The Hopf-Pioncaré Index Theorem

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These notes were compiled by Assaf Bar-Natan at the Canada/USA Mathcamp in the summer of 2019. For questions, corrections, or comments, send an email to assafbarnatan@gmail.com

Day 1: The Euler Characteristic of S^2

Oh, take me hold me
Remember what you told me
You'd meet me at the dreamer's ball
I'll meet you at the dreamer's ball

– Queen, "Dreamers Ball"

In this class, everything can be freely deformed. That means that, to us, a triangle is three points with lines connecting them, and these lines can be as squiggly as we want.

Definition 1. A **surface** is a collection of points (in $\mathbb{R}^3$, for example, but later we won't care about this) where when we zoom in on each point, the surface looks like a piece of $\mathbb{R}^2$.

This definition is a bit wishy-washy, and not very formal. Instead, in this class, we will be thinking of surfaces as objects glued together from smaller pieces, ie, polygons.

Exercise 1. Eric wants to glue polygons together to make a surface. He starts off with a bunch of polygons, and glues them along their edges. What rules must Eric follow so that the glued-together shape satisfies the axioms of being a surface?

Exercise 2. Define a **surface with boundary** using polygonal gluings. A polygonal gluing that makes up a surface X will henceforth be called a **tiling of** X.

Exercise 3. Find three different ways to glue together polygons to get a sphere. For each of these, compute:

- The total number of vertices,
- The total number of edges,
- The total number of polygons (or, faces).

Exercise 4. What pattern do you see? Use this to define the Euler Characteristic (χ) of the sphere. It should be a number that is associated to the sphere, that will use a tiling, but not depend on which one you use.

We will now try and fail to prove that the Euler Characteristic does not depend on the tiling. Don't worry though, we will still need this machinery later.

Definition 2. Let $\mathcal{T}$ be a tiling that makes up a surface X. A **refinement** of $\mathcal{T}$ is a tiling $\mathcal{S}$ satisfying:

- $\mathcal{S}$ makes up X
- Every vertex in $\mathcal{T}$ is a vertex in $\mathcal{S}$
- Every edge in $\mathcal{T}$ is a union of edges in $\mathcal{S}$.
- Every face in $\mathcal{T}$ is a union of faces in $\mathcal{S}$.

Exercise 5. Prove that if $\mathcal{S}$ is a refinement of $\mathcal{T}$, then the Euler Characteristic as computed by $\mathcal{T}$ is the same as the Euler Characteristic as computed by $\mathcal{S}$.

Exercise 6. Let $\mathcal{T}_1$ and $\mathcal{T}_1$ be two gluings of polygons that make up X. Assume that there exists a tiling of $\mathcal{S}$ that is a refinement of both $\mathcal{T}_1$ and $\mathcal{T}_2$. Show that the Euler Characteristic as computed by $\mathcal{T}_1$ is the same as the one computed by $\mathcal{T}_2$.

Exercise 7. Find two tilings of a square that do not have a mutual refinement.
Hint: how would you go about constructing a mutual refinement? What conditions have to be met?

We will now give a rigorous proof that the Euler Characteristic of the sphere does not depend on the tiling. The proof for other surfaces will have to wait for later.

Exercise 8. By removing edges and points one by one, prove that the Euler characteristic of the disk does not depend on the tiling.

Exercise 9. Use the above to prove that the Euler Characteristic of the sphere does not depend on the tiling.

Exercise 10. Prove the **combinatorial Gauss Bonnet theorem**:

Theorem 3. *Let $\mathcal{T}$ be a tiling of a surface X that has no boundary, and where all of the tiles are triangles. If v is a vertex in the tiling, we define $\kappa(v)$ to be the number of faces attached to the vertes. Then*

$$6\chi(X) = \sum_{v \in V}(6 - \kappa(v))$$

Definition 4. We say that a tiling is *reduced* if pairs of tiles share at most one edge.

Exercise 11. Prove that any reduced tiling of the sphere must contain at least one pentagon, one square, or one triangle. What is the minimal number of each that the tiling must have?

Exercise 12. A soccer ball is comprised of hexagonal tiles of fabric and pentagonal tiles of fabric. Assuming that the tiling is reduced[1], how many pentagons are there in it?

[1]which is it, if you look up a picture of a soccer ball

Exercise 13. Generalize the combinatorial Gauss Bonnet theorem to surfaces with boundary.

