Lower bounds based on restrictions

Restrictions

- A (**random**) **restriction** is a (random) subset R of $\{0,1\}^n$
- When R is a subcube of $\{0,1\}^n$, identify with a function $\{x_1,\ldots,x_n\} \to \{0,1,\star\}$ (each coordinate fixed to 0 or 1 or free)
- For $0 \le p \le 1$, let $\mathbf{R}_p$ denotes the p-**random restriction**

$$\mathbf{R}_p(x_i) = \begin{cases} \star & \text{with prob. } p \\ 0 & \text{with prob. } (1-p)/2 \\ 1 & \text{with prob. } (1-p)/2 \end{cases}$$

independently for each variable x_i

Lower Bounds from Restrictions

- A restriction $R \subseteq \{0,1\}^n$ can be applied to both
  - Boolean functions $f : \{0,1\}^n \to \{0,1\}$
  - Boolean circuits C (by syntactic simplification)

- Recipe for lower bounds:

  Show that $C \restriction R$ becomes "simple", while $f \restriction R$ remains "complex" (with high prob. if R is random)

Types of Restrictions $R \subseteq \{0,1\}^n$
(increasing order of generality)

- subcube $x_i = 0,\ x_i = 1$
- mon. projection $x_i = 0,\ x_i = 1,\ x_i = x_j$
- projection $x_i = 0,\ x_i = 1,\ x_i = x_j,\ x_i \neq x_j$
- affine $x_{i\_1} \oplus \cdots \oplus x_{i\_k} = 0,$

  $x_{i\_1} \oplus \cdots \oplus x_{i\_k} = 1$
- low-degree variety $P(x_1,\ldots,x_n) = 0$ where $\deg(P) \leq d$

Circuit Complexity

Circuit Complexity

- Studies the complexity of specific problems (e.g. PARITY, MATRIX MULTIPLICATION, etc.) in ***combinatorial models of computation***, most importantly Boolean circuits
- Goal is to prove ***unconditional lower bounds***, which do not rely on any unproven assumptions

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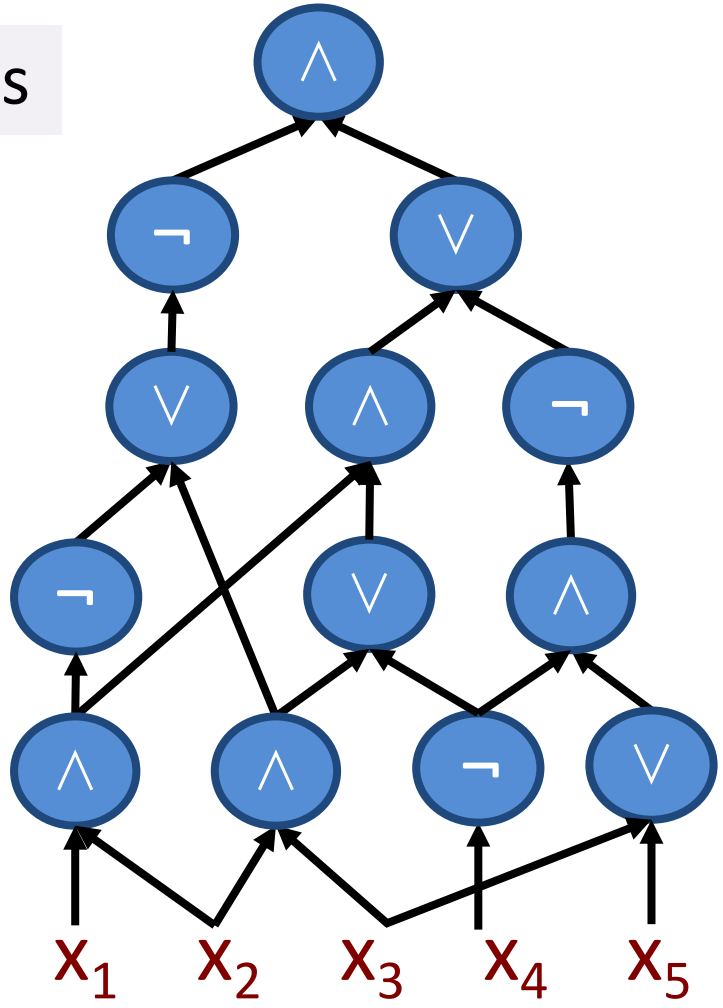
- Go

a **problem** (i.e. decision problem) is represented by a sequence of boolean functions $f_n : \{0,1\}^n \rightarrow \{0,1\}$

Boolean Circuits

size = # of AND and OR gates



Boolean Circuits

- An n-variable Boolean circuit computes an n-variable Boolean function $\{0,1\}^n \rightarrow \{0,1\}$
- A problem is "solved" by a sequence of Boolean circuits $C_1, C_2, \ldots, C_n, \ldots$ if C_n computes the appropriate function $\{0,1\}^n \rightarrow \{0,1\}$

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in contrast to *uniform* models of computation (e.g. Turing machines) where a single algorithm solves the problem on all instances

Circuit Size

- The **circuit size** of a function $f : \{0,1\}^n \to \{0,1\}$ is the minimum # of AND/OR gates in a circuit computing f
- Theorem [Shannon 1949, Lupanov 1958]

  Almost all Boolean functions have circuit size $\Theta(2^n/n)$
- The goal in Circuit Complexity is proving lower bounds for ***explicit*** Boolean functions (e.g. k-CLIQUE)

Circuit Size

- Theorem [Schnorr 1976, Fischer-Pippenger 1979]

  Turing mach. time $T(n)$ $\Longrightarrow$ circuit size $O(T(n)*\log T(n))$

- Corollary

  A ***super-polynomial lower bound*** on the circuit size of any function in NP (i.e. $NP \not\subseteq P/poly$) implies $P \neq NP$

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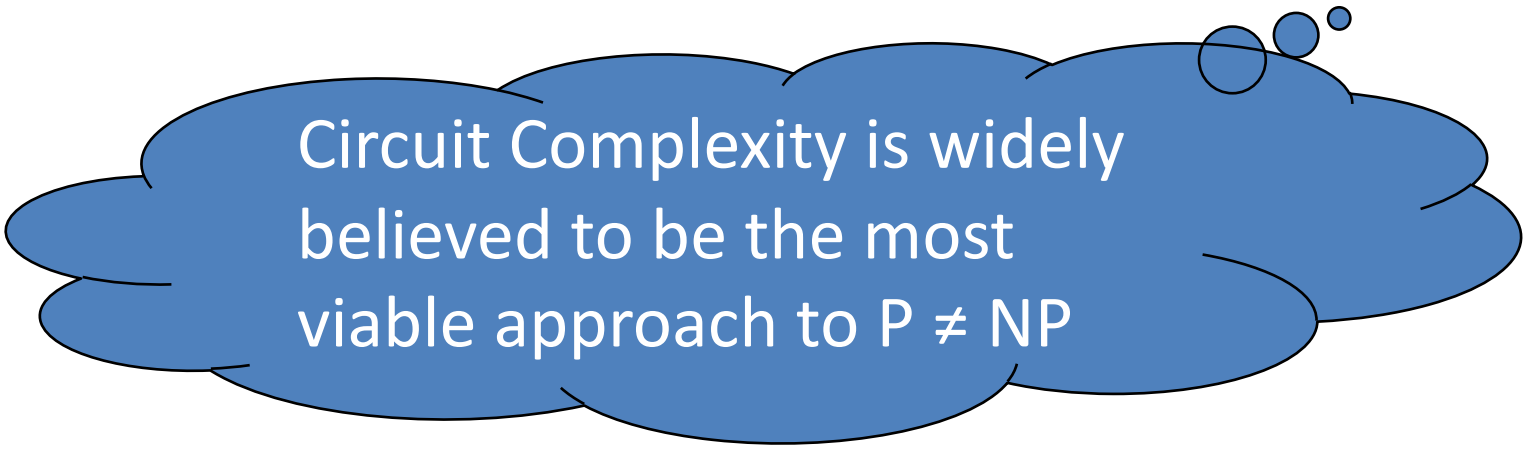

Circuit Size

- Holy Grail ($P \neq NP$)

  Prove a ***super-polynomial lower bound*** on the circuit size of any problem in NP

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- Best known lower bound

$3n - O(1)$	1976	[Schnorr]
$4n - O(1)$	1991	[Zwick]
$4.5n - o(n)$	2001	[Lachish-Raz]
$5n - o(n)$	2002 - today	[Iwama-Morizumi]

- Circuit
- e
circuit

3.01n for circuits in the *full binary basis* (all fan-in 2 gates)

[Find-Golovnev-Hirsch-Kulikov '16]

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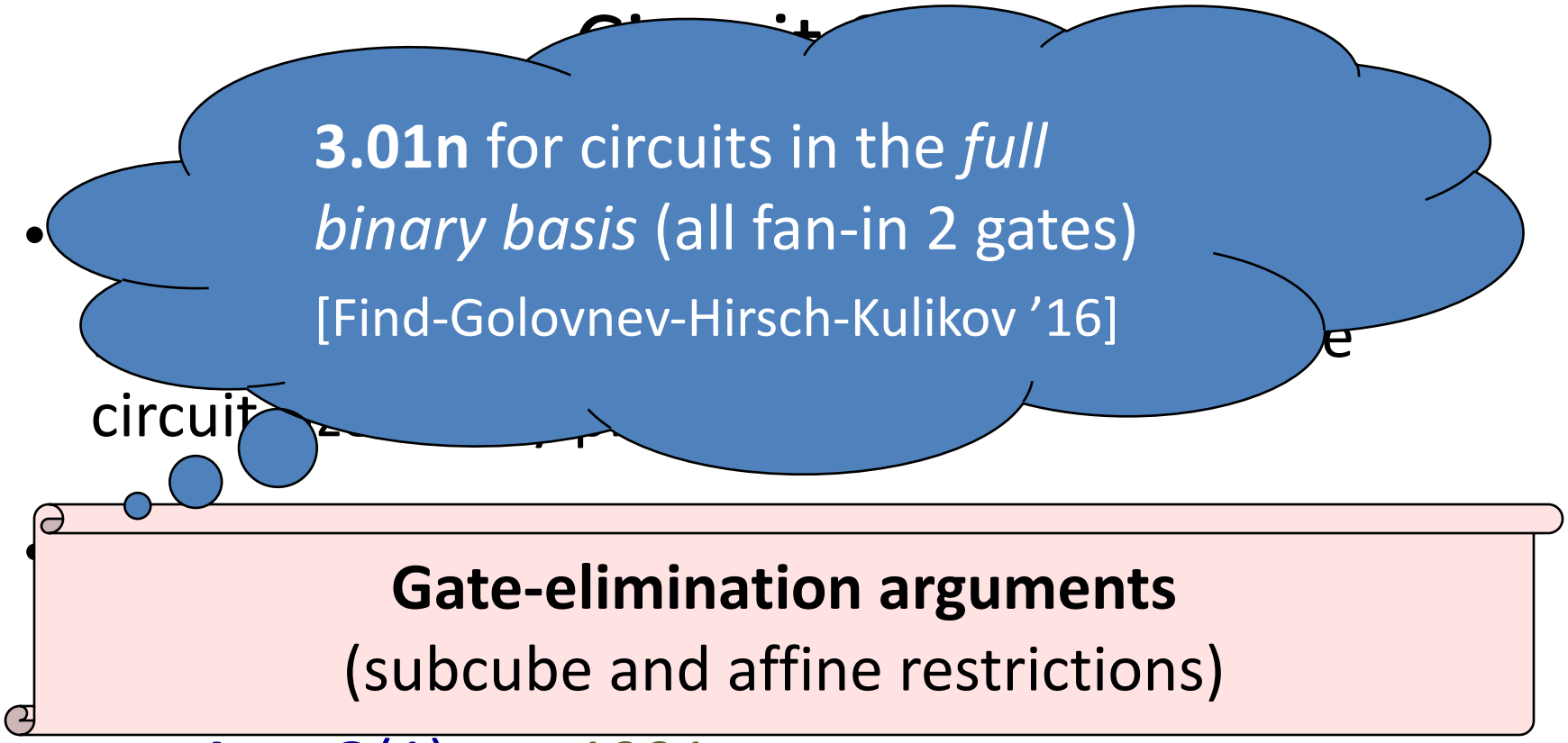
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Gate-elimination arguments
(subcube and affine restrictions)
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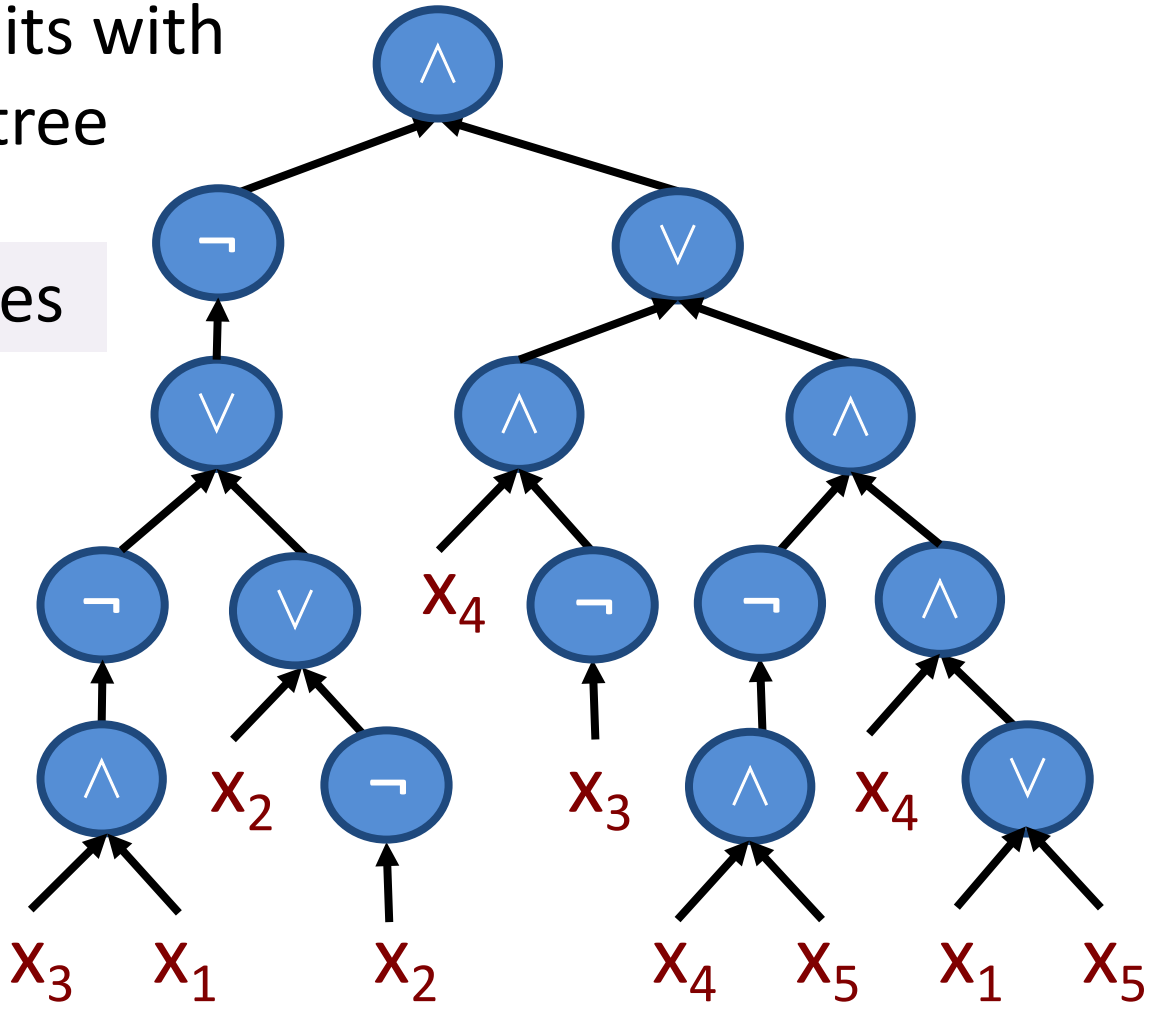
(DeMorgan) Formulas

Formulas are circuits with the structure of a tree

leafsize = # of leaves



(DeMorgan) Formulas

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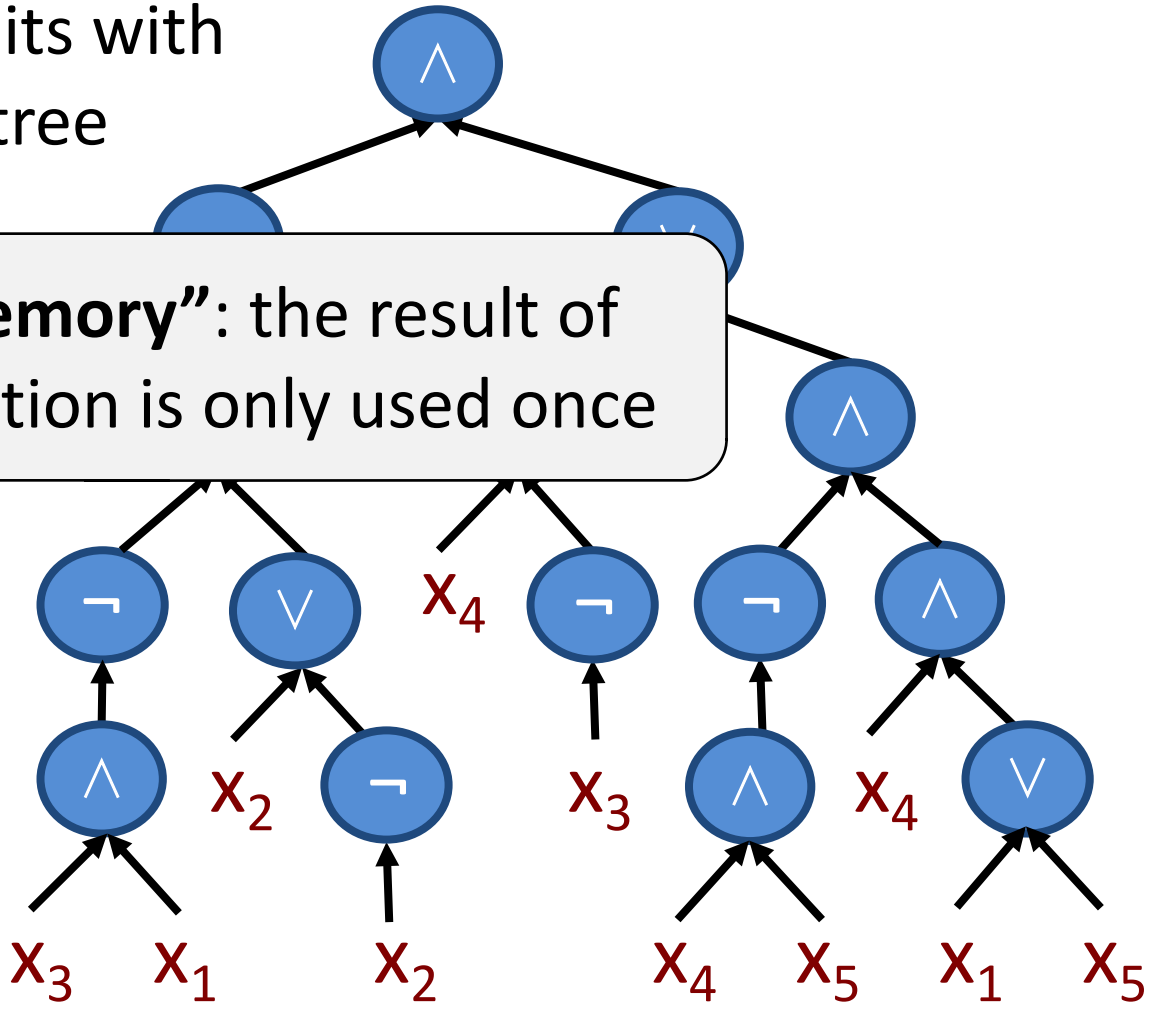
Formulas lack "memory": the result of each sub-computation is only used once



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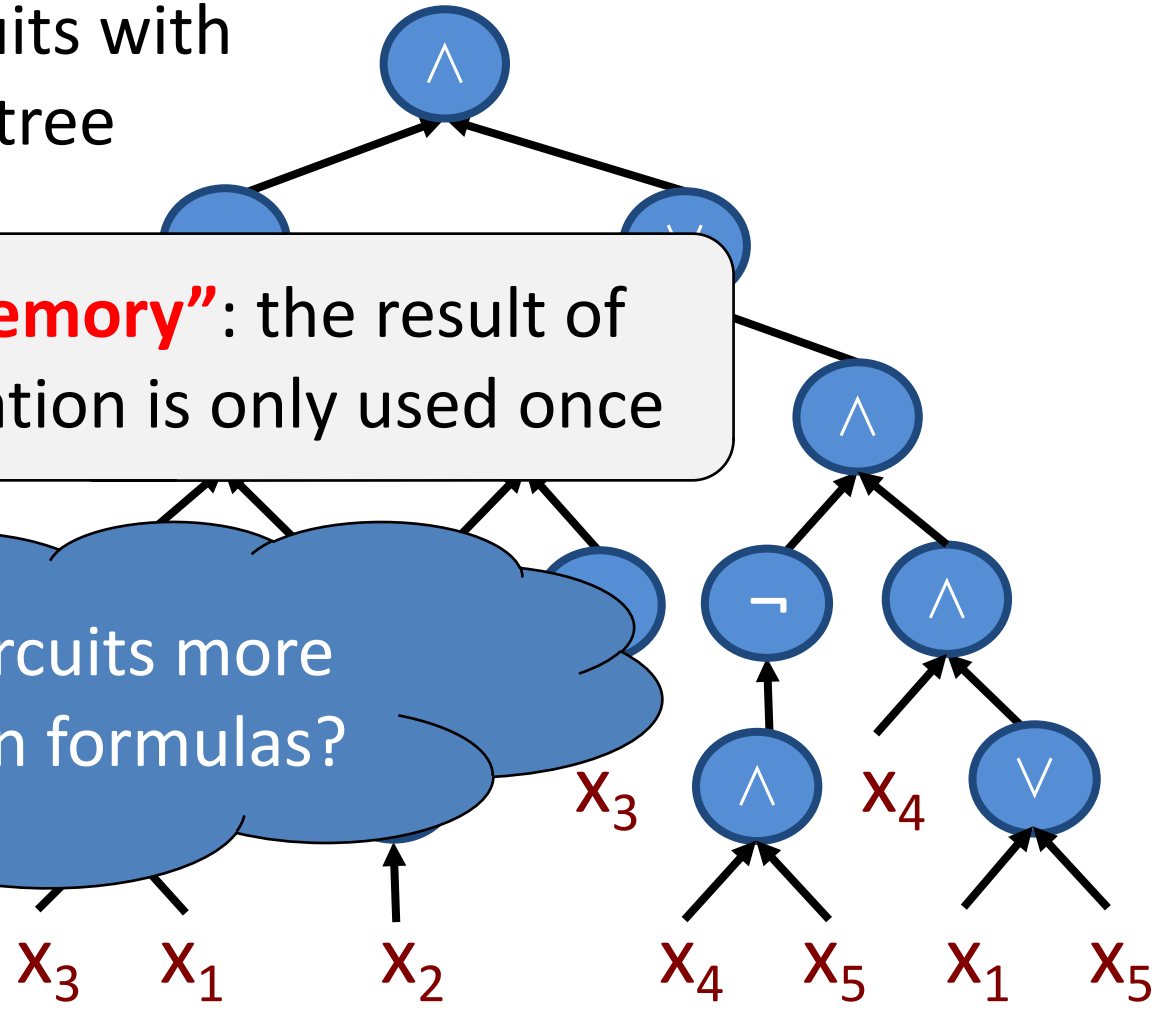
Open: Are circuits more powerful than formulas?



Formulas vs. Circuits

- A Pret-ty Holy Grail ($NC^1 \neq P$)

  Prove that **poly-size circuits** are strictly more powerful than **poly-size formulas**

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- Best known formula size lower bound

$n^{1.5 - o(1)}$	1961	[Subbotovskaya]
n^2	1971	[Khrapchenko]
$n^{2.5 - o(1)}$	1991	[Andreev]
$n^{3 - o(1)}$	1998 - today	[Hastad]

(log-factor improvement [Tal'14])

Formulas vs. Circuits

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  Prove that **poly-size circuits** are strictly more powerful than **poly-size formulas**

Shrinkage of DeMorgan formulas
(simplification under p-random restrictions)

n^2	1971	[Khrapchenko]
$n^{2.5 - o(1)}$	1991	[Andreev]
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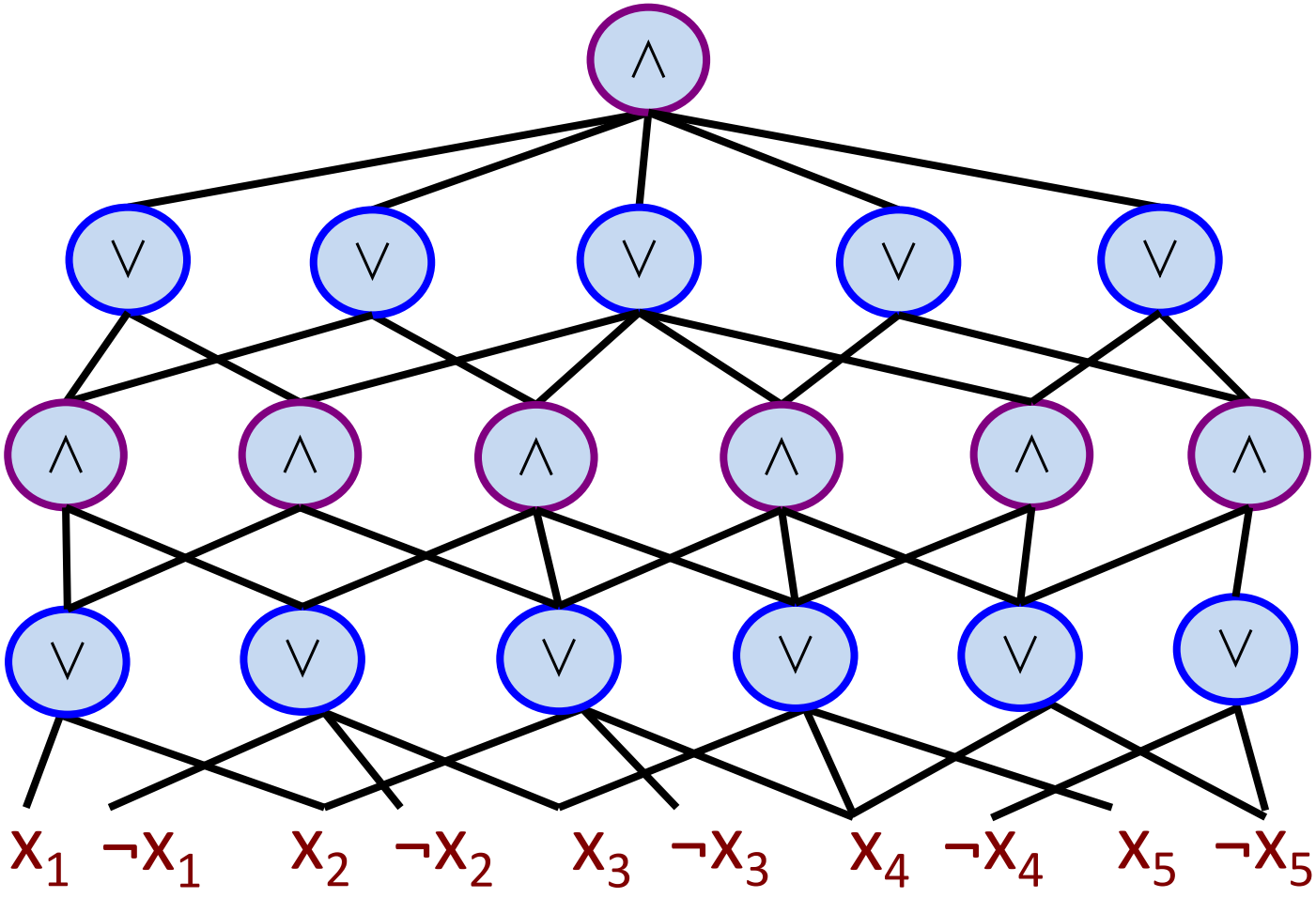
(log-factor improvement [Tal'14])

Restricted Classes (AC^0, monotone, etc.)

