Welcome to MAT135 LEC0501 (Assaf)

Now is a good time to think about the midterm!

S6.4 – The Other Fundamental Theorem – The Construction Theorem

Assaf Bar-Natan

"Try to change.
I try to change.
I make a list of all the ways to change my ways.
But I stay the same,
I stay the s-ame."

–"Try To Change", Mother Mother

Jan. 22, 2020

Functions Defined by Integrals

We've encountered many functions in MAT135, and in this course:

- Some functions are given as tables, verbally, or as a graph...
- Some functions are defined algebraically or geometrically: trigonometric, polynomials, exponents..
- Some functions are inverses or compositions of others: $\sin(e^{3x+5})$, $\log(x)$, $\log^2(x)$...

Functions Defined by Integrals

We've encountered many functions in MAT135, and in this course:

- Some functions are given as tables, verbally, or as a graph...
- Some functions are defined algebraically or geometrically: trigonometric, polynomials, exponents..
- Some functions are inverses or compositions of others: $\sin(e^{3x+5})$, $\log(x)$, $\log^2(x)$...

Today: Functions defined as integrals of other functions:

$$f(x) = \int_a^x g(t)dt$$

where a is some constant.

Functions Defined by Integrals

Some examples:

$$Si(x) = \int_0^x \frac{\sin(t)}{t} dt$$

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$li(x) = \int_0^x \frac{1}{\log(t)} dt$$

(log is the natural logarithm here)

Submissions Closed

A table of values of a function $p(t)$ is shown below. Consider the function $S(y) = \int_8^y p(t)\,dt$. Which of the following is the best estimate for $S(5)$, given the information provided

t	5	8	10	12
$p(t)$	10	7	3	1

✓ **50%** Answered Correctly

A	-22.5	106
B	-9	67
C	9	27
D	22.5	10

Invalid date | Segment Results | Compare with session | **Show percentages** | **Hide Graph** | **Condense Text**

210/210 answered | Ask Again

Open | Closed | Responses | Correct | 100%

What's The Difference

Let's say that we have a function, $f(x)$. In groups, write an explanation of the difference between:

- A definite integral of f.
- The antiderivatives of f.
- A function defined by an integral of f.

What's The Difference

Let's say that we have a function, $f(x)$. In groups, write an explanation of the difference between:

- A definite integral of f.
- The antiderivatives of f.
- A function defined by an integral of f.

Hint: think about the definitions!

What's The Difference

Let's say that we have a function, $f(x)$. In groups, write an explanation of the difference between:

- A definite integral of f. **This is a number.**
- The antiderivatives of f. **This is a family of functions whose derivative is f.**
- A function defined by an integral of f. **This is a function defined by an expression of the form $\int_a^x f(t)dt$.**

Hint: think about the definitions!

The Construction Theorem

Let $f(t)$ be a continuous function defined everywhere, and we will write $F(x) = \int_a^x f(t)dt$.

The Construction Theorem

Let $f(t)$ be a continuous function defined everywhere, and we will write $F(x) = \int_a^x f(t)dt$.

Write the limit definition of the derivative of F

The Construction Theorem

Let $f(t)$ be a continuous function defined everywhere, and we will write $F(x) = \int_a^x f(t)dt$. We have:

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$$

We can rewrite this as:

$$F'(x) = \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t)dt$$

The Construction Theorem

Let $f(t)$ be a continuous function defined everywhere, and we will write $F(x) = \int_a^x f(t)dt$. We have:

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

We can rewrite this as:

$$F'(x) = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)dt$$

Explain why we can do this to your neighbour

The Construction Theorem

Let $f(t)$ be a continuous function defined everywhere, and we will write $F(x) = \int_a^x f(t)dt$. We have:

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)dt$$

The Construction Theorem

Let $f(t)$ be a continuous function defined everywhere, and we will write $F(x) = \int_a^x f(t)dt$. We have:

$$F'(x) = \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)dt$$

Draw a picture representing the integral above as a small rectangle. What does the limit above equal to?

The Construction Theorem

Theorem

(Construction Theorem, or, the Second Fundamental Theorem of Calculus)
If f is continuous, then the function defined by the integral $F(x) = \int_a^x f(t)dt$ satisfies $F'(x) = f(x)$.

