ converges for any x, so no matter what we substitute into cos, the Taylor series will converge.

Radius of Convergence

The Taylor series for $\cos(x)$ converges for any x, so no matter what we substitute into cos, the Taylor series will converge.

If two functions have Taylor series which converge everywhere, then their product also has a Taylor series that converges everywhere

This is just an application of the product formula for Taylor series (Example 4)

Computing Series Using Derivatives

We are going to compute the series:

$$\sum_{n=1}^{\infty} \frac{3^{2n-1} 2n}{n!}$$

- Identify constants, indices, and patterns in the series
- Substitute a variable into the series
- Interpret each term as an integral or derivative of something
- Integrate or differentiate term by term to get a new series
- Do you recognize the new series? If not, go back to step 2.
- Write the new series in closed form, and interpret the original series as its derivative or integral

Computing Series Using Derivatives

$$\sum_{n=1}^{\infty} \frac{3^{2n-1}2n}{n!}$$

The constants here are $2, 3$ (and 1). The index is n. We will try replacing all instances of 3 with the variable x:

$$\sum_{n=1}^{\infty} \frac{x^{2n-1}2n}{n!}$$

Computing Series Using Derivatives

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Q: Can we interpret each term as the derivative of something?

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$$\sum_{n=1}^{\infty} \frac{x^{2n-1} 2n}{n!}$$

Q: Can we interpret each term as the derivative of something?

$$\frac{x^{2n-1} 2n}{n!} = \frac{d}{dx} \left(\frac{x^{2n}}{n!} \right)$$

Computing Series Using Derivatives

Remembering that we are evaluating when $x = 3$...

$$\sum_{n=1}^{\infty} \frac{x^{2n-1}2n}{n!} = \sum_{n=1}^{\infty} \frac{d}{dx}\left(\frac{x^{2n}}{n!}\right) = \frac{d}{dx}\sum_{n=1}^{\infty}\left(\frac{x^{2n}}{n!}\right)$$

Q: Do you recognize this series?

Computing Series Using Derivatives

Remembering that we are evaluating when $x = 3$...

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Q: Do you recognize this series?

$$\sum_{n=1}^{\infty} \frac{x^{2n-1} 2n}{n!} = \frac{d}{dx}\sum_{n=1}^{\infty}\left(\frac{x^{2n}}{n!}\right) = \frac{d}{dx}\left(e^{x^2} - 1\right)$$

Computing Series Using Derivatives

Remembering that we are evaluating when $x = 3$...

$$\sum_{n=1}^{\infty} \frac{x^{2n-1}2n}{n!} = \frac{d}{dx}\left(e^{x^2} - 1\right) = 2xe^{x^2}$$

Q: What is $\sum_{n=1}^{\infty} \frac{3^{2n-1}2n}{n!}$?

