THE Intuitive Proof of the Hairy Ball Theorem

July 4, 2019

Definition 1. A function is **continuous** if a small change in input yields a small change in output.

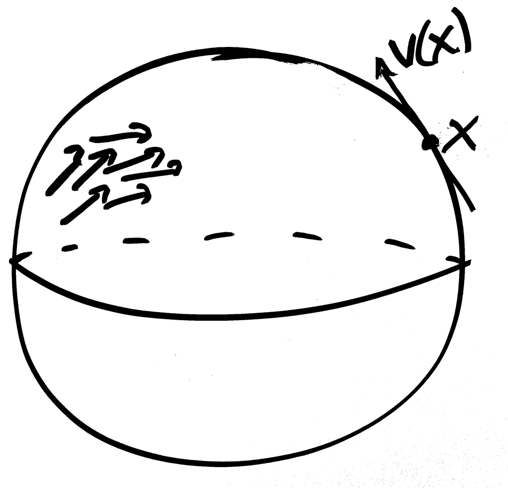
Exercise 1. Prove that a continuous function from $\mathbb{Z}$ to $\mathbb{Z}$ is constant.

Exercise 2. Use the above definition to prove that the composition of two continous functions is a continuous function.

Exercise 3. Consider the sphere, $S^2 \subset \mathbb{R}^3$. Find two continuous bijections $f : S^2 \to S^2$ and $g : S^2 \to S^2$.

Exercise 4. Write a formula that describes a nonconstant continuous function $f : S^2 \to \mathbb{R}$.

Definition 2. A often-used analogy of a **vector field** on a surface is wind velocity at every point on that surface. A **vector field** on S^2 is a continuous function $v : S^2 \to \mathbb{R}^3$ such that $v(x)$ is ____________________. We say that v is nonvanishing if ____________________.
Hint: Consult the following picture:



Exercise 5. Draw a vector field on S^2 that vanishes at:

- exactly 2 points,
- exactly 3 points,
- *(bonus)* exactly 1 point,
- *(impossible)* exactly 0 points.

Note 1. We now assume that v is a nonvanishing vector field on S^2.

Exercise 6. Let N be the north pole of S^2. Show that we can assume that $v(N) = (1, 0, 0)$.

Exercise 7. Cut open a small hole around N. Draw a picture of what v looks like around this small hole.

Exercise 8. Put the sphere on the table, put your hands into the hole, and grasp the sides of the hole. Everyone in your group should then pull together, to stretch the sphere out flat onto the table.

- Draw a picture of this process.
- Give a name to this process.
- What 2-D shape does the sphere become after this process?

Exercise 9. If we move the sphere, we can move v with it. If we **deform** the sphere (ie, squish it, stretch parts of it), we can also move v along the deformation. Draw what v looks like around the boundary of the sphere after the process above.

Note 2. Call this new shape C, and call the vector field on it w.

Exercise 10. Prove that w is a continuous nonvanishing vector field on C.

Exercise 11. Parametrize the boundary of C as a function of θ, as $0 \leq \theta \leq 1$, and plot the direction of w (in degrees) as a function of θ.
Hint: $2\pi = 0$.

Definition 3. The **winding number** of w around the boundary of C is described by the formula ______________________.

Exercise 12. Show that as we continuously shrink the path tracing the boundary of C, the winding number of w around this path changes continuously.
Hint: Look at the second exercise

Exercise 13. Show that after shrinking for enough time, the winding number around the path becomes 0.

Exercise 14. Prove the hairy ball theorem

Bonus: A Second Proof

There is a second, less intuitive, and a bit less hand-waivey proof of the hairy ball theorem, using Euler Characteristics. The following exercises will guide you through it.

Definition 4. A triangulation of the sphere is a division of the sphere into triangles, the faces of which are disjoint. As an example, think of the tetrahedron, the octahedron, or the icosahedron.

Exercise 15. For the tetrahedron, octahedron, and icosahedron, compute

$$V - E + F$$

Where V is the total number of vertices, E is the number of edges, and F is the number of faces of the triangles. Keep in mind that when two triangles share a vertex or an edge, it should only be counted once.

Exercise 16. Using the technique from Exercise 8, prove that $V - E + F$ does not depend on the triangulation of the sphere.
hint: remove edges and vertices one by one

Exercise 17. Assume that S^2 has a nonvanishing continuous vector field v. Convince yourself that we can find a triangulation of S^2 with the following properties:

- The edges are never parallel to the vector field
- The triangles are sufficiently small so that the vector field is approximately constant on each one.

Exercise 18. Using the triangulation above, put a positive charge on every vertex, a negative charge at the center of each edge, and a positive charge on each face. What is the total charge on the sphere?

Exercise 19. Think of v as wind on the sphere, and let it blow the charges just a little bit. Compute the total charge on the sphere by computing the charge on each triangle.

Exercise 20. Prove the hairy ball theorem.

Note 3. The number 2 appeared quite often today. This is not a coincidence. For more information, come to the Hopf-Poincaré index theorem class happening in week 4