We can also think of a surface in the following way:

Definition 5. A **surface** is a subset $X \subset \mathbb{R}^n$ such that for every point $x \in X$, there exists a smooth map $f : \mathbb{R}^2 \to \mathbb{R}^n$ such that

1. df is injective at every point
2. There exists an open set $V \subset \mathbb{R}^n$ such that $f(\mathbb{R}^2) = V \cap X$
3. f is bijective and has a continuous inverse

Exercise 14. Use the above definition to prove that the sphere is a surface.

Day 2: Classification of Surfaces

Keep coming up with love but it's so slashed and torn
Why, why, why? Love, love, love, love, love
Insanity laughs under pressure we're breaking

– Queen, "Under Pressure"

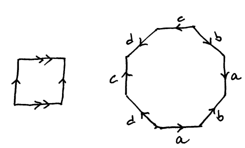
Yesterday, we proved the following theorem:

Theorem 6. *For any polygonal tiling of S^2, $V - E + F = 2$. In other words, the Euler characteristic of the sphere is* 2.

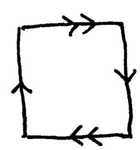
In this class, we will classify all compact surfaces. This will give us a framework to show that the Euler characteristic does not depend on the tiling, but that will have to come later.

Definition 7. We define the **connect sum** of two surfaces X_1 and X_2 as the surface obtained by cutting a small circle in X_1 and in X_2, and gluing the rims to each other.

Exercise 1. Identify the following shapes:



Definition 8. The **projective plane** is the surface obtained by the tiling:



The theorem we will prove is:

Theorem 9. *If X is a compact surface, then X is homotopy equivalent to a connect sum of projective planes and tori.*

Exercise 2. Given a tiling of a compact surface X, construct a tiling of X composed of a single polygon P, where the gluing is along the boundary of P.

In other words, by erasing interior edges, it's enough for us to examine polygons whose edges are glued. We label the edges with letters to keep track of what gets glued to what.

Definition 10. Such a diagram is called a **planar diagram**.

Note that if P is a polygon glued to make X, then we can cut P along any diagonal and reglue the pieces by pre-existing letters.

To prove the theorem, we will first prove the lemma:

Lemma 11. *Let X be a compact surface. Then there exist $m, n \in \mathbb{N}$ such that X is homotopy equivalent to a planar diagram where the letters along the boundary spell*

$$a_1 a_1 a_2 a_2 \cdots a_n a_n c_1 d_1 c_1^{-1} d_1^{-1} c_2 d_2 c_2^{-1} d_2^{-1} \cdots c_m d_m c_m^{-1} d_m^{-1}$$

Moreover, any planar diagram can be converted into a planar diagram of this sort via a finite number of cuts and pastes. Moreover, this form minimizes the total number of labels.

The proof of this lemma is left to the end of the worksheet.

Exercise 3. Assume that P is a planar diagram as in the above lemma. Show that after gluing, there is only one vertex.

Exercise 4. Assume that P is a planar diagram as in the above lemma. Show that there is no cutting and pasting operation that you can do that reduces the number of edges. Conclude the last part of the lemma.

Exercise 5. Conclude that the planar diagrams in the form described by the lemma classify compact surfaces. That is, conclude that any surface is cut-and-paste equivalent to exactly one polygonal gluing as in the lemma. Is cut-and-paste equivalence the same thing as homotopy equivalence?

Exercise 6. Show that if X_1 has a planar diagram with a string of letters w_1, and X_2 has a planar diagram with a string of letters w_2, then the planar diagram given with string of letters $w_1 w_2$ gives the surface $X_1 \# X_2$.

