

## ON THE $AC^0$ COMPLEXITY OF SUBGRAPH ISOMORPHISM\*

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**Abstract.** Let  $P$  be a fixed graph (hereafter called a “pattern”), and let  $\text{SUBGRAPH}(P)$  denote the problem of deciding whether a given graph  $G$  contains a subgraph isomorphic to  $P$ . We are interested in  $AC^0$ -complexity of this problem, determined by the smallest possible exponent  $C(P)$  for which  $\text{SUBGRAPH}(P)$  possesses bounded-depth circuits of size  $n^{C(P)+o(1)}$ . Motivated by the previous research in the area, we also consider its “colorful” version  $\text{SUBGRAPH}_{\text{col}}(P)$  in which the target graph  $G$  is  $V(P)$ -colored, and the average-case version  $\text{SUBGRAPH}_{\text{ave}}(P)$  under the distribution  $G(n, n^{-\theta(P)})$ , where  $\theta(P)$  is the threshold exponent of  $P$ . Defining  $C_{\text{col}}(P)$  and  $C_{\text{ave}}(P)$  analogously to  $C(P)$ , our main contributions can be summarized as follows: (1)  $C_{\text{col}}(P)$  coincides with the treewidth of the pattern  $P$  up to a logarithmic factor. This shows that the previously known upper bound by Alon, Yuster, and Zwick [*J. ACM*, 42 (1995), pp. 844–856] is almost tight. (2) We give a characterization of  $C_{\text{ave}}(P)$  in purely combinatorial terms up to a multiplicative factor of 2. This shows that the lower bound technique of Rossman [*Proceedings of the 40th ACM Symposium on Theory of Computing*, 2008, pp. 721–730] is essentially tight for any pattern  $P$  whatsoever. (3) We prove that if  $Q$  is a minor of  $P$ , then  $\text{SUBGRAPH}_{\text{col}}(Q)$  is reducible to  $\text{SUBGRAPH}_{\text{col}}(P)$  via a linear-size monotone projection. At the same time, we show that there is *no* monotone projection whatsoever that reduces  $\text{SUBGRAPH}(M_3)$  to  $\text{SUBGRAPH}(P_3 + M_2)$  ( $P_3$  is a path on three vertices,  $M_k$  is a matching with  $k$  edges, and “+” stands for the disjoint union). This result strongly suggests that the colorful version of the subgraph isomorphism problem is much better structured and well-behaved than the standard (worst-case, uncolored) one.

**Key words.** constant-depth circuits, subgraph isomorphism, threshold exponent, treewidth

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**1. Introduction.** The *subgraph isomorphism problem* takes as its input two graphs  $H$  and  $G$  and asks us to determine whether  $G$  contains a subgraph (not necessarily induced) isomorphic to  $H$ . This is one of the most basic NP-complete problems that includes  $\text{CLIQUE}$  and  $\text{HAMILTONIAN CYCLE}$  as special cases, and little more can be said about its complexity in full generality.

A significant body of research, motivated both by the framework of parameterized complexity and practical applications, has been devoted to the case when the graph  $H$  is fixed and possesses some useful structure (see, e.g., the sources [3, 10, 11, 21, 22, 24] related to the subject of our paper). To stress its nature in this situation, the graph  $H$  is traditionally called a *pattern* and designated by the letter  $P$ ; we also follow this convention and denote by  $\text{SUBGRAPH}(P)$  the corresponding restriction of the general subgraph isomorphism problem.

The sources above (among many others!) provide quite nontrivial improvements on the obvious time bound  $O(n^{|V(P)|})$  in many cases of interest. But for *unconditional*

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lower bounds we, given our current state of knowledge, have to resort to restricted models, and, indeed, a substantial amount of work has been done here in the context of both bounded-depth circuits and monotone circuits. In this paper we focus on the former model.

As for upper bounds, it was observed by Amano [4] that the color-coding algorithm by Alon, Yuster, and Zwick [3] can be adapted to our context and gives  $AC^0$  circuits for  $SUBGRAPH(P)$  of size<sup>1</sup>  $\tilde{O}(n^{tw(P)+1})$ , where  $tw(P)$  is the treewidth of the pattern  $P$ . Our paper is motivated by the following natural question: *How tight is this bound?* Or, in other words, we ask the following.