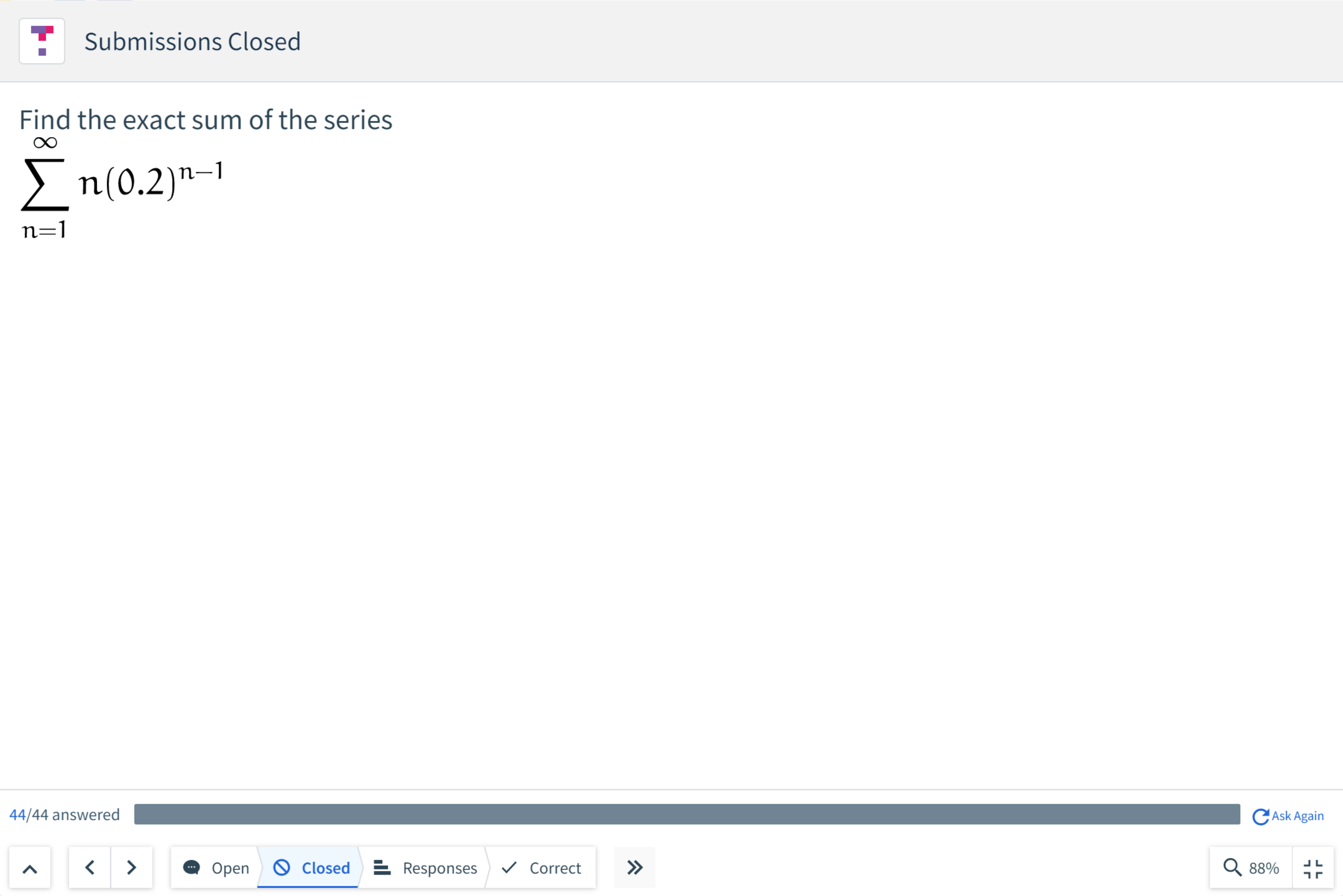
Computing Series Using Derivatives

Remembering that we are evaluating when $x = 3$...

$$\sum_{n=1}^{\infty} \frac{x^{2n-1}2n}{n!} = \frac{d}{dx}\left(e^{x^2} - 1\right) = 2xe^{x^2}$$

Q: What is $\sum_{n=1}^{\infty} \frac{3^{2n-1}2n}{n!}$? We plug in $x = 3$ to get:

$$\sum_{n=1}^{\infty} \frac{3^{2n-1}2n}{n!} = 6e^9$$



Submissions Closed

Find the exact sum of the series

$$\sum_{n=1}^{\infty} n(0.2)^{n-1}$$

44/44 answered

The Series $\sum n(0.2)^{n-1}$

Folllowing the steps we've outlined, replace 0.2 with x, and get:

$$\sum_{n=1}^{\infty} nx^{n-1}$$

The Series $\sum n(0.2)^{n-1}$

Folllowing the steps we've outlined, replace 0.2 with x, and get:

$$\sum_{n=1}^{\infty} nx^{n-1}$$

Interpret each term as a derivative to get:

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{d}{dx}\sum_{n=1}^{\infty} x^n = \frac{d}{dx}\left(\frac{1}{1-x} - 1\right)$$

The Series $\sum n(0.2)^{n-1}$

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Finally, differentiate and plug in $x = 0.2$:

$$\sum_{n=1}^{\infty} n(0.2)^{n-1} = \frac{1}{(1-(0.2))^2} = 1.5625$$

The Series $\sum n(0.2)^{n-1}$

Folllowing the steps we've outlined, replace 0.2 with x, and get:

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Interpret each term as a derivative to get:

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Finally, differentiate and plug in $x = 0.2$:

$$\sum_{n=1}^{\infty} n(0.2)^{n-1} = \frac{1}{(1-(0.2))^2} = 1.5625$$

Everything worked because $|x| < 1$, so the series above converge

Takeaway

When we have a series, we can plug in a variable, x, then interpret it as a derivative or an integral of series that we know

Plans for the Future

For next time:

Watch the uploaded Section 10.3 video, and go over Taylor Solutions to ODEs

Welcome to MAT136 LEC0501 (Assaf)

Today: ODEs
Friday: Review

COURSE EVALUATIONS!!!!!!

