Perspectives on Cohomology: Čech

I wanna take your hand - lead you from this place
Gonna leave it all behind
(Check out) Check out of this rat race

– Queen, "Ride the Wild Wind"

1 The Setting

The motivating idea behind Čech cohmology is to stitch things together to get a local-to-global thing that will be cohomological. In order to get anything cohomology-related, we better have a **chain complex** with **boundary operator** δ:

$$\to C^n \to C^{n+1} \to \cdots$$

so that we can define $H^k = \frac{ker(\delta)}{Im(\delta)}$. To do this in our framework, we first introduce the main players in this play.

Definition 1. A **covering** of a space X is a set of open subsets of X whose union is X.

Example 2. The circle is covered by three intervals.

Definition 3. A **presheaf of abelian groups** over a topological space X is an association of an abelian group to every open set U of X, denoted by $F(U)$. We also require that if $V \subset U$ is open, then we have a restriction map $F(U) \to F(V)$ satisfying some reasonable properties.

Example 4.

- We can take the same group over every open set, and let the restriction just be the identity.
- If M is a manifold, then we can look at the presheaf $\mathcal{C}^\infty(M)$, where $F(U)$ is the set of all smooth functions on U with addition. The restriction map is then the usual function restriction.
- In the same way as above, we can define $\mathcal{C}^{\infty *}(M)$, the presheaf of nonvanishing functions under multiplication, or $\mathcal{O}(M)$ the presheaf of holomorphic functions, etc..

Definition 5. Let $\mathcal{U}$ be an open covering of M, and let $\mathcal{F}$ be a presheaf on M. A n-**simplex** σ on $\mathcal{U}$ is a collection of $n+1$ sets in $\mathcal{U}$ which have a nonempty intersection. We turn this into a vector space by formal addition.

Definition 6. We define the j-**boundary** of a simplex $\sigma = (U_0, \ldots, U_n)$ by

$$\partial_j \sigma = (U_0, \ldots, \hat{U}_j, \ldots, U_n)$$

Exercise 1. Define the **boundary** of σ by $\partial\sigma = \sum_{j=0}^{n}(-1)^{j+1}\partial_j\sigma$. Prove that $\partial\partial\sigma = 0$ for any simplex σ.

2 The Chain Complex

A q-cochain of $\mathcal{U}$ with coefficients in $\mathcal{F}$ is a map which associates with each q-simplex σ an element of $F(|\sigma|)$. We denote the set of all q-cochains of $\mathcal{U}$ by $C^q(\mathcal{U}, \mathcal{F})$, and make it into an abelian group via formal addition.

Example 7. If $\mathcal{F}$ is $\mathbb{Z}/2$, then a 0-cochain is a choice of ± 1 for every set in $\mathcal{U}$. An explicit example is

$$3(A, 1) - 1(B, -1) + (C, 1)$$

We think of a cochain as a kind of "function", and we will denote it by f. It is a function in the sense that it takes a simplex and returns a group element, or at least an element in $F(|\sigma|)$ if we're looking at a presheaf.

Definition 8. Let $f \in C^n(\mathcal{U}, \mathcal{F})$. We define the coboundary map $\delta f \in C^{n+1}(\mathcal{U}, \mathcal{F})$ by seeing how it acts on an $n+1$-simplex:

$$(\delta f)(\sigma) = \sum_{j=0}^{n+1}(-1)^{j+1} f(\partial_j \sigma)|_{|\partial_j \sigma|}$$

Exercise 2. $\delta^2 = 0$

I think we need an example:

Example 9. Consider the presheaf $\mathcal{C}^\infty$, and let $\{U_i\}$ be the covering of M. A 0-cochain, f, is a collection of functions $\{f_i : U_i \to \mathbb{R}\}$. We would like to apply $g = \delta f$ to a 1-cycle. We write:

$$g((U_i, U_j)) = f(U_i)|_{U_i \cap U_j} - f(U_j)|_{U_i \cap U_j}$$

In other words, on the simplex $U_i \cap U_j$, the cochain $\{g_{ij}\}$ looks like $g_{ij} = f_i - f_j$.

Example 10. Similarly, if $\{U_i\}_i$ is a covering of M, and $g_{ij} : U_i \cap U_j \to \mathbb{R}$ is a cochain, then on triple intersections,

$$\delta\{g_{ij}\} = \{g_{jk} - g_{ik} + g_{ij} : U_i \cap U_j \cap U_k \neq \emptyset\}$$

Definition 11. Let $\mathcal{U}$ be a covering of M and let $\mathcal{F}$ be a presheaf on M. We define:

$$\check{H}^k(\mathcal{U}, \mathcal{F}) = \frac{Ker(\delta^k)}{Im(\delta^{k-1})}$$

Where we think of $\delta^k : C^k \to C^{k+1}$

3 Double Covers

Why all of this? It turns out that we can use these cohomology groups to classify things about M.

