Welcome to MAT136 LEC0501 (Assaf)

`https://tinyurl.com/Unit2-3CIQ`

Taylor Expansions and ODEs

Assaf Bar-Natan

"The game has been disbanded
My mind has been expanded"

–"Rose Tint My World", Susan Sarandon, et. al.

March 3, 2020

Critical Incident Questionnaire

`https://tinyurl.com/Unit2-3CIQ`

What is a Solution?

How do we solve an ODE?

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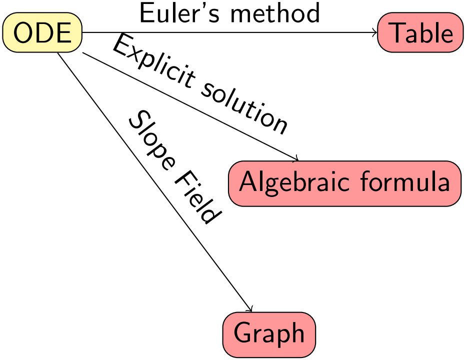


ODE

Euler's method

Table

Explicit solution

Algebraic formula

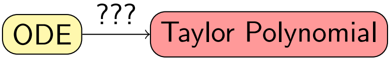
Slope Field

Graph

Takeaway

Depending on the ODE, and depending on how we want our solution presented, we use different solution strategies.

A New Way: Taylor Solutions



ODE

???

Taylor Polynomial

Key idea: Express a function as a Taylor polynomial, and solve for the coefficients.

A First Example

$$\frac{dy}{dx} = y$$
$$y(0) = 2$$

We compute the Taylor expansion of y around 0:

$$y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

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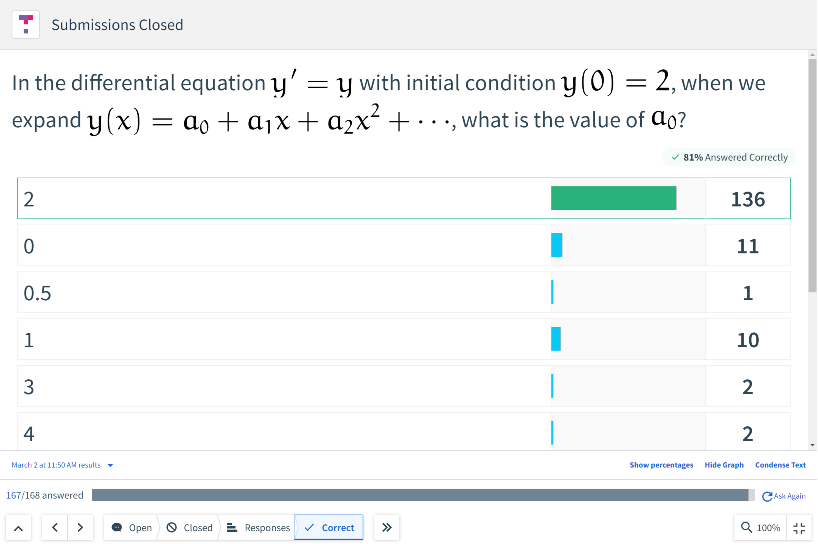
Sanity check: What is the formula for a_2? $a_2 = \frac{y''(0)}{2}$
Now, we differentiate:

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$



Submissions Closed

In the differential equation $y' = y$ with initial condition $y(0) = 2$, when we expand $y(x) = a_0 + a_1x + a_2x^2 + \cdots$, what is the value of a_0?

✓ **81%** Answered Correctly

2	136
0	11
0.5	1
1	10
3	2
4	2

March 2 at 11:50 AM results

167/168 answered

A First Example

$$\frac{dy}{dx} = y \qquad y(0) = 2$$

We have:

$$y(x) = 2 + a_1x + a_2x^2 + a_3x^3 + \cdots$$
$$y' = a_1 + 2a_2x + 3a_3x^2 + \cdots$$

To solve the equation, we set them to be equal to each other!

$$\begin{aligned} a_1 + 2a_2x + 3a_3x^2 + \cdots = y' = y \\ =a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \end{aligned}$$

A First Example

$$\frac{dy}{dx} = y \qquad y(0) = 2$$

We have:

$$\begin{aligned} y(x) &= 2 + a_1x + a_2x^2 + a_3x^3 + \cdots \\ y' &= a_1 + 2a_2x + 3a_3x^2 + \cdots \end{aligned}$$

To solve the equation, we set them to be equal to each other!

$$\begin{aligned} &a_1 + 2a_2x + 3a_3x^2 + \cdots = y' = y \\ =&a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \end{aligned}$$

Q: What is a_2 in terms of a_1?