**Question 1.** Is it possible to give good general lower bounds on the  $AC^0$  complexity of  $SUBGRAPH(P)$  in terms of the treewidth of  $P$  only?

Prior to our work, Rossman [27] answered this question in the affirmative for the case of a  $k$ -clique by proving a lower bound of  $\Omega(n^{k/4})$  on the  $AC^0$  complexity of  $SUBGRAPH(K_k)$ . Generalizing Rossman's method, Amano [4] gave a general lower bound that holds for arbitrary patterns  $P$ . It in particular implied an  $n^{\Omega(k)}$  lower bound (and, thus, an affirmative answer to Question 1) for the  $k \times k$  grid  $G_{k,k}$ : this result is very interesting since  $G_{k,k}$  is the "canonical" example of a sparse graph with large treewidth.

Before discussing our results, it will be convenient to introduce the following handy notation: given a pattern  $P$ , we let  $C(P)$  be the minimal real number  $c \geq 0$  for which  $SUBGRAPH(P)$  is solvable on  $n$ -vertex graphs by  $AC^0$  circuits of size  $n^{c+o(1)}$ . In this notation, the previous results mentioned above can be stated as  $C(P) \leq tw(P) + 1$  ([3, 4],  $P$  any pattern),  $C(K_k) \geq k/4$  [27], and  $C(G_{k,k}) \geq \Omega(k)$  [4].

**Our contributions.** We explicitly formulate and study two modifications that already played a great role in the previous research. The first of them is the *colorful  $P$ -subgraph isomorphism problem*  $SUBGRAPH_{col}(P)$  in which the target graph  $G$  comes with a coloring  $\chi : V(G) \rightarrow V(P)$  (that without loss of generality (w.l.o.g.) can and will be assumed to be a graph homomorphism), and we are looking only for properly colored  $P$ -subgraphs. Let  $C_{col}(P)$  be defined analogously to  $C(P)$ . Then the very first thing done by the algorithm of Alon, Yuster, and Zwick is a simple reduction from  $SUBGRAPH(P)$  to  $SUBGRAPH_{col}(P)$ , thus establishing  $C(P) \leq C_{col}(P)$ . After that they work exclusively with the colorful version that leads to

$$C(P) \leq C_{col}(P) \leq tw(P) + 1.$$

We settle in the affirmative (up to a logarithmic factor) our motivating Question 1 for the colorful version by proving the following theorem.

**THEOREM 1.1.**  $C_{col}(P) \geq \Omega(tw(P)/\log tw(P))$ .

By previous work of Marx [21], it was known that  $SUBGRAPH_{col}(P)$  has no  $n^{o(tw(P)/\log tw(P))}$  time algorithm unless the exponential time hypothesis (ETH) fails. Theorem 1.1 establishes the same lower bound *unconditionally* for  $AC^0$  circuits. (We say more about Marx's result and related work of Alon and Marx [1] in section 6.)

We show that the colorful version is quite well-behaved by proving that it is minor-monotone: if  $Q$  is a minor of  $P$ , then  $C_{col}(Q) \leq C_{col}(P)$  (Theorem 5.1).<sup>2</sup> Whether

<sup>1</sup>" $\tilde{O}$ " is the "soft" version of the "big- $O$ " notation that ignores not only constant but polylogarithmic multiplicative factors as well.

<sup>2</sup>It is worth observing that this fact, along with the recent result [8] by Chekura and Chuzhoy and Amano's bound  $C_{col}(G_{k,k}) \geq \Omega(k)$  [4], already implies the weaker bound  $C_{col}(P) \geq tw(P)^{\Omega(1)}$ . But the exponent given by this approach will be disappointingly small.

a similar result holds for  $C(P)$  is open, but we give strong evidence (Theorem 5.6) that even if this is true, the proof will most likely require totally different techniques. One possible interpretation is that perhaps the colorful version is in fact a cleaner and more natural model to study than the standard (uncolored) version. We also observe that if the pattern  $P$  is a core (i.e., every homomorphism from  $P$  to  $P$  is an automorphism), then  $C(P) = C_{\text{col}}(P)$  and thus our lower bound from Theorem 1.1 transfers to the uncolored case. What happens to  $C(P)$  at the opposite side of the spectrum, say, for bipartite patterns  $P$ , remains wide open.