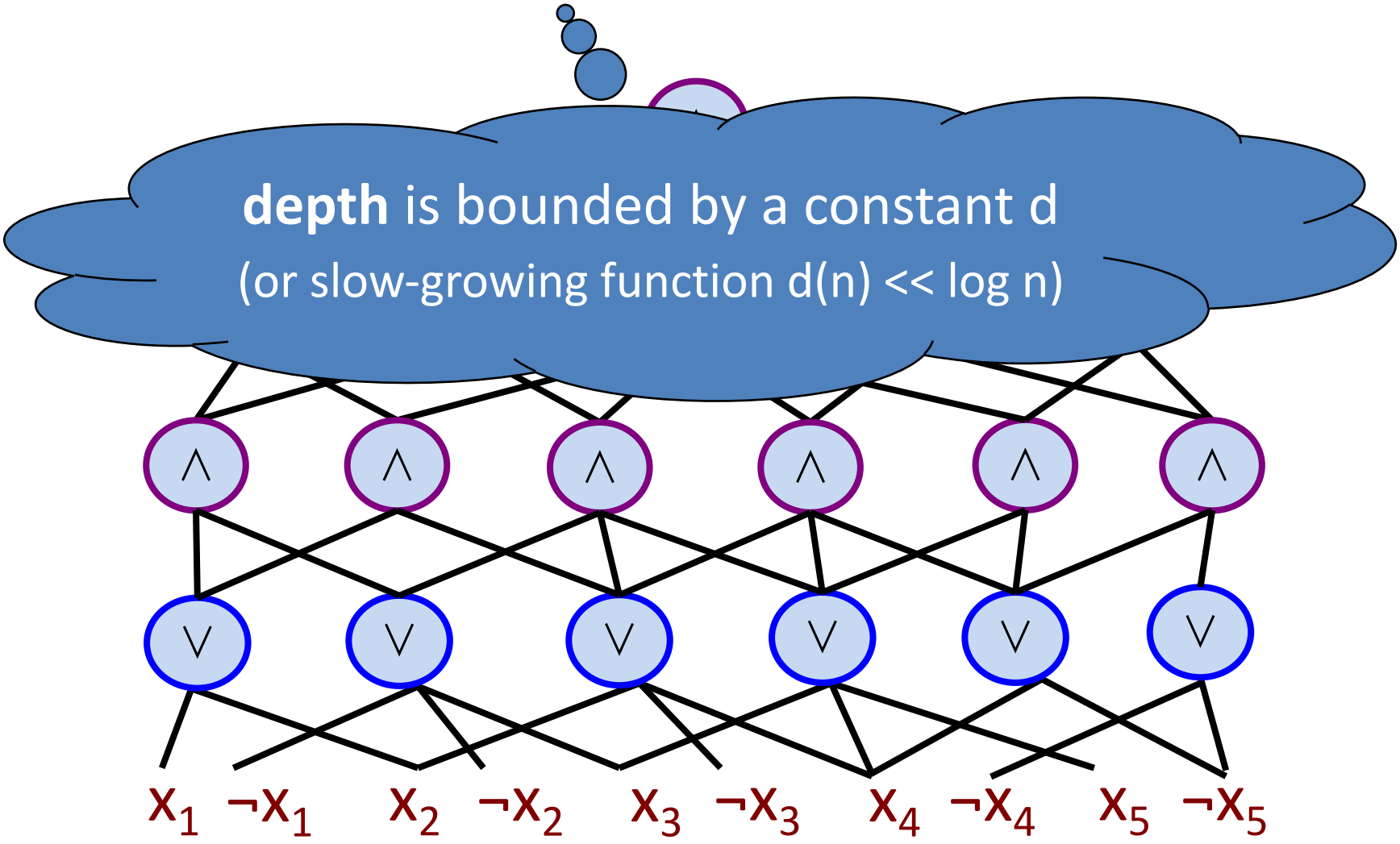
Restricted Classes

- **AC^0 setting** (fast parallel computation)
  constant-depth, unbounded fan-in AND/OR gates
- **monotone setting**
  negation-free (no NOT gates)
- **arithmetic (+, ×), tropical (min, +), …**

AC^0 Circuits



AC^0 Circuits



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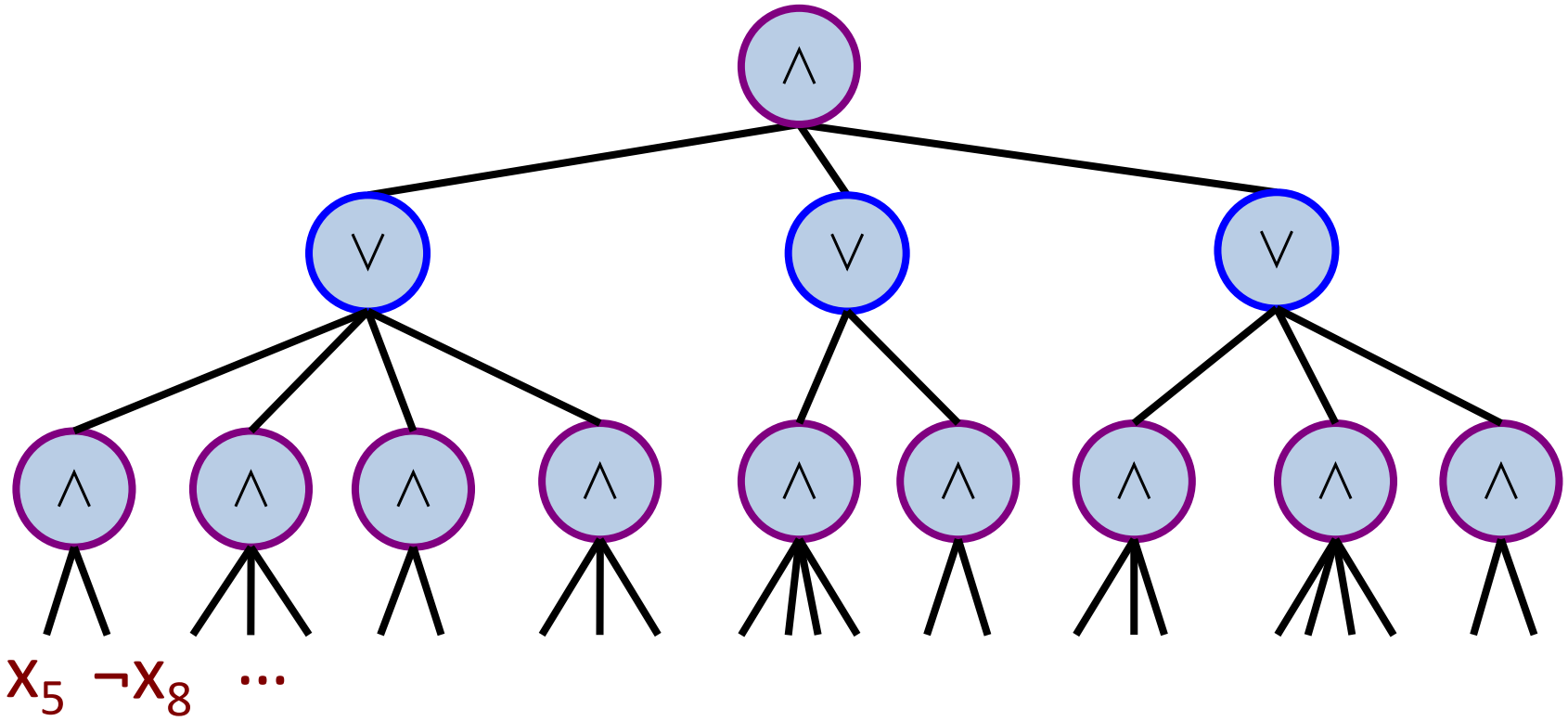

AC^0 Lower Bounds

- **Exponential lower bounds** known since the 1980's: the depth-d AC^0 circuit size $PARITY_n$ is $2^{\Theta(n^{1/(d-1)})}$

[Ajtai, Furst-Saxe-Sipser, Yao, Hastad]

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Switching Lemma
(simplification under p-random restrictions)

Lower Bound Techniques

- **counting**
  - almost all Boolean functions are complex
  - circuit size hierarchy theorem
- **gate-elimination arguments [restriction based]**
  - best lower bounds for *unrestricted* circuits and formulas
- **switching lemmas [restriction based]**
  - best lower bounds against AC^0
- **polynomial method**
  - best lower bounds against $AC^0[\oplus]$

Monotone Lower Bounds

$$mAC^0 \subset mNC^1 \subset mL \subset mNL \subset mNC \subset mP \subset mNP \subset \cdots$$

- We know essentially all separations among interesting monotone classes, via a multitude of techniques

Gate Elimination Arguments & Shrinkage

Restrictions

- Consider a Boolean function

$$f : \{0,1\}^n \rightarrow \{0,1\}$$

- A **restriction** (on the variables of f) is a function

$$R : \{x_1,\ldots,x_n\} \rightarrow \{0,1,\star\}$$

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equivalently, a **partial function** from $\{x_1,\ldots,x_n\}$ to $\{0,1\}$

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$$R \quad \star 1 \star \star 1\,0 \star 1 \star 1\,0\,0 \star \star 0 \star 0 \star \star \star 0 \star 0$$

$$f \restriction R\,(0 \quad 1\,0 \quad\quad 0 \quad 0 \quad\quad\quad 1\,1 \quad 1 \quad 1\,0\,1 \quad 1 \quad)$$

$$f(0\,1\,1\,0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,1\,0\,1\,0\,1\,0\,1\,0)$$

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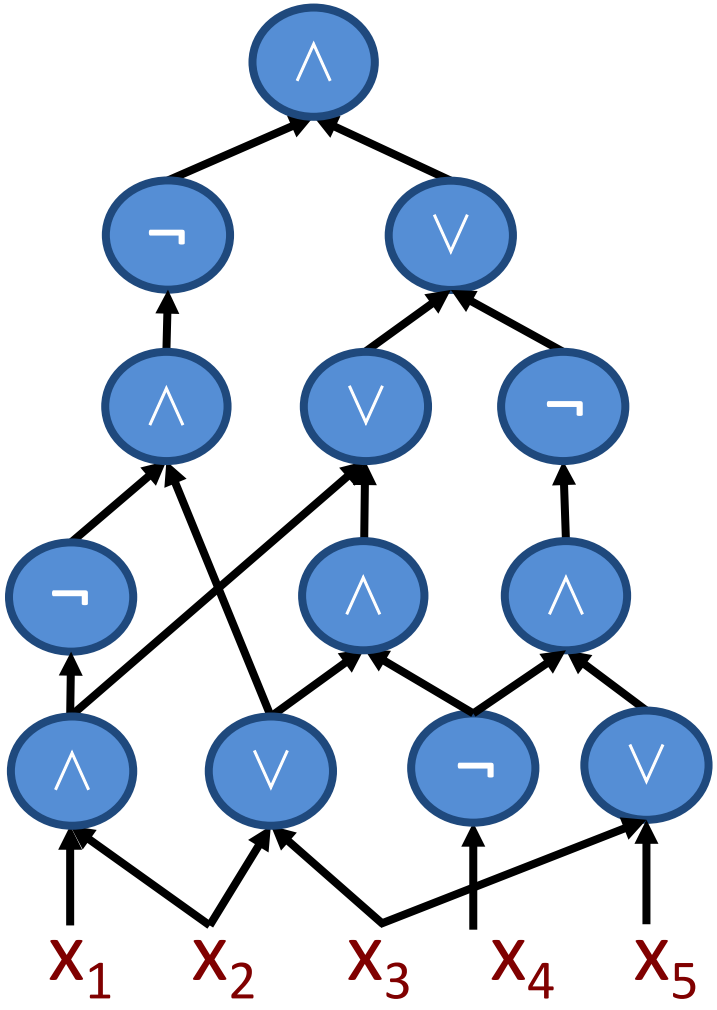
$$f \restriction R : \{0,1\}^{\mathrm{Stars}(R)} \to \{0,1\}$$

- Can also apply R *syntactically* to circuits (and other objects)

Restricting a Circuit

- Consider the 1-bit restriction

  $R = \{ x_2 \mapsto 1 \}$



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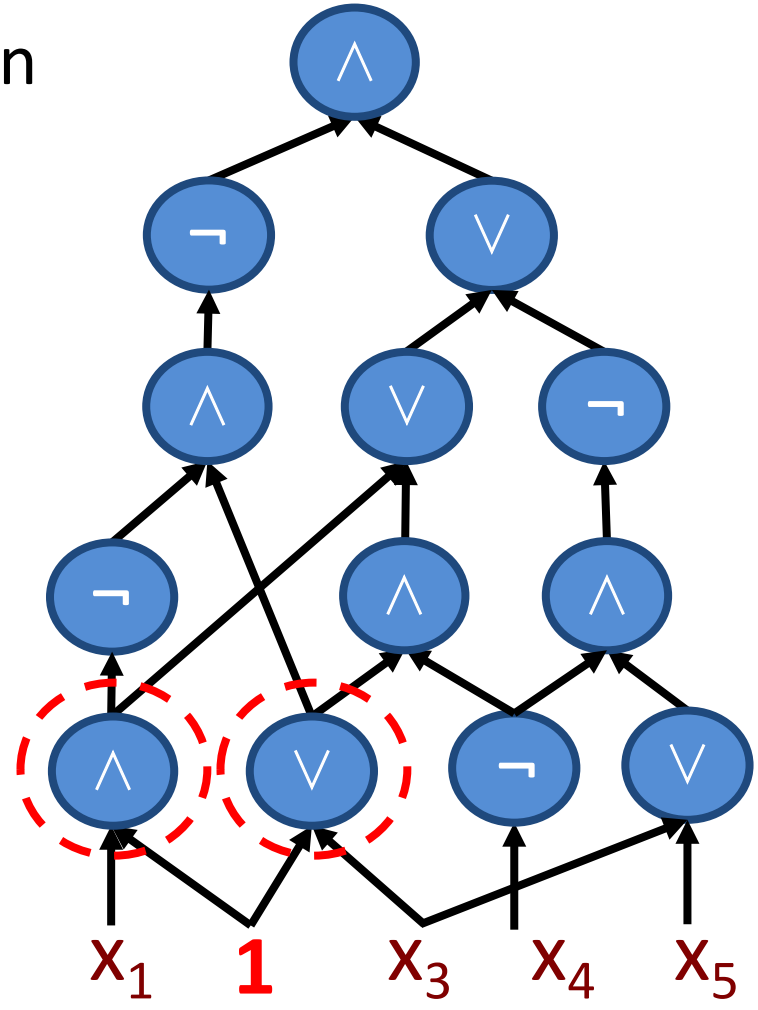

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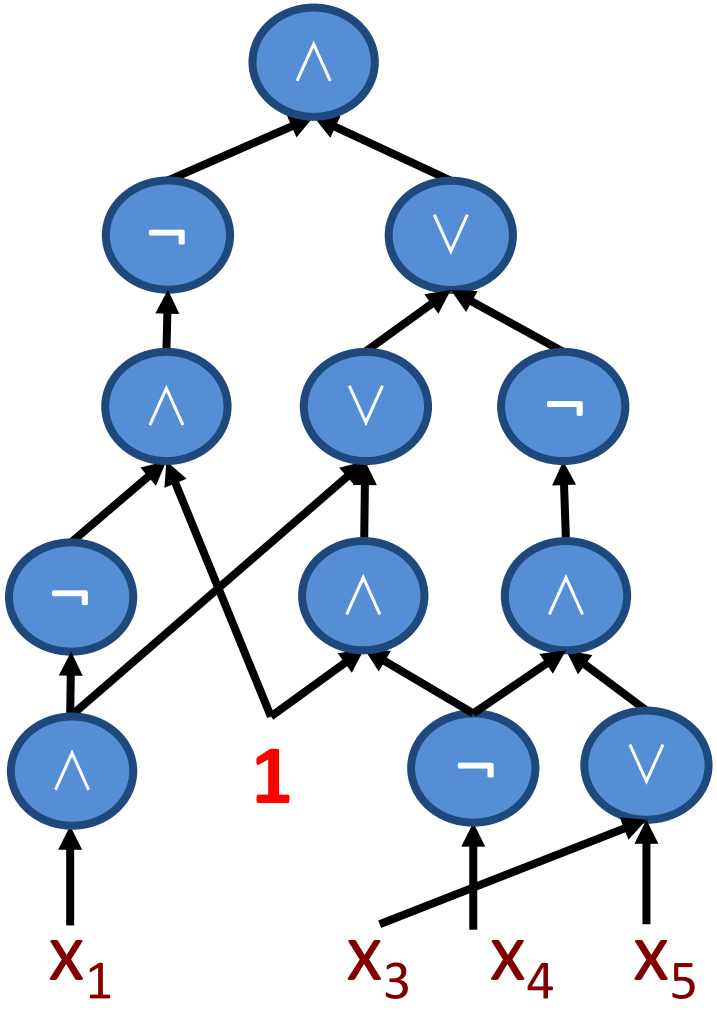

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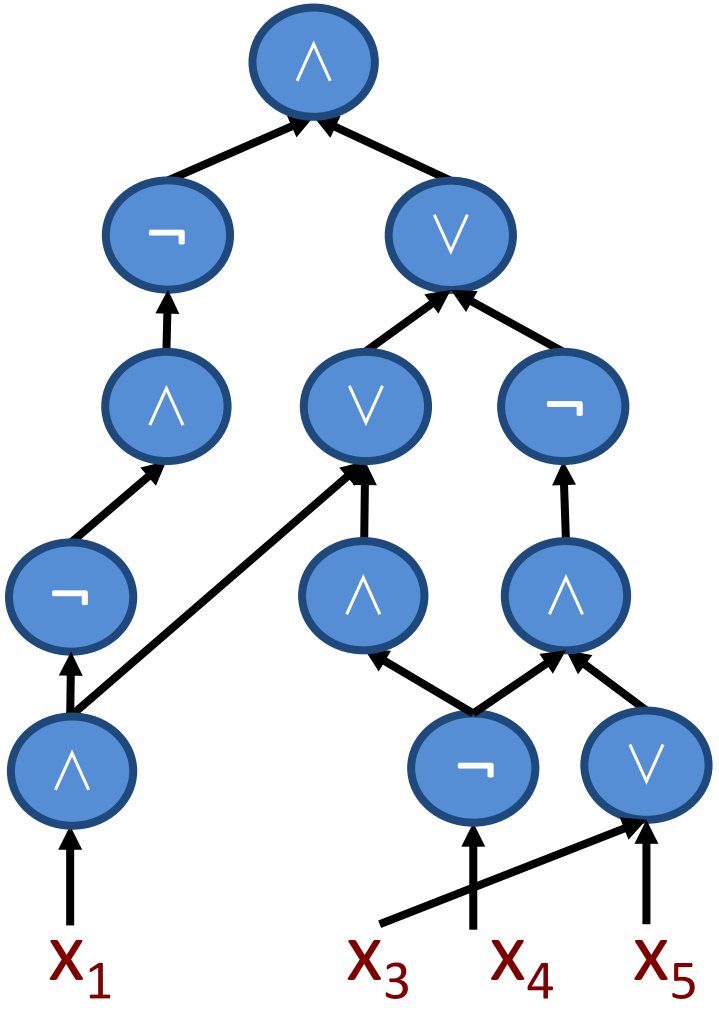

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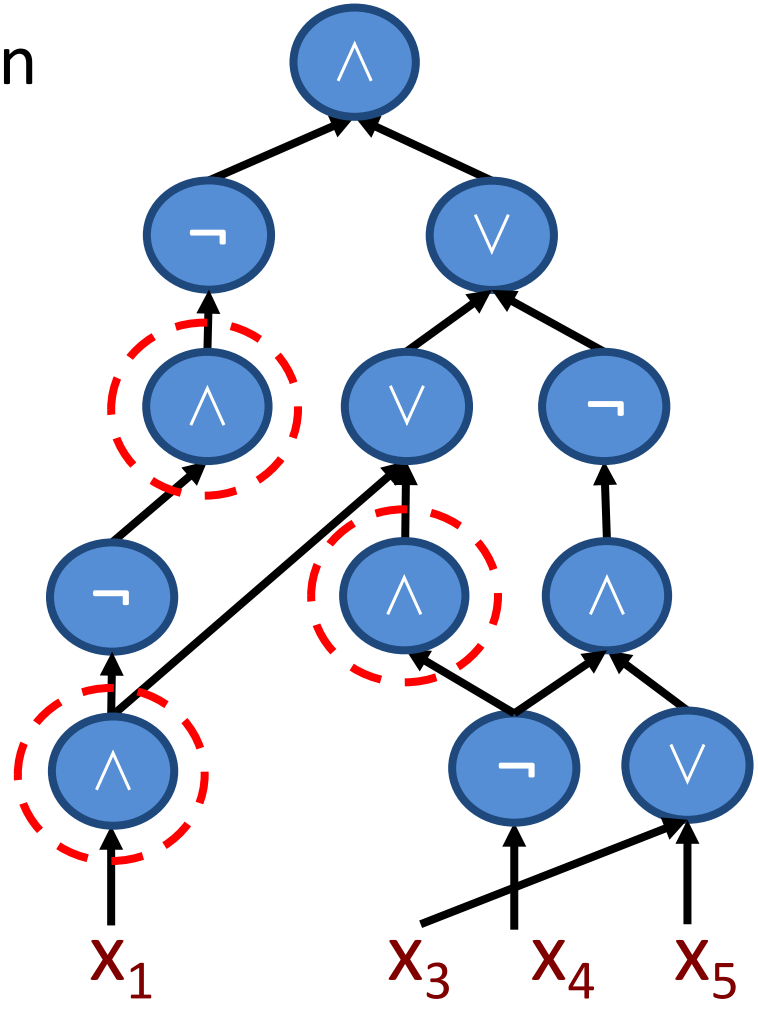

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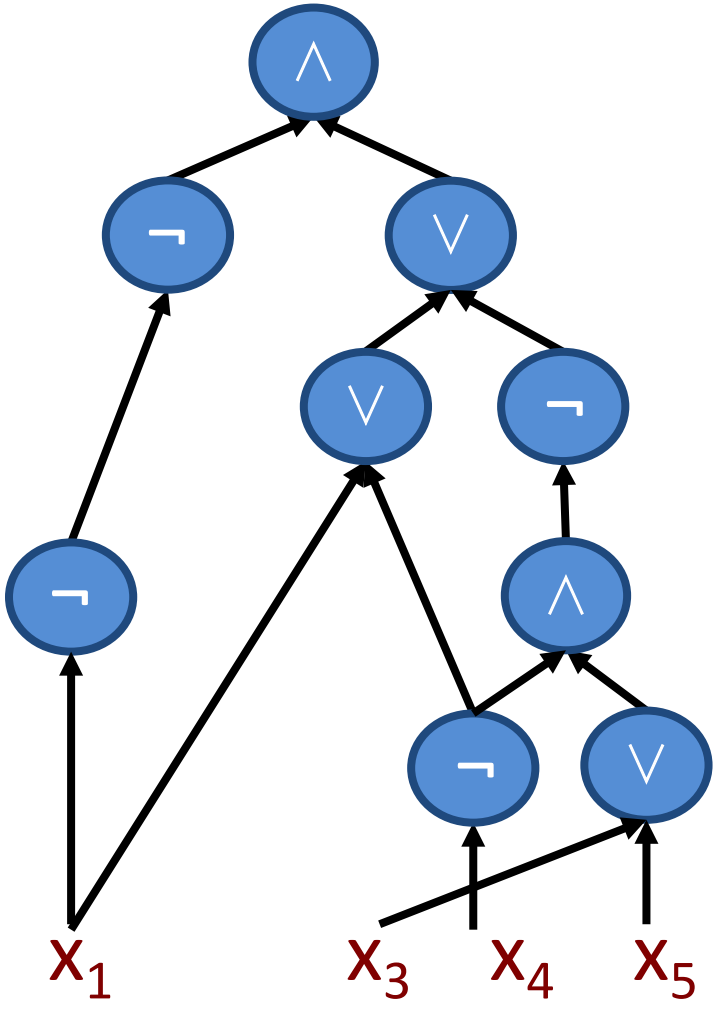

Gate Elimination

- Lemma [Schnorr '76]

  If a circuit C (in basis $\{AND_2, OR_2, NOT\}$) computes $PARITY_n$ ($n \geq 2$), then there exists a 1-bit restriction R killing at least 3 AND/OR gates of C (i.e. $size(C \restriction R) \leq size(C) - 3$)

- Corollary

  $PARITY_n$ has circuit size at least $3n - 3$. Moreover, matching upper bound.

