THE ARITHMETIC AND ALGEBRA OF GEORGE SPENCER-BROWN

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Abstract. While this paper is largely an exposition of the mathematical portion of Spencer-Brown's book, *The Laws of Form* [1], it also provides a fully canonical form for arbitrary elements of the Spencer-Brown algebra (theorems 11.3 and 11.6) and various related results. An epilogue contains some brief remarks concerning 'complex' extensions of the Spencer-Brown theory.

Introduction

This work arose from attempts to come to terms with the first ten chapters of Spencer-Brown's book *The Laws of Form*, [1], although the initial portion still seem rather opaque. So, although there may be an occasional comment on that earlier portion, the real starting point of this essay is somewhere around chapter 4, where his philosophical principles have been crystallized into mathematics.

The exposition portion of this paper (§2–§10) introduces *arrangement graphs* (§3), a companion notation to Spencer-Brown's expressions. These graphs are useful in understanding his ideas, in part because of the terminology they provide. For completeness, the appendix gives proofs of other identities in [1] that are not needed for this exposition.

In §11, what appears to be a new canonical form for expressions in the Spencer-Brown algebra is given, along with various consequences.

In the epilogue the Varela-Kauffman ([2], [3]) extension of the Spencer-Brown algebra is considered briefly. This is a complete algebra that, in a certain sense, 'complexifies' the Spencer-Brown algebra by the introduction of a third 'truth value' *paradox* which, like $\sqrt{-1}$, is the solution of an equation that has no solution in the Spencer-Brown context.

I would like to thank John Mason, Joy Anderson, Mary Sharpe and especially Gerard Kuiken for their comments on drafts of this work.

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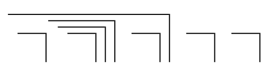
1. AN INFORMAL LOOK AT ARRANGEMENTS

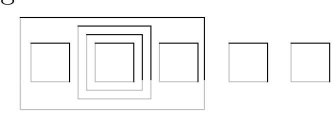
Let A be a finite collection of non-empty rectangular regions in the plane. We say that A is an *arrangement* if it is *perfectly continent,*[1] meaning that if any two of these rectangles overlap, then one is contained within the other.

The real subject under review here is the study of things related to the *pattern of inclusions* that an arrangement determines. A pattern of inclusions is just the information telling us which rectangles are included in which other rectangles, ignoring both their size and disposition in the plane. Such a pattern can be very complicated; how could one provide such information? Spencer-Brown's *expressions* do exactly that. We note the obvious fact that while an arrangement determines a pattern of inclusions, there are many different arrangements that have the same pattern of inclusions. For example, moving an arrangement by translating and rotating it gives a new arrangement with the same pattern of inclusions as the original.

As Louis Kauffman has pointed out, Spencer-Brown's notation $\urcorner$ for the basic element of a pattern of inclusions (refered to as a *cross*), is modeled on rectangular subsets of the plane. For example, Spencer-Brown's expression



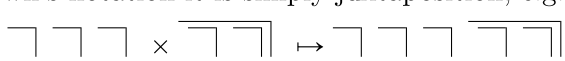
can be seen as an arrangement in the informal sense of this section as



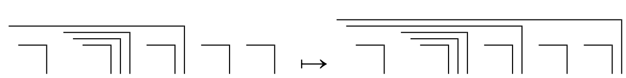
Spencer-Brown includes also the empty arrangement – the one with no rectangles – denoting it by an empty space. We denote it by $\cdot$.

There are two fundamental operations that act on arrangements: *multiplication* and *crossing*. These, with the collection of all the possible patterns of inclusions, form the ensemble that is the *Spencer-Brown arithmetic*.

To *multiply* together two arrangements in the plane we simply place one of them in the left half of the plane and the other in the right half of the plane. As an operation on patterns of inclusions, this is clearly associative and commutative and uniquely yields a new pattern of inclusions. In terms of Spencer-Brown's notation it is simply juxtaposition, e.g.,



To *cross* an arrangement is to add to it one more rectangle that contains the whole of the original arrangement. This operation also induces an operation on the associated patterns of inclusion. Here is an example given in Spencer-Brown's notation:



[1] Spencer-Brown introduces this term without explanation on page 1 of his book *Laws of Form*. I understand it in the sense of *containing perfectly*.

2. THE SPENCER-BROWN ARITHMETIC

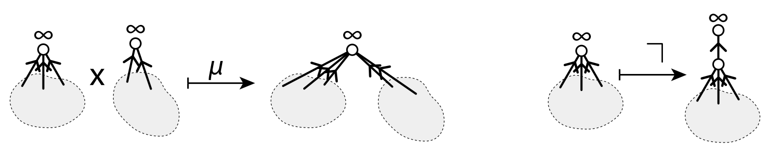
The set of all patterns of inclusion may be formalized as an arrangement graph.

2.1 **Definition**. An *arrangement graph* is a rooted tree, with root denoted by ∞. We let $\mathbb{F}$ denote the set of all arrangement graphs. ❀

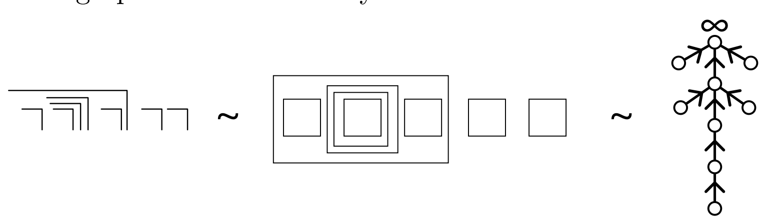
Note that the edges of a rooted tree are canonically oriented by the direction pointing toward the root. We will regard the root as being at the *top* of the tree. From this point of view a vertex is above (below) another if it is closer (further) to (from) the root.

Clearly an element of $\mathbb{F}$ determines a pattern of inclusions in which the vertices u, v etc. correspond to rectangles R_u, R_v in the plane with $R_u \subset R_v$ if and only if $[u, v]$ is an oriented edge.

Elements of $\mathbb{F}$ are multiplied together by identifying their roots. Then multiplication is an associative and commutative operation $\mu : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$. The map $\urcorner : \mathbb{F} \to \mathbb{F}$ is defined by gluing an additional edge to ∞ and replacing the old ∞ by a new one at the other end of the new edge. These operations are illustrated in the following diagram.



The correspondences among Spencer-Brown's notation, arrangements and arrangement graphs are indicated by:



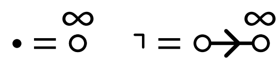
Because the Spencer-Brown and graphical notations are equivalent we will use whichever is most convenient in the sequel.

Note that the root corresponds to the whole plane, the finite vertices correspond to rectangles, and the arrows correspond to crosses (the boundaries of the rectangles).

2.2 **Definition**. The *Spencer-Brown arithmetic* consists of the set $\mathbb{F}$ of arrangement graphs with the operations of multiplication and crossing. ❀

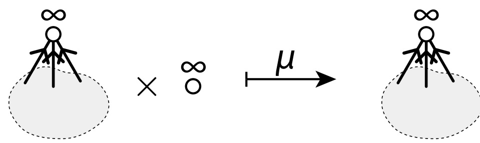
2.3 **Definition**. The *depth of a vertex u* in an arrangement graph is defined to be the number of edges (i.e., the number of arrows) in the shortest chain from α to ∞. The *depth of an arrangement graph* is the maximum of the depths of its vertices. ❀

2.4 **Definition**. The *simple* arrangement graphs are the following two rooted trees to which we give the names $\cdot$ and $\rceil$.



❀

Note that in multiplication, $\cdot$ always acts as the identity.

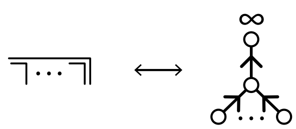


2.5 **Proposition.** (Factorization) Let $e \in \mathbb{F}$. Then e can be written as a product of factors $e = e_1 \ldots e_n$ where for each $k = 1 \ldots n$, e_k has a single edge leading to ∞.

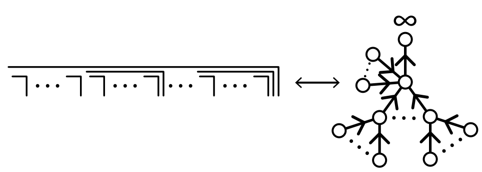
Proof. If there are n edges meeting ∞ let e_k denote the subtree of e that connects to ∞ through the kth of these n edges. Then $e = e_1 \ldots e_n$. The e_k are the *factors* of e. ■

2.6 **Definition**. (Leading Factors) If $e \in \mathbb{F}$ has depth d, the *leading factors* of e are the factors of depth d. ❀

2.7 **Remark**. Let $e \in \mathbb{F}$. The leading factors of depth two have the form:



and those of depth three have the form:



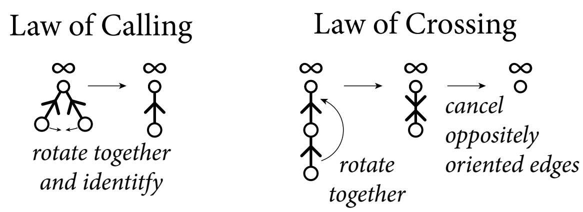
❖

3. THE LAWS OF CALLING AND CROSSING

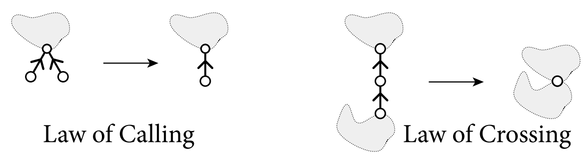
Spencer-Brown introduced an equivalence relation on his expressions, which we denote $\mathfrak{i}$, generated by the following two elementary relations:

> The Law of Calling (I1): $\rceil\rceil = \rceil$; i.e., whenever the form $\rceil\rceil$ appears in an expression it can be replaced by $\rceil$.
>
> The Law of Crossing (I2): $\overline{\rceil}\!\rceil = \cdot$; i.e., whenever the form $\overline{\rceil}\!\rceil$ appears in an expression, it can be replaced by $\cdot$.

