Welcome to MAT136 LEC0501 (Assaf)

COURSE EVALUATIONS ARE OPEN! PLEASE PLEASE PLEASE SUBMIT THEM!!!

Especially important: feedback about online learning, the transition process, the overall course structure given the situation, and your suggestions for next semester if we still need to do things online by then.

S10.2 – Taylor Series – Back Again

Assaf Bar-Natan

"I never knew I'd love this world they've let me into
And the memories were lost long ago
So I'll dance with these beautiful ghosts"

–"Beautiful Ghosts (Cats movie)", Taylor Swift

March 25, 2020

Review: Taylor Polynomials

Recall that if f is some function, we can approximate f around a using a Taylor polynomial:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

where x is close to a.

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where x is close to a.
Use the following Geogebra applet to investigate what happens when n gets big:

https://www.geogebra.org/m/s9SkCsvC

The Taylor Series

Take a Taylor polynomial to the extreme, and use a **power series** to approximate f:

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

This is called the **Taylor Series of f at $x = a$**

Submissions Closed

True or False: A Taylor series always converges

A True

B False

C Depends on the function

0/8 answered

Open | Closed | Responses | Correct | 100%

Takeaway

The Taylor series is a power series, so, just like any power series, it might converge for some values of x and diverge for other values of x.

Submissions Closed

For what values of x is it possible that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots ?$$

A This is true for all values of x because of the Taylor series formula

B This may only be true for $x > -1$ because of our graphical evaluation (geogebra)

C This may only be true for $-1 < x < 1$ because the series doesn't have a finite value for other values of x

0/8 answered

Open Closed Responses Correct 100%

Takeaway

We use the ratio test to check when a Taylor series converges

Submissions Closed

For which values of x is it possible that $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

A This is true for all values of x because of the Taylor series formula.

B This appears to be true for all values of x based on the graphical evaluation (geogebra).

C This may only be true for $-5 < x < 5$ because the series doesn't have a value for other values of x

D For all values of x, because of the ratio test

0/34 answered

Open | Closed | Responses | Correct | 100%

The Miracle of Taylor Series

I'd like to use my Taylor series to approximate sin(1000000). I know:

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

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Let me try again....

$$\sin(1000000) \approx 1000000 - \frac{(1000000)^3}{6} \approx -1.6 \times 10^{17}$$

Wow! That's even worse than the first time... Let me try again...

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$$\sin(1000000) \approx 1000000 - \frac{(1000000)^3}{6} + \frac{(1000000)^5}{5!} \approx 8.3 \times 10^{27}$$

The Miracle of Taylor Series

My approximations on the previous slide were trash.
$-1 < \sin(1000000) < 1$, but I kept getting absurdly high numbers.
Q: What is something I can do to get good approximations of $\sin(1000000)$?

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Q: What is something I can do to get good approximations of $\sin(1000000)$?

- I could choose to expand at a point close to 1000000
- I could keep going, and take many many many more terms in the series
- I could use the fact that sin is periodic, and get that $\sin(1000000) = \sin(x)$, where $-\pi < x < \pi$. Then I could approximate.

Takeaway

Some functions have Taylor series that have an infinite radius of convergence (eg: sin, cos, e^x). For these functions, the Taylor series always converges, but it might converge very slowly!

Check that sin, cos, and e^x indeed have this property:
`https://www.geogebra.org/m/s9SkCsvC`

Euler's Identity

We know:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

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Let $i = \sqrt{-1}$ be an imaginary number (don't worry about it, just pretend that all algebra works the same, but $i^2 = -1$).

Q: Write the Taylor series for e^{ix}.

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These look related

Let $i = \sqrt{-1}$ be an imaginary number (don't worry about it, just pretend that all algebra works the same, but $i^2 = -1$).

Q: Write the Taylor series for e^{ix}.

$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

Euler's Identity

We know:

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \cdots$$

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Q: Relate e^{ix}, $\cos(x)$, and $\sin(x)$ using the above.

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$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Q: Relate e^{ix}, $\cos(x)$, and $\sin(x)$ using the above. Hint: multiply $\sin(x)$ by i...

$$e^{ix} = \cos(x) + i\sin(x)$$

Q: Compute $e^{i\pi}$.

$$e^{i\pi} = -1$$

A good explanation of this:
`https://www.youtube.com/watch?v=v0YEaeIClKY`

Plans for the Future

For next time:

Watch the week 11 videos, do WeBWork 10.3, actively read section 10.3

Submissions Closed

Since

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots,$$

$$-1 = \frac{1}{1-2} = 1 + 2 + 4 + 8 + \cdots.$$

✓ **52%** Answered Correctly

A	True	62
B	False	67

Invalid date ▾ | Segment Results | Compare with session | Show percentages | Hide Graph | Condense Text

129/129 answered | Ask Again

Open | Closed | Responses | ✓ Correct | 72%

Submissions Closed

Let

$$g(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Is $x = 2$ in the domain of $g(x)$?

✓ **41%** Answered Correctly

A	Yes: the series converges by the Ratio Test	53
B	Yes, the series converges by the Integral Test	30
C	No, the series diverges by the Ratio Test	40
D	No, the series diverges by the Integral Test	5

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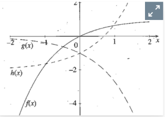
128/128 answered | Ask Again

Open | Closed | Responses | Correct | 72%

Submissions Closed

The graphs of 3 functions are shown below. For which functions is
$$-1 + 0.3x - 0.1x^2 + 0.08x^3 + \cdots$$
the Taylor series around $x = 0$?



✓ **12%** Answered Correctly

A	f(x)	25
B	g(x)	30
C	h(x)	27
D	it could be more than one of these functions	30
E	it cannot be any of these functioons	15

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127/127 answered | Ask Again

Open | Closed | Responses | ✓ Correct | 72%

Submissions Closed

Compute

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}$$

using a Taylor series approximation.

✓ **14%** Answered Correctly

1	17
-0.25	1
0	35
0.21	1
0.5	45
1.32079632	1
2	12

Invalid date

Show percentages **Hide Graph** **Condense Text**

123/123 answered

Ask Again

Open Closed Responses Correct

72%