Gate Elimination

- More sophisticated gate elimination arguments give the best lower bounds:

  $5n - o(n)$ $\{AND_2, OR_2, NOT\}$ basis

  [Iwama-Lachish-Morizumi-Raz '02]

  $\approx 3.01n$ full binary basis

  [Find-Golovnev-Hirsch-Kulikov '16]

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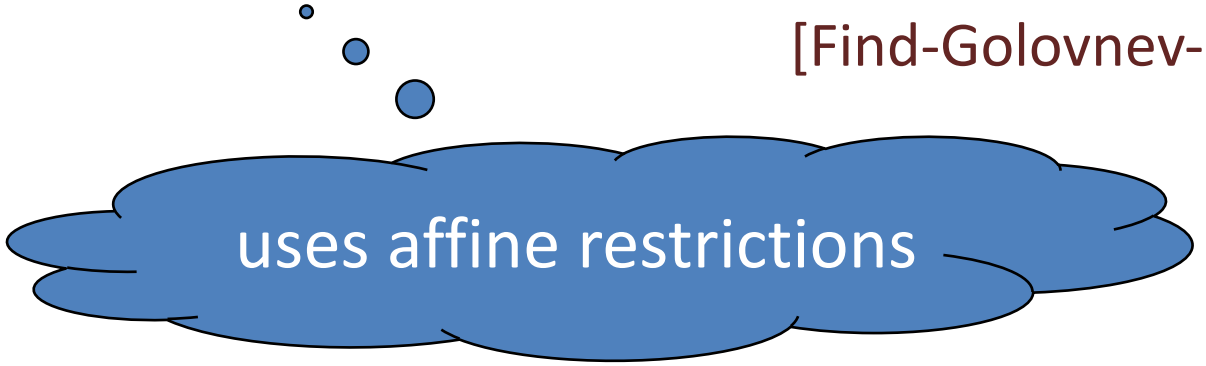

Gate Elimination

- Theorem [Chaudhuri-Radhakrishnan '96]

  $n^{1 + 1/\exp(d)}$ lower bound on the depth-d AC^0 circuit size of APPROX-MAJORITY via ***deterministic restrictions*** (greedily apply the best 1-bit restriction)

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- Theorem [Koppary-Srinivasan '12]

  Similar lower bound for $AC^0[\oplus]$ circuits via ***deterministic low-degree-variety restrictions*** (method of "certifying polynomials")

p-Random Restriction $\mathbf{R}_p$

- For $0 \le p \le 1$, let $\mathbf{R}_p$ denotes the p**-random restriction**

$$\mathbf{R}_p(x_i) = \begin{cases} \star & \text{with prob. } p \\ 0 & \text{with prob. } (1-p)/2 \\ 1 & \text{with prob. } (1-p)/2 \end{cases}$$

  independently for each variable index $i \in [n]$

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independently for each variable index $i \in [n]$

<u>Convention</u>:

Random objects written in **boldface**

Effect of $\mathbf{R}_p$

- $\mathbf{R}_p$ simplifies Boolean functions computed by small:
  - DeMorgan formulas
  - decision trees
  - AC^0 circuits
- Certain Boolean functions, like $PARITY_n$, maintain their complexity under $\mathbf{R}_p$
- Ergo, *lower bounds!*

Effect of $\mathbf{R}_p$ on DeMorgan Formulas

- Subbotovskaya '61

  If F is an n-variable DeMorgan formula, then

  $$\mathrm{Ex}[\ \mathrm{leafsize}(F \restriction \textbf{random 1-bit rest.})\ ] \leq (1-n)^{1.5}\,\mathrm{leafsize}(F)$$

- As a consequence,

  $$\mathrm{Ex}[\ \mathrm{leafsize}(F \restriction \mathbf{R}_p)\ ] \leq O(p^{1.5}\,\mathrm{leafsize}(F) + 1)$$

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Effect of $\mathbf{R}_p$ on DeMorgan Formulas

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  If F is an n-variable DeMorgan formula, then

  Ex[leafsize(F ↾ **random 1-bit rest.**)]

  Known as the ***shrinkage exponent*** of DeMorgan formulas

  leafsize(F) + 1)

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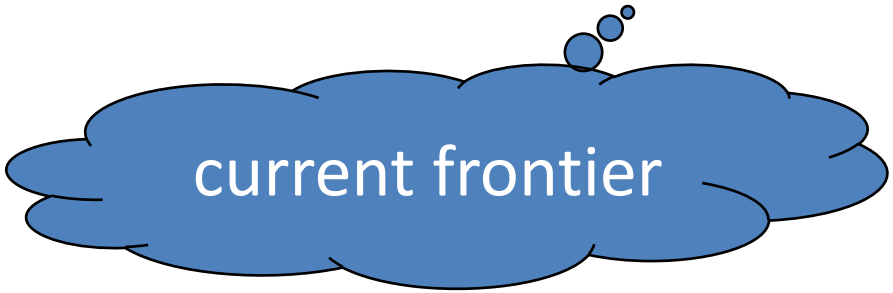
Effect of $\mathbf{R}_p$ on DeMorgan Formulas

- Implies lower bounds:

$$\text{leafsize}(\text{PARITY}_n) = \Omega(n^2)$$

$$\text{leafsize}(\text{ANDREEV}_n) = \tilde{\Omega}(n^3)$$



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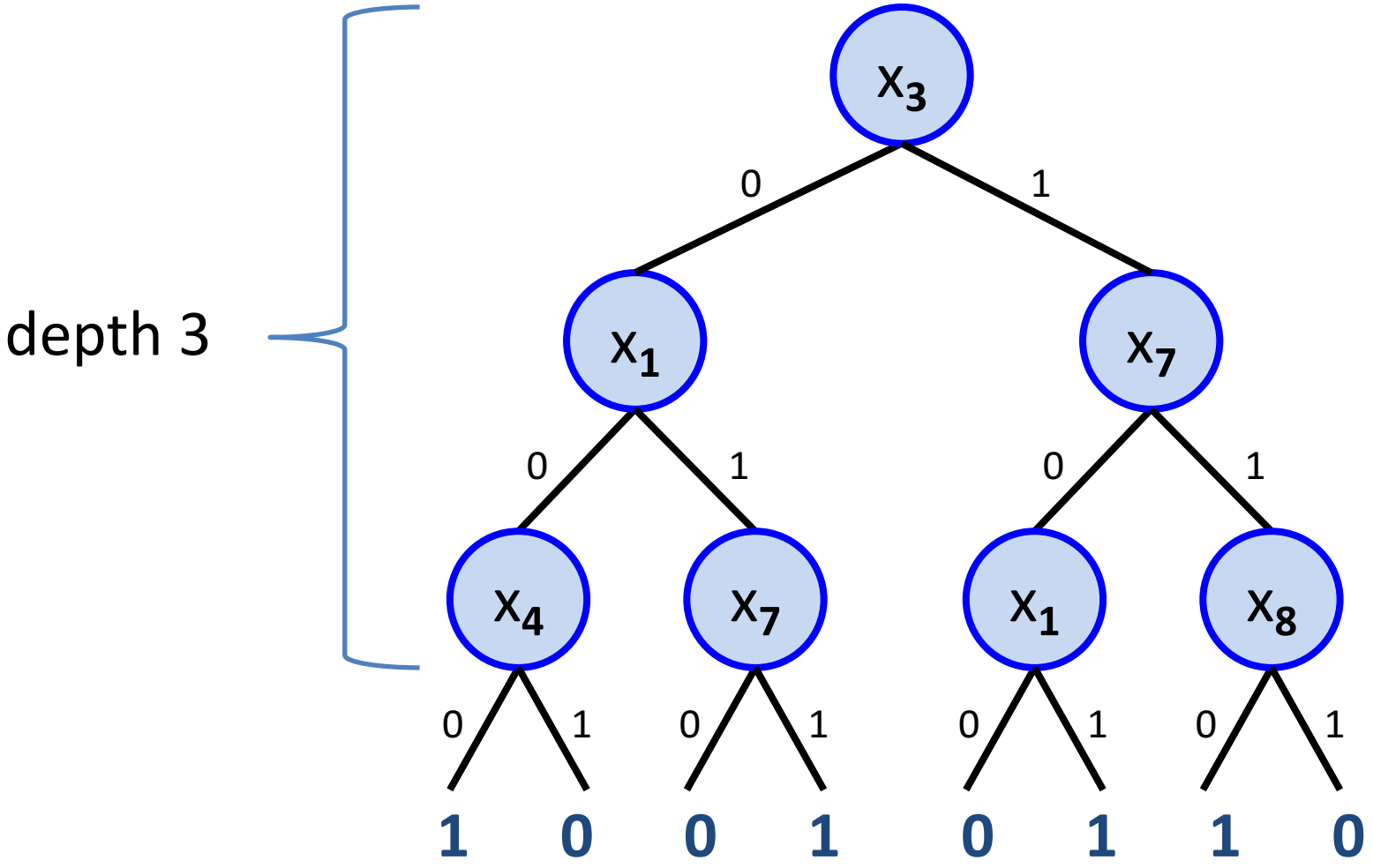
Effect of $\mathbf{R}_p$ on *Monotone* Formulas

- Open Question What is the shrinkage exponent of **monotone formulas** (basis $\{AND_2, OR_2\}$)?
- Conjecture Equals the shrinkage exponent of **read-once formulas** (≈ 3.27) [Hastad-Razborov-Yao '97]

The Switching Lemma

Decision Trees



Decision Trees

The ***decision-tree depth*** of a Boolean function

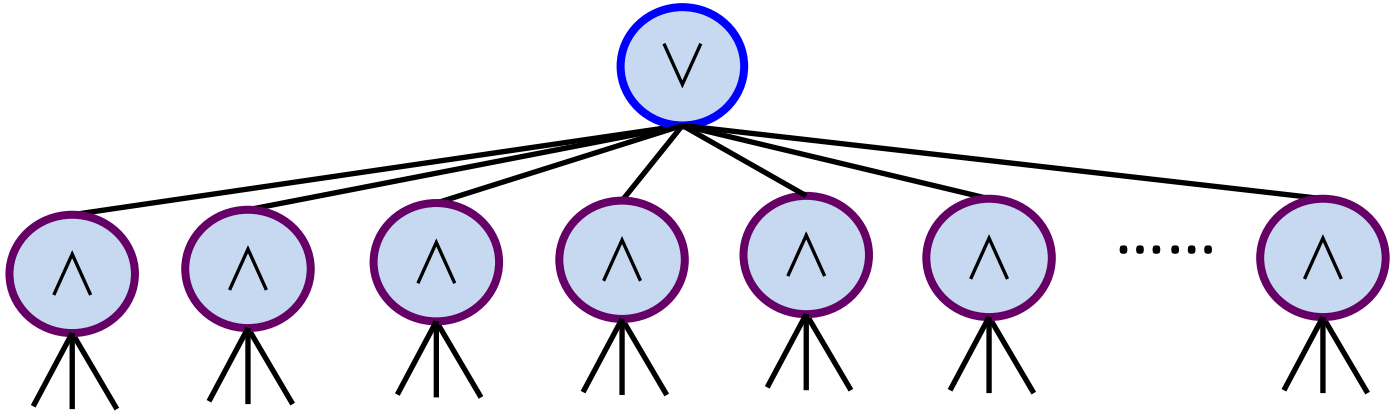
$$f : \{0,1\}^n \rightarrow \{0,1\}$$

is the minimum depth of a decision tree that computes f.

- $DT_{depth}(PARITY_n) = DT_{depth}(AND_n) = n$
- $DT_{depth}(f) = 0 \Leftrightarrow$ f is constant

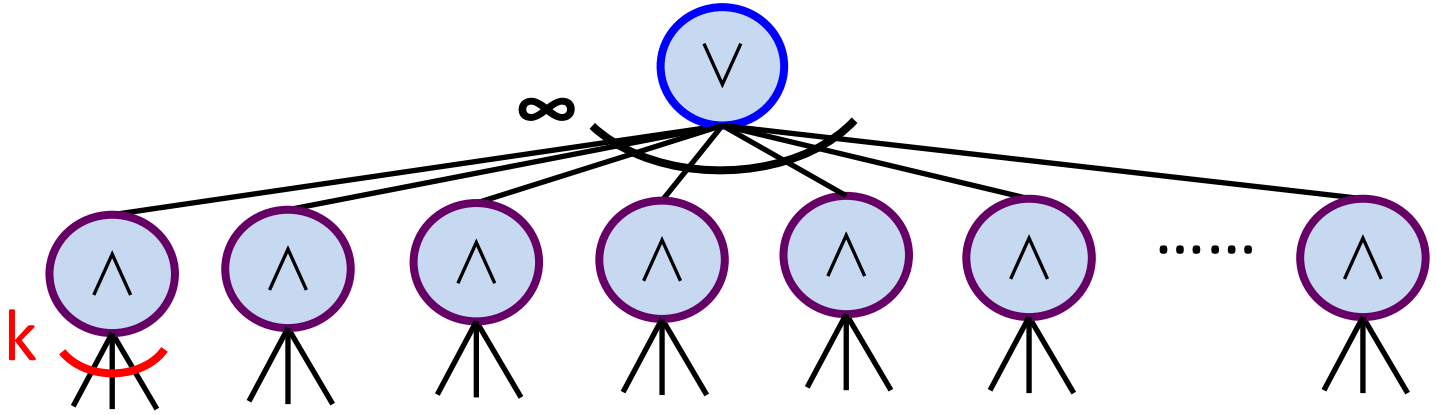
Depth-2 Formulas (DNFs and CNFs)

- **DNF** = disjunctive normal form (OR-AND formula)
- **CNF** = conjunctive normal form (AND-OR formula)



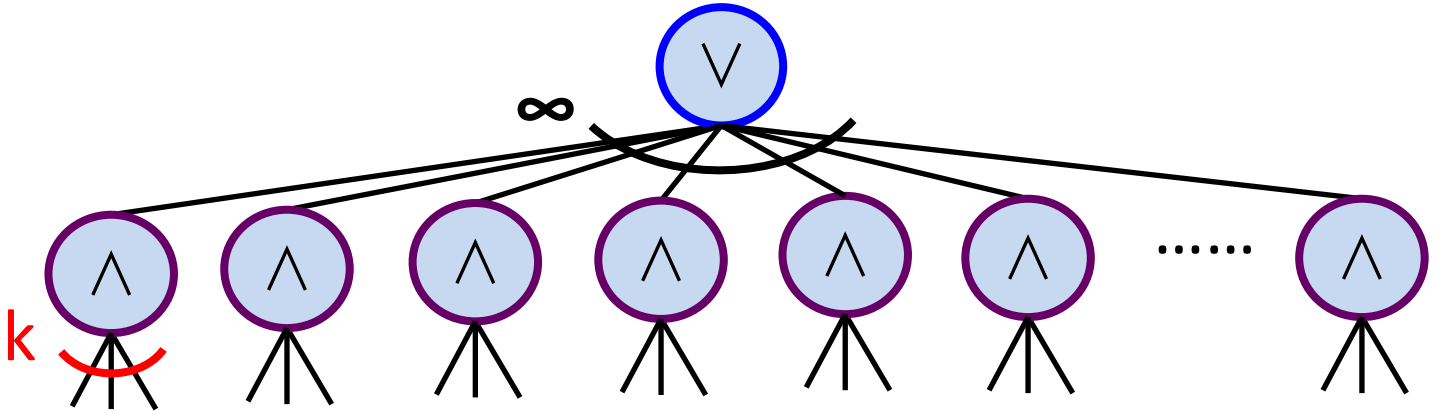
Depth-2 Formulas (DNFs and CNFs)

- **DNF** = disjunctive normal form (OR-AND formula)
- **CNF** = conjunctive normal form (AND-OR formula)
- **width** = bottom fan-in (max # of variables in a clause)



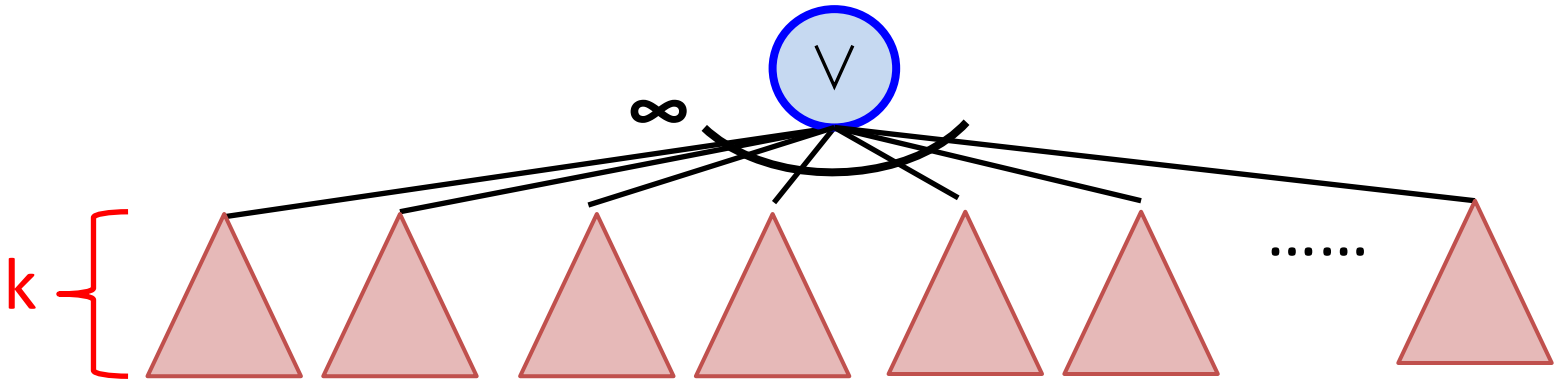
Depth-2 Formulas (DNFs and CNFs)

- **k-DNF** = width-k DNF
- **k-CNF** = width-k CNF



Depth-2 Formulas (DNFs and CNFs)

- **k-DNF** = width-k DNF = OR_∞ of depth-k DTs
- **k-CNF** = width-k CNF = AND_∞ of depth-k DTs



Depth-2 Formulas (DNFs and CNFs)

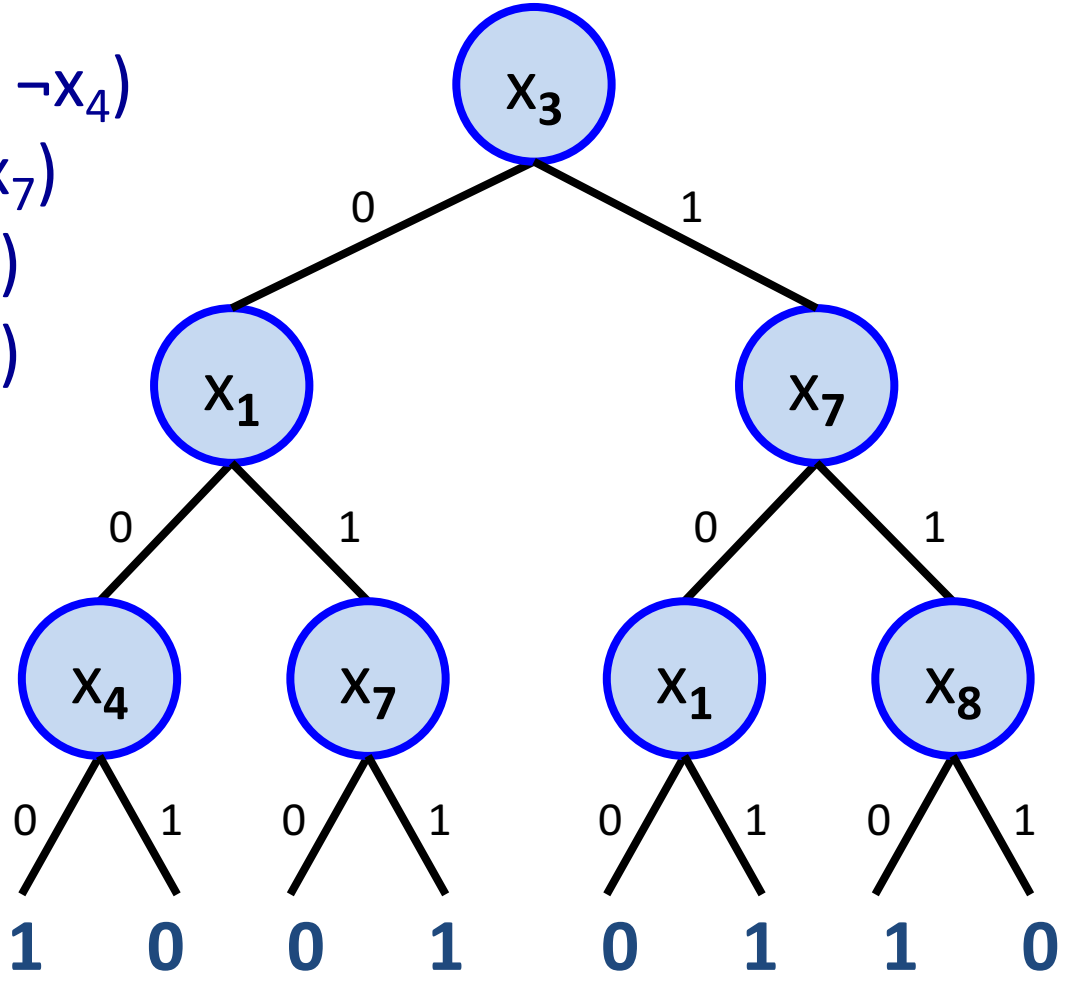
- **k-DNF** = width-k DNF = OR_∞ of depth-k DTs
- **k-CNF** = width-k CNF = AND_∞ of depth-k DTs

- Every depth-k DT is equivalent to a k-DNF and a k-CNF

- Weak converse: If a Boolean function is equivalent to a k-DNF and an ℓ-CNF, then it is equivalent to a DT of depth $k\ell$

Decision Tree to DNF

$$(\neg x_3 \wedge \neg x_1 \wedge \neg x_4) \vee (\neg x_3 \wedge x_1 \wedge x_7) \vee (x_3 \wedge x_7 \wedge x_1) \vee (x_3 \wedge x_7 \wedge x_8)$$



Decision Tree to DNF

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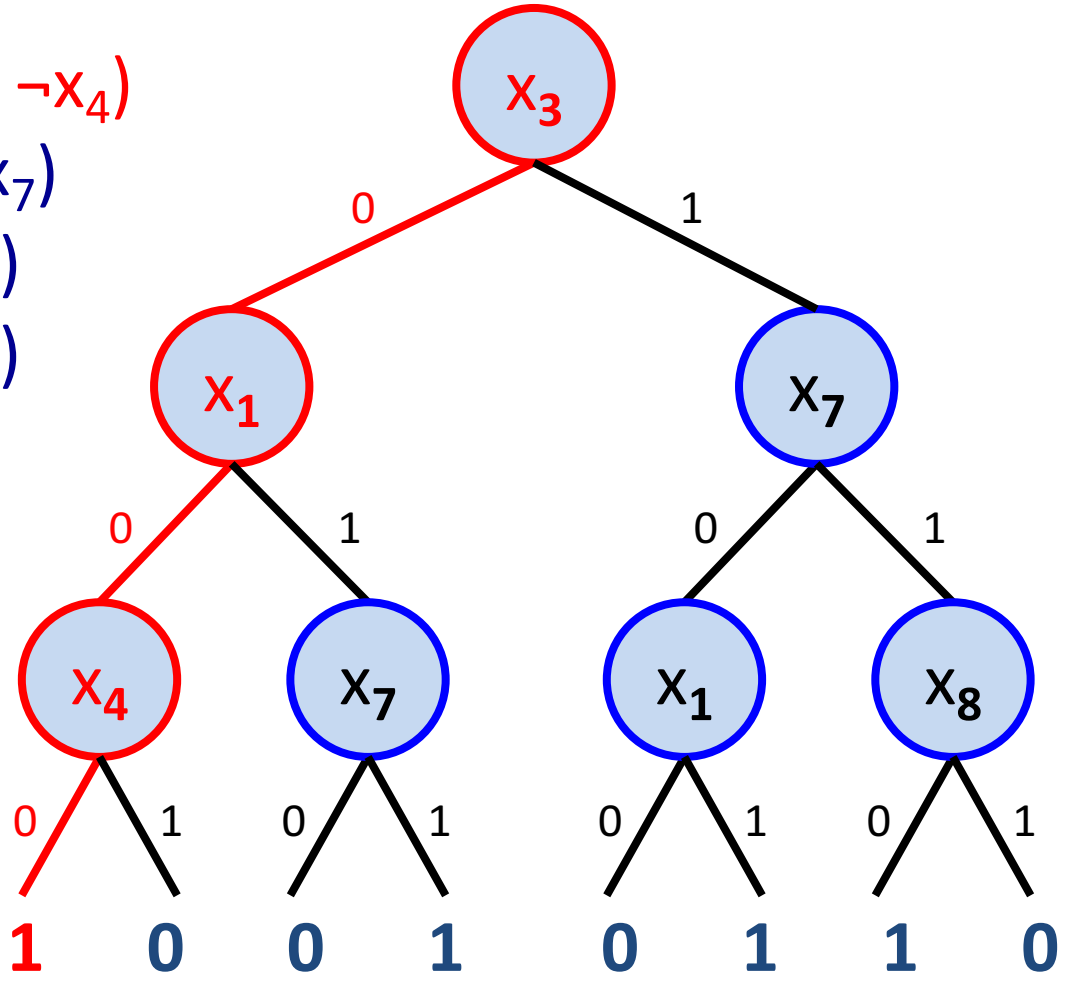

Decision Tree to DNF

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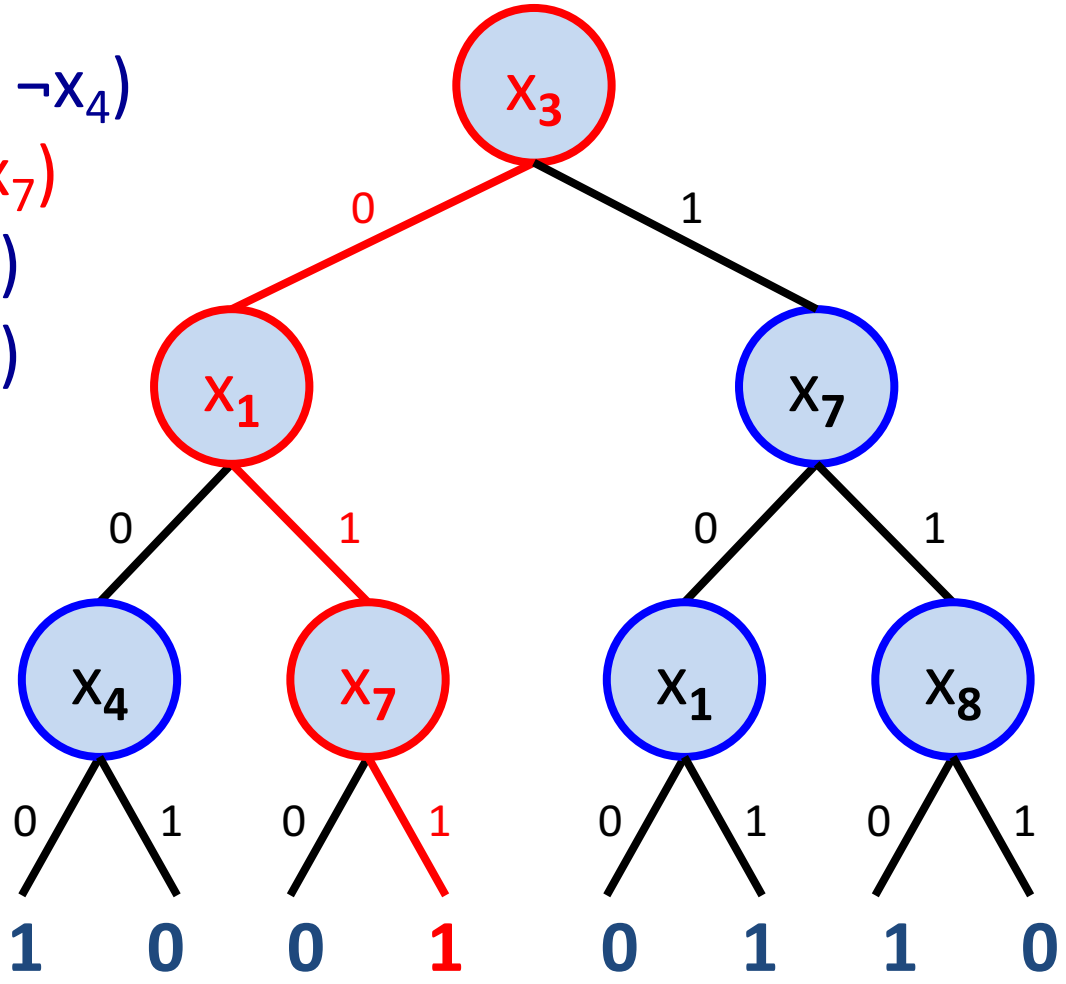

