Welcome to MAT136 LEC0501 (Assaf)

Today: ODEs

Friday: Review

COURSE EVALUATIONS!!!!!!

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Taylor Expansions and ODEs – Part 2

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"It goes, all my troubles on a burning pile
All lit up and I start to smile
If I, catch fire then I change my aim
Throw my troubles at the pearly gates"

–"Burning Pile", Mother Mother

April 1, 2020

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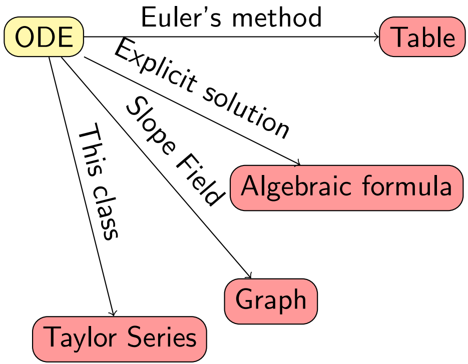
Last time: What is a Solution?

How do we solve an ODE?



Depending on the ODE, and depending on how we want our solution presented, we use different solution strategies.

Today's Main Idea

Express a function as a Taylor polynomial, and solve for the coefficients.

A First Example

We will try to solve:

$$y'' - 2y' + y = 0$$
$$y(0) = 0 \qquad y'(0) = 1$$

This equation is **not** separable, and we do not have other techniques to solve it.

The Steps

- Write the solution (which we want to find) as a Taylor series
- Find the Taylor series for every term in the differential equation
- Group together like terms
- Write out the differential equation as a Taylor series equation
- Solve for the coefficients
- (Hopefully) Identify the Taylor series as a known function

Step 1: Writing Taylor series

We will try to solve:

$$y'' - 2y' + y = 0$$
$$y(0) = 0 \qquad y'(0) = 1$$

We write

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

From now on, assume that our solution has this form.

Step 2: Find the Taylor Series of the other terms

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

Q: What are the formulas for y'' and $2y'$?

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Q: What are the formulas for y'' and $2y'$?

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$2y' = 2\sum_{n=1}^{\infty} n a_n x^{n-1}$$

Step 3: Group Terms

$$y'' - 2y' + y = 0$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = 2a_2 + 6a_3 x + \cdots$$

$$2y' = \sum_{n=1}^{\infty} 2na_n x^{n-1} = 2a_1 + 4a_2 x + 6a_3 x^2 + \cdots$$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

We want to add up these series. **Q:** What is the constant term in $y'' - 2y' + y$?

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We want to add up these series. **Q:** What is the constant term in $y'' - 2y' + y$?

A: The constant term is $2a_2 - 2a_1 + a_0$

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A: The linear term is $6a_3 x - 4a_2 x + a_1 x$

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A: The quadratic term is $12a_4x^2 - 6a_3x^2 + a_2x^2$

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Q: In general, what is the coefficient of x^n?

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Q: In general, what is the coefficient of x^n?

$$(n+2)(n+1)a_{n+2}x^n - 2(n+1)a_{n+1}x^n + a_n x^n$$

Step 4: Write the Equation as a series

We know: $y'' - 2y' + y = 0$, so when expressing this equation as a series, we get:

$$0 = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - 2(n+1)a_{n+1}x^n + a_n x^n$$

This means that every coefficient here needs to be 0.
In other words:

$$(n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} + a_n = 0$$

Step 5: Solve for the coefficients

$$(n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} + a_n = 0$$

This means:

$$0 = 2a_2 - 2a_1 + a_0$$

Q: Knowing $y(0) = 0$ and $y'(0) = 1$, what does this tell us about a_0 and a_1?

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This means:

$$0 = 2a_2 - 2a_1 + a_0$$

Q: Knowing $y(0) = 0$ and $y'(0) = 1$, what does this tell us about a_0 and a_1? **A:** $a_0 = 0$ and $a_1 = 1$

Q: What is a_2?

Takeaway

Using an initial condition and the differential equation, we can write the solution as a Taylor polynomial, and solve for the coefficients

This is an entirely new way to solve ODEs!

Step 5: Solving For the Coefficients

$$0 = (n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} + a_n$$

Or:

$$a_{n+2} = \frac{2a_{n+1}}{(n+2)} - \frac{a_n}{(n+1)(n+2)}$$

We know $a_0 = 0$, $a_1 = 1$, and $a_2 = 1$. Use this to find a_3 and a_4.

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We know $a_0 = 0$, $a_1 = 1$, and $a_2 = 1$. Use this to find a_3 and a_4. The sequence turns out to be...

$$0, 1, 1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}$$

Interlude: Properties of the solution

Knowing that $a_0 = 0$, $a_1 = 1$, and $a_2 = 1$, what does this tell us about the shape of the solution at $x = 0$?

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The solution is positive, increasing, and concave up at $x = 0$

Takeaway

We can get information about convexity or other properties of the function by looking at the coefficients of its solution. For example, increasing, decreasing, concavity,...

Step 6: Identify the Function

The solution to the differential equation:

$$y'' - 2y' + y = 0$$
$$y(0) = 0y'(0) = 1$$

is:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

We've checked, and saw that $a_{n+1} = \frac{1}{n!}$, and $a_0 = 0$ So:

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^{n+1}$$

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Q: Can you identify this function?

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Q: Can you identify this function?

A: This is xe^x.

Plans for the Future

For next time:

Pick a lesson in the course. Write down the list of concepts, and how they connect to each other