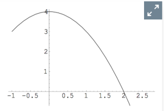
Takeaway

Functions defined by integrals are antiderivatives of the integrands

Submissions Closed

Below is the graph of a function f. Let $g(x) = \int_0^x f(t)\,dt$.

Then for $0 < x < 2$, $g(x)$ is:



✓ **83%** Answered Correctly

A	increasing and concave up	10
B	increasing and concave down	173
C	decreasing and concave up	9
D	decreasing and concave down	17

Invalid date · Segment Results · Compare with session · **Show percentages** · **Hide Graph** · **Condense Text**

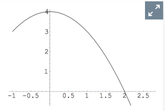
209/209 answered · Ask Again

Open · Closed · Responses · ✓ Correct · 100%

Submissions Closed

Below is the graph of a function f. Let $g(x) = \int_0^x f(t)\,dt$.

Then:



✓ **88%** Answered Correctly

A	$g(0) = 0, g'(0) = 0, g'(2) = 0$	14
B	$g(0) = 0, g'(0) = 4, g'(2) = 0$	183
C	$g(0) = 1, g'(0) = 0, g'(2) = 1$	2
D	$g(0) = 0, g'(0) = 0, g'(2) = 1$	10

Invalid date | Segment Results | Compare with session | **Show percentages** | **Hide Graph** | **Condense Text**

209/209 answered | Ask Again

Open | Closed | Responses | Correct | 100%

Hard Derivatives

We define:

$$F(x) = \int_5^{e^x} \frac{\sin(t)}{t} dt$$

Our goal is to find $F'(x)$.

Hard Derivatives

We define:

$$F(x) = \int_5^{e^x} \frac{\sin(t)}{t} dt$$

Our goal is to find $F'(x)$.

- Use $Si(x)$, and the net change theorem to write $F(x)$ explicitly.

Hard Derivatives

We define:

$$F(x) = \int_5^{e^x} \frac{\sin(t)}{t} dt$$

Our goal is to find $F'(x)$.

- Use $Si(x)$, and the net change theorem to write $F(x)$ explicitly.
- Use differentiation rules to compute $F'(x)$

Hard Derivatives

We define:

$$F(x) = \int_5^{e^x} \frac{\sin(t)}{t} dt$$

Our goal is to find $F'(x)$.

- Use $Si(x)$, and the net change theorem to write $F(x)$ explicitly.
- Use differentiation rules to compute $F'(x)$
- Use the construction theorem to simplify

Bonus: replace $\frac{\sin(t)}{t}$ with $\sin(t^3)$. How does your solution change?

Hard Derivatives

We define:

$$F(x) = \int_5^{e^x} \frac{\sin(t)}{t} dt$$

Our goal is to find $F'(x)$.

- Use $Si(x)$, and the net change theorem to write $F(x)$ explicitly. $F(x) = Si(e^x) - Si(5)$
- Use differentiation rules to compute $F'(x)$. **By the chain rule:** $F'(x) = Si'(e^x) \cdot e^x$
- Use the construction theorem to simplify. **Since** $Si(x)$ **is an antiderivative of** $\frac{\sin(x)}{x}$**, we get:** $F'(x) = \frac{\sin(e^x)}{e^x} e^x = \sin(e^x)$

Bonus: replace $\frac{\sin(t)}{t}$ with $\sin(t^3)$. How does your solution change?
We get $\sin(e^{3x})e^x$

Submissions Closed

If $f(t) = \int_t^7 \cos x dx$, then:

✓ 63% Answered Correctly

A	$f'(t) = \cos(t)$	29
B	$f'(t) = \sin(t)$	16
C	$f'(t) = \sin(7) - \sin(t)$	17
D	$f'(t) = -\cos(t)$	127
E	$f'(t) = -\sin(t)$	12

Invalid date | Segment Results | Compare with session | Show percentages | Hide Graph | Condense Text

201/201 answered | Ask Again

Open | Closed | Responses | ✓ Correct | 100%

Plans for the Future

For next time:

WeBWork 7.1 and read section 7.1