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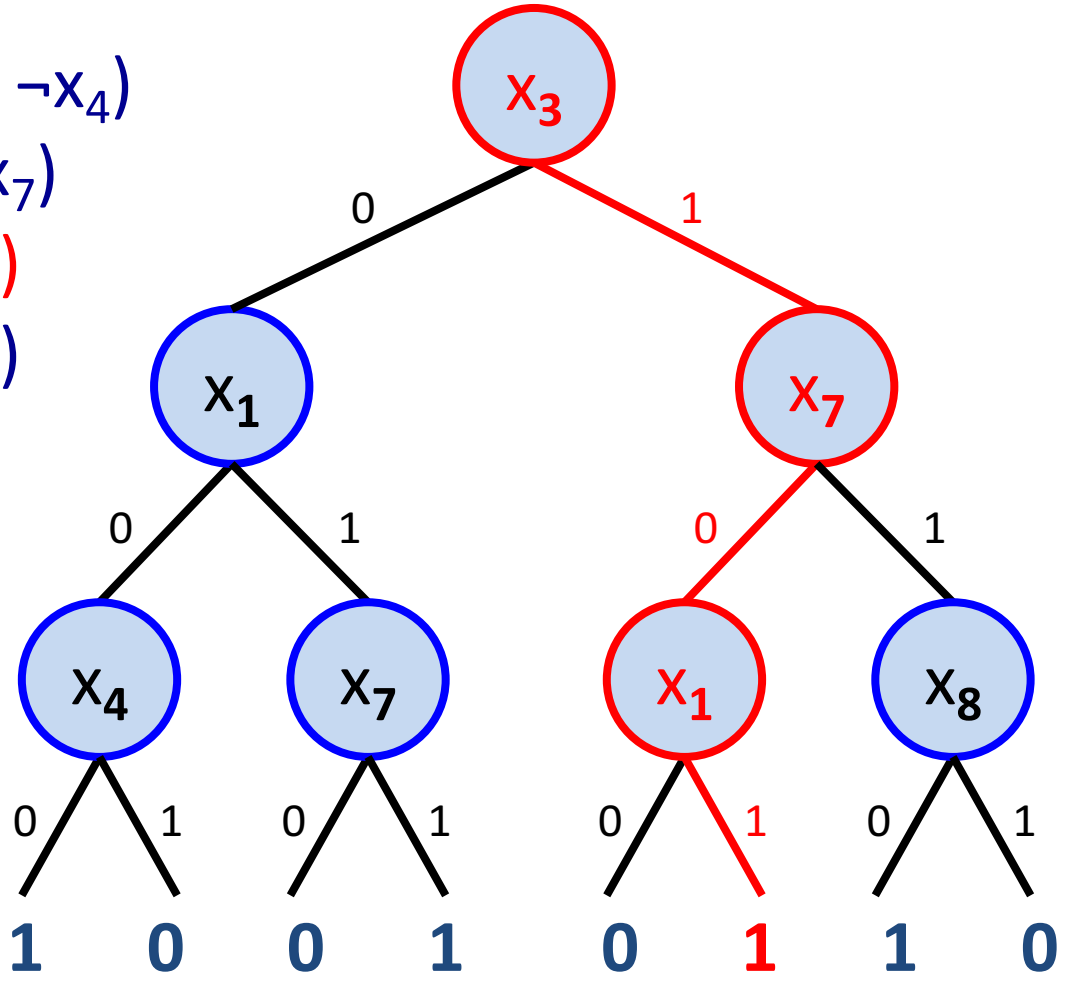

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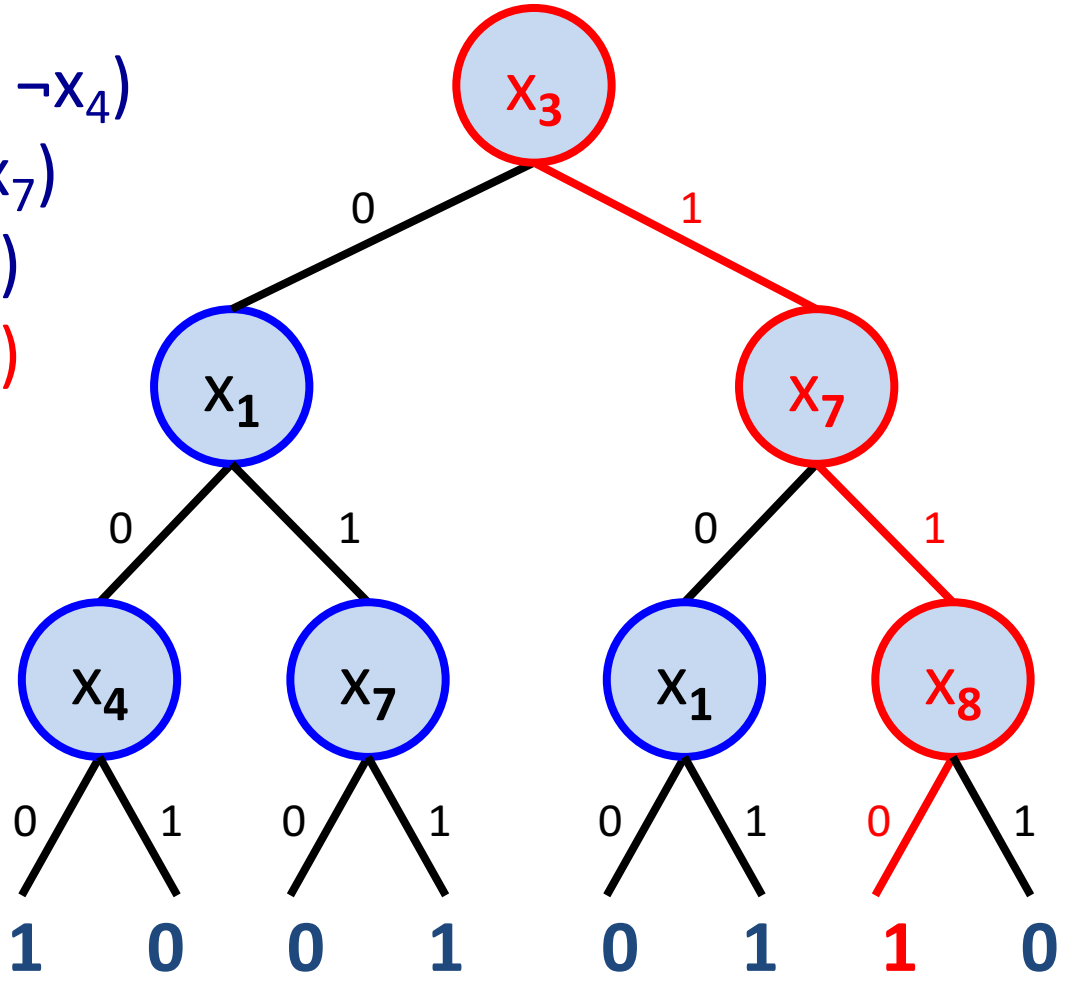

k-DNF Switching Lemma

Hastad's Switching Lemma (1986)

If F is a k-DNF (i.e. OR_∞ of depth-k decision trees), then

$$\Pr[\, DT_{depth}(F \restriction \mathbf{R}_p) \geq t \,] \leq (5pk)^t$$

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$\leq 2^{-t}$ when $p = 1 / 10k$

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Dual CNF version

If F is a k-CNF (i.e. AND_∞ of depth-k decision trees), then

$$\Pr[\, DT_{depth}(F \restriction \mathbf{R}_p) \geq t \,] \leq (5pk)^t$$

k-DNF Switching Lemma

Hastad's Switching Lemma (1986)

If F is a k-DNF (i.e. OR_∞ of depth-k decision trees), then

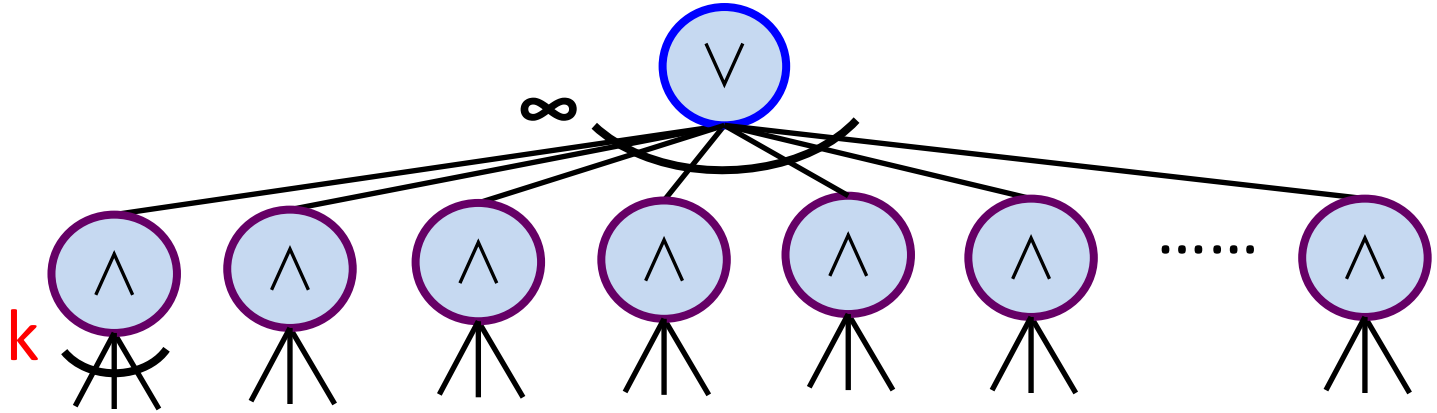
$$\Pr[\, DT_{depth}(F \restriction \mathbf{R}_p) \geq t \,] \leq (5pk)^t$$

Corollary (usual statement of the S.L.)

If F is a k-DNF, then

$$\Pr[\, F \restriction \mathbf{R}_p \text{ is not equivalent to a } t\text{-CNF} \,] \leq (5pk)^t$$

k-DNF Switching Lemma



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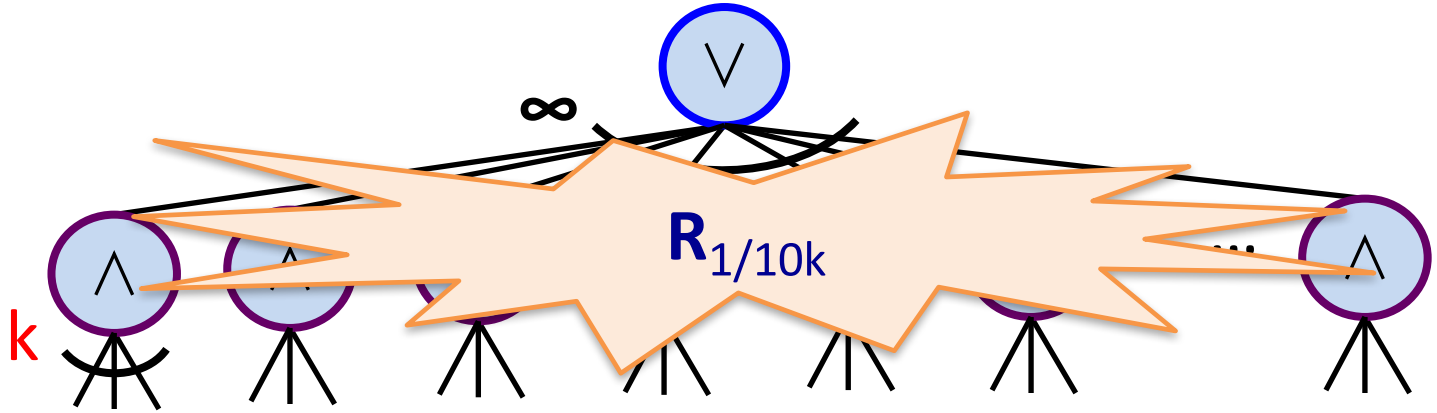
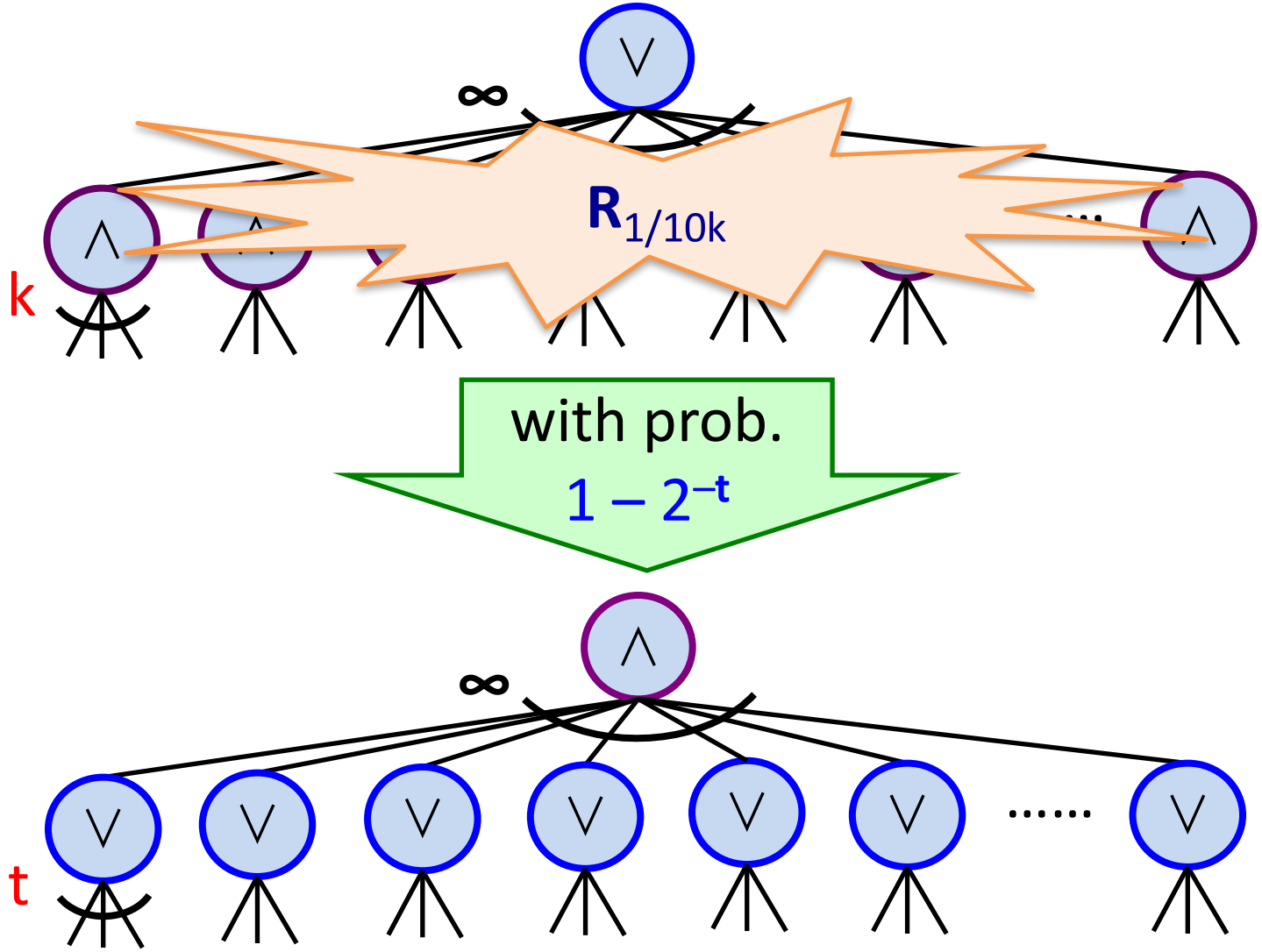
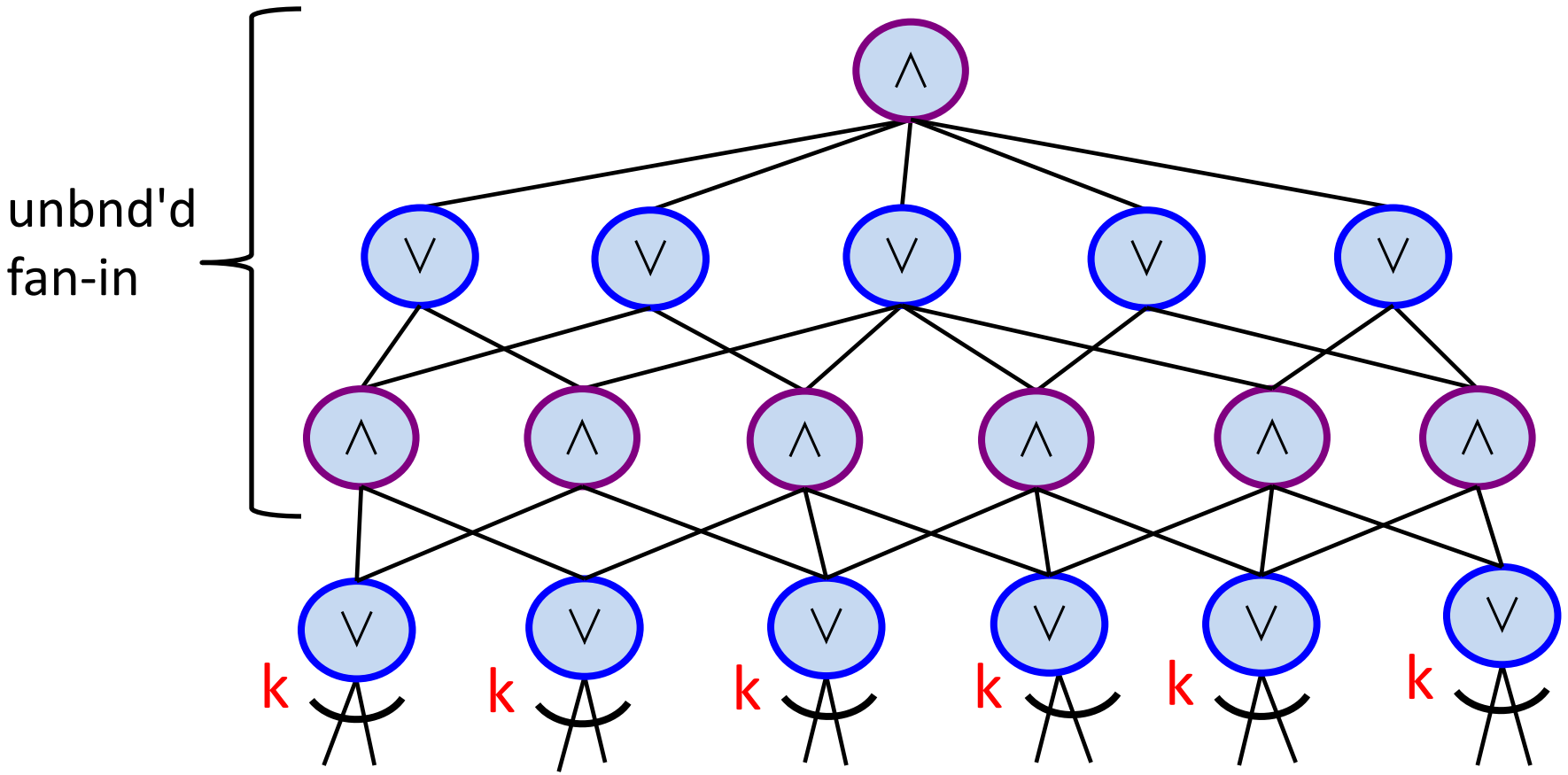


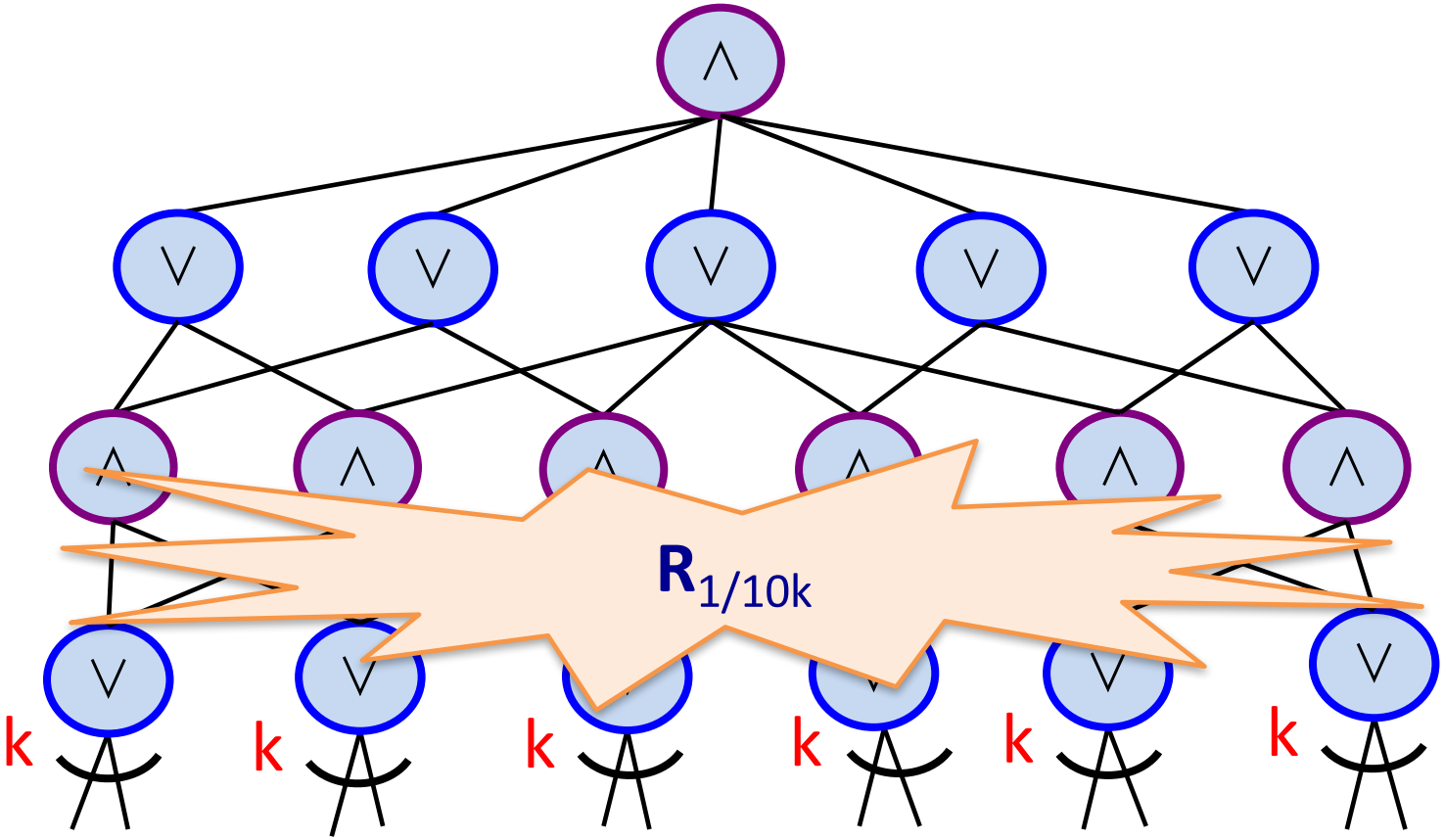
k-DNF Switching Lemma



Depth Reduction

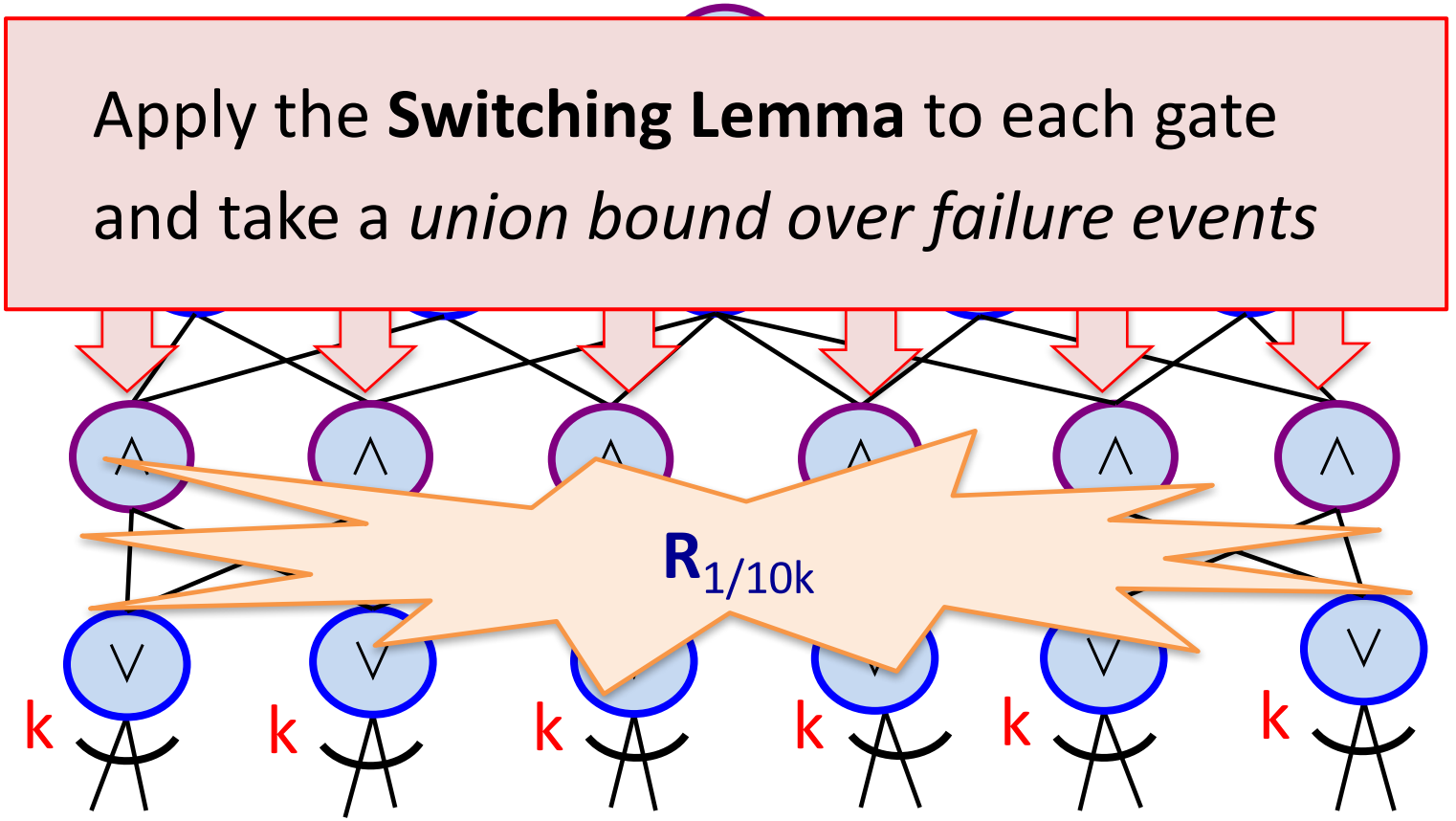


Depth Reduction

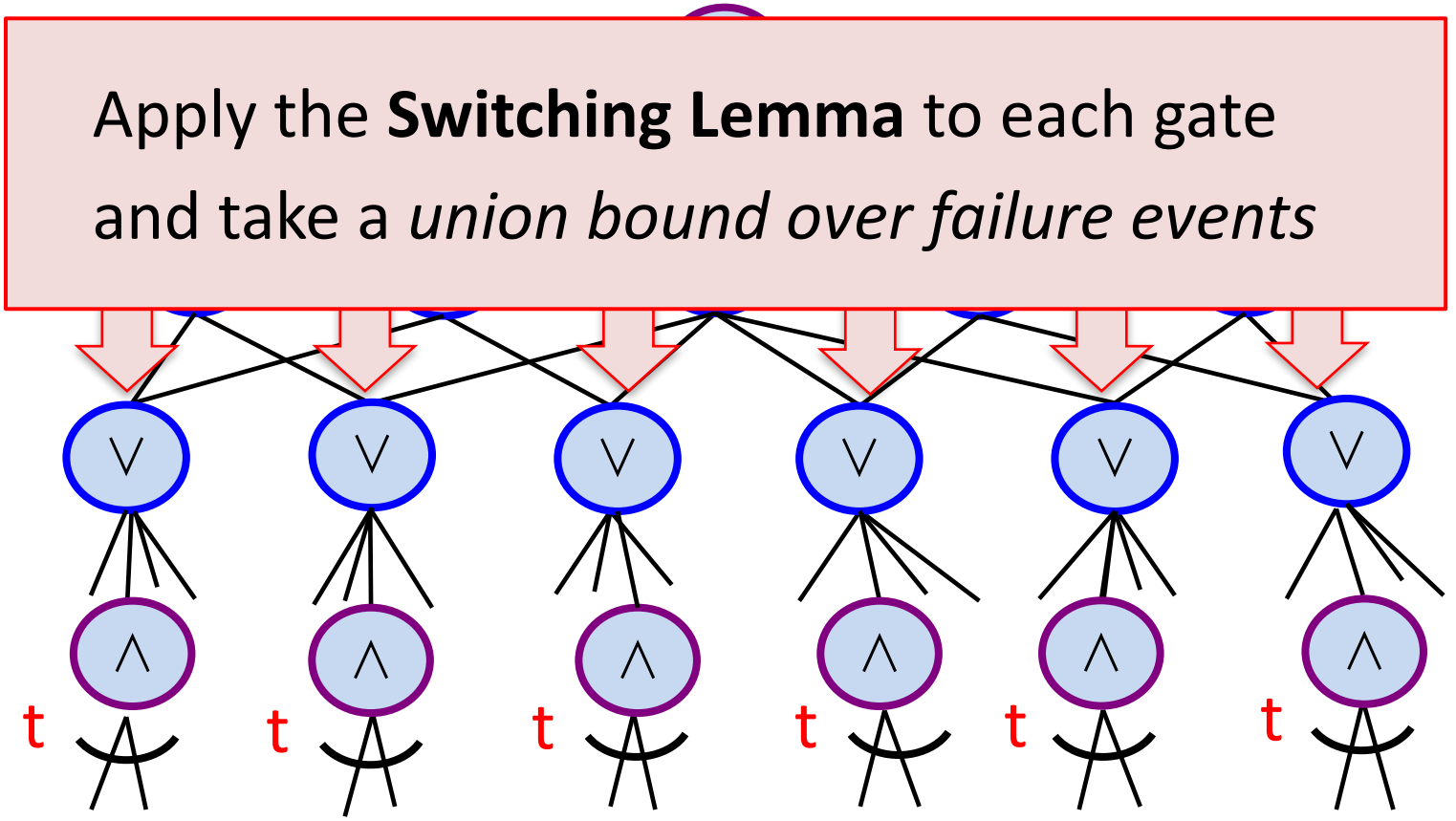


Depth Reduction

Apply the **Switching Lemma** to each gate and take a *union bound over failure events*