`http://uoft.me/openevals`

Taylor Expansions and ODEs – Part 2

Assaf Bar-Natan

“It goes, all my troubles on a burning pile
All lit up and I start to smile
If I, catch fire then I change my aim
Throw my troubles at the pearly gates”

–“Burning Pile”, Mother Mother

April 1, 2020

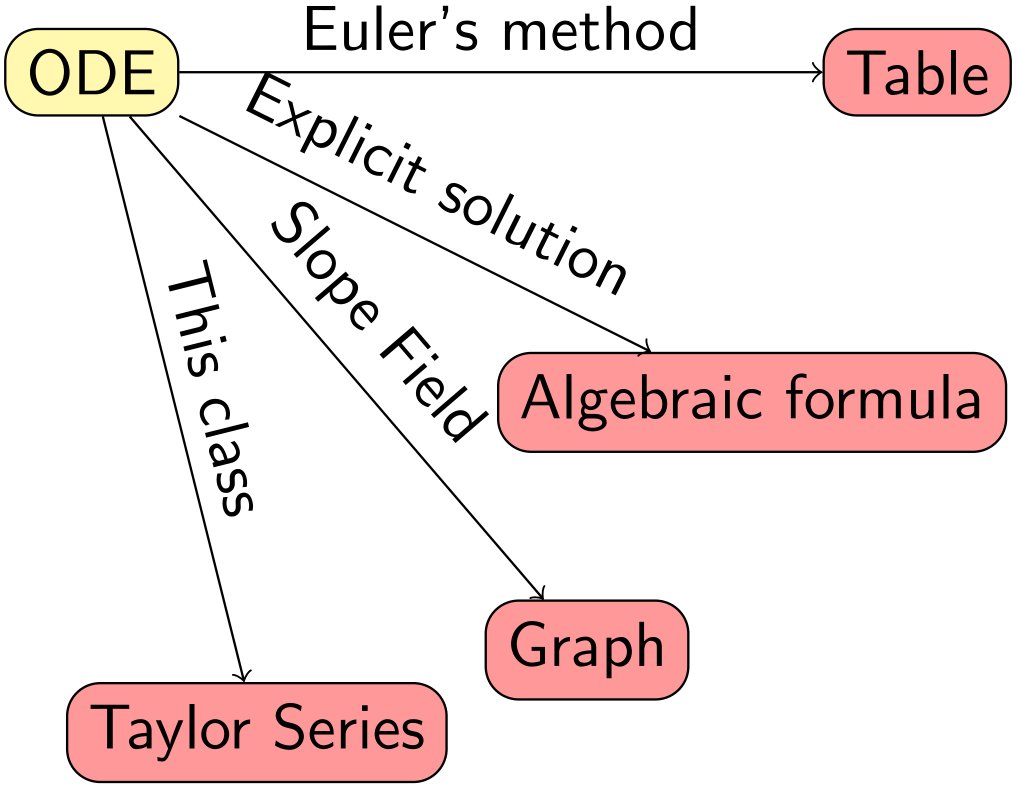
Welcome to MAT136 LEC0501 (Assaf)

COURSE EVALUATIONS!!!!!!

`http://uoft.me/openevals`

Last time: What is a Solution?

How do we solve an ODE?


ODE

Euler's method

Table

Explicit solution

Algebraic formula

Slope Field

Graph

This class

Taylor Series

Depending on the ODE, and depending on how we want our solution presented, we use different solution strategies.

Today's Main Idea

Express a function as a Taylor polynomial, and solve for the coefficients.

A First Example

We will try to solve:

$$y'' - 2y' + y = 0$$

$$y(0) = 0 \qquad y'(0) = 1$$

This equation is **not** separable, and we do not have other techniques to solve it.

The Steps

- Write the solution (which we want to find) as a Taylor series
- Find the Taylor series for every term in the differential equation
- Group together like terms
- Write out the differential equation as a Taylor series equation
- Solve for the coefficients
- (Hopefully) Identify the Taylor series as a known function

Step 1: Writing Taylor series

We will try to solve:

$$y'' - 2y' + y = 0$$
$$y(0) = 0 \qquad y'(0) = 1$$

We write

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

From now on, assume that our solution has this form.

Step 2: Find the Taylor Series of the other terms

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Q: What are the formulas for y'' and $2y'$?

Step 2: Find the Taylor Series of the other terms

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Q: What are the formulas for y'' and $2y'$?

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$2y' = 2\sum_{n=1}^{\infty} n a_n x^{n-1}$$

Step 3: Group Terms

$$y'' - 2y' + y = 0$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = 2a_2 + 6a_3 x + \cdots$$

$$2y' = \sum_{n=1}^{\infty} 2na_n x^{n-1} = 2a_1 + 4a_2 x + 6a_3 x^2 + \cdots$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

We want to add up these series. **Q:** What is the constant term in $y'' - 2y' + y$?

Step 3: Group Terms

$$y'' - 2y' + y = 0$$

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We want to add up these series. **Q:** What is the constant term in $y'' - 2y' + y$?

A: The constant term is $2a_2 - 2a_1 + a_0$

Step 3: Group Terms

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$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = 2a_2 + 6a_3 x + \cdots$$

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We want to add up these series. **Q:** What is the linear term in $y'' - 2y' + y$?