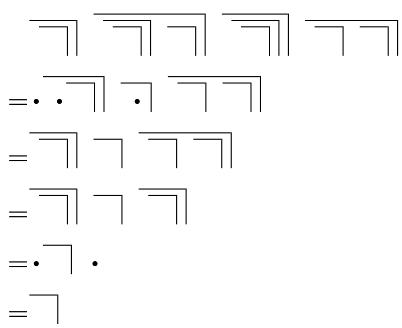
The simplest cases of these two laws appear in arrangement graphs of three vertices:



More generally, if these pictures appear in the context of a larger graph, similar operations obtain. For the law of calling, the two lower vertices must be minimal, and for the law of crossing the middle (hinge) vertex must meet only the two edges shown. These are illustrated in the following pictures.



3.1 **Example.** I1 and I2 can be used to simplify expressions in the free arithmetic as follows:



(I2) three times

(• is the multiplicative identity)

(I1)

(I2) twice

(• is the multiplicative identity) ◆

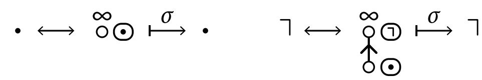
In the next section we show (proposition 4.6) that, using the two laws (I1) and (I2), *every* expression in $\mathbb{F}$ can be transformed into either • or ⌉.

4. THE SPENCER-BROWN INVARIANT

4.1 **Definition.** $\mathbb{V} = \{\bullet, ⌉\} \subset \mathbb{F}$ is the set of *values* of the Spencer-Brown arithmetic. ❀

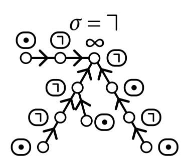
4.2 **Definition.** The *Spencer-Brown invariant* $\sigma : \mathbb{F} \to \mathbb{V}$ is defined by an inductive process that marks each of the vertices with one or another of the elements of $\mathbb{V}$, and defines $\sigma(A)$ to be the value at ∞. The induction starts by labeling all the minimal vertices as •. The remaining vertices are then labelled inductively as follows. If we have already labelled all the immediate predecessors of a vertex α, and if all of those labels agree, then we label α oppositely; otherwise we label α by ⌉. ❀

4.3 **Example.** The following diagrams calculate the invariants of $\cdot$ and $\urcorner$. (For clarity we display the labels within cartouches.)



Note that this shows σ splits the inclusion $\iota : \mathbb{V} \subset \mathbb{F}$ in the sense that $\sigma \circ \iota = \mathrm{id}_{\mathbb{V}}$. ◆

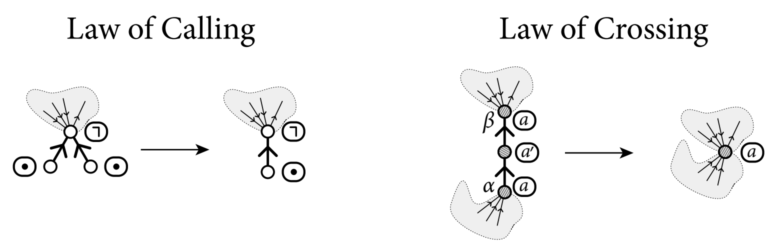
4.4 **Example.** Here is another example of calculating the Spencer-Brown invariant.



◆

4.5 **Proposition.** The operations (I1) and (I2) leave the Spencer-Brown invariant unchanged.

Proof. The following graphs are labelled for calculating the Spencer-Brown invariant:

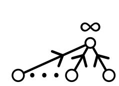


In the diagram for the law of crossing, when the procedure for labeling vertices arrives at the vertex α it receives (say) a label $a \in \mathbb{V}$ from what is below it. Continuing the labeling, the next vertex receives the label a' (the opposite of a) and the vertex β receives the label a again. It is therefore clear that the Spencer-Brown invariant will be the same in both cases of the before and after pictures. ■

4.6 **Proposition.** Let $e \in \mathbb{F}$ be any expression. Applying the laws I1 and I2, e can be reduced to $\sigma(e)$.

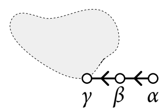
Proof. Let G be the arrangement graph of e and let $\ell(G)$ be the number of vertices in G. First we show that if $\ell(G) \geqslant 3$ then the laws I1 and I2 allow us to reduce $\ell(G)$.

If the depth is 1, then the graph must have the form:



and so $\ell(G)$ can be reduced by (I1), the law of calling.

If the depth is greater than 1, consider a vertex α at the greatest depth, along with its successor β. If there is a *second* vertex below β then it must also be minimal and so, as above, the number of vertices can be reduced by 1. However if α is the only vertex below β, considering also the successor γ of β, we have the following picture of the arrangement graph.



This can be reduced by (I2), of the law of crossing.

Thus, by induction, we see that every arrangement graph G can be reduced to an arrangement graph of at most two vertices, i.e., to graphs corresponding to either $\cdot$ or $\urcorner$, which are their own Spencer-Brown invariants. And since, by proposition 4.5, the Spencer-Brown invariant is unchanged under the operations (I1) and (I2), the result must be $\sigma(e)$. ■

Then the following corollary is obvious.

4.7 **Corollary**. $\mathbb{F}/\mathfrak{i} = \mathbb{V}$.

5. THE SPENCER-BROWN ALGEBRA

In ordinary arithmetic, an arithmetic expression becomes an algebraic expression upon introducing variables (or monomials) at various places in the expression.[2] The variables are regarded as placeholders that may be replaced, either by an arithmetic expression, or by another algebraic expression. The analogue for arrangement graphs is to decorate some of the vertices by variables or monomials.

5.1 **Definition**. (Algebraic Arrangement Graph) An *algebraic* arrangement graph consists of an arrangement graph in which some (or all, or none) of the vertices have been decorated by marking them with monomials. Two algebraic arrangement graphs are the same if they correspond to the same elements of $\mathbb{F}$ and have the same monomial markings. As a *set*, the Spencer-Brown algebra $\mathbb{A}$ is the set of all such graphs. ❀

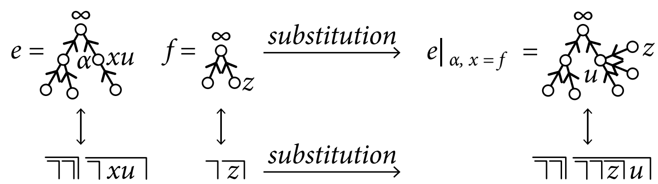
5.2 **Definition**. (Spencer-Brown Algebra) The set $\mathbb{A}$ becomes the Spencer-Brown *algebra* when the operations of multiplication and cross are included. These are straightforward extensions of those operations as they exist in $\mathbb{F}$ with the understanding that the monomial at the root of a product is the product of the monomials at the roots of the factors.[3] ❀

[2] E.g., $3 + 6 + 2 \rightsquigarrow 3x^2 + 6y + 2xz$.

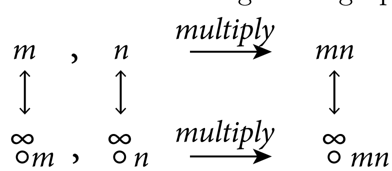
[3] Other than this special treatment of monomials at ∞, both multiplication, and the action of $\urcorner$, work in $\mathbb{A}$ exactly as they did in $\mathbb{F}$, with the finite vertices carrying their monomials along with them.

5.3 **Definition**. (Substitution) If e is an arrangement graph with a vertex α decorated by x (or by a monomial containing x), then substituting an arrangement graph f for x at α means removing the variable x from the monomial at α and gluing a copy of f onto e by identifying the root of f to α. We denote the result of the substitution by $e|_{\alpha,x=f}$. Making this substitution at *every* appearance of x in e is denoted by $e|_{x=f}$. ❀

Here is a picture showing how substitution works for an arrangement graph and its corresponding Spencer-Brown expression:



Here is the simplest picture of multiplying monomials at infinity, both as Spencer-Brown expressions and as arrangement graphs.