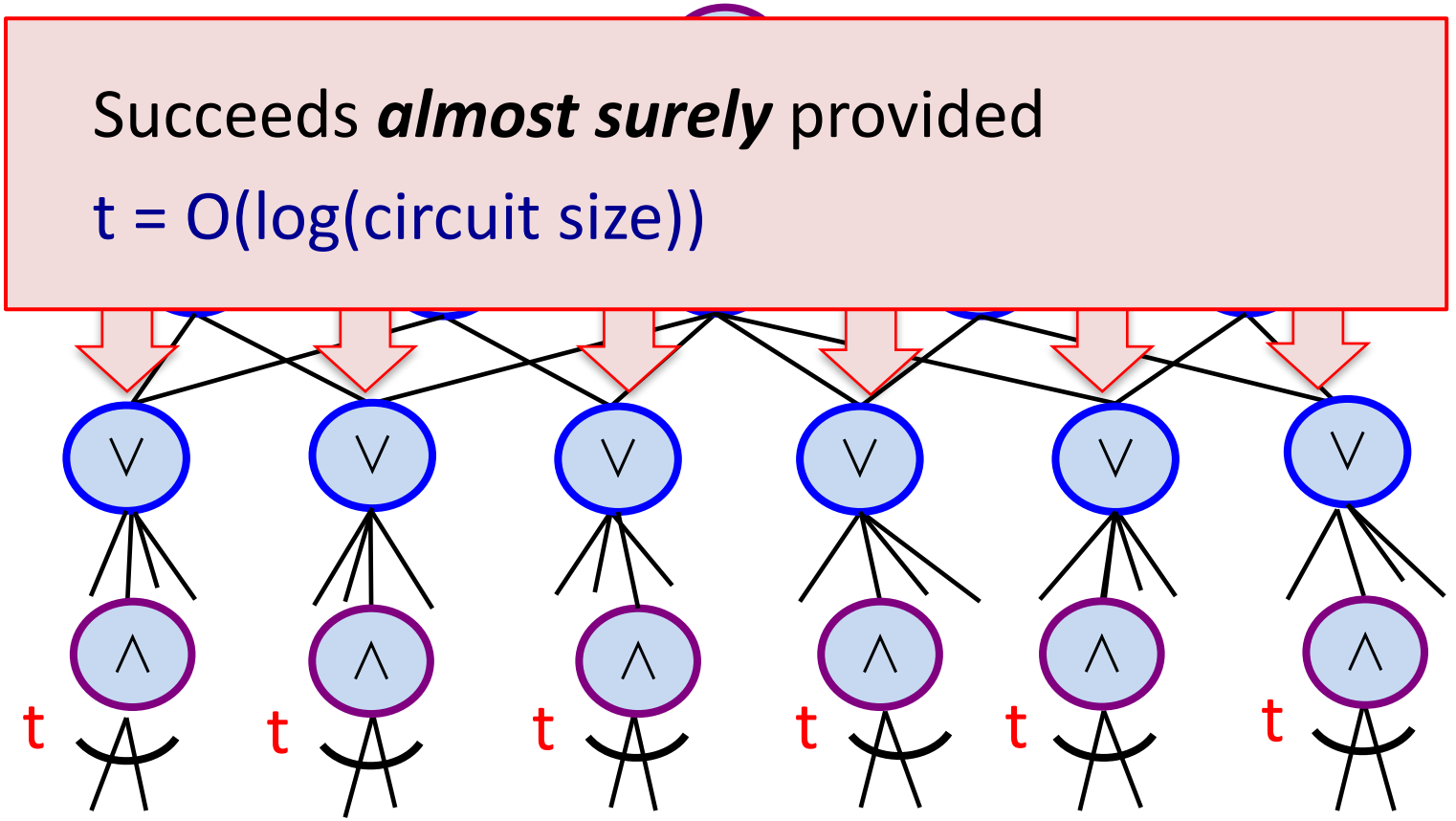
Depth Reduction



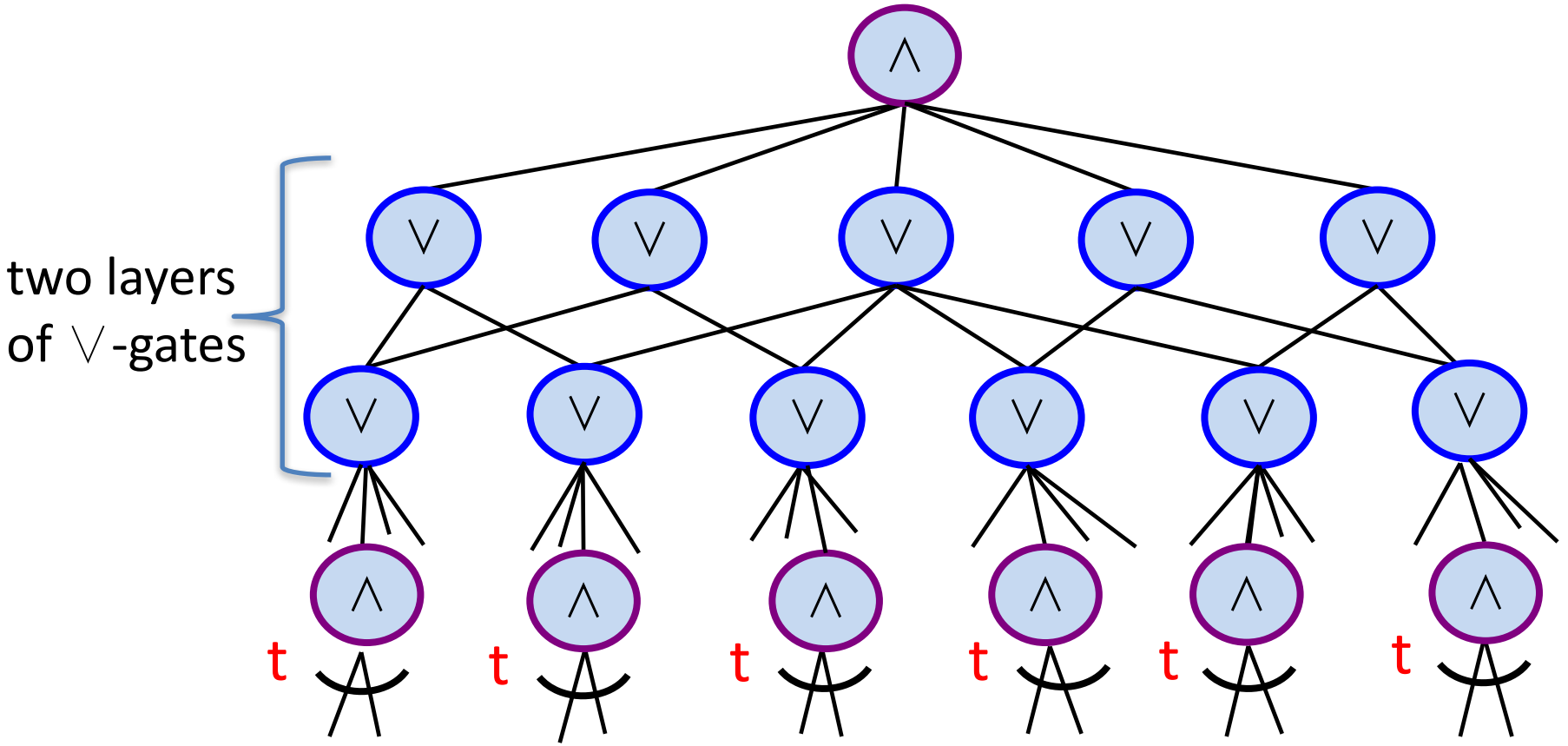
Depth Reduction

Succeeds ***almost surely*** provided

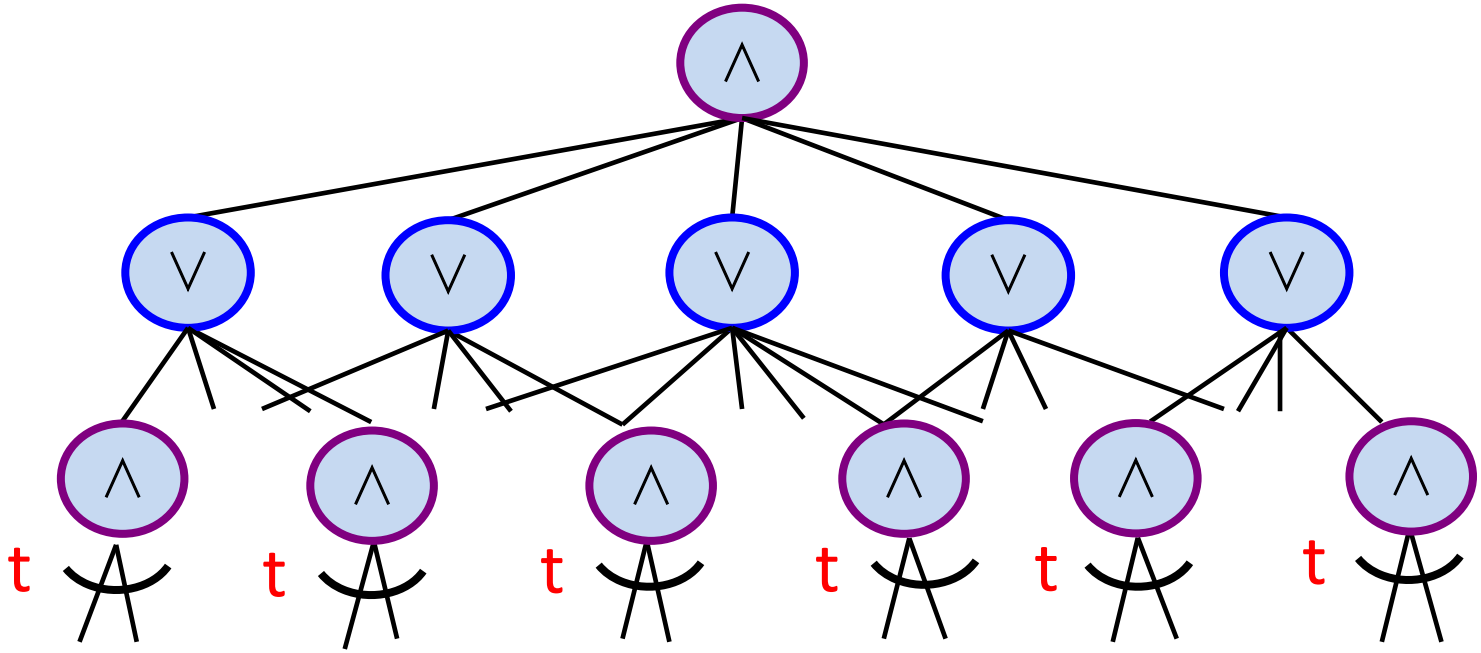
t = O(log(circuit size))



Depth Reduction



Depth Reduction



PARITY Lower Bound

Theorem [Hastad '86]

Depth d+1 **circuits** for $PARITY_n$ have size $\exp(\Omega(n^{1/d}))$

Matching Upper Bound

$PARITY_n$ has depth d+1 **circuits** of size $\exp(O(n^{1/d}))$

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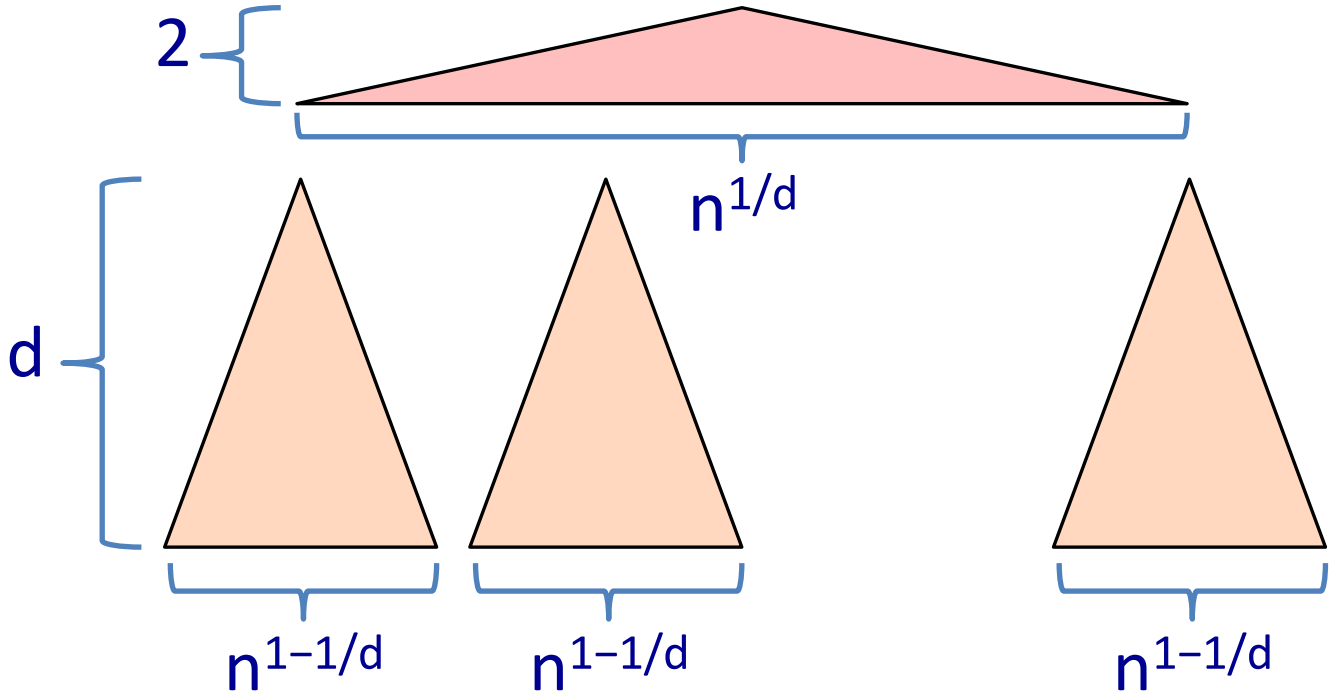
Matching Upper Bound

$PARITY_n$ has depth $d+1$ **circuits** of size $\exp(O(n^{1/d}))$

- depth 2 circuits of size $O(2^n)$ (brute-force CNF/DNF)
- for $d+1 \geq 3$, divide and conquer:



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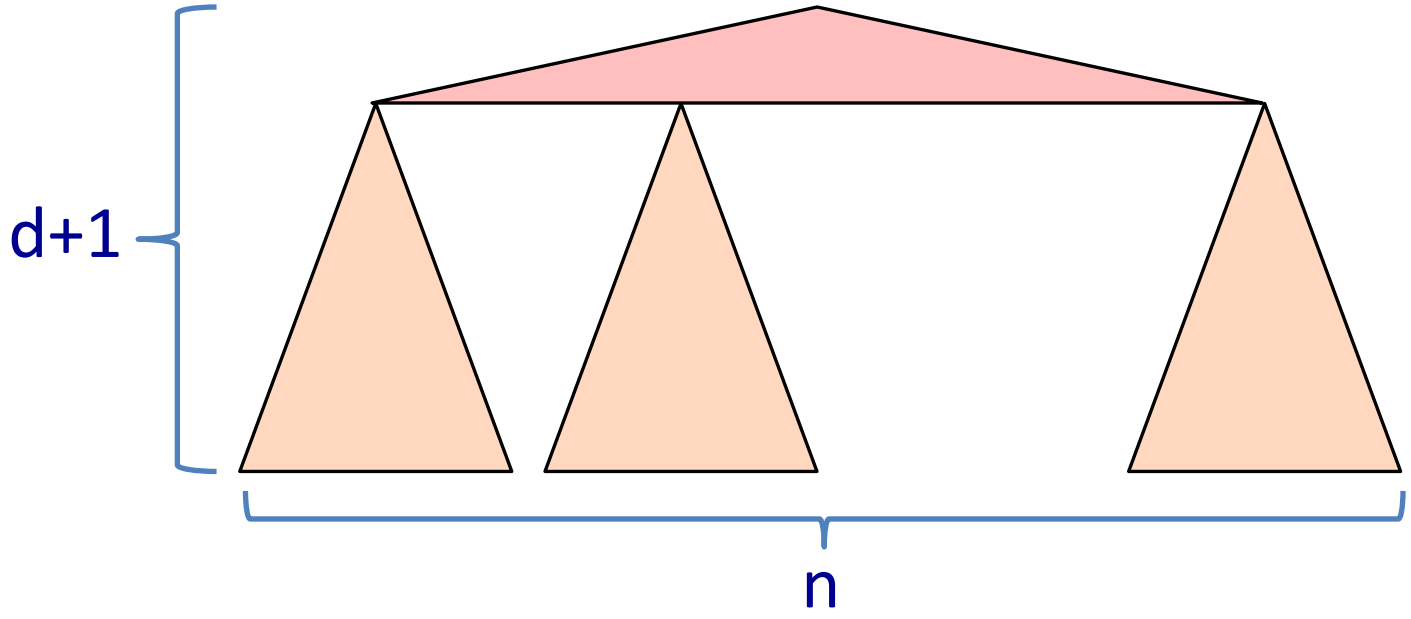
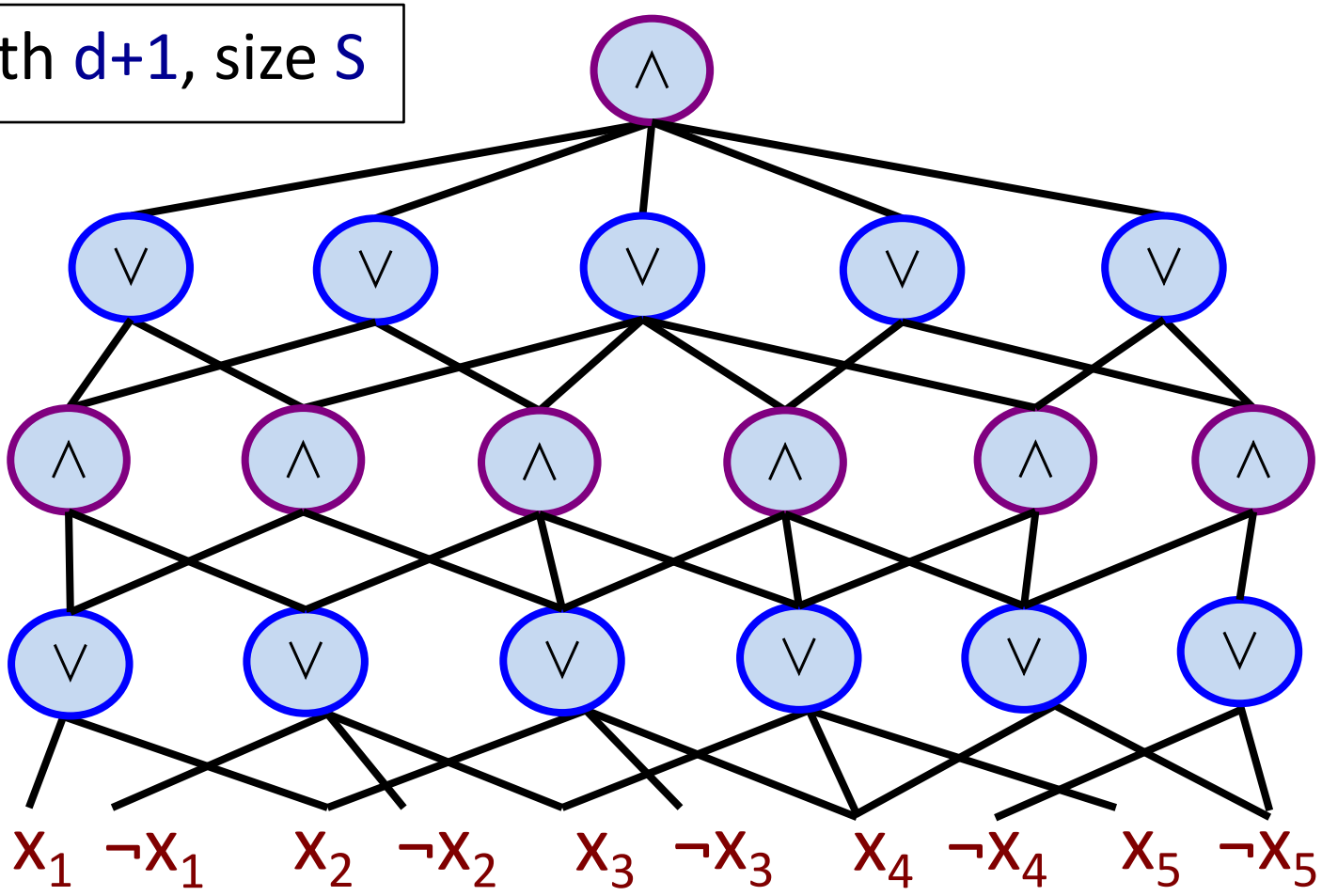
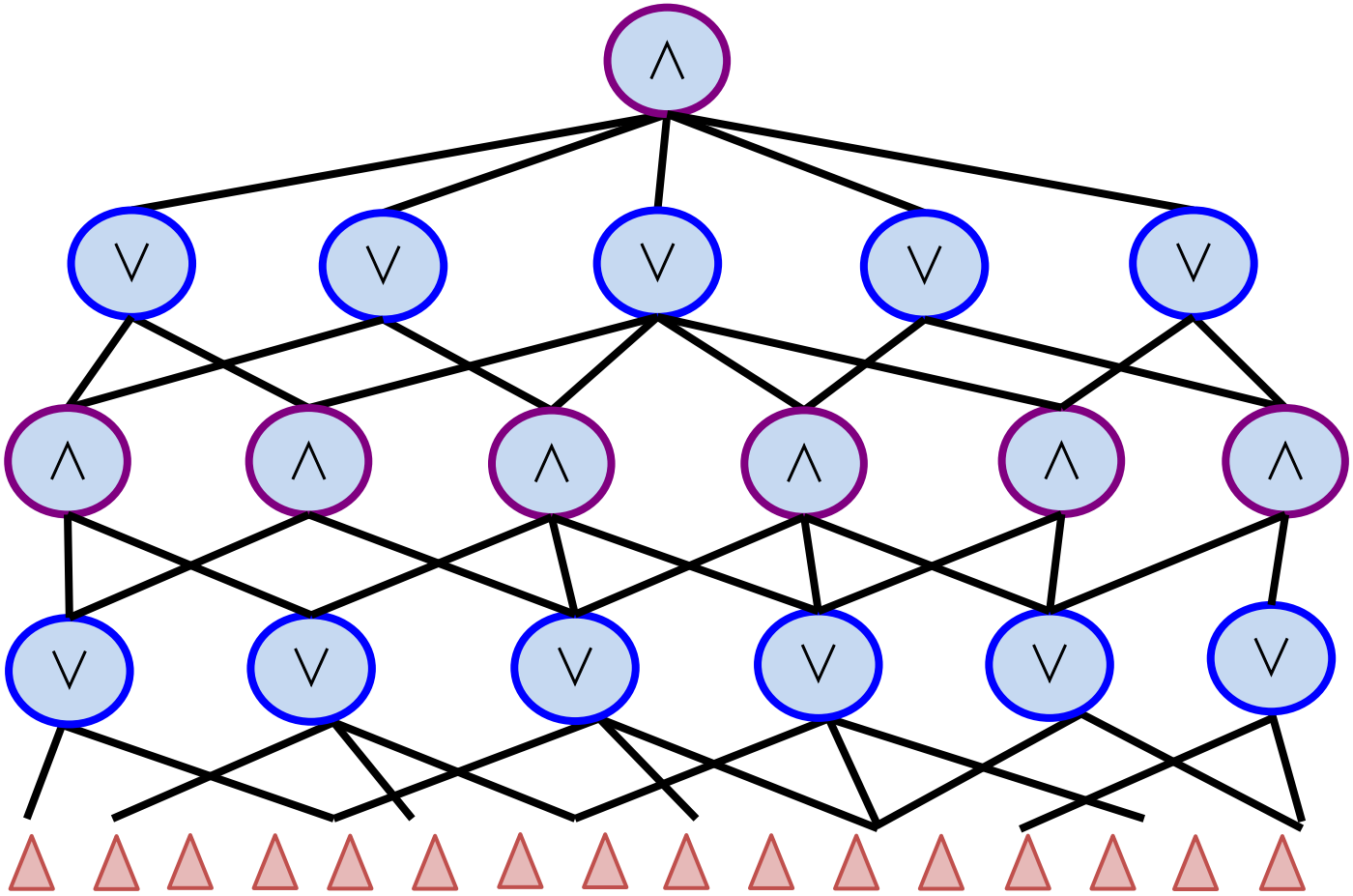