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We want to add up these series. **Q:** What is the linear term in $y'' - 2y' + y$?

A: The linear term is $6a_3 x - 4a_2 x + a_1 x$

Step 3: Group Terms

$$y'' - 2y' + y = 0$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = 2a_2 + 6a_3 x + \cdots$$

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We want to add up these series. **Q:** What is the quadratic term in $y'' - 2y' + y$?

Step 3: Group Terms

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$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = 2a_2 + 6a_3 x + \cdots$$

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We want to add up these series. **Q:** What is the quadratic term in $y'' - 2y' + y$?

A: The quadratic term is $12a_4x^2 - 6a_3x^2 + a_2x^2$

Step 3: Group Terms

$$y'' - 2y' + y = 0$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = 2a_2 + 6a_3 x + \cdots$$

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Q: In general, what is the coefficient of x^n?

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Q: In general, what is the coefficient of x^n?

$$(n+2)(n+1)a_{n+2}x^n - 2(n+1)a_{n+1}x^n + a_n x^n$$

Step 4: Write the Equation as a series

We know: $y'' - 2y' + y = 0$, so when expressing this equation as a series, we get:

$$0 = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - 2(n+1)a_{n+1}x^n + a_n x^n$$

This means that every coefficient here needs to be 0.

In other words:

$$(n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} + a_n = 0$$

Step 5: Solve for the coefficients

$$(n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} + a_n = 0$$

This means:

$$0 = 2a_2 - 2a_1 + a_0$$

Q: Knowing $y(0) = 0$ and $y'(0) = 1$, what does this tell us about a_0 and a_1?

Step 5: Solve for the coefficients

$$(n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} + a_n = 0$$

This means:

$$0 = 2a_2 - 2a_1 + a_0$$

Q: Knowing $y(0) = 0$ and $y'(0) = 1$, what does this tell us about a_0 and a_1? **A:** $a_0 = 0$ and $a_1 = 1$
Q: What is a_2?

Takeaway

Using an initial condition and the differential equation, we can write the solution as a Taylor polynomial, and solve for the coefficients

This is an entirely new way to solve ODEs!

Step 5: Solving For the Coefficients

$$0 = (n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} + a_n$$

Or:

$$a_{n+2} = \frac{2a_{n+1}}{(n+2)} - \frac{a_n}{(n+1)(n+2)}$$

We know $a_0 = 0$, $a_1 = 1$, and $a_2 = 1$. Use this to find a_3 and a_4.

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Or:

$$a_{n+2} = \frac{2a_{n+1}}{(n+2)} - \frac{a_n}{(n+1)(n+2)}$$

We know $a_0 = 0$, $a_1 = 1$, and $a_2 = 1$. Use this to find a_3 and a_4. The sequence turns out to be...

$$0, 1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}$$

Interlude: Properties of the solution

Knowing that $a_0 = 0$, $a_1 = 1$, and $a_2 = 1$, what does this tell us about the shape of the solution at $x = 0$?

Interlude: Properties of the solution

Knowing that $a_0 = 0$, $a_1 = 1$, and $a_2 = 1$, what does this tell us about the shape of the solution at $x = 0$?

The solution is positive, increasing, and concave up at $x = 0$

Takeaway

We can get information about convexity or other properties of the function by looking at the coefficients of its solution. For example, increasing, decreasing, concavity,...

Step 6: Identify the Function

The solution to the differential equation:

$$y'' - 2y' + y = 0$$

$$y(0) = 0y'(0) = 1$$

is:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

We've checked, and saw that $a_{n+1} = \frac{1}{n!}$, and $a_0 = 0$ So:

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^{n+1}$$

Step 6: Identify the Function

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Q: Can you identify this function?

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$$y(0) = 0y'(0) = 1$$

is:

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^{n+1}$$

Q: Can you identify this function?
A: This is xe^x.

Plans for the Future

For next time:

Pick a lesson in the course. Write down the list of concepts, and how they connect to each other

Welcome to MAT136 LEC0501 (Assaf)

COURSE EVALUATIONS!!!!!!