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Q: What is a_2 in terms of a_1? $a_2 = \frac{a_1}{2}$

Sanity Check

We have that for the differential equation $y' = y$,

$$\frac{y''(0)}{2} = a_2 = \frac{a_1}{2}$$

but we also know: $a_1 = y'(0)$ (Taylor polynomial) and: $y'(0) = y''(0)$ (y is a solution to the ODE).

This coincides with what we have!

A First Example

$$\frac{dy}{dx} = y$$
$$y(0) = 2$$

We have:

$$y(x) = 0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$
$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

Check that $a_n = \frac{2}{n!}$
Q: What is $y(x)$?

A First Example

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$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

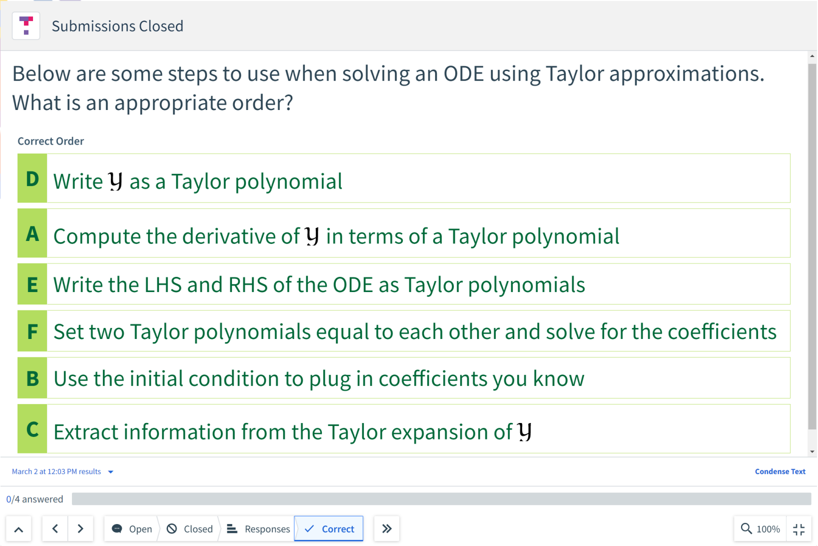
Check that $a_n = \frac{2}{n!}$
Q: What is $y(x)$?

$$y(x) = \sum_{n=0}^{\infty} \frac{2}{n!} x^n = 2e^x$$

Takeaway

Using an initial condition and the differential equation, we can write the solution as a Taylor polynomial, and solve for the coefficients

This is an entirely new way to solve ODEs!



Submissions Closed

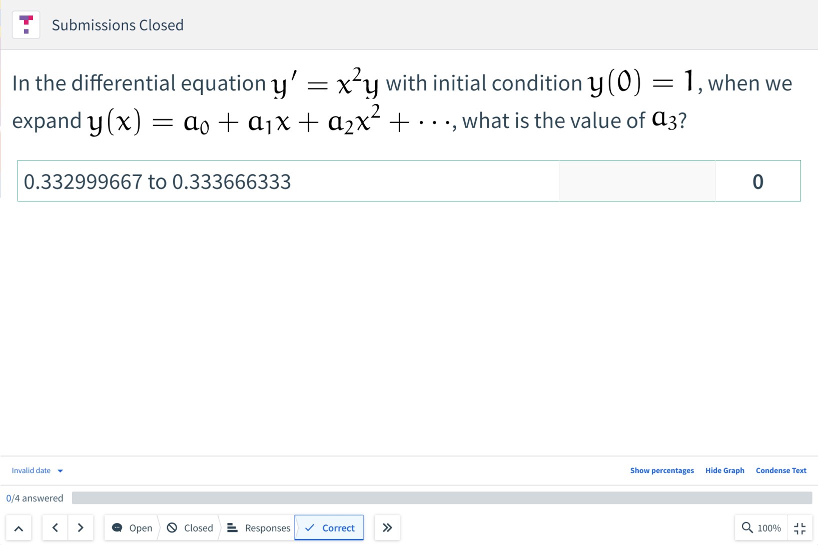
Below are some steps to use when solving an ODE using Taylor approximations. What is an appropriate order?