PARITY Lower Bound

depth d+1, size S

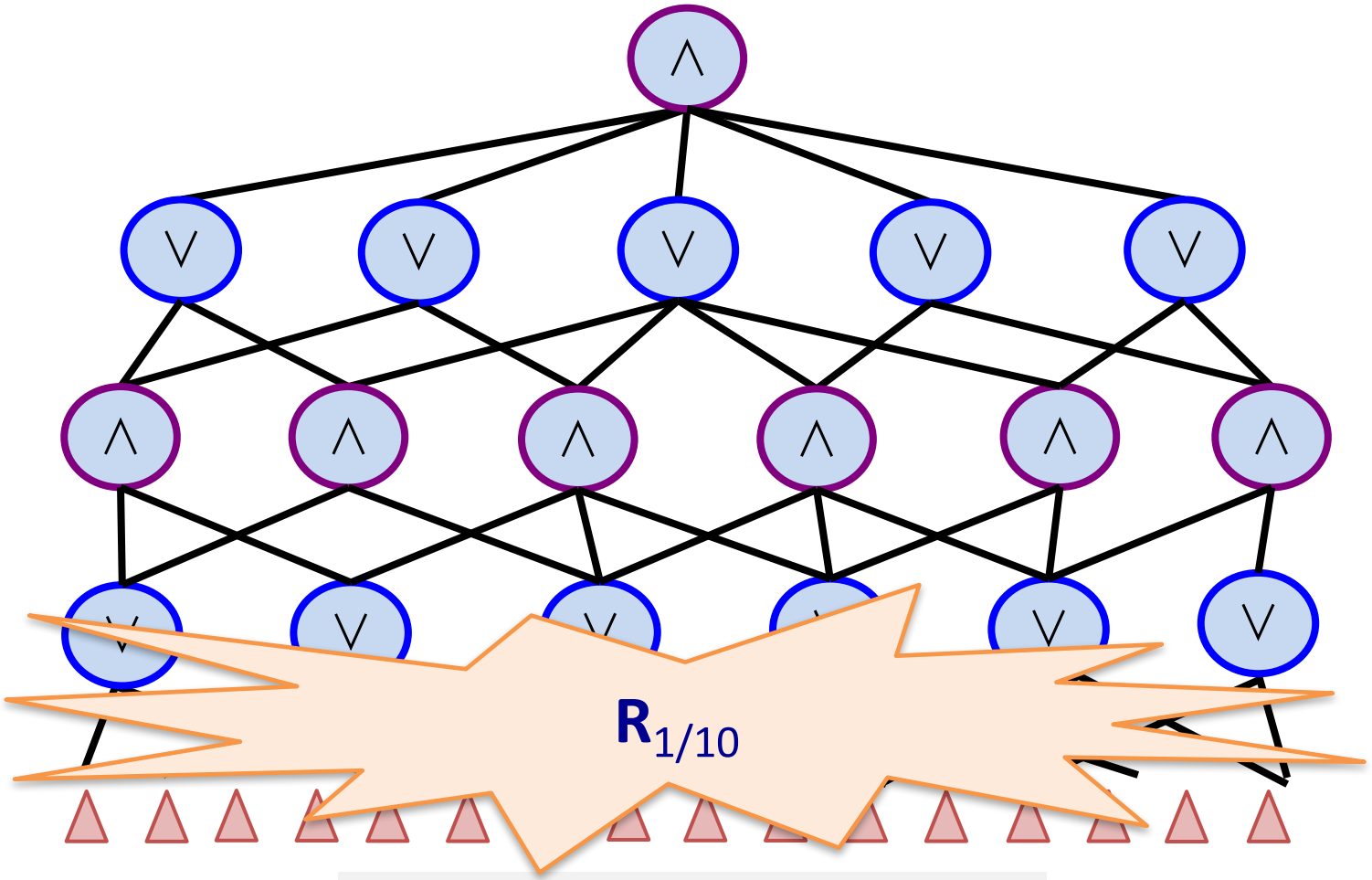


PARITY Lower Bound



depth-1 decision trees

PARITY Lower Bound



depth-1 decision trees

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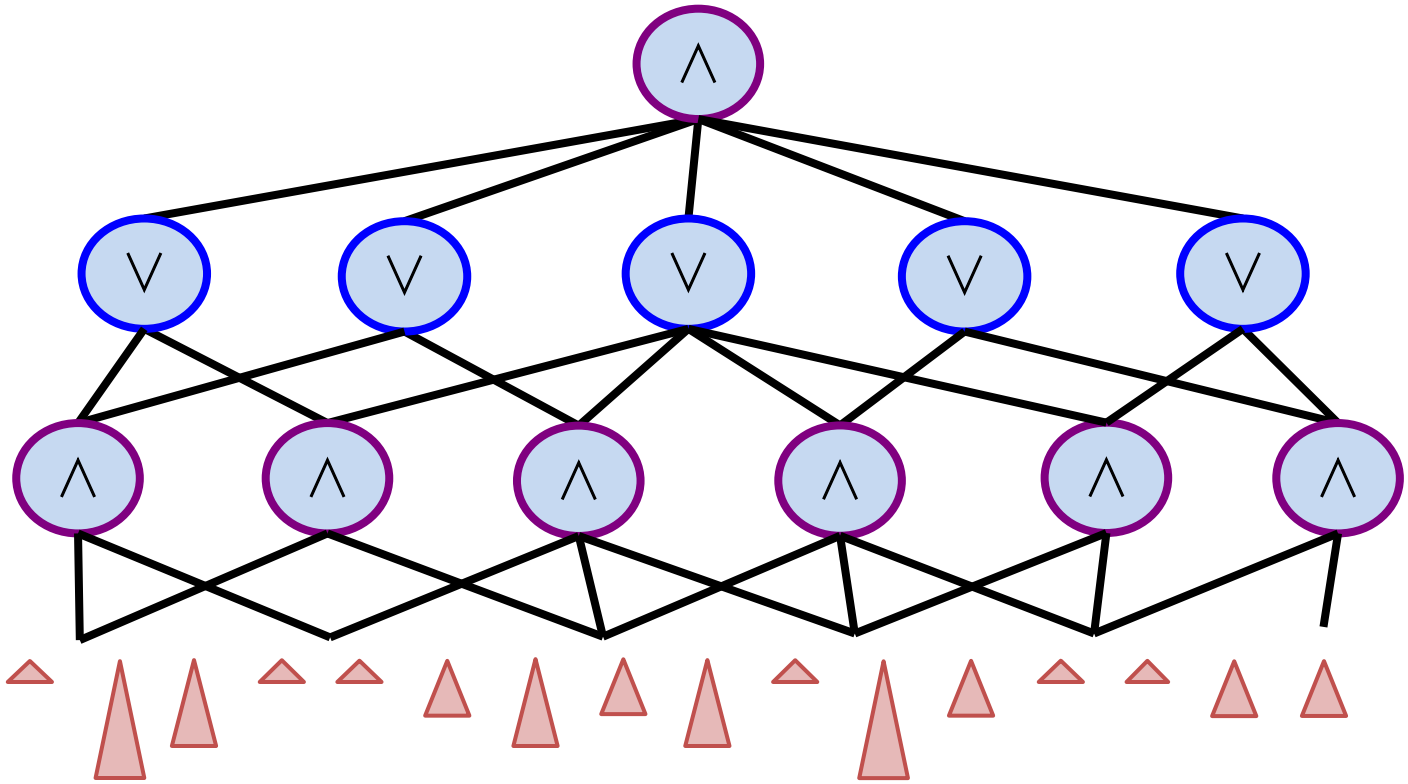
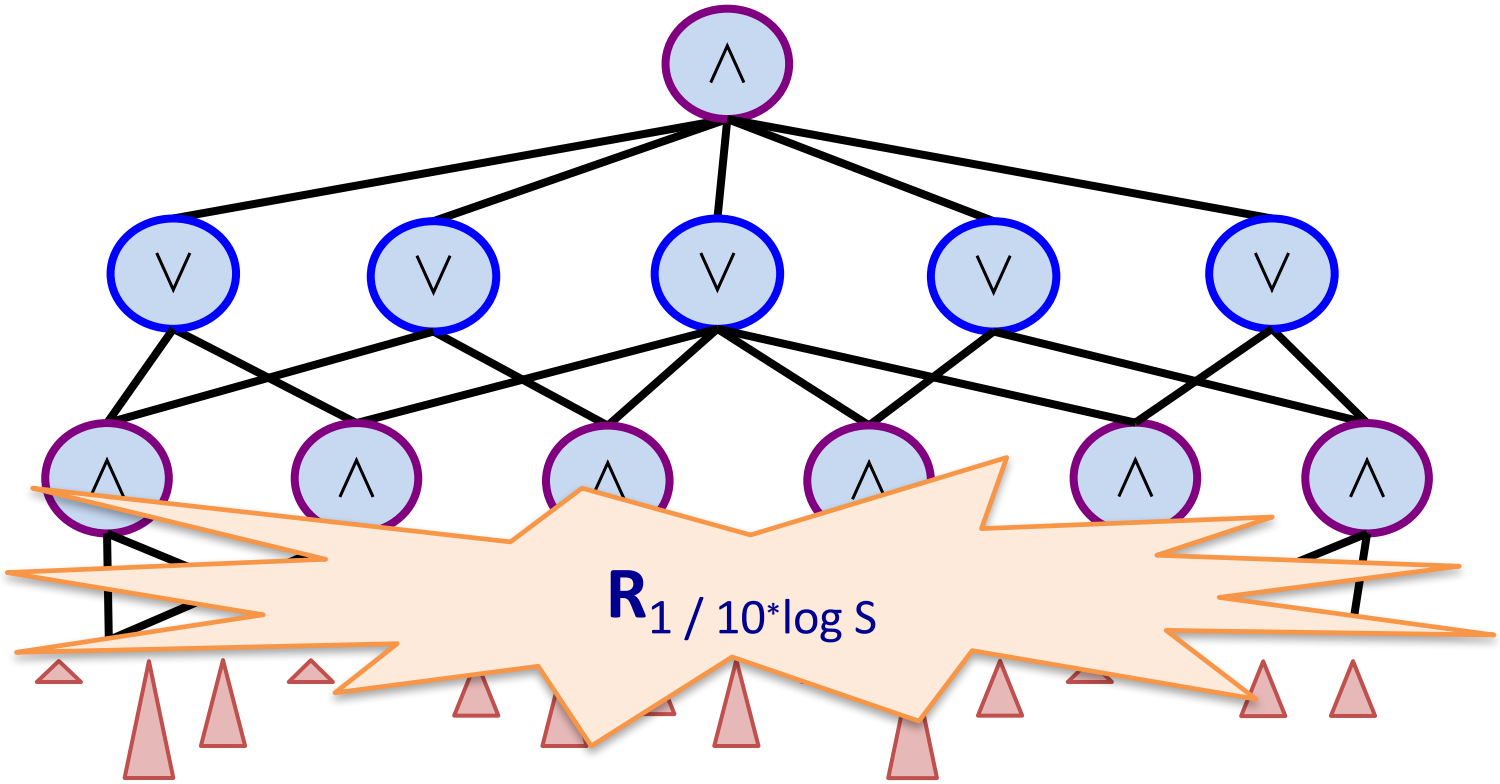


depth O(log S) decision trees (w.h.p.)

PARITY Lower Bound



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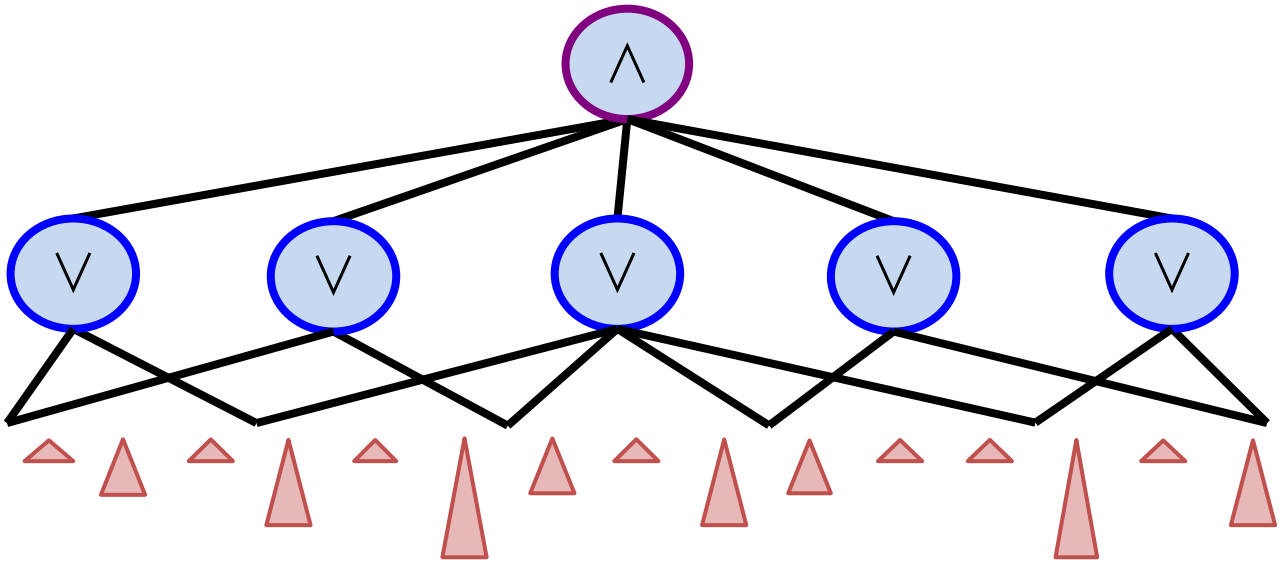

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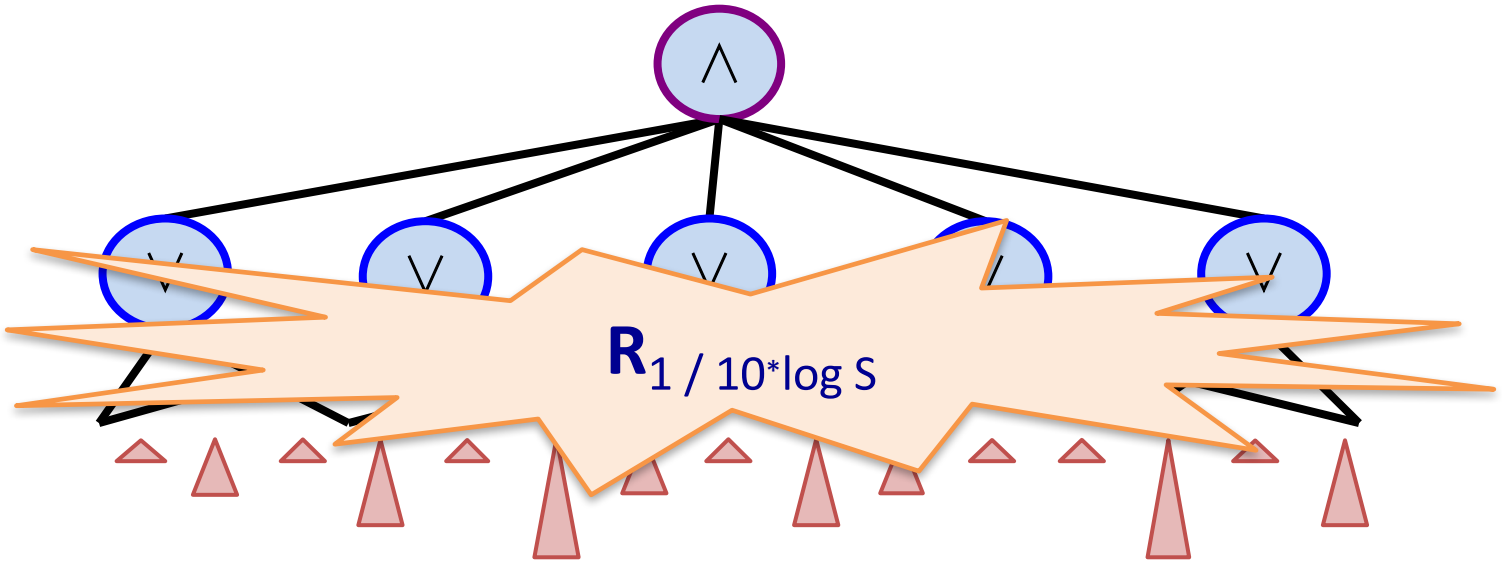

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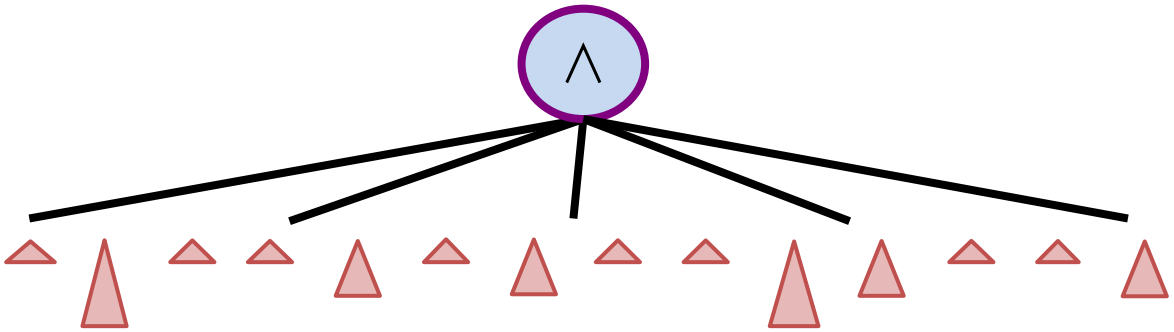
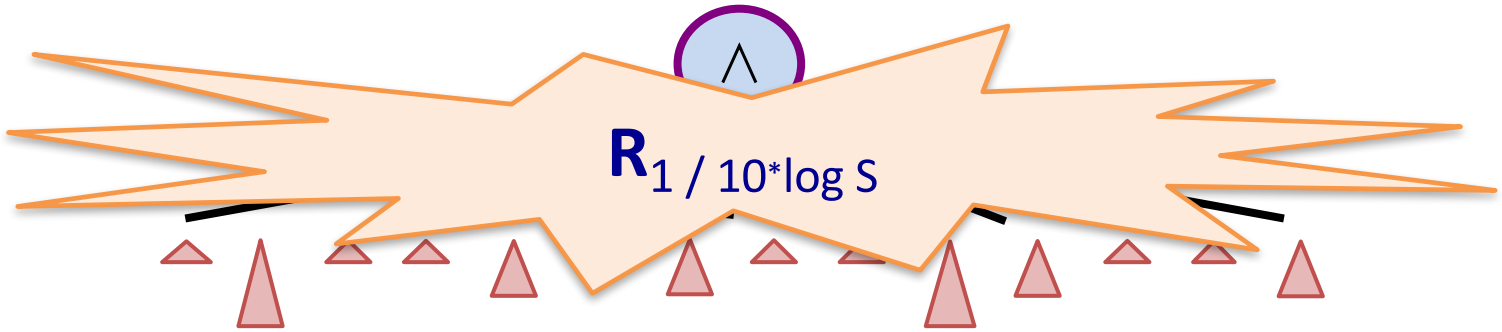


depth O(log S) decision trees (w.h.p.)

PARITY Lower Bound



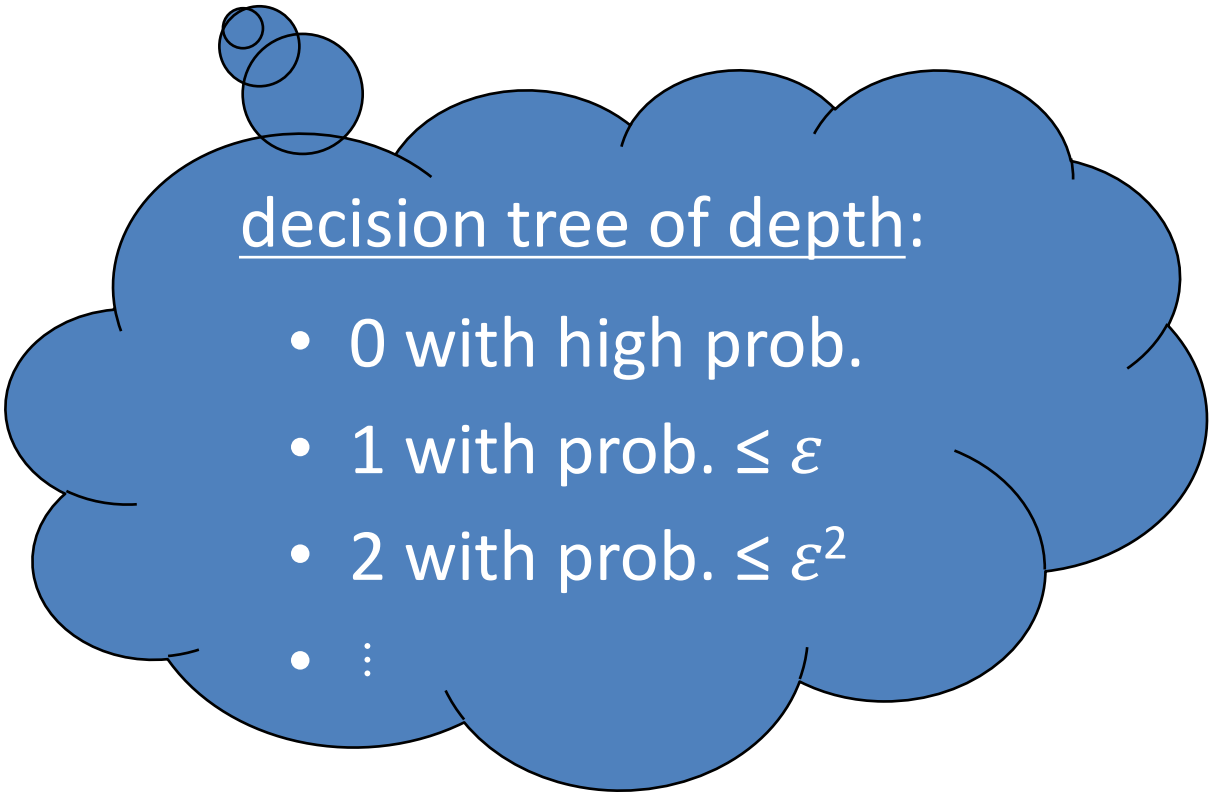
depth O(log S) decision trees (w.h.p.)

PARITY Lower Bound

constant function (w.h.p.)

PARITY Lower Bound

constant function (w.h.p.)



PARITY Lower Bound

- Started with AC^0 circuit of depth $d+1$ and size S
- Applied a sequence of restrictions

$$\mathbf{R}_{1/10}, \underbrace{\mathbf{R}_{1/(10^*\log S)}, \mathbf{R}_{1/(10^*\log S)}, \ldots, \mathbf{R}_{1/(10^*\log S)}}_{d \text{ times}}$$

  Combined restriction: $\mathbf{R}_{1/O(\log S)^d}$

- Circuit reduces to a **constant** (0 or 1) with high prob.

PARITY Lower Bound

- (AC^0 circuit of depth $d+1$ and size S) $\upharpoonright$ $\mathbf{R}_{1/O(\log S)^d}$ is almost surely **constant**
- On the other hand, $PARITY_n \upharpoonright \mathbf{R}_p$ is almost surely **non-constant** for $p = \omega(1/n)$

PARITY Lower Bound

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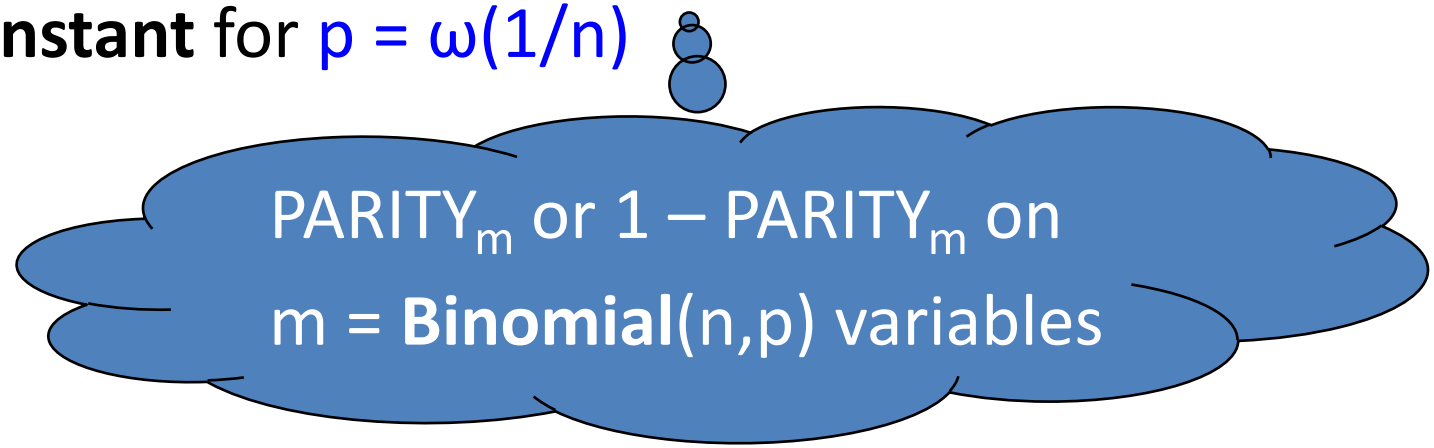
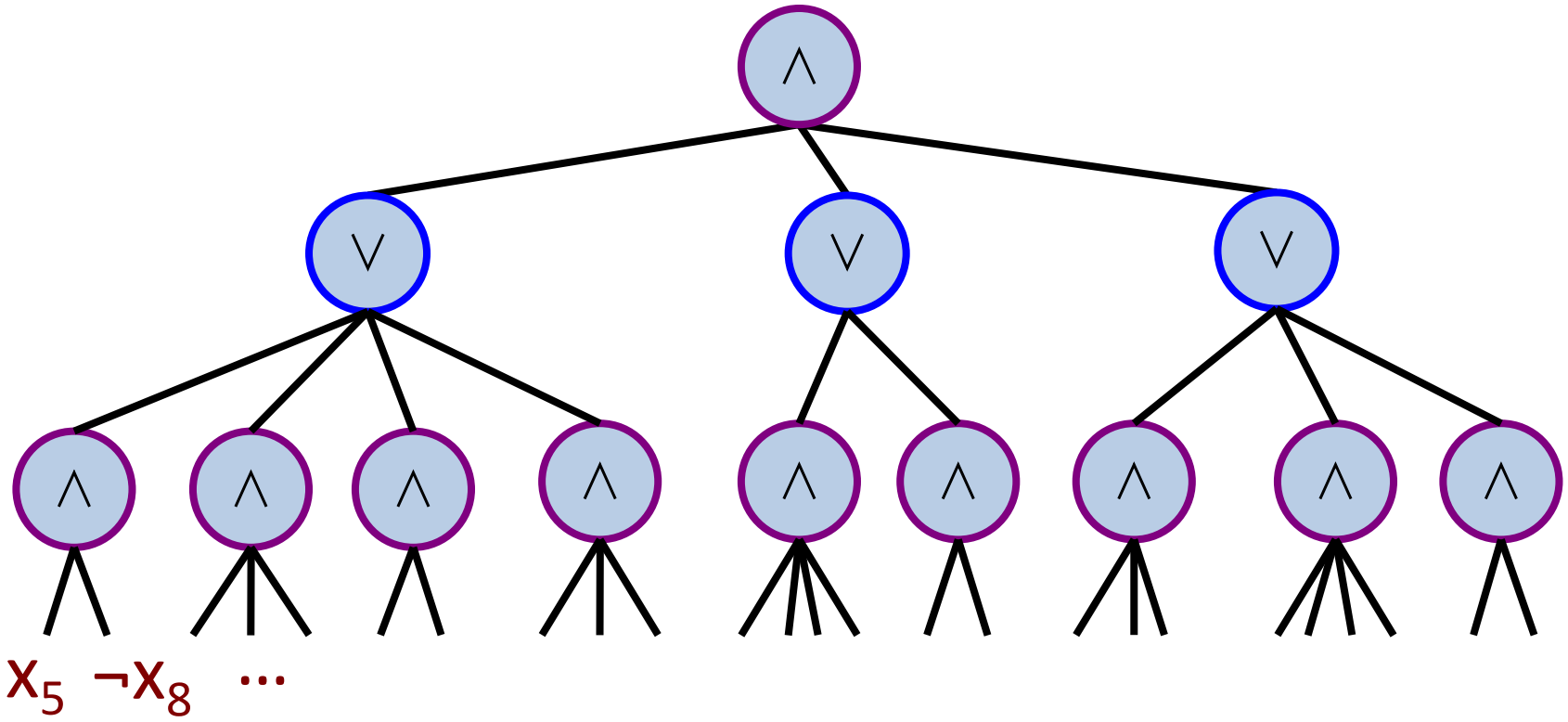