`http://uoft.me/openevals`

Review Session

Assaf Bar-Natan

"You vitriolic, patriotic, slam fight, bright light
Feeling pretty psyched
It's the end of the world as we know it
It's the end of the world as we know it
It's the end of the world as we know it and I feel fine"

–"It's the End of the World as we Know it", R.E.M

April 3, 2020

Today's Plan

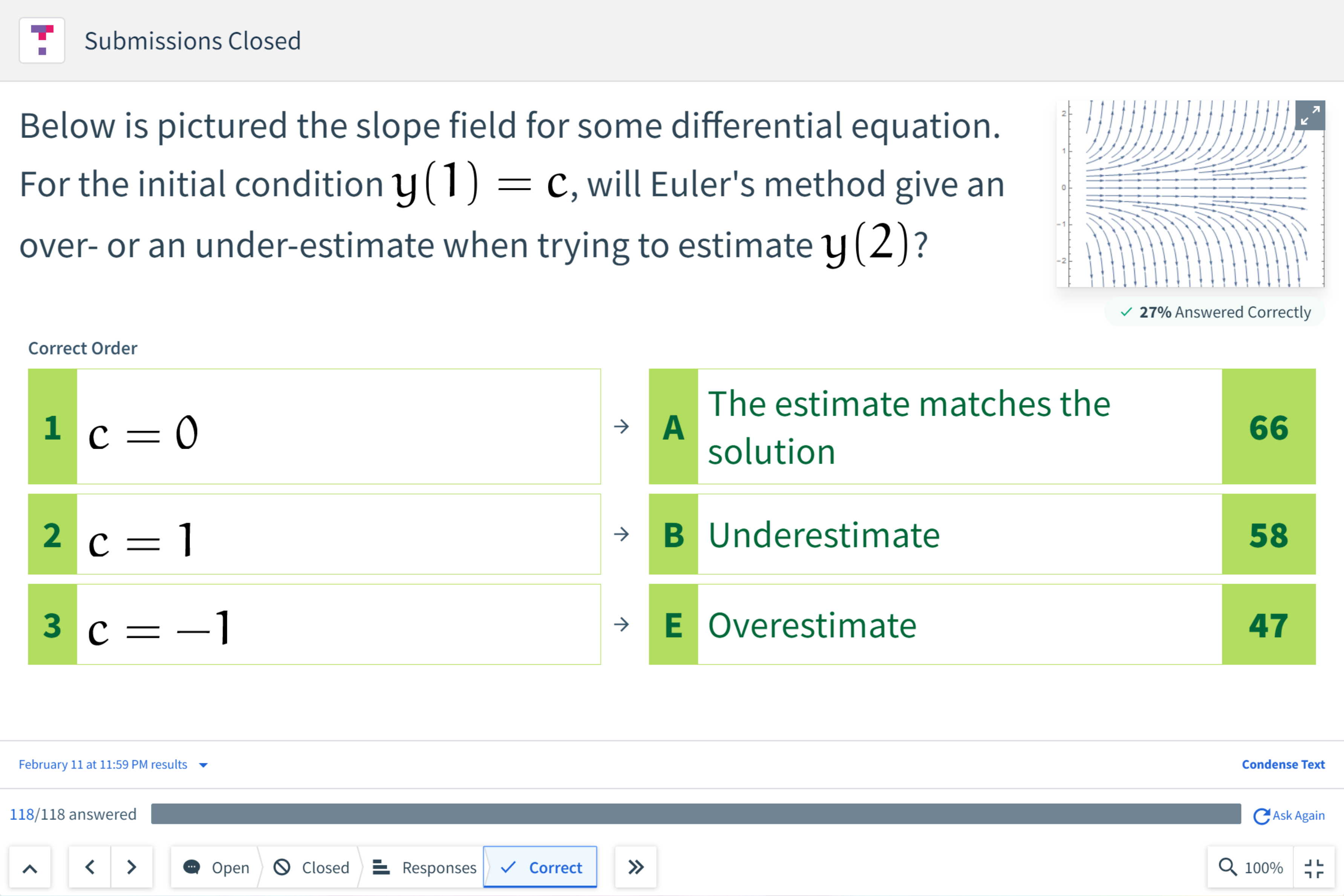
Here's what we will do today. For every unit, you will:

- Pick a TopHat question or textbook question that you found challenging or informative
- Identify the goals and ideas behind that question
- Share where this goal fits in the bigger picture
- Add it to the communal concept map: `https://witeboard.com/8e0615c0-7548-11ea-afcf-8f5dcb39ca2d`
- Then I will do the same.

We will do this for units $3, 4, 5, 6$, as these are the units that were not covered in the midterm (YOU STILL NEED TO STUDY THEM)

Unit 3 – Differential Equations

- Pick a TopHat question or textbook question that you found challenging or informative
- Identify the goals and ideas behind that question
- Share where this goal fits in the bigger picture
- Add it to the communal concept map: `https://witeboard.com/8e0615c0-7548-11ea-afcf-8f5dcb39ca2d`



Below is pictured the slope field for some differential equation. For the initial condition $y(1) = c$, will Euler's method give an over- or an under-estimate when trying to estimate $y(2)$?