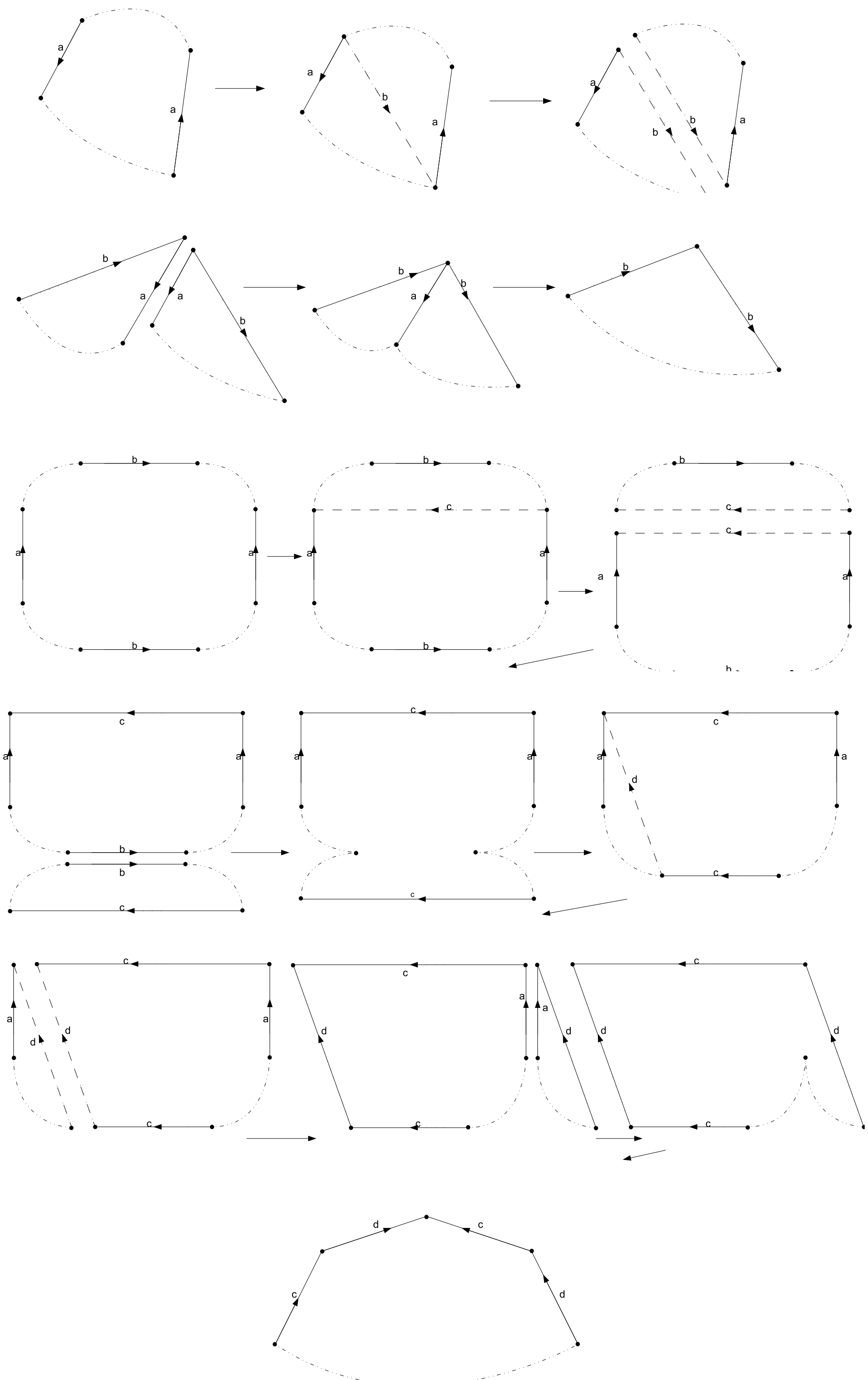
Exercise 7. Prove the classification theorem

Proving the Lemma

Exercise 8. Show that any planar diagram can be cut and pasted to get a planar diagram for which all of the vertices are glued into a single vertex.
Hint: colour the vertices by their gluing, and find a cut that reduces the total number of one colour and increases the total of another. What happens when only one vertex of a colour remains? How many edges are glued?

We now wish to show that there exists a finite sequence of cut-and-paste operations that can bring the boundary of the polygon to the form that we want. Note that we have already proved uniqueness. To do this, consult the diagrams on the next page:

Exercise 9. Consult the following pictures to finish the proof of the lemma:



Day 3: A Perfect Invariant

We gonna tear it up
Stir it up
Break it up - baby

– Queen, "Tear It Up"

Yesterday, we proved the following theorem:

Theorem 12. *Any compact surface is a connect sum of tori and projective planes, or is a sphere.*

Today we will talk about orientability and the Euler characteristic. We will show that the Euler characteristic is an invariant - that is, that it does not depend on the tiling. It is true for surfaces (and topological spaces in general), but we will only show it for orientable surfaces.

Definition 13. (intuitive) A compact surface X is called **orientable** if it has a well-defined inside and outside.

Exercise 1. Assume that the tiles used to construct a compact surface X have front and back faces, coloured red and green. Come up with a criterion on the gluing of the tiles that would ensure that X is orientable.

Exercise 2. Classify all compact orientable surfaces.

Exercise 3. Find all possible values of $\chi(X)$ where X is a compact orientable surface.

Theorem 14. *If $\mathcal{T}$ and $\mathcal{S}$ are two tilings of a compact orientable surface X, then the Euler characteristic as measured by either of them is equal.*

Proof.

