# Welcome to MAT135 LEC0501 (Assaf)

Challenge: compute the integral:

$$\int Si(x)dx$$

The integral from last class: $\int xSi(x)dx$. Integrate by parts, letting $u = Si(x)$ and $v' = x$. This gives:

$$\int xSi(x)dx = \frac{x^2}{2}Si(x) - \frac{1}{2}\int x\sin(x)dx$$

Integrating by parts again yields:

$$\int xSi(x)dx = \frac{x^2}{2}Si(x) - \sin(x) + x\cos(x) + C$$

# Computer Algebra Systems & Taylor Approximations

Assaf Bar-Natan

"It's automated computer speech
It's automated computer speech
It's a Casio on a plastic beach
It's a Casio"

–"Plastic Beach", Gorillaz

Jan. 29, 2020

# Functions Defined by Integrals

Recall: we can define functions using integrals. For example:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$F(x) = \int_0^x (1+t)e^t \sqrt{1+t^2 e^{2t}} dt$$

# Functions Defined by Integrals

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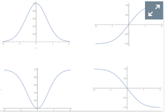
Today: explore how to work with these functions and with computer algebra systems.

Submissions Closed

Which of the following may be a plot of

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$



✓ 57% Answered Correctly

| | | |
|---|---|---|
| A | Top left | 24 |
| B | Top right | 109 |
| C | Bottom left | 41 |
| D | Bottom right | 18 |

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192/192 answered · Ask Again

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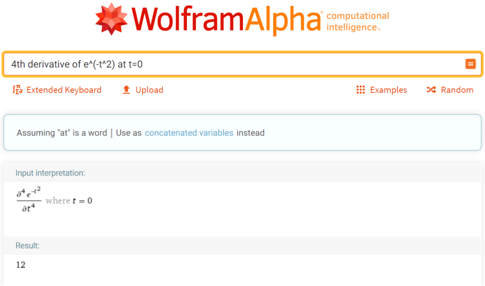
# Taylor Approximations Using C.A.S

The third-order Taylor approximation of a function, $f$ around 0 is given by:

$$T_3(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3$$

WolframAlpha can be used to compute derivatives quickly!



Use the third-order Taylor approximation of $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ to estimate $erf(0.5)$.

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$$erf(x) \approx \frac{2}{\sqrt{\pi}} \left( x - \frac{x^3}{3} \right)$$
$$erf(05) \approx 0.517$$

Compute $erf(0.5)$ directly using WolframAlpha.

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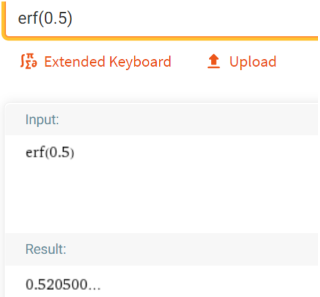


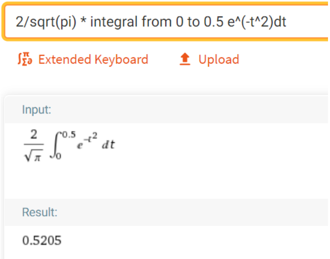

# Takeaway

**Computer algebra systems can do some of the work for us, even if we have to stitch it together at the end.**

Submissions Closed

Which of the following is an antiderivative of $(1+t)e^t\sqrt{1+t^2e^{2t}}dt$? You may use any computer algebra system to solve this.

✓ 7% Answered Correctly

| | | |
|---|---|---|
| A | $\frac{1}{2}(te^t\sqrt{1+t^2e^{2t}}+t^2e^t\sqrt{1+t^2e^{2t}}$ | 81 |
| B | $\frac{1}{2}(te^t\sqrt{1-t^2e^{2t}}-\cosh^{-1}(te^t))$ | 37 |
| C | $\frac{1}{2}(te^t\sqrt{1+t^2e^{2t}}+\sinh^{-1}(te^t)$ | 14 |
| D | I can't use WolframAlpha for this. | 68 |

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# What Went Wrong?

WolframAlpha could not solve:

$$\int (1+t)e^t\sqrt{1+t^2e^{2t}}dt$$

The integral was too complex. What can we do?

# What Went Wrong?

WolframAlpha could not solve:

$$\int (1+t)e^t\sqrt{1+t^2e^{2t}}dt$$

The integral was too complex. What can we do?

- Ask for a definite integral instead.
- Use a Taylor polynomial to estimate the integrand
- Change the input to something nicer

Submissions Closed

For which value of $n$ do we have

$$2000 > \int_0^n (1+t)e^t\sqrt{1+t^2e^{2t}}\,dt > 1000?$$

✓ **66%** Answered Correctly

| | | |
|---|---|---|
| A | 1 | 11 |
| B | 2 | 52 |
| C | 3 | 132 |
| D | 4 | 6 |

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# Takeaway

**Computer algebra systems can do definite integrals like it's nobody's business. Remember: it's just sums!**

# Using Taylor Polynomials on the Integrand

Idea: use a Taylor polynomial to approximate $(1+t)e^t\sqrt{1+t^2e^{2t}}$ around $t=0$, then integrate that. If the polynomial and the function are close, then their integrals will be close too.