5.4 **Remark**. The notion of depth of an expression in $\mathbb{F}$ applies without change to $\mathbb{A}$. ❖

5.5 **Remark**. (Expressions of Depth 2) The discussion in §2 tells us every expression in $\mathbb{F}$ of depth 2 is a *product* of factors of the form

$$(\alpha)\,\bullet,\ (\beta)\,\overline{\;}\rceil,\ \text{and}\ (\gamma)\,\overline{\overline{\;}\rceil\ \dots\ \overline{\;}\rceil}\rceil$$

Similarly, every expression in $\mathbb{A}$ of depth 2 is a *product* of expressions of the following kinds:

$$(\alpha)\ m,\ \ (\beta)\ \overline{m}\rceil,\ \ (\gamma)\ \overline{m\,\overline{m}\rceil\ \dots\ \overline{m}\rceil}\rceil$$

where the ms are monomials (but not necessarily the same monomials!) ❖

5.6 **Remark**. (Expressions of Depth 3) The same discussion tells that every expression $e \in \mathbb{F}$ of depth 3 is a *product* of expressions of the form

$$\overline{\overline{\;}\rceil\ \dots \overline{\;}\rceil\ \left(\overline{\overline{\;}\rceil \dots \overline{\;}\rceil}\rceil\right) \dots \left(\overline{\overline{\;}\rceil\ \dots \overline{\;}\rceil}\rceil\right)}\rceil$$

and expressions of depth < 3. This implies that every expression of depth 3 in $\mathbb{A}$ is a *product* of expressions of the form

$$\overline{m\,\overline{m}\rceil\ \dots\ \overline{m}\rceil\ \left(\overline{m\,\overline{m}\rceil \dots\ \overline{m}\rceil}\rceil\right) \dots \left(\overline{m\,\overline{m}\rceil\ \dots\ \overline{m}\rceil}\rceil\right)}\rceil$$

and factors of depth at most 2. ❖

6. EVALUATION EQUIVALENCE

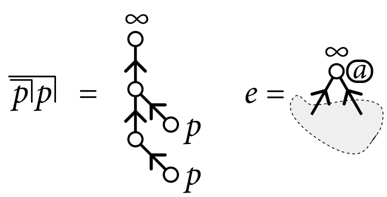
6.1 Definition. (Evaluation Equivalence) Let $e, f \in \mathbb{A}$ involve only the variables $x_1, \ldots, x_n$. Write $\vec{x} = (x_1, \ldots, x_n)$ so $e = e(\vec{x}), f = f(\vec{x})$, and let $\vec{a} = (a_1, \ldots, a_n) \in \mathbb{V}^n$. Define *evaluation equivalence* by

$$e \sim f \iff e(\vec{a}) = f(\vec{a}) \qquad \text{for all } \vec{a} \in \mathbb{V}^n$$ ❀

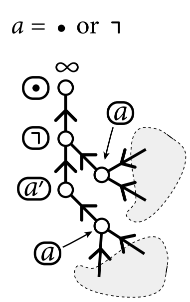
Note that because $\mathbb{V}^n$ has 2^n elements, the relation $e \sim f$ can be verified by checking 2^n cases. Clearly $\sim$ determines an equivalence relation on $\mathbb{A}$.

6.2 Proposition. Let p be a variable. Then $\overline{\overline{p}| \; p}| \sim \bullet$.

Proof. Let us compute the Spencer-Brown invariant when we replace the variable p by an expression $e \in \mathbb{F}$ with $\sigma(e) = \alpha \in \mathbb{V}$. Let $\alpha' \in \mathbb{V}$ denote the opposite invariant.



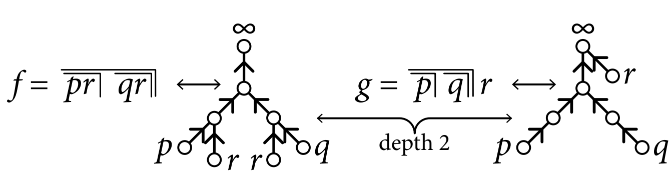
When we substitute e for p the resulting graph is already sufficiently labelled to calculate its Spencer-Brown invariant:



The vertex just below ∞ has a label conflict, so it is labelled ¬; then ∞ is labelled $\bullet$ (which is the value of $\bullet$.) ■

6.3 Proposition. $\overline{\overline{pr}| \;\; \overline{qr}|}| \sim \overline{\overline{p}| \;\; \overline{q}|}| \; r$.

Proof. Consider the arrangement graphs for these two expressions:



Let us compute the Spencer-Brown invariants of these expressions when the variable r is replaced by an expression $e \in \mathbb{F}$. There are two cases, $\sigma(e) = \cdot$ and $\sigma(e) = \urcorner$.

If $\sigma(e) = \cdot$, it forces the labels at depth 2 in the graph for f to be $\urcorner$, which yields $\sigma(f) = \urcorner$. For the expression g, since the label at r is $\cdot$, so $\sigma(g) = \urcorner$.

On the other hand, if $\sigma(e) = \urcorner$, then the labeling of both arrangement graphs is as if the vertices labelled r are absent, in which case the graphs are identical and so have identical Spencer-Brown invariants. ■

6.4 **Proposition.** (Decomposition) Let $e \in \mathbb{A}$ be an arrangement graph with a vertex α specified, and let v be a variable not appearing in e.

- Let e_0 denote the arrangement graph consisting of the vertex α in e as root, and everything below it.
- Let e_1 be the arrangement graph obtained from e by removing everything below α, and marking α with the variable v (or including v as a factor if there is already a monomial present there).

Then $e = e_1|_{v=e_0}$.[4]

Proof. This is obvious from the definition of substitution. ■

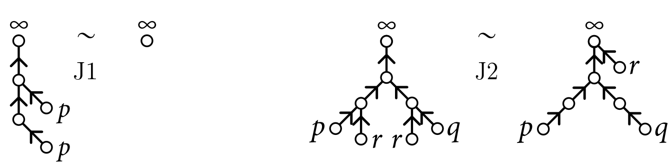
7. THE QUOTIENT ALGEBRA $\mathbb{A}/\mathfrak{j}$

The algebra $\mathbb{A}/\mathfrak{j}$ is derived from the Spencer-Brown algebra $\mathbb{A}$ by dividing by the equivalence relation $\mathfrak{j}$ generated by the two relations (called by Spencer-Brown *initials*):

(J1) $$\overline{\overline{p}\rceil\; p}\rceil \sim_{\mathfrak{j}} \cdot$$ (Position)

(J2) $$\overline{\overline{pr}\rceil\; \overline{qr}\rceil}\rceil \sim_{\mathfrak{j}} \overline{\overline{p}\rceil\; \overline{q}\rceil}\rceil\; r$$ (Transposition)

Here are the graphical versions of these:



These elementary relations are to be roughly understood as "Whenever these elementary graphs appear as portions of larger graphs which are identical except for these portions, the two larger graphs are again equivalent." Precisely stated in terms of substitution we have:

$$e|_{\alpha,z=\text{LHS (or RHS)}} = e|_{\alpha,z=\text{RHS (or LHS)}}$$

[4]Note that if α is either a minimal vertex (so that $e_1 = e$) or the infinite vertex (so that $e_0 = e$), the decomposition is trivial.

where e is some expression and LHS and RHS denote the left and right hand sides of either (J1) or of (J2).

Spencer-Brown derives twelve identities as consequences of J1 and J2. Of these we prove here only five: C1, C2, C3, C7 and T10. Only these are needed in the remainder of this paper.[5]

(C1) $\overline{\,\overline{\,a\,}\rceil\,}\rceil = a.$

Proof.

$$
\begin{aligned}
\overline{\,\overline{\,a\,}\rceil\,}\rceil &= \overline{\,\overline{\,\overline{\,a\,}\rceil\,}\rceil\ \overline{\,a\,}\rceil\,}\rceil\ \overline{\,\overline{\,a\,}\rceil\,}\rceil && \text{(J1) (with } \overline{\,a\,}\rceil = p\text{)}\\
&= \overline{\,\overline{\,\overline{\,a\,}\rceil\ \overline{\,\overline{\,a\,}\rceil\,}\rceil\,}\rceil\ \overline{\,a\ \overline{\,\overline{\,a\,}\rceil\,}\rceil\,}\rceil\,}\rceil && \text{(J2) (with } \overline{\,\overline{\,a\,}\rceil\,}\rceil = r\\
&= \overline{\,\overline{\,a\ \overline{\,\overline{\,a\,}\rceil\,}\rceil\,}\rceil\,}\rceil && \text{(J1)}\\
&= \overline{\,\overline{\,\overline{\,\overline{\,a\,}\rceil\,}\rceil\ \overline{\,a\,}\rceil\,}\rceil\ \overline{\,\overline{\,a\,}\rceil\ a\,}\rceil\,}\rceil && \text{(J1) (and a rearrangement)}\\
&= \overline{\,\overline{\,\overline{\,\overline{\,a\,}\rceil\,}\rceil\,}\rceil\ \overline{\,\overline{\,\overline{\,a\,}\rceil\,}\rceil\,}\rceil\,}\rceil\ a && \text{(J2)}\\
&= a && \text{(J1)} \qquad \blacksquare
\end{aligned}
$$

(C2) $\overline{\,ab\,}\rceil\ b = \overline{\,a\,}\rceil\ b$

Proof.

$$
\begin{aligned}
\overline{\,ab\,}\rceil\ b &= \overline{\,\overline{\,\overline{\,a\,}\rceil\,}\rceil\ \overline{\,\overline{\,b\,}\rceil\,}\rceil\,}\rceil\ b && \text{(C1) (twice)}\\
&= \overline{\,\overline{\,\overline{\,a\,}\rceil\ b\,}\rceil\ \overline{\,\overline{\,b\,}\rceil\ b\,}\rceil\,}\rceil && \text{(J2)}\\
&= \overline{\,\overline{\,\overline{\,a\,}\rceil\ b\,}\rceil\,}\rceil && \text{(J1)}\\
&= \overline{\,a\,}\rceil\ b && \text{(C1)} \qquad \blacksquare
\end{aligned}
$$

(C3) $\overline{}\rceil\ a = \overline{}\rceil$

Proof.

$$
\begin{aligned}
\overline{}\rceil\ a &= \overline{\,a\,}\rceil\ a && \text{(C2)}\\
&= \overline{\,\overline{\,\overline{\,a\,}\rceil\ a\,}\rceil\,}\rceil && \text{(C1)}\\
&= \overline{}\rceil && \text{(J1)} \qquad \blacksquare
\end{aligned}
$$

The following "Echelon Identity" is important because it equates an expression of depth 3 to one of depth 2. It will be used in propositions 9.1 and 9.2 to reduce the depth of every expressions to one of depth 2.