PARITY Lower Bound

- (AC^0 circuit of depth $d+1$ and size S) $\not\upharpoonright$ $\mathbf{R}_{1/O(\log S)^d}$ is almost surely **constant**
- On the other hand, $PARITY_n \not\upharpoonright \mathbf{R}_p$ is almost surely **non-constant** for $p = \omega(1/n)$
- Therefore, depth $d+1$ circuits for $PARITY_n$ require size $\exp(n^{1/d})$

Recall: AC^0 Formulas

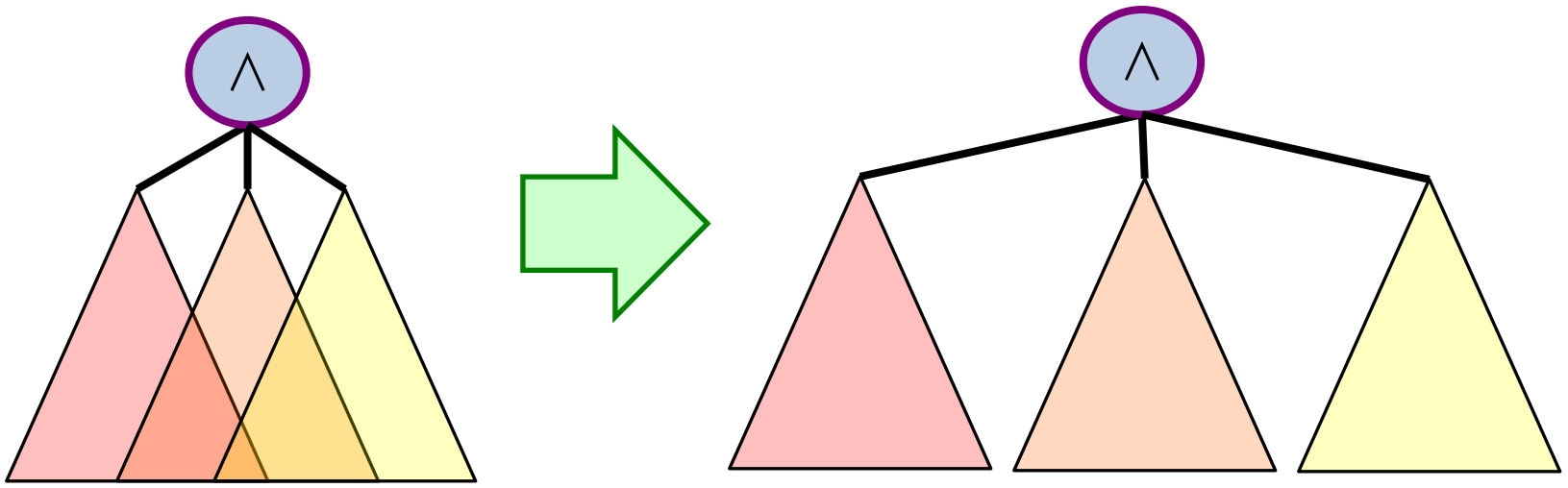


Upper Bound

PARITY has depth d+1 **circuits** of size $\exp(O(n^{1/d}))$

Upper Bound

PARITY has depth d+1 **circuits** of size $\exp(O(n^{1/d}))$
and depth d+1 **formulas** of size $\exp(O(dn^{1/d}))$



Upper Bound

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Theorem [Hastad '86]

Depth d+1 **circuits** for PARITY have size $\exp(\Omega(n^{1/d}))$

Upper Bound

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Theorem [Hastad '86]

Depth d+1 **circuits** for PARITY have size $\exp(\Omega(n^{1/d}))$

Theorem [R.'15]

Depth d+1 **formulas** for PAR. have size $\exp(\Omega(dn^{1/d}))$

Proof of the Switching Lemma

Canonical Decision Tree

- For a DNF formula F and restriction R, we define the **canonical decision tree** of F↾R, denoted CanDT(F↾R), as follows:

1. If F↾R is identically 0 or 1, then CanDT(F↾R) outputs 0 or 1 without making any queries.
2. Otherwise, let C be the first term (conjunction of literals) in F such that C↾R is not fixed to 0 or 1.

   Query all free variables of C↾R. Then proceed as CanDT(F↾R') where R' is the extension of R that includes answers to the queried variables.

- Fix a k-DNF F and $\ell \geq 1$

Switching Lemma:

$$\Pr[\ \mathrm{depth}(\mathrm{CanDT}(F \restriction \mathbf{R}_p)) \geq \ell\ ] = O(pk)^{\ell}$$

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Switching Lemma:

$$\Pr[\ \text{depth}(\text{CanDT}(F \restriction \mathbf{R}_p)) \geq \ell\ ] = O(pk)^{\ell}$$

Obs: $\text{depth}(\text{CanDT}(F \restriction \mathbf{R}_p)) < \ell$

Implies $\text{DT}_{\text{depth}}(F \restriction \mathbf{R}_p) < \ell$

- Fix a k-DNF F and $\ell \geq 1$
- BAD := { restrictions R | depth(CanDT(F $\restriction$ R)) $\geq \ell$ }

Switching Lemma:

$$\Pr[\ \text{depth}(\text{CanDT}(F \restriction \mathbf{R}_p)) \geq \ell\ ] = O(pk)^{\ell}$$

- Fix a k-DNF F and $\ell \geq 1$
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- Goal. Pr[$\mathbf{R}_p \in$ BAD] = $O(pk)^{\ell}$

Switching Lemma:

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Key idea. We associate each R $\in$ BAD with a restriction R* such that

- Fix a k-DNF F and $\ell \geq 1$
- BAD := { restrictions R | depth(CanDT(F ↾ R)) ≥ ℓ }
- Goal. Pr[$\mathbf{R}_p \in$ BAD] = $O(pk)^{\ell}$

Key idea. We associate each R ∈ BAD with a restriction R* such that

① |Stars(R*)| = |Stars(R)| − ℓ

R ⋆ 1 ⋆ ⋆ 1 0 ⋆ 1 ⋆ 1 0 0 ⋆ ⋆ 0 ⋆ 0 ⋆ ⋆ ⋆ 0 ⋆ 0

R* ⋆ 1 ⋆ 0 1 0 ⋆ 1 ⋆ 1 0 0 1 ⋆ 0 ⋆ 0 ⋆ 0 ⋆ 0 ⋆ 0

(ℓ = 3)

- Fix a k-DNF F and $\ell \geq 1$
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in particular, Pr[$\mathbf{R}_p$ = R*] / Pr[$\mathbf{R}_p$ = R] = $((1-p)/2p)^{\ell}$

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Key idea. We associate each $R \in$ BAD with a restriction R* such that

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in particular, $\Pr[\mathbf{R}_p = R^*] / \Pr[\mathbf{R}_p = R] = ((1-p)/2p)^\ell$

for any restriction ρ with s stars,

$\Pr[\mathbf{R}_p = \rho] = p^s (1-p)^{n-s}$

- Fix a k-DNF F and $\ell \geq 1$
- BAD := { restrictions R | depth(CanDT(F $\restriction$ R)) $\geq \ell$ }
- Goal. Pr[$\mathbf{R}_p \in$ BAD] = $O(pk)^{\ell}$

Key idea. We associate each R $\in$ BAD with a restriction R* such that

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② the map R $\mapsto$ R* is $(4k)^{\ell}$-to-1

- Fix a k-DNF F and $\ell \geq 1$
- BAD := { restrictions R | depth(CanDT(F $\restriction$ R)) $\geq \ell$ }
- Goal. Pr[$\mathbf{R}_p \in$ BAD] = $O(pk)^\ell$

Key idea. We associate each R $\in$ BAD with a restriction R* such that

① |Stars(R*)| = |Stars(R)| – ℓ

② the map R $\mapsto$ R* is $(4k)^\ell$-to-1

any restriction ρ equals R* for at most $(4k)^\ell$ distinct R $\in$ BAD

Key idea. We associate each $R \in \text{BAD}$ with a restriction R^* such that

① $|\text{Stars}(R^*)| = |\text{Stars}(R)| - \ell$

② the map $R \mapsto R^*$ is $(4k)^{\ell}$-to-1

Key idea. We associate each $R \in \text{BAD}$ with a restriction R^* such that

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$$\Pr[\, \mathbf{R}_p \in \text{BAD} \,] = \sum_{R \in \text{BAD}} \Pr[\, \mathbf{R}_p = R \,]$$

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$$\Pr[\, \mathbf{R}_p \in BAD \,]$$

$$= \sum_{R \in BAD} \Pr[\, \mathbf{R}_p = R \,]$$

$$= \sum_{R \in BAD} (2p/(1-p))^\ell \Pr[\, \mathbf{R}_p = R^* \,]$$

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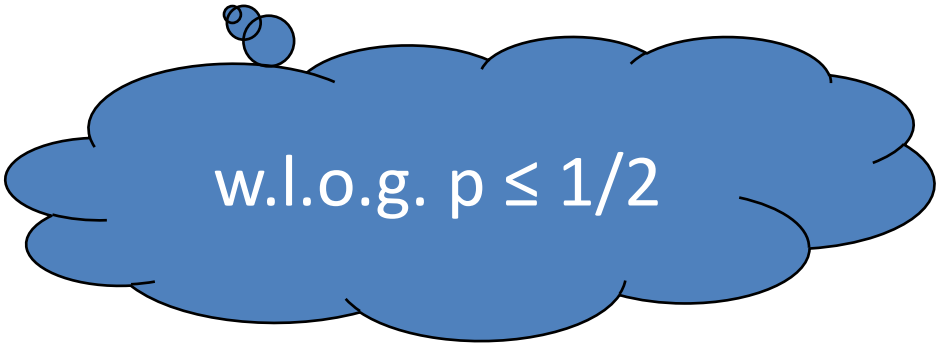
$$\leq (4p)^\ell \sum_{R \in \text{BAD}} \Pr[\, \mathbf{R}_p = R^* \,]$$



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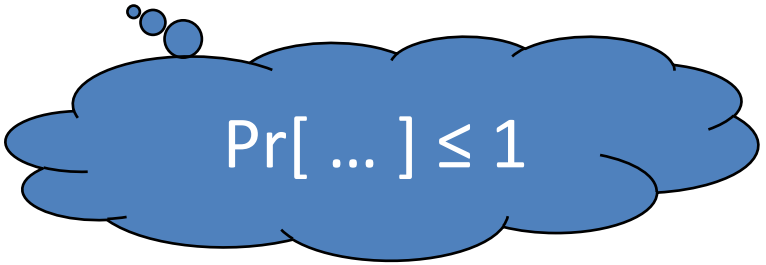

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&\quad \le (16pk)^\ell
\end{aligned}$$

Q.E.D.

Switching Lemma:

$$\Pr[\ \text{depth}(\text{CanDT}(F \restriction \mathbf{R}_p)) \geq \ell\ ] = O(pk)^{\ell}$$

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&\quad \leq (4p)^{\ell} \textstyle\sum_{R \in \text{BAD}} \Pr[\ \mathbf{R}_p = R^*\ ] \\
&\quad \leq (4p)^{\ell} (4k)^{\ell} \Pr[\ \mathbf{R}_p \in \{R^* \mid R \in \text{BAD}\}\ ] \\
&\quad \leq (16pk)^{\ell}
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$$= \sum_{R \in \text{BAD}} \Pr[\ \mathbf{R}_p = R\ ]$$

$$= \sum_{R \in \text{BAD}} (2p/(1-p))^{\ell} \Pr[\ \mathbf{R}_{1-p} = R^*\ ]$$

$$\leq (4p)^{\ell} \Pr[\ \mathbf{R}_{1-p} \in \{R^* \mid R \in \text{BAD}\}\ ]$$

$$\leq (16pk)^{\ell}$$

more careful analysis gives $(5pk)^{\ell}$

Q.E.D.

Key idea. We associate each $R \in \text{BAD}$ with a restriction R^* such that

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Key idea. We associate each $R \in$ BAD with a pair (R*,Code(R)) such that

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③

④

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③ $\mathrm{Code}(R) \in (\{0,1\}^2 \times [k])^\ell$

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in particular, Code(R) has $(4k)^\ell$ possible values

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③ & ④ imply ②

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④ the map $R \mapsto (R^*, \mathrm{Code}(R))$ is 1-to-1

intuitively, Code(R) is a "recipe" for inverting $R \mapsto (R^*, \mathrm{Code}(R))$ given knowledge of R^*

$R \mapsto (R^*, \mathrm{Code}(R))$

k = 3, $\ell = 4$

$R \mapsto (R^*, \mathrm{Code}(R))$

$$F = x_1 x_2 \neg x_3 \vee \neg x_1 x_3 x_5 \vee x_2 \neg x_4 x_5 \vee x_3 x_4 \neg x_6 \vee x_1 \neg x_4 \neg x_7$$

$k = 3,\ \ell = 4$

$R \mapsto (R^*, \mathrm{Code}(R))$

$$F = x_1\, x_2\, \neg x_3 \vee \neg x_1\, x_3\, x_5 \vee x_2\, \neg x_4\, x_5 \vee x_3\, x_4\, \neg x_6 \vee x_1\, \neg x_4\, \neg x_7$$

$R = \{\, x_1 \mapsto 1,\ x_4 \mapsto 0 \,\}$

$k = 3,\ \ell = 4$

$R \mapsto (R^*, \mathrm{Code}(R))$

$$F\restriction R \quad \overset{1}{x_1}\ x_2\ \neg x_3 \ \vee\ \neg x_1\ x_3\ x_5 \ \overset{0}{\vee}\ x_2\ \neg x_4\ x_5 \ \overset{1}{\vee}\ x_3\ x_4\ \neg x_6 \ \overset{0}{\vee}\ x_1\ \neg x_4\ \neg x_7 \quad \overset{1\ \ 1}{}$$

$R = \{\, x_1 \mapsto 1,\ x_4 \mapsto 0 \,\}$

k = 3, $\ell = 4$

$R \mapsto (R^*, \mathrm{Code}(R))$

1 0 1 0 1 1

$F \restriction R \; x_1 \; x_2 \; \neg x_3 \vee \neg x_1 \; x_3 \; x_5 \vee x_2 \; \neg x_4 \; x_5 \vee x_3 \; \neg x_6 \vee x_1 \; \neg x_4 \; \neg x_7$

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$k = 3,\ \ell = 4$

$R \mapsto (R^*, \mathrm{Code}(R))$

$F \restriction R$ $x_1\ x_2\ \neg x_3 \vee \neg x_1\ x_3\ x_6 \vee x_2\ \neg x_4\ x_5 \vee x_3\ x_5\ \neg x_6 \vee x_1\ \neg x_4\ \neg x_7$

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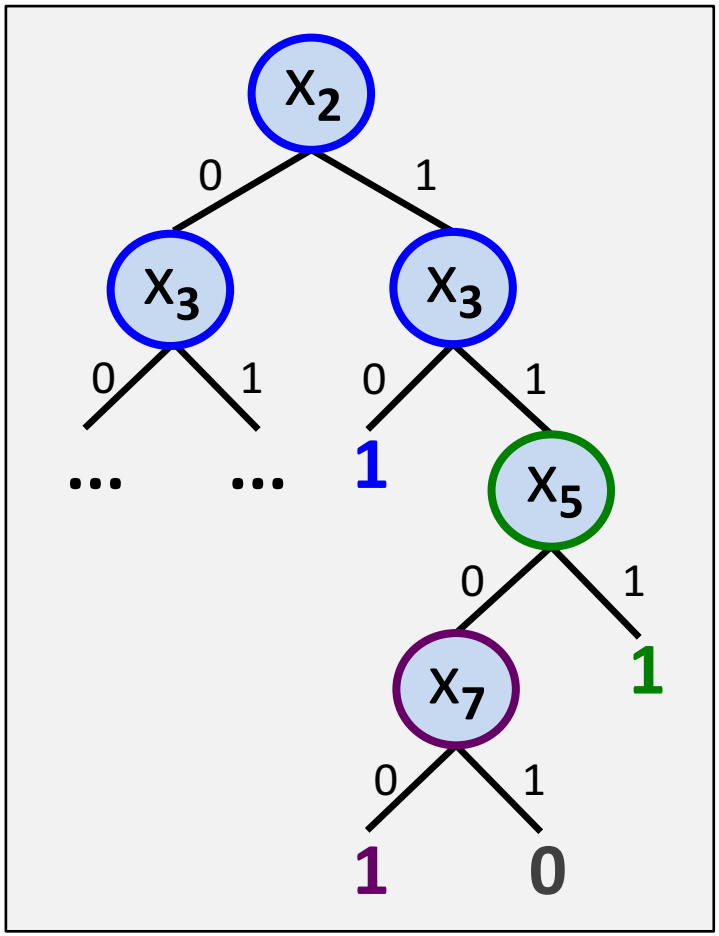
CanDT($F \restriction R$) =



k = 3, $\ell = 4$

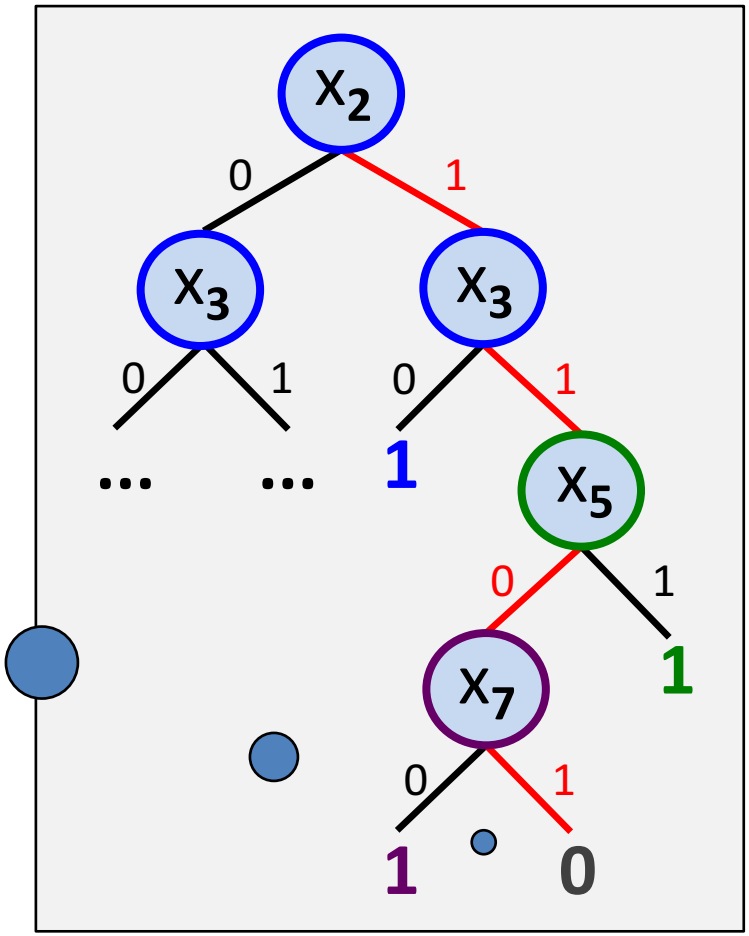
$R \mapsto (R^*, \mathrm{Code}(R))$

$F \restriction R$ $x_1\ x_2\ \neg x_3 \vee \neg x_1\ x_3\ x_6 \vee x_2\ \neg x_4\ x_5 \vee x_3\ \neg x_5\ \neg x_6 \vee x_1\ \neg x_4\ \neg x_7$

=

$R = \{ x_1 \mapsto 1, x_4 \mapsto 0 \}$

$R \in$ BAD, since
$\mathrm{CanDT}(F \restriction R) \geq \ell$



k = 3, $\ell = 4$

$R \mapsto (R^*, \mathrm{Code}(R))$

$F \upharpoonright R$ $x_1\ x_2\ \neg x_3 \vee$ [illegible] $\vee\ x_2\ \neg x_4\ x_5 \vee$ [illegible] $\vee\ x_1\ \neg x_4\ \neg x_7$

=

$R = \{ x_1 \mapsto 1, x_4 \mapsto 0 \}$

$R^{\sim} = \{ x_1 \mapsto 1, x_4 \mapsto 0,$
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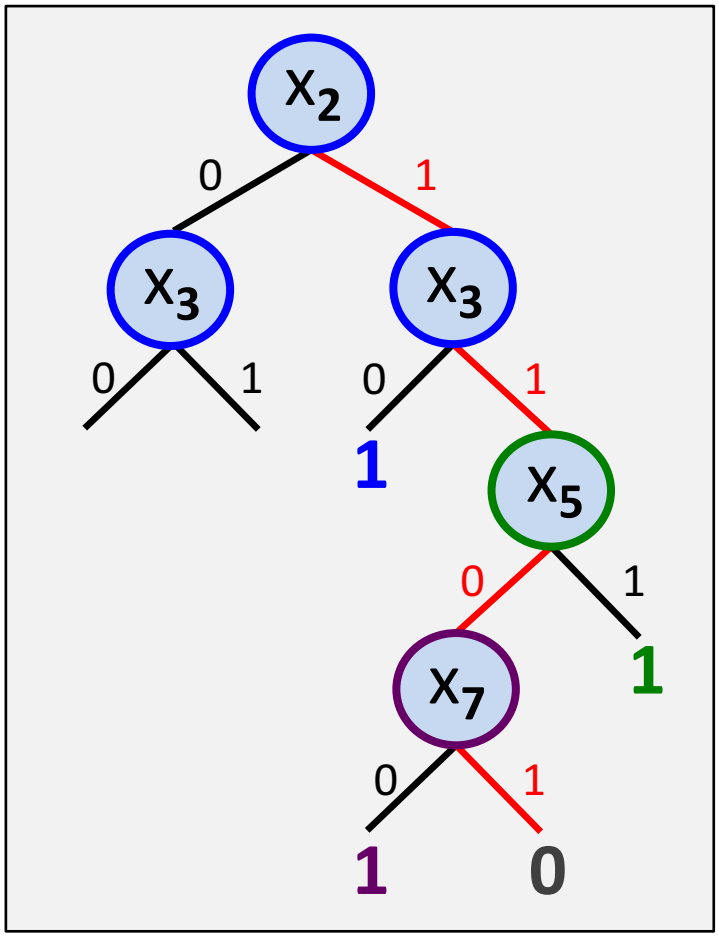

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$R \mapsto (R^*, \mathrm{Code}(R))$

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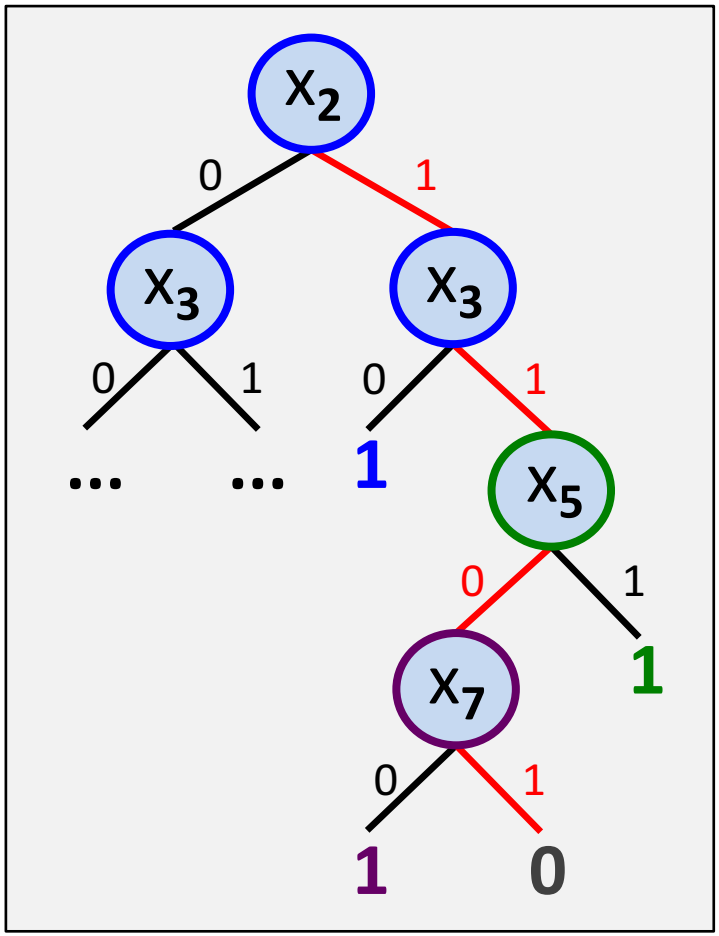

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$F \upharpoonright R = x_1\, x_2\, \neg x_3 \vee \neg x_1\, x_3\, x_6 \vee x_2\, \neg x_4\, x_5 \vee x_3\, \neg x_5\, \neg x_6 \vee x_1\, \neg x_4\, \neg x_7$

$R = \{ x_1 \mapsto 1, x_4 \mapsto 0 \}$

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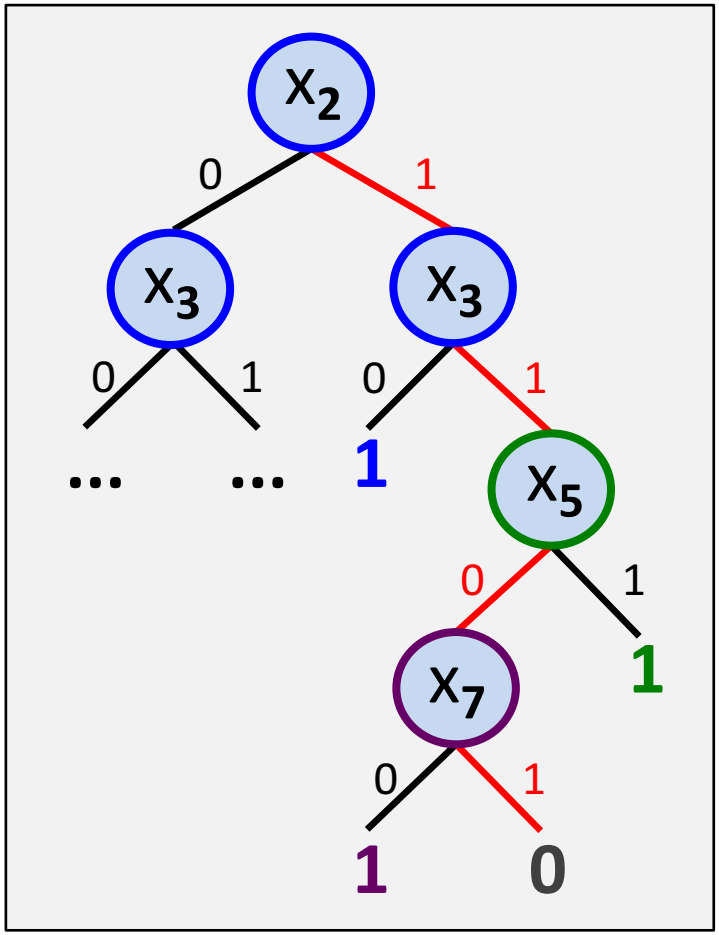

k = 3, ℓ = 4

R ↦ (R*,Code(R))