- Embed X in $\mathbb{R}^3$

- Refine $\mathcal{T}$ until we can approximate curves.

- Cut along a curve, and glue in faces.

- Apply induction

$\square$

Exercise 4. Piecing everything we have, prove the following:

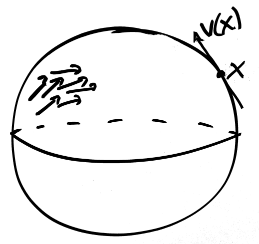
Theorem 15. *Let X and Y be a compact orientable surfaces. Then $\chi(X) = \chi(Y)$ if and only if X and Y are homotopically equivalent.*

In other words, the Euler characteristic is a **perfect invariant** for compact orientable surfaces.

Warm-up to Hopf-Poincaré

Definition 16. A often-used analogy of a **vector field** on a surface is wind velocity at every point on that surface. A **vector field** on a surface X is a continuous function $v : X \to \mathbb{R}^3$ such

that $v(x)$ is ____________________. We say that v is nonvanishing if ____________________.
Hint: Consult the following picture:



Exercise 5. Assume that X has a nonvanishing continuous vector field v. Convince yourself that we can find a triangulation of X with the following properties:

- The edges are never parallel to the vector field
- The triangles are sufficiently small so that the vector field is approximately constant on each one.

Exercise 6. Using the triangulation above, put a positive charge on every vertex, a negative charge at the center of each edge, and a positive charge on each face. What is the total charge on X?

Exercise 7. Think of v as wind on X, and let it blow the charges just a little bit. Compute the total charge on the sphere by computing the charge on each triangle.

Exercise 8. Prove the hairy ball theorem.

Wait, what? The hairy ball theorem? We actaully proved a stronger theorem:

Theorem 17. *If X admits a nonvanishing vector field, then $\chi(X) = 0$.*

The Hopf-Poincaré index theorem is a generalization of this, relating to the **index** of a vector field.

A Bit on Three-Manifolds

Exercise 9. Define a three-manifold (ie, a 3-D surface) using tilings. Define a compact manifold similarly.

Exercise 10. Define the Euler characteristic of a compact 3-manifold.

Fact 18. *The Euler characteristic is an invariant, meaning that it does not depend on the tiling of the* 3*-manifold.*

It turns out that the Euler characteristic is not a perfect invariant. In fact,

Theorem 19. *Let M be a compact* 3*-manifold. Then $\chi(M) = 0$.*

Exercise 11. Prove the above theorem.
Hint: show that $\chi(M) = -\chi(M)$

Corollary 20. *(hard - we can try doing this later) Every* 3*-manifold admits a nonvanishing vector field.*

Remark 21. This is a very soft example of what's called Poincaré Duality, which states that if M is an n-dimensional compact manifold, then, under certain conditions (orientability being one of them), there is an isomorphism $H^k(M) \cong H_{n-k}(M)$, where $H_i(M)$ and $H^i(M)$ are the homology and cohomology groups of M. This symmetry allows one to show that $\chi(M) = 0$ for any compact odd-dimensional orientable manifold.

Day 4: The Hopf Poincaré Index Formula

Ride the wild wind
(Live life on the razors edge) hey hey hey
Gonna ride the whirlwind
It ain't dangerous - enough for me

– Queen, "Ride the Wild Wind"

Yesterday, we proved two big theorems:

Theorem 22. *Let Σ_g be the connect sum of g tori. Then $\chi(\Sigma_g) = 2 - 2g$ (for any tiling ofΣ_g)*

Theorem 23. *If a compact orientable surface X admits a nonvanishing vector field (is* ***combable****), then $\chi(X) = 0$.*

But what happens if we're allowed to vanish or be undefined at one or two points?