(C7) $\overline{\,a\ \overline{\,b\ \overline{\,c\,}\rceil\,}\rceil\,}\rceil = \overline{\,a\ \overline{\,b\,}\rceil\,}\rceil\ \overline{\,ac\,}\rceil$

[5]The remaining seven are proved in the Appendix, §13.

Proof.

$$\overline{a\,\overline{b\,\overline{c}|}|}| \;=\; \overline{a\,\overline{\overline{\overline{b}|}|\,\overline{c}|}|}| \qquad \text{(C1)}$$
$$=\; \overline{\overline{\overline{a\,\overline{b}|}|\,\overline{a\,c}|}|}| \qquad \text{(J2)}$$
$$=\; \overline{a\,\overline{b}|}|\,\overline{a\,c}| \qquad \text{(C1)} \qquad \blacksquare$$

(T10) $\overline{\overline{a_1}|\ \dots\ \overline{a_{n-1}}|\ \overline{a_n}|}|\ r = \overline{\overline{a_1 r}|\ \dots\ \overline{a_{n-1}r}|\ \overline{a_n r}|}|$

Proof. If $n = 1$ this is just $\overline{\overline{a}|}|\ r = ar = \overline{\overline{ar}|}|$ (C1) , twice)

If $n = 2$ this is J2.

If $n > 2$, and the result holds for $k < n$ then

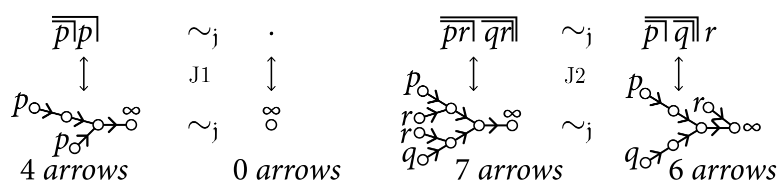
$$\overline{\overline{a_1}|\ \dots\ \overline{a_{n-2}}|\ \overline{a_{n-1}}|\ \overline{a_n}|}|\ r$$
$$= \overline{\overline{a_1}|\dots\ \overline{a_{n-2}}|\ \overline{\overline{\overline{a_{n-1}}|\ \overline{a_n}|}|}|}|\ r \qquad \text{(C1)}$$
$$= \overline{\overline{a_1 r}|\dots\ \overline{a_{n-2}r}|\ \overline{\overline{\overline{a_{n-1}}|\ \overline{a_n}|}|\ r}|}| \qquad \text{by induction}$$
$$= \overline{\overline{a_1 r}|\dots\ \overline{a_{n-2}r}|\ \overline{\overline{\overline{\overline{a_{n-1}r}|\ \overline{a_n r}|}|}|}|}| \qquad \text{(J2)}$$
$$= \overline{\overline{a_1 r}|\dots\ \overline{a_{n-2}r}|\ \overline{a_{n-1}r}|\ \overline{a_n r}|}| \qquad \text{(C1)} \qquad \blacksquare$$

8. INDEPENDENCE

The multitude of identities in $\mathbb{A}/\mathrm{j}$ raises the question whether (J1) has (J2) as a consequence, or whether (J2) has (J1) as a consequence. Neither of these is true as shown in:

8.1 **Proposition.** The relations (J1) and (J2) are independent.

Proof. The proof depends on 'counting arrows' in the arrangement graphs corresponding to these identities.



(J1) cannot be derived from (J2) because (J2) always leaves at least six arrows after its use, whereas (J1) can reduce the number of arrows to zero.

(J2) cannot be derived from (J1) because (J1) does not alter the number (mod 4) of arrows, whereas (J2) can change the number of arrows by one. $\blacksquare$

9. SPENCER-BROWN'S STANDARD FORM

9.1 **Proposition.** Every expression of depth 3 in $\mathbb{A}$ is equivalent, mod $\mathfrak{j}$, to a product of expressions of depth at most 2.

Proof. Remark 5.6 says that every expression of depth 3 in $\mathbb{A}$ is a product of an expression of depth at most 2 and additional factors of the form

$$\overline{m\,\overline{m}|\ \dots\ \overline{m}|\ \left(\overline{m\,\overline{m}|\dots\overline{m}|}|\right)\dots\left(\overline{m\,\overline{m}|\ \dots\ \overline{m}|}|\right)}|$$

In such an additional factor there are, let us say, q subexpressions of the form

$$e=\left(\overline{m\,\overline{m}|\ \dots\ \overline{m}|}|\right)$$

Let us write this additional factor in the form

$$a=\overline{\underbrace{m\,\overline{m}|\ \dots\ \overline{m}|\ \left(\overline{m\,\overline{m}|\ \dots\ \overline{m}|}|\right)\dots}_{A}\left(\overline{\underbrace{m\,\overline{m}|\ \dots\ \overline{m}|}_{B}\ \overline{m}|}|\right)}|$$

where A contains $q-1$ subexpressions of the form e and $B=m\,\overline{m}|\ \dots\ \overline{m}|$. Then a may be written as

$$a=\overline{A\,\overline{B\,\overline{m}|}|}|\ =\overline{A\,B}|\ \overline{Am}|\quad \text{mod } \mathfrak{j}\quad \text{(Echelon identity)}$$

where both factors on the right have forms similar to a, but in which the number of subexpressions of the form e has dropped to $q-1$.

Repeating this argument inductively we can eliminate all of the subexpressions of the form e, leaving only an expression of depth 2. ■

The next result completes the proof of Spencer-Brown's Theorem 14.

9.2 **Proposition.** Let $e \in \mathbb{A}$. Then there is an $f \in \mathbb{A}$ of depth at most 2 with $e = f$ mod $\mathfrak{j}$.

Proof. Let e be any expression of depth $d \geqslant 3$. Let ℓ be the number of vertices at depth d, i.e., minimal vertices. Choose one of them, say α, and let β be the (unique) vertex three above it. (This is possible, since $d \geqslant 3$.) Then, as in proposition 6.3, we may decompose e as $e = e_1|_{\beta=e_0}$, where e_0 consists of the vertex α (playing the role of ∞ for e_0) and everything below it. Then e_0 has depth 3, and so, by proposition 9.1, is equivalent, mod $\mathfrak{j}$, to an expression f of depth 2. This reduces the number ℓ of vertices of e at depth d. Continuing inductively, we can reduce the number of vertices at depth d to zero, so that the expression e is equivalent to one of depth at most $d-1$. The whole argument may now be repeated inductively to show e can be represented by an expression of depth at most 2. ■

The following two results prepare for the proof of Spencer-Brown's Theorem 15 (our Theorem 9.7)

9.3 **Lemma.** $\overline{a\,\overline{bv}|}|\ =\overline{a\,\overline{b}|}|\ \overline{a\,\overline{v}|}|$

Proof.

$$
\begin{aligned}
\overline{a\,\overline{bv}\rceil}\rceil &= \overline{a\,\overline{b\,\overline{w}\rceil}\rceil}\rceil && \text{where } w = \overline{v}\rceil \\
&= \overline{a\,\overline{b}\rceil}\rceil\ \ \overline{aw}\rceil && \text{Echelon Identity} \\
&= \overline{a\,\overline{b}\rceil}\rceil\ \ \overline{a\,\overline{v}\rceil}\rceil && \text{substitution}
\end{aligned}
$$

■

9.4 **Lemma**. Let $e \in \mathbb{A}$, and let v be a variable. Then e may be written, mod j, as a *product* of elements of the form

$$A\,\overline{Bv}\rceil\ \ \overline{C\,\overline{v}\rceil}\rceil$$

where A, B, and C are all independent of v.

Proof. Since, by theorem 9.2, e is equivalent, mod j, to an expression of depth at most 2, then, by remark 6.3, every $e \in \mathbb{A}$ is equivalent to a product of expressions of the form: $(\alpha)\ m$ $(\beta)\ \overline{m}\rceil$ $(\gamma)\ \overline{m_0\,\overline{m_1}\rceil\ \dots\ \overline{m_q}\rceil}\rceil$, where the m_j are monomials.

Now consider how v can appear in these three expressions. For those of the form (α) and (β), the monomial m either contains v or does not contain v, so the expressions (α) and (β) may be written in the following ways:

(1) n

(2) nv

(3) $\overline{n}\rceil$

(4) $\overline{nv}\rceil$

where the ns are monomials free of v.