✓ **27%** Answered Correctly

Correct Order

1	$c = 0$	→	**A**	The estimate matches the solution	**66**
2	$c = 1$	→	**B**	Underestimate	**58**
3	$c = -1$	→	**E**	Overestimate	**47**

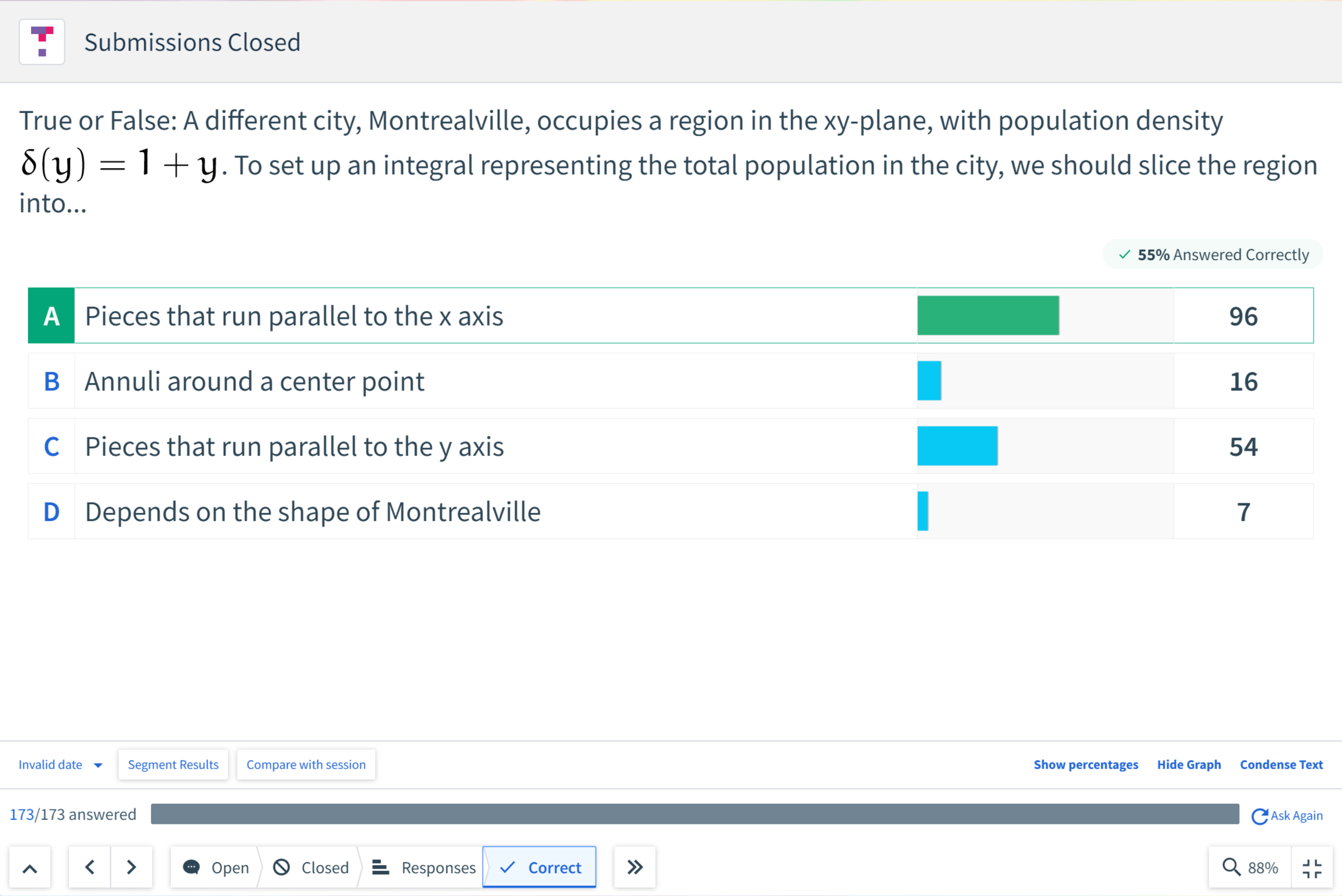
February 11 at 11:59 PM results

Condense Text

118/118 answered

Unit 4 – Slicing

- Pick a TopHat question or textbook question that you found challenging or informative
- Identify the goals and ideas behind that question
- Share where this goal fits in the bigger picture
- Add it to the communal concept map: `https://witeboard.com/8e0615c0-7548-11ea-afcf-8f5dcb39ca2d`



Submissions Closed

True or False: A different city, Montrealville, occupies a region in the xy-plane, with population density $\delta(y) = 1 + y$. To set up an integral representing the total population in the city, we should slice the region into...