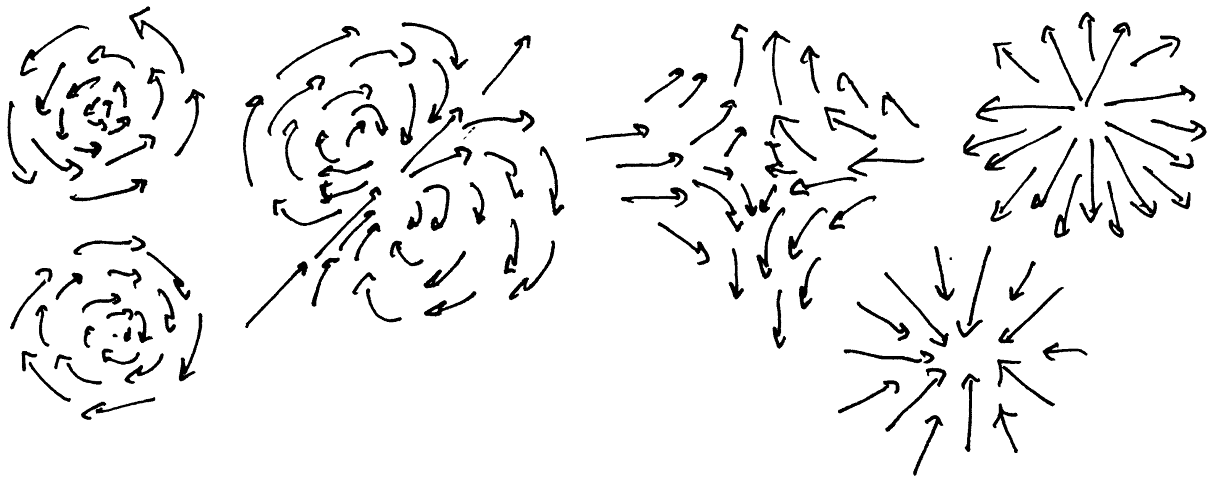
Exercise 1. Draw a vector field on S^2 that vanishes at:

- exactly 2 points,
- exactly 3 points,
- *(bonus)* exactly 1 point.

Definition 24. Let v be a vector field on an orientable surface X, and let $x \in X$. We define the **index** of v at x, $\mathrm{ind}_x(v)$ in the following manner:

- Pick a small loop around x
- compute the total change in angle as you go around the loop counter-clockwise
- Divide by 2π.

Exercise 2. Compute the index of the following vector fields at the singular points:



Exercise 3. Show that the index does not make sense on a non-orientable surface.

Exercise 4. Prove that the index is always an integer.

Exercise 5. Prove that the index does not depend on the loop you choose.
Hint: consider a continuous deformation of the loop, and note that every continuous function to $\mathbb{Z}$ *must be constant*

Exercise 6. Prove that if v is a continuous vector field that is nonvanishing at x, then $\operatorname{ind}_x(v) = 0$.

Exercise 7. Compute the indices of the vector fields you drew in Exercise 1, and add them up.

Theorem 25. *Let X be a compact orientable manifold, and let v be a vector field on X that has a finite number of vanishing points. Then*

$$\sum \mathrm{ind}_x(v) = \chi(X)$$

Proof.

- Prove that the left-hand side does not depend on the vector field.

- Find a specific vector field for which the sum of the indices is χ.

□

Day 5: Lie Groups

So dear friends your love is gone
Only tears to dwell upon
I dare not say as the wind must blow

– Queen, "Dear Friends"

A Bit on Lie Groups

Definition 26. A **compact manifold** of dimension n is an object obtained by gluing n-dimensional polyhedra (ie, simplices, or cubes) in a way that does not leave any $n-1$-dimensional faces unglued.

Exercise 1. By induction, prove that S^n, then n-dimensional sphere, is a manifold.

Exercise 2. Generalize the definition of the Euler characteristic, and compute it for S^n.

Exercise 3. Let M be a manifold. Prove that if M is combable, then $\chi(M) = 0$.

Fact 27. *The converse of the above is true. That is, if $\chi(M) = 0$, then M is combable.*

Definition 28. A **Lie group** is a manifold G, together with the following:

- An associative continuous map $\times : G \times G \to G$ (continuous in both co-ordinates).
- An identity element $x_0 \in G$, such that $x_0 \times x = x \times x_0 = x$ for all $x \in G$.
- A continuous map $i : M \to M$ such that $i(x) \times x = x \times i(x) = x_0$ for all $x \in G$.

Exercise 4. Prove that S^1 can be given the structure of a Lie group.

Theorem 29. *Let G be a Lie group of dimension n, then G is combable. Moreover, G is orientable. Moreover, G is **parallelizable**, that is, it admits n vector fields that are linearly independent at every point.*

Proof. We will only show that G is parallelizable. To do this, we take n linearly independent vectors at the identity, and move them around with the group multiplication. $\square$

Exercise 5. Classify all 2-dimensional compact Lie groups

Exercise 6. Show that $GL_n(\mathbb{R})$ is a Lie group and compute its dimension.

Exercise 7. Show that $\mathbb{R}P^3$, the real projective plane, defined by $S^3/(x \sim -x)$ is parallelizable.
Hint: Use the fact that S^3 is a Lie group