F ⊢ R $x_1\ x_2\ \neg x_3 \vee \neg x_1\ x_3\ x_6 \vee x_2\ \neg x_4\ x_5 \vee x_3\ x_5\ \neg x_6 \vee x_1\ \neg x_4\ \neg x_7$

=

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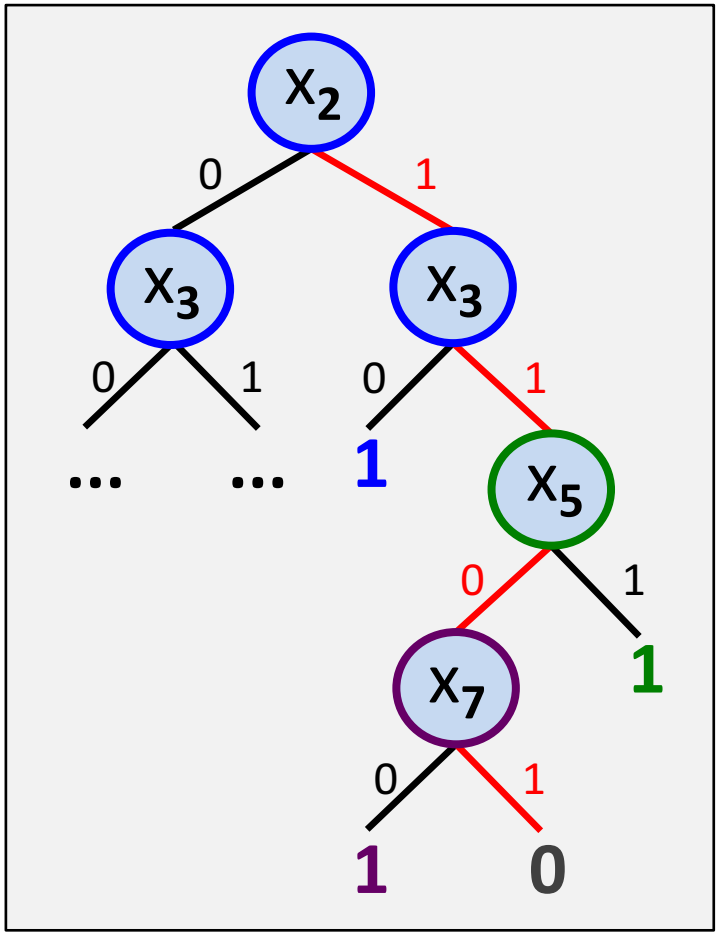

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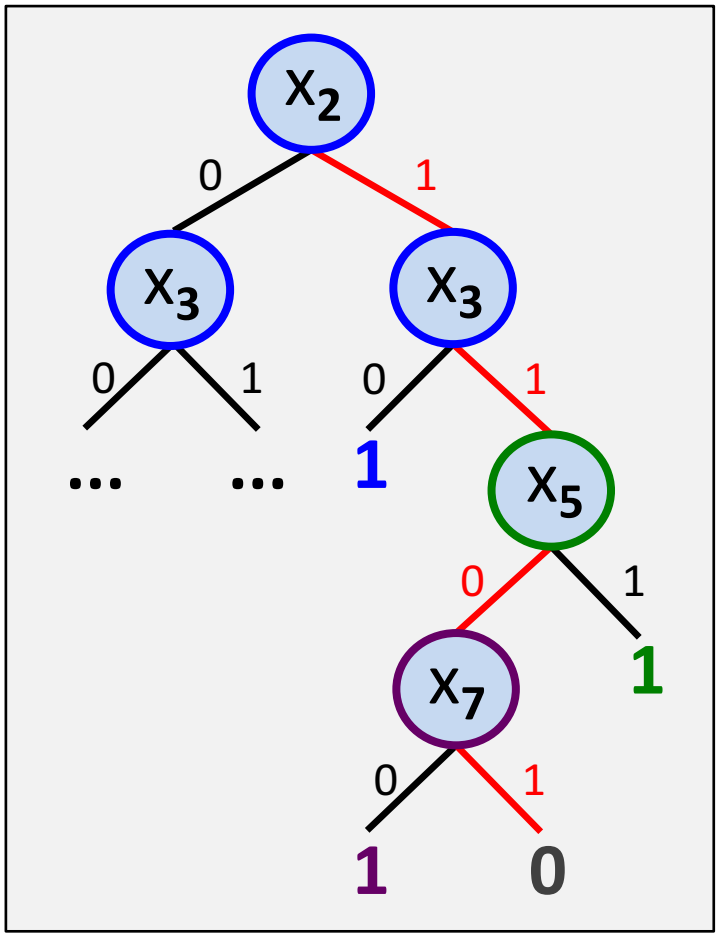

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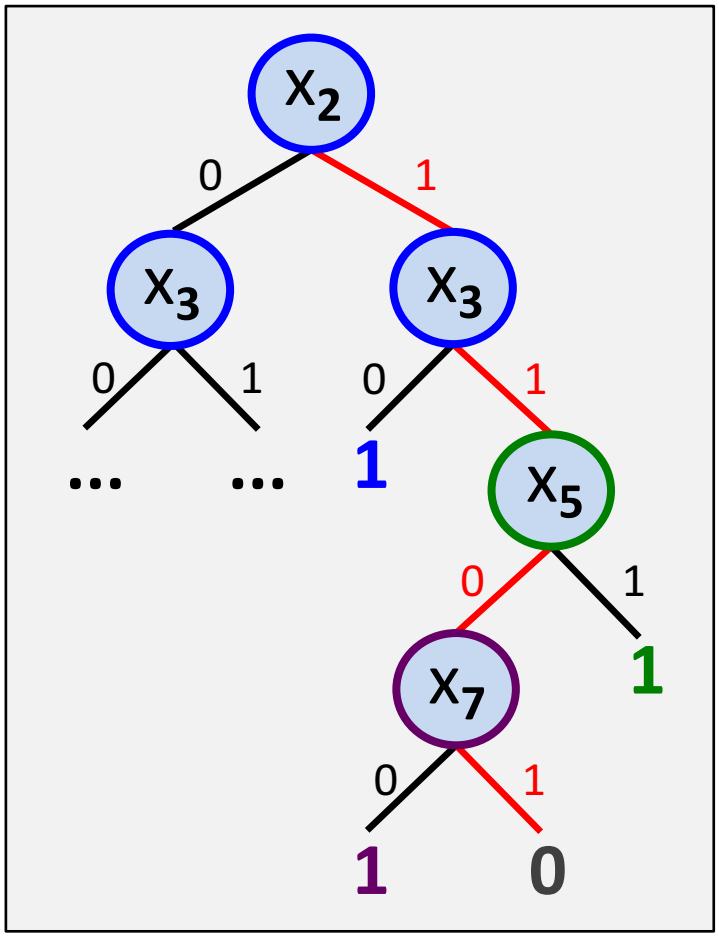

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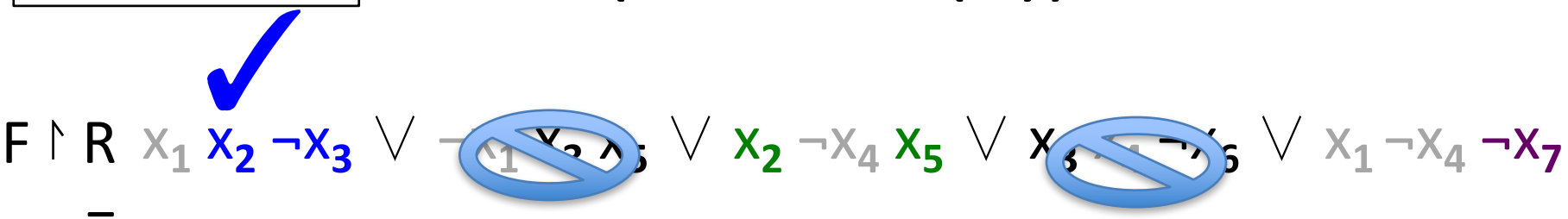
$R \mapsto (R^*, \mathrm{Code}(R))$



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$R^* = \{ x_1 \mapsto 1, x_4 \mapsto 0, x_2 \mapsto 1, x_3 \mapsto 0, x_5 \mapsto \quad x_7 \mapsto \quad \}$

Code(R) says:

- find the first satisfied term of F ⊦ R*
- the "long path" begins with

  $x_2 \mapsto 1, x_3 \mapsto 1$

  (i.e. var_2 of term ↦ 1 and var_3 of term ↦ 1)
- …

$k = 3,\ \ell = 4$

$R \mapsto (R^*, \mathrm{Code}(R))$

1 0 1

$F \restriction R$ $x_1\ x_2\ \neg x_3 \vee \neg x_1\ x_3\ x_6 \vee x_2\ \neg x_4\ x_5 \vee x_3\ \neg x_5\ \neg x_6 \vee x_1\ \neg x_4\ \neg x_7$

=

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$x_5 \mapsto \quad x_7 \mapsto \ \}$

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$x_5 \mapsto \quad x_7 \mapsto \ \}$

Fix variables according to the beginning of the long path

k = 3, $\ell = 4$

$R \mapsto (R^*, \mathrm{Code}(R))$

$F \upharpoonright R = \; \cdots \vee x_2 \neg x_4 x_5 \vee \cdots \vee x_1 \neg x_4 \neg x_7$

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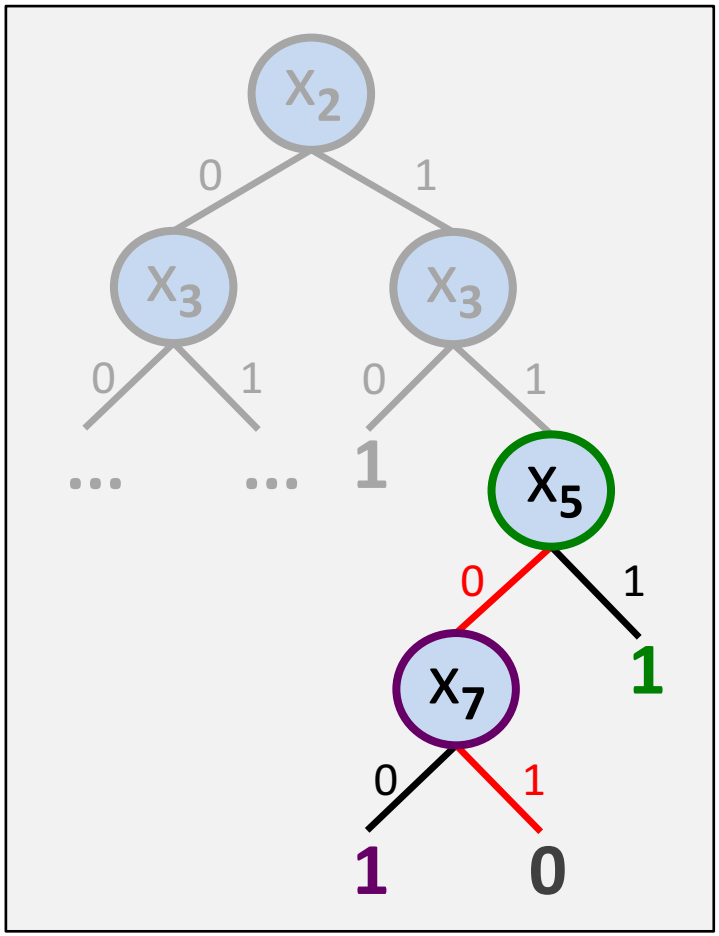

k = 3, ℓ = 4

$R \mapsto (R^*, \mathrm{Code}(R))$

$F \upharpoonright R$ = $\bigvee \bigvee x_2\, \neg x_4\, x_5 \bigvee \bigvee x_1\, \neg x_4\, \neg x_7$

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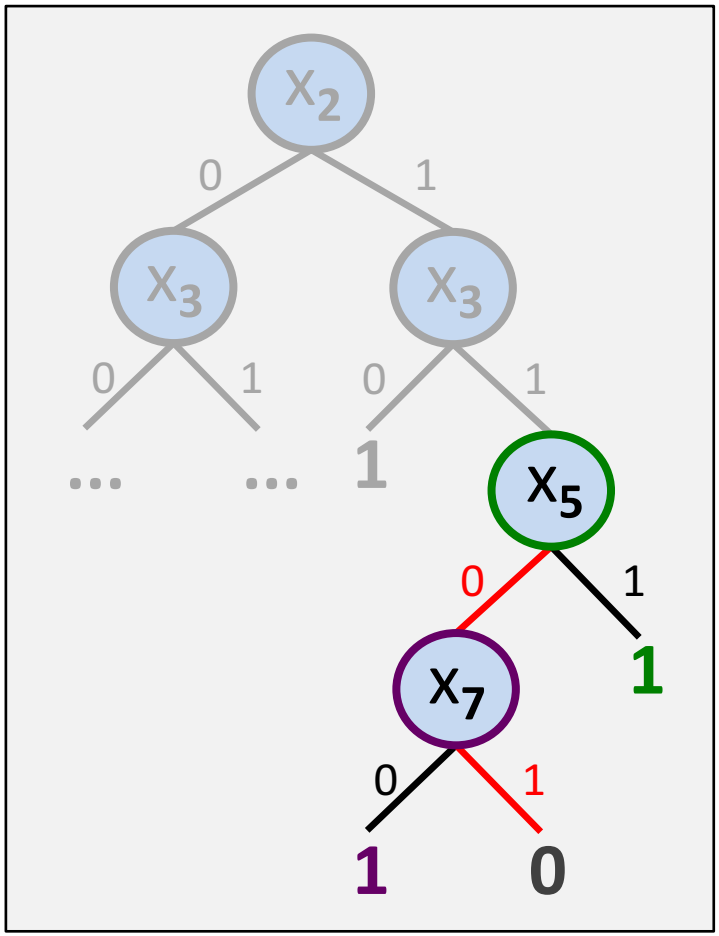
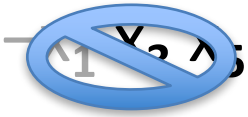


k = 3, $\ell = 4$

$R \mapsto (R^*, \mathrm{Code}(R))$



$F \restriction R \quad \ldots \vee \neg x_1 \ldots \vee x_2\ \neg x_4\ x_5 \vee \ldots \neg x_6 \vee x_1\ \neg x_4\ \neg x_7$

=

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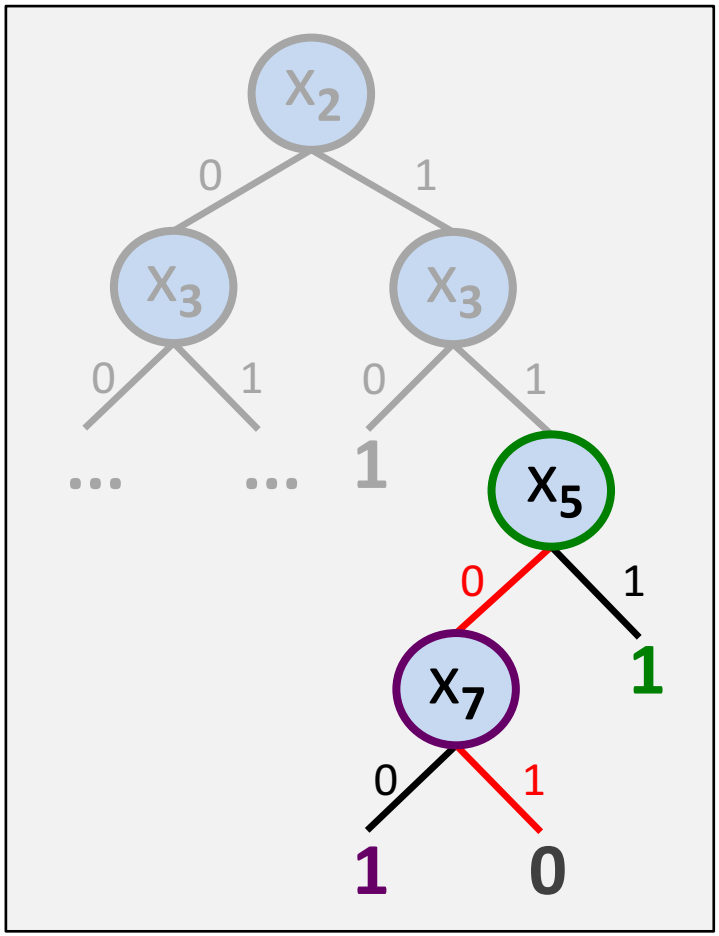
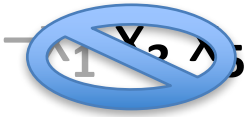


k = 3, $\ell = 4$

$R \mapsto (R^*, \mathrm{Code}(R))$

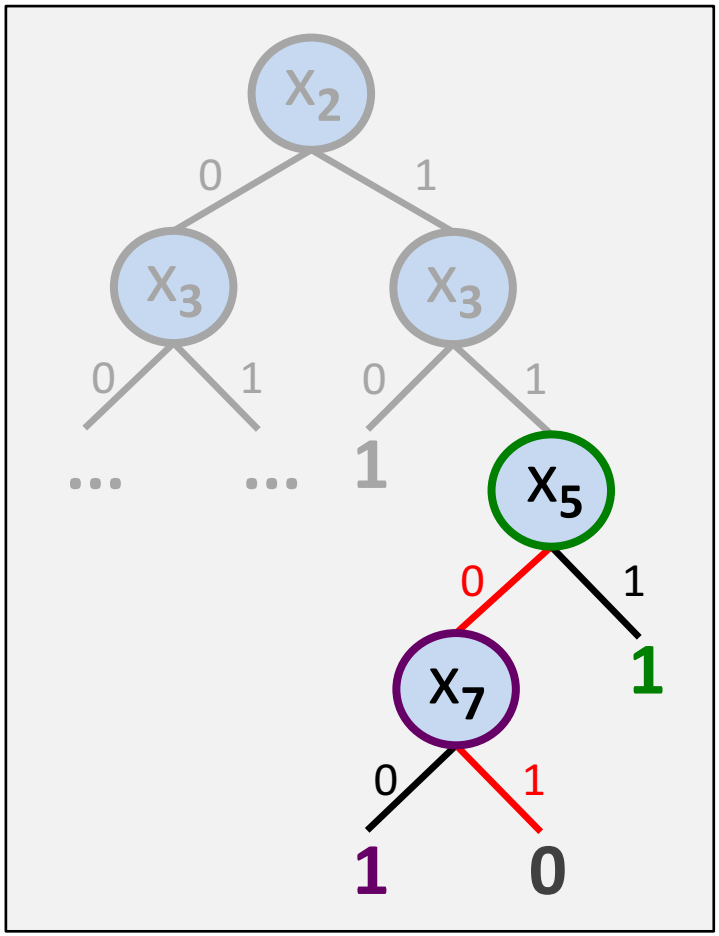




$F \restriction R = \quad \vee \quad \vee x_2 \neg x_4 x_5 \vee \quad \vee x_1 \neg x_4 \neg x_7$

$R = \{ x_1 \mapsto 1, x_4 \mapsto 0 \}$

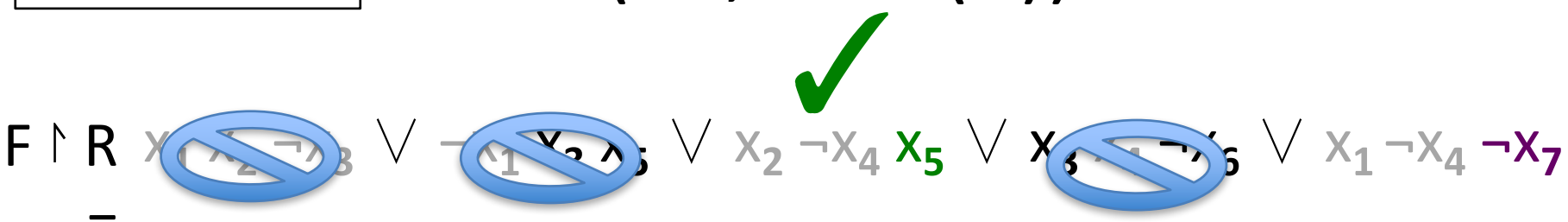
$R\sim = \{ x_1 \mapsto 1, x_4 \mapsto 0, x_2 \mapsto 1, x_3 \mapsto 1, x_5 \mapsto 0 \; x_7 \mapsto \; \}$

$R^* = \{ x_1 \mapsto 1, x_4 \mapsto 0, x_2 \mapsto 1, x_3 \mapsto 0, x_5 \mapsto 1 \; x_7 \mapsto \; \}$



k = 3, ℓ = 4

$R \mapsto (R^*, \mathrm{Code}(R))$



$R = \{ x_1 \mapsto 1, x_4 \mapsto 0 \}$

$R\sim = \{ x_1 \mapsto 1, x_4 \mapsto 0, x_2 \mapsto 1, x_3 \mapsto 1, x_5 \mapsto 0 \; x_7 \mapsto \; \}$

$R^* = \{ x_1 \mapsto 1, x_4 \mapsto 0, x_2 \mapsto 1, x_3 \mapsto 0, x_5 \mapsto 1 \; x_7 \mapsto \; \}$

Code(R) next says:

- find the next satisfied term of $F \vdash R^*$ (overwriting $x_2 \mapsto 1$, $x_3 \mapsto 1$)
- the "long path" continues

  $x_5 \mapsto 0$

  (i.e. var_3 of term $\mapsto 0$)
- …

k = 3, $\ell = 4$

$R \mapsto (R^*, \mathrm{Code}(R))$

0

$F \restriction R$ = $\neg x_1 \neg x_2 \neg x_3 \vee \neg x_1 x_3 x_5 \vee x_2 \neg x_4 x_5 \vee x_3 \neg x_5 \neg x_6 \vee x_1 \neg x_4 \neg x_7$

$R = \{ x_1 \mapsto 1, x_4 \mapsto 0 \}$

$R\sim = \{ x_1 \mapsto 1, x_4 \mapsto 0,$
$x_2 \mapsto 1, x_3 \mapsto 1,$
$x_5 \mapsto 0 \quad x_7 \mapsto \quad \}$

$R^* = \{ x_1 \mapsto 1, x_4 \mapsto 0,$
$x_2 \mapsto 1, x_3 \mapsto 0,$
$x_5 \mapsto 1, x_7 \mapsto \quad \}$

Fix variables according to the long path

k = 3, ℓ = 4

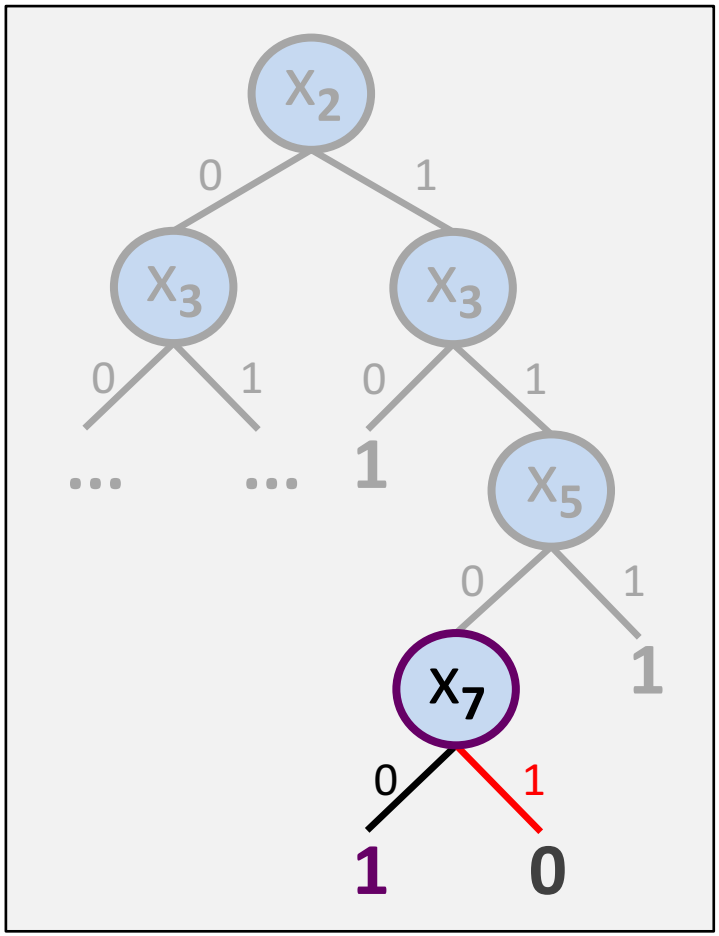
R ↦ (R*,Code(R))

F ⊢ R = $x_1 \neg x_4 \neg x_7$

$R = \{ x_1 \mapsto 1, x_4 \mapsto 0 \}$

$R\sim = \{ x_1 \mapsto 1, x_4 \mapsto 0,$
$x_2 \mapsto 1, x_3 \mapsto 1,$
$x_5 \mapsto 0, x_7 \mapsto \quad \}$

$R^* = \{ x_1 \mapsto 1, x_4 \mapsto 0,$
$x_2 \mapsto 1, x_3 \mapsto 0,$
$x_5 \mapsto 1, x_7 \mapsto \quad \}$



k = 3, $\ell = 4$

$R \mapsto (R^*, \mathrm{Code}(R))$

$$F = x_1 x_2 \neg x_3 \vee \neg x_1 x_3 x_5 \vee x_2 \neg x_4 x_5 \vee x_3 x_4 \neg x_6 \vee x_1 \neg x_4 \neg x_7$$

$R = \{ x_1 \mapsto 1, x_4 \mapsto 0 \}$

$R\sim = \{ x_1 \mapsto 1, x_4 \mapsto 0,$
$x_2 \mapsto 1, x_3 \mapsto 1,$
$x_5 \mapsto 0, x_7 \mapsto 1 \}$

$R^* = \{ x_1 \mapsto 1, x_4 \mapsto 0,$
$x_2 \mapsto 1, x_3 \mapsto 0,$
$x_5 \mapsto 1, x_7 \mapsto 0 \}$

$\mathrm{Code}(R) \in (\{0,1\}^2 \times [k])^{\ell}$
given knowledge of R^* (and F), follow these instructions to recover R (and along the way R~)

k = 3, $\ell = 4$

$R \mapsto (R^*, \text{Code}(R))$

$$F = x_1 x_2 \neg x_3 \vee \neg x_1 x_3 x_5 \vee x_2 \neg x_4 x_5 \vee x_3 x_4 \neg x_6 \vee x_1 \neg x_4 \neg x_7$$

$R = \{ x_1 \mapsto 1, x_4 \mapsto 0 \}$

$R\sim = \{ x_1 \mapsto 1, x_4 \mapsto 0,$
$x_2 \mapsto 1, x_3 \mapsto 1,$
$x_5 \mapsto 0, x_7 \mapsto 1 \}$

$R^* = \{ x_1 \mapsto 1, x_4 \mapsto 0,$
$x_2 \mapsto 1, x_3 \mapsto 0,$
$x_5 \mapsto 1, x_7 \mapsto 0 \}$

$\underline{\text{Code}(R) \in (\{0,1\}^2 \times [k])^\ell}$
given knowledge of R^* (and F), follow these instructions to recover R (and along the way R~)

- ✓ R^* has ℓ fewer stars than R
- ✓ $R \mapsto (R^*, \text{Code}(R))$ is 1-to-1