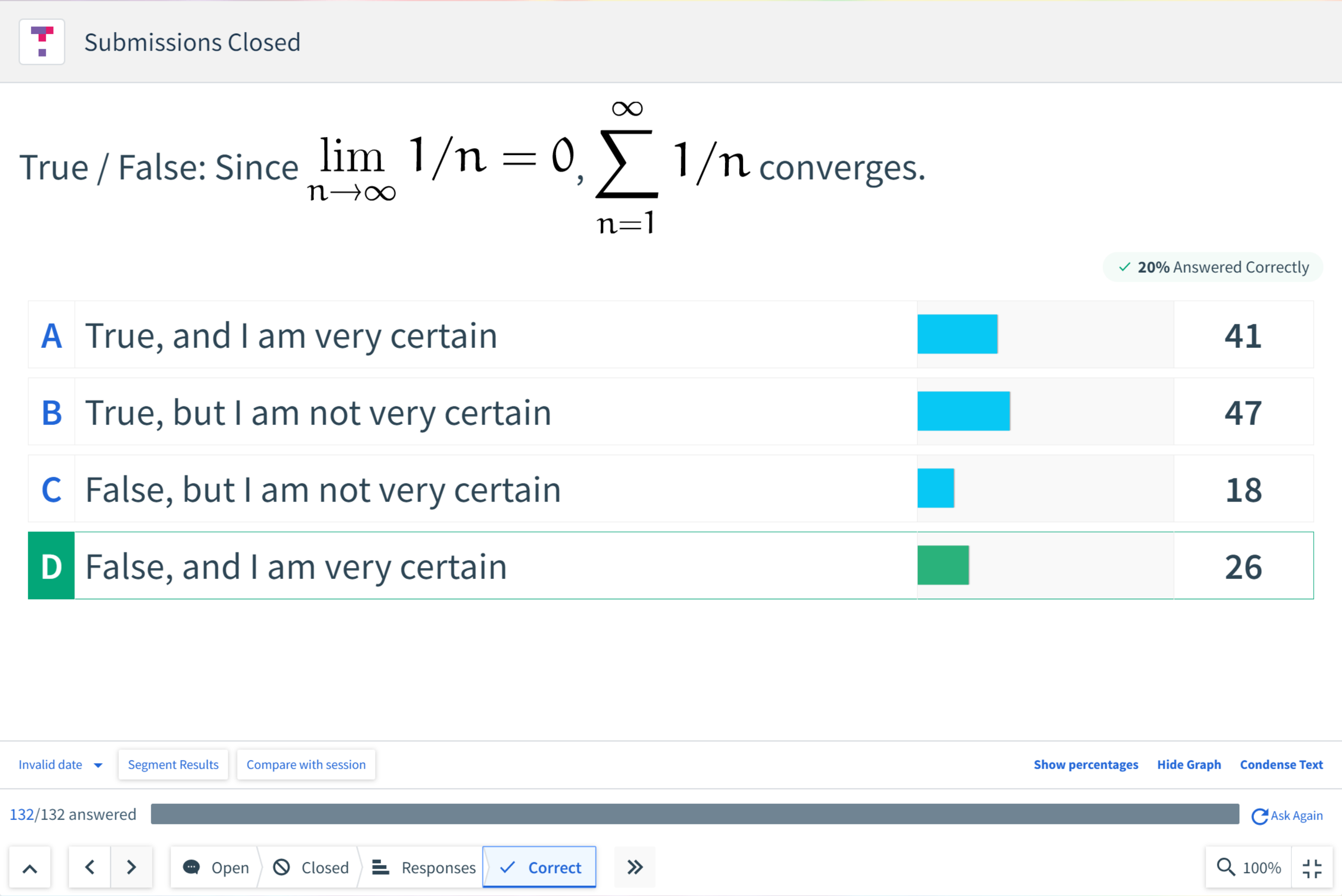
✓ **55%** Answered Correctly

A	Pieces that run parallel to the x axis	96
B	Annuli around a center point	16
C	Pieces that run parallel to the y axis	54
D	Depends on the shape of Montrealville	7

173/173 answered

Unit 5 – Sequences and Series

- Pick a TopHat question or textbook question that you found challenging or informative
- Identify the goals and ideas behind that question
- Share where this goal fits in the bigger picture
- Add it to the communal concept map: `https://witeboard.com/8e0615c0-7548-11ea-afcf-8f5dcb39ca2d`



Submissions Closed

True / False: Since $\lim_{n\to\infty} 1/n = 0$, $\sum_{n=1}^{\infty} 1/n$ converges.