All lower bounds surveyed above, including our proof of Theorem 1.1, were actually achieved in the context of average-case complexity. Prior to our work, the only distribution that was considered for this purpose is the *Erdős-Rényi* model  $G(n, n^{-\theta(P)})$ , where  $\theta(P)$  is the uniquely defined *threshold exponent* for which the probability of containing a copy of  $P$  is bounded away from 0 and 1 (see [19] or section 2.4 below). Accordingly, we define  $C_{\text{ave}}(P)$  analogously to  $C(P)$ , but only require that our circuit outputs the correct answer asymptotically almost surely (a.a.s.) when the input is drawn from  $G(n, n^{-\theta(P)})$ . Clearly,  $C_{\text{ave}}(P) \leq C(P)$  so the whole picture now looks like

$$C_{\text{ave}}(P) \leq C(P) \leq C_{\text{col}}(P) \approx tw(P),$$

where  $\approx$  means approximation within a logarithmic factor. Also,  $C_{\text{ave}}(K_k) \geq k/4$  [27] and  $C_{\text{ave}}(G_{k,k}) \geq \Omega(k)$  [4], where  $K_k$  is the complete graph on  $k$  vertices and  $G_{k,k}$  is the  $k$ -by- $k$  grid.

We explicitly define a combinatorial parameter  $\kappa(P)$  and prove the following.

**THEOREM 1.2.**  $\kappa(P) \leq C_{\text{ave}}(P) \leq 2\kappa(P) + O(1)$ .

In other words, we give lower and upper bounds on the average-case  $\text{AC}^0$  complexity for an arbitrary pattern  $P$ , matching within a quadratic factor. The proof of Theorem 1.2 exploits a duality in the definition of  $\kappa(P)$ , which has equivalent min-max and max-min formulations (the former suited to upper bounds and the latter to lower bounds). The lower bound  $C_{\text{ave}}(P) \geq \kappa(P)$  generalizes the proof of  $C_{\text{ave}}(K_k) \geq k/4$  in Rossman [27] and improves a previous lower bound of Amano [4] for general patterns  $P$ . (A detailed comparison with previous work is given in section 2.6 following the definition of  $\kappa(P)$ .)

Let us say a few words about the proof of Theorem 1.1. Being itself a worst-case lower bound, it is obtained as the maximum of a family of average-case lower bounds with respect to  $P$ -colored random graphs. These random graphs generalize Erdős-Rényi random graphs in the  $P$ -colored setting by allowing different edge probabilities according to the color classes of vertices, and we believe that this generalization may be of independent interest. Each  $P$ -colored random graph in this family is parameterized by a point in a certain convex polytope, denoted  $\theta_{\text{col}}(P)$ . We rely on results of [12, 21] that characterize the treewidth of  $P$  in terms of the existence of an appropriate concurrent flow on  $P$ , which we convert to a suitable point in  $\theta_{\text{col}}(P)$ .

Finally, it is also worth noting that lower bounds in Theorems 1.1 and 1.2 (and hence all our structural conclusions) hold even if we allow circuits of a superconstant depth  $d(n)$ , as long as  $d(n) \leq o(\log n / \log \log n)$ .

The paper is organized as follows. In section 2 we give the necessary definitions and preliminaries; in particular, in section 2.5 we present the parameters  $\kappa(P)$  and  $\kappa_{\text{col}}(P)$  that are our main technical tools in this paper. Section 3 is devoted to the proof of Theorem 1.2, and it also paves the way to the proof of Theorem 1.1 that, up to a certain point, goes in parallel to the former. The proof of Theorem 1.1 is

completed in section 4. Section 5 contains our structural results about the behavior of SUBGRAPH( $P$ ) and SUBGRAPH<sub>col</sub>( $P$ ) with respect to minors and subgraphs. The paper is concluded with a brief discussion and a list of open problems in section 6.