Definition 12. A **double cover** of M is a manifold E with a surjective map $\pi : E \to M$ that is 2 to 1, and such that if U is a small open set in M, then $\pi^{-1}(U)$ is homeomorphic to two disjoint copies of U.

Example 13. Two disjoint circles are a double cover of the circle. Another double cover of the circle is the boundary of the mobius strip, where π is the "shrinking" map that sends the boundary of the strip to its middle.

We will show that a double cover of M can be thought of as an element in $\check{H}^1(\mathcal{U}, \mathbb{Z}/2)$, and vice-versa. In other words, $\check{H}^1(\mathcal{U}, \mathbb{Z}/2)$ classifies all double covers of M.

Let E be a double cover of M, and let $\mathcal{U} = \{U_i\}$ be an open covering so that $\pi^{-1}(U_i)$ is homeomorphic to two disjoint copies of U_i[1], label them $+1$ and -1. We'd like to get a 1-cocycle. To do this, we define

$$g_{ij}(U_i \cap U_j) = \begin{cases} 1 & \text{the labelings of } \pi^{-1}(U_i) \text{ and } \pi^{-1}(U_j) \text{ agree} \\ -1 & \text{otherwise} \end{cases}$$

[1] Any **good** covering has this property. A good covering is one where all of the sets are homeomorphic to balls, and all of their intersections are too. It turns out that such coverings do exist.

Exercise 3. Check that this is a cocycle.

We need to check that this is well-defined. That is, that if $\{f_i, U_i\}$ is some 0-chain, and $\delta\{f_i\} = \{f_i f_j^{-1}, U_i \cap U_j\}$, then multiplying g_{ij} by $f_i f_j^{-1}$ does not change the double cover. Indeed, if $f_i = -1$, then all of the transition functions g_{ij} get multiplied by -1, which essentially just means that we've fipped the labels of $+1$ and -1. Hey! This also means that the labels were arbitrary when going into cohomology!

Now, note that we can use a cohomology class to stitch together a double cover using the g_{ij}'s as a gluing guideline. Thus,

Theorem 14. *The set of double-coverings of M is given by $\check{H}^1(\mathcal{U}, \mathbb{Z}/2)$.*

It turns out that if M is a closed orientable manifold, then $\check{H}^k(\mathcal{U}, \mathbb{Z}/2) = H^1(M, \mathbb{Z}/2)$, so this proves that the circle has only two double covers, which were the ones shown above.

3.1 Line Bundles

Definition 15. A **line bundle** over M is a manifold E together with a surjective map $\pi : E \to M$ such that $\pi^{-1}(x)$ has the structure of a real 1-dimensional vector space.

We can think of a line bundle in terms of coverings $\{U_\alpha\}$ in the following manner:

Example 16. Let $\{U_\alpha\}$ be a covering of M. A line bundle can be thought of as the disjoint union $\sqcup U_\alpha \times \mathbb{R}$, but where we glue the fibers together along $U_\alpha \cap U_\beta$ via some linear isomorphism on each fiber. But a linear isomorphism is just a nonzero scalar multiplication!

Thus, a line bundle can be thought of as a covering, together with a collection of functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{R}^*$. We check that $\delta\{g_{\alpha\beta}\} = 0$. Indeed, $\delta\{g_{\alpha\beta}\} = \{g_{\beta\gamma} g_{\alpha\gamma}^{-1} g_{\alpha\beta}\}$ which corresponds to tracking the multiplication constant when we go from α to β to γ to α again, which is just the identity map!

Next, note that multiplying $g_{\alpha\beta}$ by $f_\gamma f_\alpha^{-1}$ maintains the vector bundle shape - it's just a reparametrization of the fiber, like in the double cover case!

Thus, we get an association of a line bundle E to an element in $\check{H}^1(\mathcal{U}, \mathcal{C}^{\infty *})$.

Now, if $\{g_{\alpha\beta}\} \in \check{H}^1$, we can use them to glue together a line bundle by reversing the process above.

Similarly, if $E \cong F$, then the linear isomorphism on the fibers is given by a nonzero scalar multiple at every point, so in fact, the cohomology class we get is unique up to isomorphism of the line bundle.