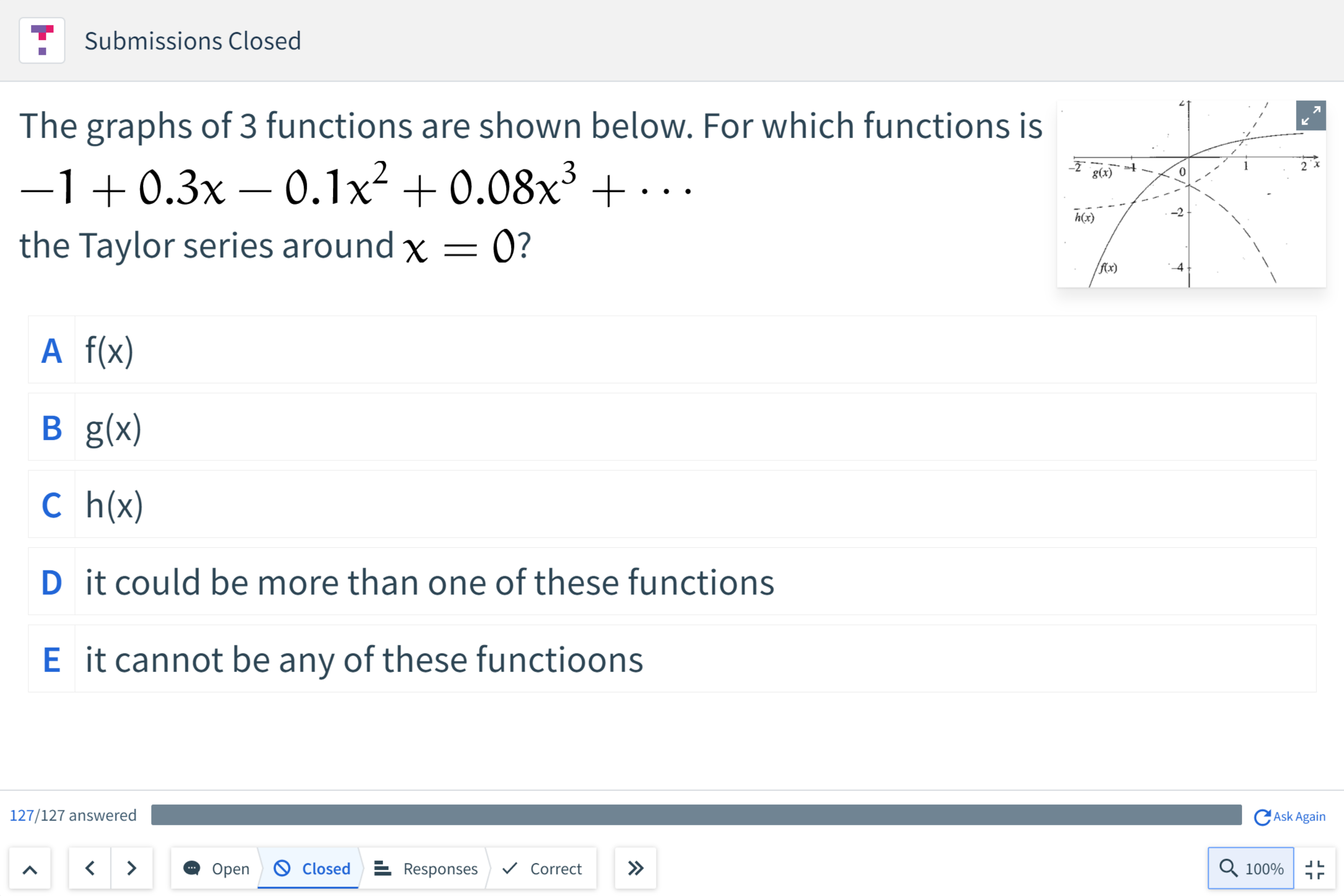
✓ **20%** Answered Correctly

A	True, and I am very certain	41
B	True, but I am not very certain	47
C	False, but I am not very certain	18
D	False, and I am very certain	26

132/132 answered

Unit 6 – Taylor Series & Taylor Polynomials

- Pick a TopHat question or textbook question that you found challenging or informative
- Identify the goals and ideas behind that question
- Share where this goal fits in the bigger picture
- Add it to the communal concept map: `https://witeboard.com/8e0615c0-7548-11ea-afcf-8f5dcb39ca2d`



The graphs of 3 functions are shown below. For which functions is

$$-1 + 0.3x - 0.1x^2 + 0.08x^3 + \cdots$$

the Taylor series around $x = 0$?

A f(x)

B g(x)

C h(x)

D it could be more than one of these functions

E it cannot be any of these functioons

127/127 answered

Resource Reminder

In addition to everything on the main site:

- Lec. 16 Study Tips TopHat Discussion
- Your groups from lecture
- Assaf will post a list of **ALL** course learning objectives together
- Old TopHat questions

Plans for the Future

For next time:

There is no next time. I'm going to miss you. I only wish I could have said goodbye in person