**2. Definitions and preliminaries.** Let  $[k] := \{1, \dots, k\}$ , and let  $\binom{X}{2}$  be the family of all two-element subsets of  $X$ .

**2.1. Graphs.** We start off with terminology and notation for graphs. Throughout this paper, *graphs* are finite simple graphs  $G = (V(G), E(G))$ , where  $E(G)$  is a subset of  $\binom{V(G)}{2}$ . We often write  $v(G)$  for  $|V(G)|$  and  $e(G)$  for  $|E(G)|$ .

A graph  $H$  is a *subgraph* of  $G$ , denoted  $H \subseteq G$ , if  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ . For arbitrary  $G$  and  $H$ ,  $G+H$  and  $G \times H$  respectively denote the *disjoint union* and *Cartesian product* of graphs  $G$  and  $H$  (where  $E(G \times H) := \{\{(v, v'), (w, w')\} : \{v, w\} \in E(G) \text{ and } \{v', w'\} \in E(H)\}$ ).

A *homomorphism* from  $G$  to  $H$  is a function  $\varphi : V(G) \rightarrow V(H)$  such that  $\{\varphi(v), \varphi(w)\} \in E(H)$  for all  $\{v, w\} \in E(G)$ . A graph  $G$  is a *core* if every homomorphism from  $G$  to  $G$  is an automorphism.

The *treewidth* of  $G$  is denoted by  $tw(G)$  (for the definition and background, see, e.g., [6]). Relevant facts about treewidth will be stated where needed.

$K_k$  is a *clique* on  $k$  vertices, and  $G_{k,k}$  is a  $k \times k$  *grid*. These graphs have treewidth  $tw(K_k) = k - 1$  and  $tw(G_{k,k}) = k$ .

**2.2. Monotone projections.**

DEFINITION 2.1. *Let  $I, J$  be arbitrary sets.*

- (i) *For a function  $p : J \rightarrow I \cup \{0, 1\}$  and  $x \in \{0, 1\}^I$ , we write  $p^*(x)$  for the unique  $y \in \{0, 1\}^J$  such that  $y_j = x_{p(j)}$  if  $p(j) \in I$  and  $y_j = p(j)$  if  $p(j) \in \{0, 1\}$ .*
- (ii) *For boolean functions  $f : \{0, 1\}^I \rightarrow \{0, 1\}$  and  $g : \{0, 1\}^J \rightarrow \{0, 1\}$ , we say that  $f$  is reducible via a monotone projection to  $g$ , denoted  $f \leq_{\text{mp}} g$ , if there exists  $p : J \rightarrow I \cup \{0, 1\}$  such that  $f(x) = g(p^*(x))$  for all  $x \in \{0, 1\}^I$ . (Note that  $\leq_{\text{mp}}$  is transitive.)*

Any decision problem  $L$  can be represented as a sequence of Boolean functions  $\{L^n\}$  in  $n$  variables. We say that  $L_1$  is *reducible via a monotone projection* to another decision problem  $L_2$  if for any  $n$  there exists<sup>3</sup>  $m(n)$  such that  $L_1^n \leq_{\text{mp}} L_2^{m(n)}$ . If in addition  $m(n) \leq O(n)$ , we call this projection *linear*.

**2.3. Subgraph isomorphism problems.** Throughout this paper, the letters  $P, Q$  represent arbitrary fixed graphs that should be intuitively thought of as “patterns.”  $G$  stands for a (large) “input” graph for the  $P$ -subgraph isomorphism problem. Subgraphs of  $G$  (not necessarily induced) which are isomorphic to  $P$  will be called  *$P$ -subgraphs*.

We also consider  *$P$ -colored graphs*, defined as pairs  $(G, \chi)$ , where  $G$  is a graph and  $\chi : V(G) \rightarrow V(P)$  is a homomorphism. We usually suppress  $\chi$  and simply refer to  $G$  as  *$P$ -colored graph*. In this setting, given a subpattern  $Q \subseteq P$  (not necessarily induced), a  *$Q$ -subgraph* of  $G$  is a subgraph of  $G$  (again, not necessarily induced) that is isomorphic to  $Q$  and consistent with  $\chi$  in the sense that every vertex  $v \in V(Q)$  is mapped to a vertex in  $\chi^{-1}(v)$ .