Correct Order

- **D** Write y as a Taylor polynomial
- **A** Compute the derivative of y in terms of a Taylor polynomial
- **E** Write the LHS and RHS of the ODE as Taylor polynomials
- **F** Set two Taylor polynomials equal to each other and solve for the coefficients
- **B** Use the initial condition to plug in coefficients you know
- **C** Extract information from the Taylor expansion of y

March 2 at 12:03 PM results

0/4 answered



Submissions Closed

In the differential equation $y' = x^2 y$ with initial condition $y(0) = 1$, when we expand $y(x) = a_0 + a_1 x + a_2 x^2 + \cdots$, what is the value of a_3?

0.332999667 to 0.333666333	0

A Hard Differential Equation

$$\begin{aligned} y' &= x^2 y \\ y(0) &= 1 \end{aligned}$$

Write:

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\ x^2 y(x) &= \qquad\qquad\qquad a_0 x^2 + a_1 x^3 + a_2 x^4 + a_3 x^5 \\ y'(x) &= a_1 + 2a_2 x + 3a_3 x^2 \end{aligned}$$

We know:

$$\begin{aligned} a_0 &= 1 & a_1 &= 0 \\ a_2 &= 0 & a_3 &= \frac{1}{3} a_0 = \frac{1}{3} \end{aligned}$$

Using Taylor Solutions to Estimate

Assume that:

$$y' = x^2 y$$
$$y(0) = 1$$

Use Taylor approximations to estimate $y(0.5)$.

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at $x = 0.5$, we get: $y(0.5) \approx 1.04$.

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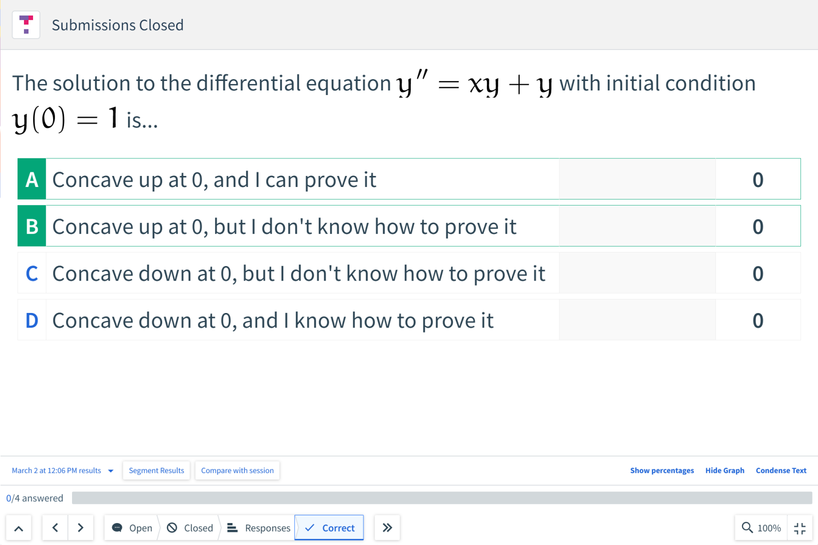
Use Taylor approximations to estimate $y(0.5)$.
We know that:

$$y(x) = 1 + 0x + 0x^2 + \frac{1}{3}x^3$$

at $x = 0.5$, we get: $y(0.5) \approx 1.04$.
The actual solution to this ODE (it's separable) is $y(x) = e^{x^3}$. How close is our estimate?



The solution to the differential equation $y'' = xy + y$ with initial condition $y(0) = 1$ is...

A	Concave up at 0, and I can prove it	0
B	Concave up at 0, but I don't know how to prove it	0
C	Concave down at 0, but I don't know how to prove it	0
D	Concave down at 0, and I know how to prove it	0

Takeaway

We can get information about convexity or other properties of the function by looking at the coefficients of its solution. For example, increasing, decreasing, concavity,...

Plans for the Future

For next time:
WeBWork 8.1 and actively read section 8.1