Similarly, for an expression of the form (γ), each of the monomials

$$m_0, m_1, \dots, m_q$$

either contains or does not contain v, so the expression (γ) may be written in one of the following four ways:

(5) $\overline{n_0\,\overline{n_1}\rceil\ \dots\ \overline{n_q}\rceil}\rceil$

(6)

$$
\begin{aligned}
&\overline{n_0\,\overline{n_1}\rceil\ \dots\ \overline{n_t}\rceil\ \overline{n_{t+1}v}\rceil\ \dots\ \overline{n_qv}\rceil}\rceil \\
&= \overline{a\,\overline{bv}\rceil}\rceil \text{ by (T10) where } a = n_0\,\overline{n_1}\rceil\ \dots\ \overline{n_t}\rceil \\
&\qquad\qquad\qquad\qquad \text{and } b = \overline{\overline{n_{t+1}}\rceil\ \dots\ \overline{n_q}\rceil}\rceil \\
&= \overline{a\,\overline{b}\rceil}\rceil\ \ \overline{a\,\overline{v}\rceil}\rceil \quad \text{(by lemma 9.3)} \\
&= A\,\overline{B\,\overline{v}\rceil}\rceil \quad \text{where } A \text{ and } B \text{ are free of } v
\end{aligned}
$$

(7) $\overline{n_0v\,\overline{n_1}\rceil\ \dots\ \overline{n_q}\rceil}\rceil = \overline{Bv}\rceil$ where B is free of v

(8)

$$\begin{aligned} &\overline{n_0 v\,\overline{n_1|}\ \ldots\ \overline{n_t|}\ \overline{n_{t+1}v|}\ \ldots\ \overline{n_q v|}\,|} \\ =\ &\overline{n_0 v\,\overline{n_1|}\ \ldots\ \overline{n_t|}\ \overline{n_{t+1}|}\ \ldots\ \overline{n_q|}\,|} \quad \text{by (C2)} \\ =\ &\overline{Bv|} \quad \text{where } B \text{ is free of } v \end{aligned}$$

where again, the n_js are all free of v. Clearly each of these eight expressions may be written as $A\,\overline{Bv|}\ \overline{C\,\overline{v|}|}$ where A, B, and C are free of v. ■

9.5 **Definition**. A subset $\mathbb{B} \subset \mathbb{A}$ is a *subalgebra* if it contains $\bullet$ and is closed under multiplication. ❀

9.6 **Proposition.** Fix a variable v and consider $\mathbb{B} \subset \mathbb{A}$ consisting of all the expressions of the form

$$A\,\overline{Bv|}\ \overline{C\,\overline{v|}|} \in \mathbb{A}$$

where the coefficients A, B, C are independent of v. Then $\mathbb{B}$ is a subalgebra.[6]

Proof. $\mathbb{B}$ contains $\bullet$ because $\bullet = \bullet\,\overline{\overline{\,|}\,v|}\ \overline{\overline{\,|}\ \overline{v|}|} \in \mathbb{B}$. $\mathbb{B}$ is closed under multiplication because the product of two element of the given form is again of the given form as the following calculation shows:

$$\begin{aligned} &A\,\overline{Bv|}\ \overline{C\,\overline{v|}|}\ A'\,\overline{B'v|}\ \overline{C'\,\overline{v|}|} && \\ =&AA'\,\overline{Bv|}\ \overline{B'v|}\ \overline{C\,\overline{v|}|}\ \overline{C'\,\overline{v|}|} && \text{Rearrange} \\ =&AA'\,\overline{\overline{\overline{Bv|}\ \overline{B'v|}|}|}\ \overline{\overline{\overline{C\,\overline{v|}|}\ \overline{C'\,\overline{v|}|}|}|} && \text{(C1)} \\ =&AA'\,\overline{\overline{\overline{B|}\ \overline{B'|}|}\,v|}\ \overline{\overline{\overline{C|}\ \overline{C'|}|}\ \overline{v|}|} && \text{(J2)} \\ \in&\mathbb{B} && \end{aligned}$$

■

The following result is Spencer-Brown's T15.

9.7 **Theorem**. (Spencer-Brown's standard form with respect to one variable) Let v be a variable and let $e \in \mathbb{A}$ be an expression. Then e may be written as

$$e = A\,\overline{Bv|}\ \overline{C\,\overline{v|}|} \quad \mod \mathfrak{j}$$

where $A, B, C \in \mathbb{A}$ are independent of v.

Proof. By lemma 9.4, e is equivalent, mod $\mathfrak{j}$, to a *product* of elements of the form

$$A\,\overline{Bv|}\ \overline{C\,\overline{v|}|} \in \mathbb{A}$$

where $A, B, C \in \mathbb{A}$ are independent of v and A is a monomial. By proposition 9.6, products of expressions of this form are again of this form. ■

---

[6] We might also define an *involutive* subalgebra as a subalgebra that is also closed under cross. Then it is easily verified that $\mathbb{B}$ is also an involutive subalgebra.

10. COMPLETENESS

In definition 6.1 we defined the evaluation equivalence relation $\sim$ on $\mathbb{A}$ using arithmetic, saying that two algebraic expressions are equivalent if all their evaluations in the value arithmetic $\mathbb{V}$ are the same. In §8 we introduced a second, algebraic, equivalence relation $\sim_{\mathrm{j}}$ on $\mathbb{A}$. Proposition 10.2 below shows these two equivalence relations are identical, which is to say that the algebra $\mathbb{A}/\mathrm{j}$ is *complete*. This means in particular that we can verify the equality of two elements of $\mathbb{A}/\mathrm{j}$ using either arithmetical evaluation or algebraic equivalence on their representatives in $\mathbb{A}$. (See example 11.8 and the appendix for various cases of this.)

The following lemma is a preparation for the proof of completeness. It is a consequence of (C9), but we give a separate proof here.

10.1 Lemma. $\overline{\overline{Bv}\rvert\ \overline{C\ \overline{v}\rvert}\rvert}\rvert = \overline{\overline{B}\rvert\ v}\rvert\ \overline{\overline{C}\rvert\ \overline{v}\rvert}\rvert$

Proof.

$$\begin{aligned}
\overline{\overline{Bv}\rvert\ \overline{C\ \overline{v}\rvert}\rvert}\rvert &= \overline{\overline{Bv}\rvert\ \overline{C}\rvert}\rvert\ \overline{\overline{Bv}\rvert\ v}\rvert && \text{(Echelon)}\\
&= \overline{\overline{Bv}\rvert\ \overline{C}\rvert}\rvert\ \overline{\overline{B}\rvert\ v}\rvert && \text{(C2)}\\
&= \overline{\overline{Bv}\rvert\ \left(\overline{\overline{B}\rvert\ v}\rvert\right)\overline{C}\rvert}\rvert\ \overline{\overline{B}\rvert\ v}\rvert && \text{(C2)}\\
&= \overline{\overline{\overline{\overline{Bv}\rvert\ \overline{\overline{B}\rvert\ v}\rvert}\rvert}\rvert\ \overline{C}\rvert}\rvert\ \overline{\overline{B}\rvert\ v}\rvert && \text{(C1)}\\
&= \overline{\overline{\overline{\overline{B}\rvert\ \overline{\overline{B}\rvert}\rvert}\rvert\ v}\rvert\ \overline{C}\rvert}\rvert\ \overline{\overline{B}\rvert\ v}\rvert && \text{(J2)}\\
&= \overline{\overline{v}\rvert\ \overline{C}\rvert}\rvert\ \overline{\overline{B}\rvert\ v}\rvert && \text{(J1)} \quad \blacksquare
\end{aligned}$$

10.2 Proposition. (Completeness) Let $f, g \in \mathbb{A}$. Then

$$f \sim_{\mathrm{j}} g \iff f \sim g$$

Proof. ($\implies$) To say that $f \sim_{\mathrm{j}} g$ is to say there is a sequence of steps

$$f = f_1 \sim_{\mathrm{j}} f_2 \sim_{\mathrm{j}} \cdots \sim_{\mathrm{j}} f_k = g$$

where each elementary step $f_i \sim_{\mathrm{j}} f_{i+1}$ has the form

$$(\star) \qquad f_i = e_i|_{\alpha_i, z_i = \text{LHS (or RHS)}} = e_i|_{\alpha_i, z_i = \text{RHS (or LHS)}} = f_{i+1}$$

where, as before, LHS and RHS stand for the two sides of either (J1) or of (J2). It is clear from propositions 6.2 and 6.3 that whenever we evaluate both sides of all the equations ($\star$), the resulting values in $\mathbb{V}$ will be the same. It follows that $f \sim g$.

($\Longleftarrow$) We prove this by induction on the total number n of variables in the two expressions f and g.

If $n = 0$ then $f, g \in \mathbb{F} \subset \mathbb{A}$, so it suffices to show that the laws of calling and crossing are consequences of (J1) and (J2). But setting $a = \urcorner$ in (C3) yields $\urcorner\,\urcorner = \urcorner$, the law of calling, and setting $a = \bullet$ in (C1) yields the equation $\overline{\urcorner}\rceil = \bullet$, the law of crossing.