We consider two versions (“uncolored” and “colored”) of the  $P$ -subgraph isomorphism problem:

<sup>3</sup>Uniformity issues do not play any role in this paper.

- $\text{SUBGRAPH}(P)$  is the problem, given a graph  $G$ , of determining whether  $G$  contains a  $P$ -subgraph.
- $\text{SUBGRAPH}_{\text{col}}(P)$  is the problem, given a  $P$ -colored graph  $(G, \chi)$ , of determining whether  $G$  contains a (properly colored)  $P$ -subgraph.

This problem is also known in the literature as the “partitioned” or “colorful” variant (see, e.g., [3, 1]), and in this paper we mostly adopt the latter term.

It will be convenient to introduce notation for the  $\text{AC}^0$  complexity of these problems. (Recall that  $\text{AC}^0$  is the class of problems solvable by polynomial-size constant-depth boolean circuits over  $\{\neg, \wedge, \vee\}$  with unbounded fan-in.)

**DEFINITION 2.2.** *Let  $C(P)$  (resp.,  $C_{\text{col}}(P)$ ) denote the infimum of all real numbers  $c > 0$  such that  $\text{SUBGRAPH}(P)$  (resp.,  $\text{SUBGRAPH}_{\text{col}}(P)$ ) is solvable (in the worst case) on  $n$ -vertex graphs by  $\text{AC}^0$  circuits of size<sup>4</sup>  $O(n^c)$ .*

Note that if  $\text{SUBGRAPH}(P)$  is reducible to  $\text{SUBGRAPH}(Q)$  via a linear monotone projection, then  $C(P) \leq C(Q)$ , and this remains true if we add the subscript  $\text{col}$  to both sides.

**LEMMA 2.3.**

1.  $C(P) \leq C_{\text{col}}(P) \leq tw(P) + 1$ .
2. If  $P$  is a core, then  $C(P) = C_{\text{col}}(P)$ .

*Proof.*

1. The second inequality  $C_{\text{col}}(P) \leq tw(P) + 1$  is by the color-coding algorithm of Alon, Yuster, and Zwick [3] (adapted to the  $P$ -colored setting), which can be implemented in  $\text{AC}^0$  as observed by Amano [4]. The first inequality  $C(P) \leq C_{\text{col}}(P)$  is also implicitly proved there by reducing  $\text{SUBGRAPH}(P)$  to  $\text{SUBGRAPH}_{\text{col}}(P)$ : the reduction searches through logarithmically many different colorings  $\chi_1, \chi_2, \dots : V(G) \rightarrow V(P)$  of the same target graph  $G$ , picked at random. An easy counting argument shows that a.a.s. every  $P$ -subgraph of  $G$  will be properly colored with respect to at least one of the colorings  $\chi_i$ .
2. This observation goes back at least to Grohe [13]. If  $P$  is a core, then  $(G, \chi) \mapsto G$  is a reduction from  $\text{SUBGRAPH}_{\text{col}}(P)$  to  $\text{SUBGRAPH}(P)$ . To see why, it suffices to show that every  $P$ -subgraph of  $G$  is properly colored with respect to every homomorphism  $\chi : G \rightarrow P$ . Suppose  $H$  is a  $P$ -subgraph of  $G$ . Then  $H = \varphi(P)$  for some one-to-one homomorphism  $\varphi : P \rightarrow G$ . Since  $P$  is a core, the homomorphism  $\chi \circ \varphi : P \rightarrow P$  is an automorphism of  $P$ . It follows that the homomorphism  $\chi|_{V(H)} : H \rightarrow P$  is one-to-one. Since  $|E(H)| = |E(P)|$ , it must be an isomorphism, that is,  $H$  is properly colored with respect to  $\chi$ .  $\square$

**2.4. The average case.** We now define the random graphs which appear in our average-case lower bounds for  $\text{SUBGRAPH}(P)$  and  $\text{SUBGRAPH}_{\text{col}}(P)$ . In the uncolored setting, we consider the Erdős–Rényi random graph  $G(n, p(n))$  for an appropriately chosen threshold function  $p(n)$ . Also, in what follows we assume that  $P$  is nonempty, that is, contains at least one edge.