# Using Taylor Polynomials on the Integrand

Idea: use a Taylor polynomial to approximate $(1+t)e^t\sqrt{1+t^2e^{2t}}$ around $t=0$, then integrate that. If the polynomial and the function are close, then their integrals will be close too.

- Find the Taylor polynomial of the function
- Use it to estimate the function for small values of $x$
- Find an antiderivative of the Taylor polynomial

# Estimating Integerals With Taylor Polynomials

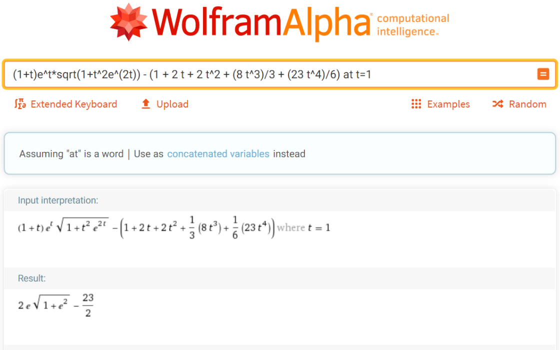
Use WolframAlpha to compute the Taylor polynomial of $(1+t)e^t\sqrt{1+t^2e^{2t}}$ around $t=0$ to fourth order.

# Estimating Integerals With Taylor Polynomials

Use WolframAlpha to compute the Taylor polynomial of $(1+t)e^t\sqrt{1+t^2e^{2t}}$ around $t=0$ to fourth order.

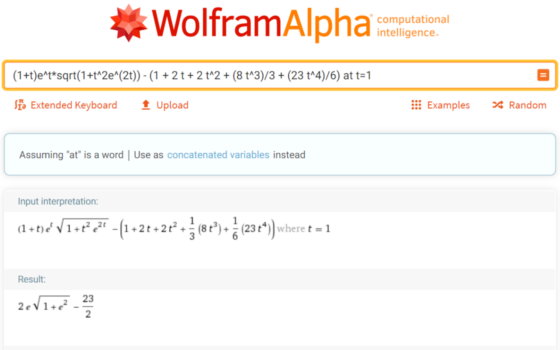
$$T_3(t) = 1 + 2t + 2t^2 + 8\frac{t^3}{3} + 23\frac{t^4}{6}$$

# Estimating Integerals With Taylor Polynomials




This error ends up being approximately 4.2.

# Estimating Integerals With Taylor Polynomials




This error ends up being approximately 4.2.
Estimate $\int_0^1 (1+t)e^t\sqrt{1+t^2e^{2t}}dt$ using the Taylor approximation you found. How good is this approximation?

# Estimating Integerals With Taylor Polynomials

$$\int_0^1 (1+t)e^t\sqrt{1+t^2e^{2t}}dt \approx \int_0^1 T_3(t)dt$$
$$= \int_0^1 \left(1+2t+2t^2+8\frac{t^3}{3}+23\frac{t^4}{6}\right)dt$$

# Estimating Integerals With Taylor Polynomials

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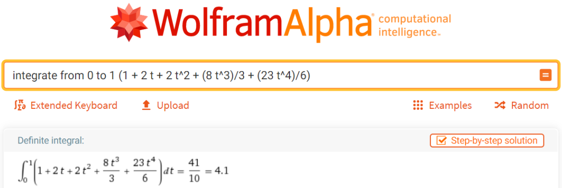


What is the true value of the integral?

# Estimating Integerals With Taylor Polynomials

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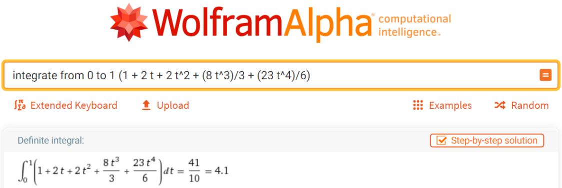


What is the true value of the integral? 4.78

## Simplifying the Integral with Substitution

We wish to compute:

$$\int (1+t)e^t \sqrt{1+t^2e^{2t}}dt$$

Make the substitution $u = te^t$.

# Simplifying the Integral with Substitution

We wish to compute:

$$\int (1+t)e^t\sqrt{1+t^2e^{2t}}dt$$

Make the substitution $u = te^t$. The integral then becomes:

$$\int \sqrt{1+u^2}du$$

Plug this integral into a computer algebra system

Submissions Closed

Which of the following is an antiderivative of $(1+t)e^t\sqrt{1+t^2e^{2t}}dt$? You may use any computer algebra system to solve this.

✓ **61%** Answered Correctly

| | Answer | Responses |
|---|---|---|
| A | $\frac{1}{2}(te^t\sqrt{1+t^2e^{2t}}+t^2e^t\sqrt{1+t^2e^{2t}}$ | 33 |
| B | $\frac{1}{2}(te^t\sqrt{1-t^2e^{2t}}-\cosh^{-1}(te^t))$ | 29 |
| C | $\frac{1}{2}(te^t\sqrt{1+t^2e^{2t}}+\sinh^{-1}(te^t)$ | 116 |
| D | I can't use WolframAlpha for this. | 12 |

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# Plans for the Future

For next time:
**WeBWork 7.6 and read section 7.6**