Now suppose the result is true for all cases of fewer than n variables. Let f and g have a total number of n variables and let v be one of these. Then, by theorem 9.7, f and g may be represented, mod $\mathfrak{j}$, by

$$A\,\overline{Bv}\rceil\;\overline{C\,\overline{v}\rceil}\rceil \quad \text{and } A'\,\overline{B'v}\rceil\;\overline{C'\,\overline{v}\rceil}\rceil$$

respectively, where $A, B, C, A', B', C' \in \mathbb{A}$ are free of v and so have fewer than n variables. Since $f \sim_{\mathfrak{j}} A\,\overline{Bv}\rceil\;\overline{C\,\overline{v}\rceil}\rceil \implies f \sim A\,\overline{Bv}\rceil\;\overline{C\,\overline{v}\rceil}\rceil$ and similarly for g, we have

$$A\,\overline{Bv}\rceil\;\overline{C\,\overline{v}\rceil}\rceil \sim f \sim g \sim A'\,\overline{B'v}\rceil\;\overline{C'\,\overline{v}\rceil}\rceil$$

Substituting $v = \bullet$ and $v = \urcorner$ in this equation yields

$$A\,\overline{B\bullet}\rceil\;\overline{C\,\overline{\bullet}\rceil}\rceil \sim A'\,\overline{B'\bullet}\rceil\;\overline{C'\,\overline{\bullet}\rceil}\rceil \quad \text{and } A\,\overline{B\urcorner}\rceil\;\overline{C\,\overline{\urcorner}\rceil}\rceil \sim A'\,\overline{B'\urcorner}\rceil\;\overline{C'\,\overline{\urcorner}\rceil}\rceil$$

or

$$A\,\overline{B}\rceil \sim A'\,\overline{B'}\rceil \quad \text{and } A\,\overline{C}\rceil \sim A'\,\overline{C'}\rceil$$

and so, since $A, B, C, A', B', C' \in \mathbb{A}$ have fewer than n variables, we have

$$(*) \qquad A\,\overline{B}\rceil \sim_{\mathfrak{j}} A'\,\overline{B'}\rceil \quad \text{and } A\,\overline{C}\rceil \sim_{\mathfrak{j}} A'\,\overline{C'}\rceil$$

Then calculating mod $\mathfrak{j}$, we have

$$\begin{aligned}
f &= A\,\overline{Bv}\rceil\;\overline{C\,\overline{v}\rceil}\rceil \\
&= A\,\overline{\overline{\overline{B}\rceil\, v}\rceil\;\overline{\overline{C}\rceil\;\overline{v}\rceil}\rceil}\rceil && \text{(C1), (Lemma 10.1)} \\
&= \overline{\overline{A\,\overline{B}\rceil\, v}\rceil\;\overline{A\,\overline{C}\rceil\;\overline{v}\rceil}\rceil}\rceil && \text{(J2)} \\
&= \overline{\overline{A'\,\overline{B'}\rceil\, v}\rceil\;\overline{A'\,\overline{C'}\rceil\;\overline{v}\rceil}\rceil}\rceil && (*) \\
&= g && \text{(reversing the first three steps)} \qquad \blacksquare
\end{aligned}$$

11. CANONICAL FORMS

11.1 **Definition.** If $\epsilon \in \mathbb{V}$ and $f \in \mathbb{A}$, define

$$\langle \epsilon \mid f \rangle = \begin{cases} f & \text{if } \epsilon = \bullet \\ \overline{f}\rceil & \text{if } \epsilon = \urcorner \end{cases}$$

More generally, if $\vec{\epsilon} = (\epsilon_1, \ldots, \epsilon_q) \in \mathbb{V}^q$ and $\vec{f} = (f_1, \ldots, f_q) \in \mathbb{A}^q$, write

$$\langle \vec{\epsilon} \mid \vec{f} \rangle = \prod_{k=1}^{q} \langle \epsilon_k \mid f_k \rangle \qquad ❀$$

11.2 **Remark.** If $\epsilon, \mu \in \mathbb{V}$ (so that $\mu \in \mathbb{V} \subset \mathbb{A}$) then

$$\langle \epsilon \mid \mu \rangle = \begin{cases} \bullet & \text{if } \epsilon = \mu \\ \urcorner & \text{if } \epsilon \neq \mu \end{cases}$$

And if $\vec{\epsilon}, \vec{\mu} \in \mathbb{V}^q$, (so that $\vec{\mu} \in \mathbb{V}^q \subset \mathbb{A}^q$) then[7]

$$(\vec{\epsilon} \mid \vec{\mu}) = \begin{cases} \bullet & \text{if } \vec{\epsilon} = \vec{\mu} \\ \urcorner & \text{if } \vec{\epsilon} \neq \vec{\mu} \end{cases}$$

That is, $\langle \epsilon \mid \mu \rangle$ and $\langle \vec{\epsilon} \mid \vec{\mu} \rangle$ behave like delta functions. ❖

11.3 **Theorem.** (General Canonical Form I) Let $e(\vec{x}, \vec{y}) \in \mathbb{A}$ be a function of the variables $x_1, \ldots, x_q$ and $y_1, \ldots, y_n$. Then

$$e(\vec{x}, \vec{y}) = \prod_{\vec{\mu} \in \mathbb{V}^q} \overline{\overline{e(\vec{\mu}, \vec{y})\,}\rvert \;\; \langle \vec{\mu} \mid \vec{x} \rangle\,}\rvert \qquad \text{mod j}$$

Proof. Let f denote the product. It suffices to show that all evaluations of e and f are identical.

Fixing an arbitrary element $\vec{\tau} = (\tau_1, \ldots, \tau_q) \in \mathbb{V}^q$ we have (mod j)

$$\begin{aligned} f \mid_{\vec{x}=\vec{\tau}} &= \prod_{\vec{\mu} \in \mathbb{V}^q} \overline{\overline{e(\vec{\mu}, \vec{y})\,}\rvert \;\; \langle \vec{\mu} \mid \vec{\tau} \rangle\,}\rvert && \text{mod j} \\ &= \overline{\overline{e(\vec{\tau}, \vec{y})\,}\rvert \;\bullet\,}\rvert \prod_{\vec{\mu} \in \mathbb{V}^q, \vec{\mu} \neq \vec{\tau}} \overline{\overline{e(\vec{\mu}, \vec{y})\,}\rvert \;\; \urcorner\,}\rvert && \text{mod j} \\ &= \overline{\overline{e(\vec{\tau}, \vec{y})\,}\rvert\,}\rvert \prod_{\vec{\mu} \in \mathbb{V}^q, \vec{\mu} \neq \vec{\tau}} \overline{\urcorner\,}\rvert && \text{mod j} \\ &= e(\vec{\tau}, \vec{y}) && \text{mod j} \end{aligned}$$

Since all evaluations of e and f are equal, by completeness, $e = f \mod$ j. ■

11.4 **Example.** (Constant Functions) How does this work for the constant functions of two variables? First consider $e(x, y) = \bullet$. We have

$$\begin{aligned} \prod_{\mu \in \mathbb{V}} \overline{\overline{e(\mu, y)\,}\rvert \;\; \langle \mu \mid x \rangle\,}\rvert &= \prod_{\mu \in \mathbb{V}} \overline{\overline{\bullet\,}\rvert \;\; \langle \mu \mid x \rangle\,}\rvert \\ &= \prod_{\mu \in \mathbb{V}} \overline{\urcorner\,}\rvert \\ &= \bullet \end{aligned}$$

[7] The product will be $\bullet$ iff each factor is $\bullet$, i.e., $\epsilon_k = \mu_k$ for each k. Otherwise the product is $\urcorner$.

Now consider $e(x,y) = \urcorner$.

$$\begin{aligned}
\prod_{\mu\in\mathbb{V}} \overline{\overline{e(\mu,y)|}\ \langle \mu\,|\,x\rangle|} &= \prod_{\mu\in\mathbb{V}} \overline{\overline{\urcorner|}\ \langle \mu\,|\,x\rangle|} \\
&= \prod_{\mu\in\mathbb{V}} \overline{\langle \mu\,|\,x\rangle|} \\
&= \overline{x|}\ \overline{\overline{x|}|} = \overline{x|}\ x = \urcorner
\end{aligned} \qquad \blacklozenge$$

The following special case of theorem 11.3 improves Spencer-Brown's Theorem 15.

11.5 **Corollary**. (Canonical form with respect to one variable) Let $e(x,\vec{y}) \in \mathbb{A}$. Then

$$e \sim \overline{Ax|}\ \overline{B\,\overline{x|}|}$$

where $A = \overline{e(\bullet,\vec{y})|}$ and $B = \overline{e(\urcorner,\vec{y})|}$. In particular, if A and B do not involve any other variables, then e is equivalent to one of the following four expressions: $\bullet$, x, $\overline{x|}$, $\overline{x|}\ x$.

Proof. Proposition 11.3 with $q = 1$ yields

$$\begin{aligned}
e &\sim \prod_{\mu\in\mathbb{V}} \overline{\overline{e(\mu,\vec{y})|}\ \langle \mu\,|\,x\rangle|} \\
&= \overline{\overline{e(\bullet,\vec{y})|}\ \langle \bullet\,|\,x\rangle|}\ \overline{\overline{e(\urcorner,\vec{y})|}\ \langle \urcorner\,|\,x\rangle|} \\
&= \overline{Ax|}\ \overline{B\,\overline{x|}|}
\end{aligned}$$

If A and B are constant, then there are four cases:

$$\begin{aligned}
A = \urcorner, B = \urcorner &\implies e \sim \overline{\urcorner x|}\ \overline{\urcorner\,\overline{x|}|} = \overline{\urcorner|}\ \overline{\urcorner|} = \bullet\bullet = \bullet \\
A = \bullet, B = \bullet &\implies e \sim \overline{\bullet x|}\ \overline{\bullet\,\overline{x|}|} = \overline{x|}\ \overline{\overline{x|}|} = \overline{x|}\ x = \urcorner \\
A = \urcorner, B = \bullet &\implies e \sim \overline{\urcorner x|}\ \overline{\bullet\,\overline{x|}|} = \overline{\urcorner|}\ \overline{\overline{x|}|} = \bullet x = x \\
A = \bullet, B = \urcorner &\implies e \sim \overline{\bullet x|}\ \overline{\urcorner\,\overline{x|}|} = \overline{x|}\ \overline{\urcorner|} = \overline{x|}\ \bullet = \overline{x|}
\end{aligned} \qquad \blacksquare$$