**DEFINITION 2.4.**

- (i) The threshold exponent of  $P$  is defined by  $\theta(P) := \min_{\substack{Q \subseteq P \\ Q \neq \emptyset}} v(Q)/e(Q)$ .
- (ii)  $P$  is balanced if  $v(P)/e(P) = \theta(P)$ .

<sup>4</sup>In this paper, the size of all constant-depth circuits is measured by the number of *gates*.

- (iii)  $P$  is strictly balanced if  $v(Q)/e(Q) > \theta(P)$  for every nonempty proper subgraph  $Q \subset P$ .
- (iv) Let  $\text{Bal}(P) := \bigcup\{Q \subseteq P : v(Q)/e(Q) = \theta(P)\}$ .

LEMMA 2.5.

1.  $P$  is balanced iff  $P = \text{Bal}(P)$ .
2. For every  $P$ ,  $\text{Bal}(P)$  is balanced and  $\theta(\text{Bal}(P)) = \theta(P)$ .

*Proof.* It suffices to show that  $\mathcal{B} := \{Q \subseteq P : Q \neq \emptyset \wedge v(Q)/e(Q) = \theta(P)\}$  is closed under unions (in fact, it is closed under intersections as well). For all  $Q_1, Q_2 \in \mathcal{B}$ , we have

$$\begin{aligned}
 (1) \quad v(Q_1 \cup Q_2) + v(Q_1 \cap Q_2) &= v(Q_1) + v(Q_2) \\
 &= \theta(P)e(Q_1) + \theta(P)e(Q_2) \\
 &= \theta(P)e(Q_1 \cup Q_2) + \theta(P)e(Q_1 \cap Q_2).
 \end{aligned}$$

By definition of  $\theta(P)$ ,

$$(2) \quad v(Q_1 \cup Q_2) \geq \theta(P)e(Q_1 \cup Q_2) \quad \text{and} \quad v(Q_1 \cap Q_2) \geq \theta(P)e(Q_1 \cap Q_2).$$

Together (1) and (2) imply that equality holds in (2), that is,  $Q_1 \cup Q_2$  and  $Q_1 \cap Q_2$  are both in  $\mathcal{B}$ . □

Recall that  $G(n, p)$  is the Erdős–Rényi random graph with vertex set  $[n]$ , in which each  $e \in \binom{[n]}{2}$  occurs as an edge independently with probability  $p$ . The next lemma states that  $p = n^{-\theta(P)}$  is a threshold function for  $\text{SUBGRAPH}(P)$  and that detecting  $P$ -subgraphs on  $G(n, n^{-\theta(P)})$  is equivalent to detecting  $\text{Bal}(P)$ -subgraphs. (Lemma 2.6.1 is a standard fact about random graphs (see [19]); Lemma 2.6.2 was proved in [7].)

LEMMA 2.6.

1.  $\Pr[G(n, n^{-\theta(P)}) \text{ has a } P\text{-subgraph}]$  is bounded away from 0 and 1.
2. Asymptotically almost surely, if  $G(n, n^{-\theta(P)})$  contains a  $\text{Bal}(P)$ -subgraph, then it contains a  $P$ -subgraph.

With slight abuse of notation, we denote by  $\text{SUBGRAPH}_{\text{ave}}(P)$  the algorithmic problem of solving  $\text{SUBGRAPH}(P)$  on  $G(n, n^{-\theta(P)})$  correctly a.a.s., that is, with probability that tends to 1 as  $n$  tends to  $\infty$ . (We remark that our results are unchanged if  $n^{-\theta(P)}$  is replaced by any other threshold function  $p(n) \in \Theta(n^{-\theta(P)})$ .) Similarly to Definition 2.2, let  $C_{\text{ave}}(P)$  be the smallest  $c > 0$  for which this problem can be solved by  $AC^0$ -circuits of size  $n^{c+o(1)}$ .

*Remark 2.7.* Obviously,  $C_{\text{ave}}(P) \leq C(P)$ , but the gap between them can be arbitrarily large. Assume, e.g., that  $P = K_4 + G_{k,k}$  where  $k \rightarrow \infty$ . Then  $\text{Bal}(P) = K_4$  and thus Lemma 2.6.2 implies that  $C_{\text{ave}}(P) = C_{\text{ave}}(K_4) \leq 4$ . On the other hand,  $\text{SUBGRAPH}(G_{k,k})$  is reduced to  $\text{SUBGRAPH}(P)$  via an obvious linear monotone projection that takes  $G$  to  $K_4 + G$ . This proves  $C(P) \geq C(G_{k,k}) \geq \Omega(k)$  by the result from [4].

One might argue that this example is not “fair” since it heavily exploits the fact that the pattern  $P$  is highly unbalanced. It is, however, possible to give nearly the same separation (albeit, more complicated) with a strictly balanced pattern  $P$ . Say, let  $d > 0$  be a sufficiently large constant, and  $V(P) = [k]$ , where  $k \gg d$ . We start building  $E(P)$  with the clique on the set  $[d]$  and then for every  $i \in \{d + 1, \dots, k\}$  pick at random  $d$  different vertices  $j_1, \dots, j_d < i$  and add all  $d$  edges  $\{j_\nu, i\}$ . Then

$P$  will be strictly balanced since every subgraph with  $v \geq d$  vertices has at most  $\frac{d(d-1)}{2} + d(v-d) = \frac{d(2v-d-1)}{d}$  edges. Taking as a union sequence (see Definition 2.11 below) the natural sequence according to the order in which  $P$  was built, we conclude (see Definition 2.12) that  $\kappa(P) \leq O(d)$ . Hence  $C_{\text{ave}}(P) \leq O(d)$  by Theorem 1.2. On the other hand, randomness in selecting the edges implies that  $tw(P) \geq \Omega(k)$  and that  $P$  is a core. From the latter fact we conclude that  $C(P) = C_{\text{col}}(P)$ , and from the former, by Theorem 1.1, that  $C_{\text{col}}(P) \geq \Omega(k/\log k)$ .

We now move on to the notion of average-case complexity for  $\text{SUBGRAPH}_{\text{col}}(P)$ . In contrast to the uncolored setting, there is no single most natural distribution on  $P$ -colored random graphs. Instead, we consider a family of  $P$ -colored random graphs, denoted  $G_{\alpha,\beta}(n)$ , which are parameterized by certain pairs of functions  $\alpha : V(P) \rightarrow [0, 1]$  and  $\beta : E(P) \rightarrow [0, 2]$  called “threshold pairs.” (Note that unlike  $G(n, p)$ , the vertex set of  $G_{\alpha,\beta}(n)$  is not  $[n]$ , but rather consists of  $|V(P)|$  disjoint parts of different sizes.)

DEFINITION 2.8 (P-colored random graph  $G_{\alpha,\beta}(n)$ ).