11.6 **Theorem**. (General Canonical Form II) Let $e \in \mathbb{A}$ contain only the variables $x_1, \ldots, x_n$ and set $\vec{x} = (x_1, \ldots, x_n)$ so that we may write $e = e(\vec{x})$. Let

$$S_e = \{\vec{\mu} \in \mathbb{V}^n \mid e(\vec{\mu}) = \urcorner\}$$

Then

$$e = \prod_{\vec{\mu}\in S_e} \overline{\langle \vec{\mu}\,|\,\vec{x}\rangle|} \qquad \mathrm{mod}\ \jmath$$

Proof. By theorem 11.3 we know that e is equivalent (mod $\mathfrak{j}$) to

$$
\begin{aligned}
\prod_{\vec{\mu}\in\mathbb{V}^n} \overline{\overline{e(\mu)}|\ \langle\vec{\mu}\,|\,\vec{x}\rangle}| &= \prod_{\vec{\mu}\in\mathbb{V}^n, e(\vec{\mu})=\bullet} \overline{\overline{e(\mu)}|\ \langle\vec{\mu}\,|\,\vec{x}\rangle}| \prod_{\vec{\mu}\in\mathbb{V}^n, e(\vec{\mu})=\urcorner} \overline{\overline{e(\mu)}|\ \langle\vec{\mu}\,|\,\vec{x}\rangle}| \\
&= \prod_{\vec{\mu}\in\mathbb{V}^n, e(\vec{\mu})=\bullet} \overline{\overline{\bullet}|\ \langle\vec{\mu}\,|\,\vec{x}\rangle}| \prod_{\vec{\mu}\in\mathbb{V}^n, e(\vec{\mu})=\urcorner} \overline{\overline{\urcorner}|\ \langle\vec{\mu}\,|\,\vec{x}\rangle}| \\
&= \prod_{\vec{\mu}\in S_e} \overline{\langle\vec{\mu}\,|\,\vec{x}\rangle}|
\end{aligned}
$$

■

11.7 **Corollary.** (Number of Expressions) $\mathbb{A}[x_1,\ldots,x_n]/\mathfrak{j}$ has 2^{2^n} elements.

Proof. Let $S \subset \mathbb{V}^n$, set $\vec{x} = (x_1,\ldots,x_n)$, and set $f_S(\vec{x}) = \prod_{\vec{\mu}\in S} \overline{\langle\vec{\mu}\,|\,\vec{x}\rangle}|$. Since $f_S(\vec{\nu}) = \urcorner$ for $\nu \in S$ and $\bullet$ otherwise, we see that the f_S are distinct. Moreover by theorem 11.6, every element of $\mathbb{A}[\vec{x}]$ is represented by some f_S. Therefore the elements of $\mathbb{A}[x_1,\ldots,x_n]/\mathfrak{j}$ are in bijective correspondence with the subsets of $\mathbb{V}^n$ of which there are 2^{2^n}. ■

11.8 **Example.** Let us calculate the canonical form for the expression

$$e(x) = x\ \overline{Ax}|\ \overline{B\ \overline{x}|}|$$

with respect to x. Then

$$e(\bullet) = \bullet\ \overline{A\bullet}|\ \overline{B\ \overline{\bullet}|}| = \overline{A}| \text{ and } e(\urcorner) = \urcorner\ \overline{A\urcorner}|\ \overline{B\ \overline{\urcorner}|}| = \urcorner$$

Thus

$$e \sim \overline{\overline{\overline{A}|}|\ x}|\ \overline{\overline{\urcorner}|\ \overline{x}|}| \sim \overline{Ax}|\ \overline{\overline{x}|}| \sim \overline{Ax}|\ x \sim \overline{A}|\ x$$

Since similarity in the Spencer-Brown algebra $\mathbb{A}$ is the same as equality in the algebra $\mathbb{A}/\mathfrak{j}$, we should be able to transform e into f using the initials (J1) and (J2) and their consequences. In fact we have:

$$
\begin{aligned}
e &= x\ \overline{Ax}|\ \overline{B\ \overline{x}|}| && \\
&= x\ \overline{A}|\ \overline{Bx\ \overline{x}|}| && \text{(C2) twice} \\
&= x\ \overline{A}|\ \overline{B\ \overline{\ }|}| && \text{(J1) and (I2)} \\
&= x\ \overline{A}|\ \overline{\overline{\ }|}| && \text{(C3)} \\
&= x\ \overline{A}| && \text{(C1)}
\end{aligned}
$$

◆

11.9 **Example.** Now let's see how the canonical form deals with the case of a function of a two variables x, y. We take

$$e(x,y) = \overline{A\ \overline{Bx\ \overline{Cy}|}|\ \overline{Dy\ \overline{Ex}|}|}|$$

where the expressions A, B, C, D may contain variables other than x and y. Then

$$
\begin{aligned}
e(\bullet,\bullet) &= \overline{A\ \overline{B\bullet\ \overline{C\bullet}|}|\ \ \overline{D\bullet\ \overline{E\bullet}|}|}| \ = \ \overline{A\ \overline{B\ \overline{C}|}|\ \ \overline{D\ \overline{E}|}|}| \\
e(\bullet,\urcorner) &= \overline{A\ \overline{B\bullet\ \overline{C\urcorner}|}|\ \ \overline{D\urcorner\ \overline{E\bullet}|}|}| \ = \ \overline{A\ \overline{B}|}| \\
e(\urcorner,\bullet) &= \overline{A\ \overline{B\urcorner\ \overline{C\bullet}|}|\ \ \overline{D\bullet\ \overline{E\urcorner}|}|}| \ = \ \overline{A\ \overline{D}|}| \\
e(\urcorner,\urcorner) &= \overline{A\ \overline{B\urcorner\ \overline{C\urcorner}|}|\ \ \overline{D\urcorner\ \overline{E\urcorner}|}|}| \ = \ \overline{A}|
\end{aligned}
$$

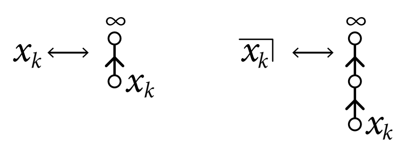
and therefore

$$
\begin{aligned}
e &= \prod_{(\mu_1,\mu_2)\in\mathbb{V}^2} \overline{\overline{e(\mu_1,\mu_2)}|\ \ \langle(\mu_1,\mu_2)\,|\,(x,y)\rangle}| \qquad \bmod \mathfrak{j} \\
&= \overline{\overline{e(\bullet,\bullet)}|\ xy}|\ \ \overline{\overline{e(\bullet,\urcorner)}|\ x\ \overline{y}|}|\ \ \overline{\overline{e(\urcorner,\bullet)}|\ \overline{x}|\ y}|\ \ \overline{\overline{e(\urcorner,\urcorner)}|\ \overline{x}|\ \overline{y}|}| \\
&= \overline{\overline{\overline{A\ \overline{B\ \overline{C}|}|\ \ \overline{D\ \overline{E}|}|}|}|\ xy}|\ \ \overline{\overline{\overline{A\ \overline{B}|}|}|\ x\ \overline{y}|}| \ \times \\
&\qquad\qquad \overline{\overline{\overline{A\ \overline{D}|}|}|\ \overline{x}|\ y}|\ \ \overline{\overline{\overline{A}|}|\ \overline{x}|\ \overline{y}|}| \\
&= \overline{A\ \overline{B\ \overline{C}|}|\ \ \overline{D\ \overline{E}|}|\ xy}|\ \ \overline{A\ \overline{B}|\ x\ \overline{y}|}|\ \ \overline{A\ \overline{D}|\ \overline{x}|\ y}|\ \ \overline{A\ \overline{x}|\ \overline{y}|}| \qquad \blacklozenge
\end{aligned}
$$

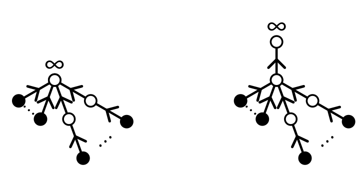
11.10 **Example.** Theorem 11.6 involves expressions

$$e_{\vec{\mu}} = \langle\vec{\mu}|\vec{x}\rangle = \prod_{k=1}^{n} \langle\mu_k|x_k\rangle = \prod_{u_k=\bullet} x_k \prod_{u_k=\urcorner} \overline{x_k}|$$

The factors on the right side of this equation have one or the other of the following two arrangement graphs.



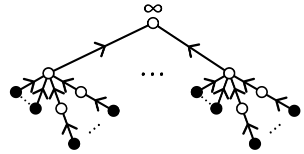
So $e_{\vec{\mu}}$ and $\overline{e_{\vec{\mu}}}|$ have, respectively, arrangement graphs of the forms



where the n black vertices denote the distinct variables $x_1, \ldots, x_n$. Since the canonical form for e is

$$e = \prod_{\vec{\mu}\in S_e} \overline{e_{\vec{\mu}}}| \qquad \bmod \mathfrak{j}$$

its arrangement graph assumes the form



$\blacklozenge$

11.11 **Corollary**. Let $f_{\vec{\mu}}(\vec{x}) = \prod_{i=1}^{n} \langle \mu_i | x_i \rangle$ where $\vec{\mu} \in \mathbb{V}^n$. Then

$$f_{\vec{\mu}}(\vec{x}) = \prod_{\vec{\nu} \neq \vec{\mu}} \overline{\langle \vec{\nu} | \vec{x} \rangle}\rceil$$

12. Epilogue

In the preface to the first American edition of [1] Spencer-Brown expresses his hope that there might be a ‘complex’ extension of his algebra based perhaps on *three* ‘truth values’

$$\mathbb{V}_{\mathbb{C}} = \{\cdot, \urcorner, \text{ə}\}$$

where ə is the solution to the liar’s paradox, defined as the solution of the equation $\overline{\text{ə}}\rceil = \text{ə}$. In his paper [2] Varela took up this direction using the multiplication table

$\times$	$\cdot$	$\urcorner$	ə
$\cdot$	$\cdot$	$\urcorner$	ə
$\urcorner$	$\urcorner$	$\urcorner$	$\urcorner$
ə	ə	$\urcorner$	ə

His next step was to provide an algebra associated to this arithmetic by replacing the Spencer-Brown axioms with

(V1) $\quad \overline{\overline{p}\rceil\; q}\rceil\; p \sim p$

(V2) $\quad \overline{\overline{pr}\rceil\; \overline{qr}\rceil}\rceil \sim \overline{\overline{p}\rceil\; \overline{q}\rceil}\rceil\; r$

(V3) $\quad \overline{p\text{ə}}\rceil\; p \sim p\text{ə}$

hoping to obtain a complete algebra. However, as Kauffman pointed out in [3], Varela’s proof of completeness fails because of a neglected term, and indeed Varela’s axioms appear to be insufficient to prove completeness. In the same paper Kaufmann succeeded in proving completeness upon replacing Varela’s axiom (V3) with:

(K3) $\quad \overline{\overline{p}\rceil\; p}\rceil\; \text{ə} = \text{ə}.$

We note that any complete set of axioms for an algebra is equivalent to any other complete set of axioms. Kaufmann’s axioms are quite simple, so it is unlikely that any simpler (but of course equivalent) axioms can be found.

It is an open question as to whether there exist useful canonical forms for the Varela-Kauffman algebra.

However, to more fully correspond with the properties of the usual complexification of the real numbers, one should require more than just completeness from an extension of the Spencer-Brown algebra. One would also want to know that the algebra is also complete in the sense that every nontrivial equation in a single variable of the form $f(x) = g(x)$ always has a solution. In the Varela-Kauffman algebra, the only values the expression $\overline{x}|\; x$ can assume are $\urcorner$ and ə. Therefore $\overline{x}|\; x = \cdot$ has no solution there. Is there a doubly complete extension of the Spencer-Brown algebra?

13. APPENDIX: MORE IDENTITIES FROM SPENCER-BROWN

In this appendix we give proofs of the remaining seven identities Spencer-Brown provides in his book. Our proofs depend upon the completeness of the Spencer-Brown algebra.

13.1 Proposition. Let $a, b, \dots$ be variables. Then:

(C4) $\overline{\overline{a}|\; b}|\; a = a$

(C5) $aa = a$

(C6) $\overline{\overline{a}|\;\overline{b}|}|\;\;\overline{\overline{a}|\; b}| = a$

(C8) $\overline{\overline{a}|\;\overline{br}|\;\overline{cr}|}| = \overline{\overline{a}|\;\overline{r}|}|\;\;\overline{\overline{a}|\;\overline{b}|\;\overline{c}|}|$

(C9) $\overline{\overline{\overline{b}|\;\overline{r}|}|\;\overline{\overline{a}|\;\overline{r}|}|\;\overline{\overline{x}|\; r}|\;\overline{\overline{y}|\; r}|}| = \overline{\overline{r}|\; ab}|\;\overline{rxy}|$

(T11) $\overline{\overline{a_1}|\;\overline{a_2 r}|\;\dots\;\overline{a_n r}|}| = \overline{\overline{a_1}|\;\overline{a_2}|\;\dots\;\overline{a_n}|}|\;\overline{\overline{a_1}|\;\overline{r}|}|$

(T12) $\overline{\overline{\overline{a_n}|\;\overline{r}|}|\;\dots\;\overline{\overline{a_1}|\;\overline{r}|}|\;\overline{\overline{x_1}|\; r}|\;\dots\;\overline{\overline{x_n}|\; r}|}| = \overline{\overline{r}|\; a_1\dots a_n}|\;\overline{r x_1\dots x_n}|$

Proof. (C4) We want to show that $\overline{\overline{a}|\; b}|\; a = \overline{\overline{a}|\; ba}|\; a$. Setting $a = \cdot$ the equation becomes $\overline{\urcorner\;\overline{b}|}| = \overline{\urcorner b}|$ which is true. Setting $a = \urcorner$ it becomes $\urcorner = \urcorner$ which is also true. Therefore (C4) is true by completeness.

(C5) We want to show that $aa = a$. But this is true for $a = \cdot$ and for $a = \urcorner$, so (C5) is true by completeness.

(C6) We are trying to show that $\overline{\overline{a}|\;\overline{b}|}|\;\;\overline{\overline{a}|\; b}| = a$. But whether $b = \cdot$ or $b = \urcorner$ the equation reads $\overline{\overline{a}|}| = a$, which is true by (C1), so the original is true by completeness.

(C8) We are trying to show that $\overline{\overline{a}|\;\overline{br}|\;\overline{cr}|}| = \overline{\overline{a}|\;\overline{r}|}|\;\;\overline{\overline{a}|\;\overline{b}|\;\overline{c}|}|$. If $r = \cdot$ the equation reads $\overline{\overline{a}|\;\overline{b}|\;\overline{c}|}| = \overline{\overline{a}|\;\overline{b}|\;\overline{c}|}|$, which is true. If $r = \urcorner$ it reads $a = a\;\overline{\overline{a}|\;\overline{b}|\;\overline{c}|}|$ which is true whether $a = \cdot$ (by C1) or $a = \urcorner$, so (C8) is true by completeness.

(C9) We want to verify $\overline{\overline{\overline{b}\rceil\ \overline{r}\rceil}\rceil\ \overline{\overline{a}\rceil\ \overline{r}\rceil}\rceil\ \overline{\overline{x}\rceil\ r}\rceil\ \overline{\overline{y}\rceil\ r}\rceil}\rceil = \overline{\overline{r}\rceil\ ab}\rceil\ \overline{rxy}\rceil$. If we set $r = \cdot$ the equation becomes $\overline{xy}\rceil = \overline{xy}\rceil$ which is true. If $r = \urcorner$, the equation becomes $\overline{ba}\rceil = \overline{ab}\rceil$, which is also true. Therefore (C9) is true by completeness.

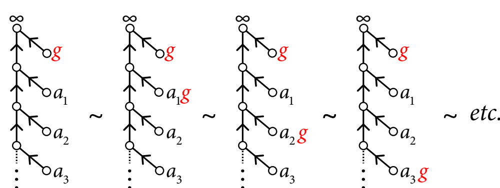
(T11) We want to verify $\overline{\overline{a}\rceil\ \overline{b_1 r}\rceil\ \dots\ \overline{b_n r}\rceil}\rceil = \overline{\overline{a}\rceil\ \overline{b_1}\rceil\ \dots\ \overline{b_n}\rceil}\rceil\ \overline{\overline{a}\rceil\ \overline{r}\rceil}\rceil$
If we set $r = \cdot$ the equation becomes $\overline{\overline{a}\rceil\ \overline{b_1}\rceil\ \dots\ \overline{b_n}\rceil}\rceil = \overline{\overline{a}\rceil\ \overline{b_1}\rceil\ \dots\ \overline{b_n}\rceil}\rceil$ which is true. If $r = \urcorner$, the equation becomes $a = \overline{\overline{a}\rceil\ \overline{b_1}\rceil\ \dots\ \overline{b_n}\rceil}\rceil\ a$ (by C1), which is also true, since it is true whether $a = \cdot$ or $a = \urcorner$. Therefore (T11) is true by completeness.

(T12) We want to verify

$$\overline{\overline{\overline{a_n}\rceil\ \overline{r}\rceil}\rceil\ \dots\ \overline{\overline{a_1}\rceil\ \overline{r}\rceil}\rceil\ \overline{\overline{x_1}\rceil\ r}\rceil\ \dots\ \overline{\overline{x_n}\rceil\ r}\rceil}\rceil = \overline{\overline{r}\rceil\ a_1 \dots a_n}\rceil\ \overline{r x_1 \dots x_n}\rceil$$

If $r = \cdot$ the equation reads $\overline{\overline{\overline{x_1}\rceil}\rceil\ \dots\ \overline{\overline{x_n}\rceil}\rceil}\rceil = \overline{x_1 \dots x_n}\rceil$, which is true by (C1). If $r = \urcorner$ the equation reads $\overline{\overline{\overline{a_n}\rceil}\rceil\ \dots\ \overline{\overline{a_1}\rceil}\rceil\ }\rceil = \overline{a_1 \dots a_n}\rceil$ which is also true by (C1). Thus (T12) is true by completeness.

(T13) Spencer-Brown's final identity is best expressed in terms of its arrangement graphs:



The first equivalence in this diagram is just (C2). The second equivalence arises in two stages. First we use (C2) to add the g next to a_2. Then we use (C2) again to remove the g next to the a_1. This can all be repeated inductively to insert gs against all the a_k for $k \leqslant n$ and then we use (C2) again to remove gs appearing beside all the a_k with $1 \leqslant k < n$. ■

References

[1] G. Spencer-Brown, *The Laws of Form*, Bantam, Massachusetts, 2nd Edition, 1973.
[2] G, Varela, *A Calculus For Self-Reference* Int. J. General Systems, vol. 2. pp.5-25, 1975.
[3] L, Kauffman, Imaginary Values in Mathematical Logic, Proc. of the 17th Intl. Symp. on Multiple-Valued Logic, Boston, MA