- (i) A threshold pair for  $P$  is a pair  $(\alpha, \beta)$  of functions  $\alpha : V(P) \rightarrow [0, 1]$  and<sup>5</sup>  $\beta : E(P) \rightarrow [0, 2]$  such that
  - $\alpha(P) = \beta(P)$ ,
  - $\alpha(Q) \geq \beta(Q)$  for all  $Q \subseteq P$ ,
 where  $\alpha(Q) := \sum_{v \in V(Q)} \alpha(v)$  and  $\beta(Q) := \sum_{e \in E(Q)} \beta(e)$ .
- (ii)  $\theta_{\text{col}}(P)$  denotes the set of threshold pairs for  $P$ . Note that  $\theta_{\text{col}}(P)$  is a polytope in  $\mathbb{R}^{V(P) \cup E(P)}$  and its section  $\{\beta : (1, \beta) \in \theta_{\text{col}}(P)\}$  is a polytope in  $\mathbb{R}^{E(P)}$ . We view elements of  $\theta_{\text{col}}(P)$  as the “ $P$ -colored” analogue of  $\theta(P)$  (see Remark 2.9 below).
- (iii) We say that  $(\alpha, \beta) \in \theta_{\text{col}}(P)$  is nontrivial if  $\alpha$  and  $\beta$  are not identically zero.
- (iv) We say that  $(\alpha, \beta) \in \theta_{\text{col}}(P)$  is strictly balanced if  $\alpha(Q) > \beta(Q)$  for every nonempty proper subgraph  $Q \subset P$ .
- (v) For all  $(\alpha, \beta) \in \theta_{\text{col}}(P)$ , let  $G_{\alpha,\beta}(n)$  denote the random graph with vertex set  $\{(v, i) : v \in V(P), 1 \leq i \leq \lfloor n^{\alpha(v)} \rfloor\}$ , where each  $\{(v, i), (w, j)\}$  with  $\{v, w\} \in E(P)$  is an edge, independently, with probability  $n^{-\beta(\{v, w\})}$ . The  $P$ -coloring of  $G_{\alpha,\beta}(n)$  is the obvious one:  $(v, i) \mapsto v$ .

Remark 2.9. Note that if  $P$  is a balanced pattern, then the pair of constant functions  $(\alpha \equiv 1, \beta \equiv \theta(P))$  is a threshold pair for  $P$ ; moreover,  $P$  is strictly balanced iff this  $(\alpha, \beta)$  is strictly balanced. Thus, Definition 2.8 is indeed a generalization of threshold exponent for balanced patterns. The following lemma makes the analogy even more clear, justifies the terminology “threshold pair,” and refines Lemma 2.6.1.

- LEMMA 2.10. For every pattern  $P$  and nontrivial threshold pair  $(\alpha, \beta) \in \theta_{\text{col}}(P)$ ,
1.  $\liminf_{n \rightarrow \infty} [G_{\alpha,\beta}(n) \text{ contains no } P\text{-subgraph}] \geq \frac{1}{e}$ ,
  2.  $\liminf_{n \rightarrow \infty} [G_{\alpha,\beta}(n) \text{ contains exactly one } P\text{-subgraph}] \geq \frac{1}{e^{|E(P)|}}$ .

The proof is included in Appendix A. With a bit of work, it is possible to completely characterize the asymptotic distribution of the number of  $P$ -subgraphs in  $G_{\alpha,\beta}(n)$ ; this distribution is a function of independent Poisson random variables (in the uncolored setting; see [7] for a characterization of the asymptotic number of  $P$ -subgraphs in  $G(n, n^{-\theta(P)})$ ).

In the context of  $\text{SUBGRAPH}_{\text{col}}(P)$ , we speak of the *average-case complexity with*

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<sup>5</sup> $n^\alpha$  and  $n^{-\beta}$  will determine the number of vertices in the colored parts and edge densities between them, respectively, whence comes our choice of normalization.

respect to  $G_{\alpha,\beta}(P)$ , meaning the size of an AC<sup>0</sup> circuit which solves SUBGRAPH<sub>col</sub>( $P$ ) on  $G_{\alpha,\beta}(P)$  with probability that tends to 1 as  $n$  tends to  $\infty$ . We do not introduce any special notation like  $C_{\alpha,\beta}(P)$  as this concept is intended to be auxiliary.

**2.5. Parameters  $\kappa(P)$  and  $\kappa_{\text{col}}(P)$ .** We now introduce the parameters  $\kappa(P)$  and  $\kappa_{\text{col}}(P)$  which figure in our lower bounds. The definitions, which might appear unmotivated at first glance, are derived from the lower bound technique of [27], which we explain in the next section.

DEFINITION 2.11 (union sequences and hitting sets). *A union sequence for  $P$  is a sequence  $Q_1, \dots, Q_t$  of subgraphs of  $P$  such that  $Q_t = P$  and for all  $1 \leq k \leq t$ , either  $Q_k$  is a single vertex or a single edge or  $Q_k = Q_i \cup Q_j$  for some  $1 \leq i < j < k$ . A hitting set for union sequences (or hitting set for short) is a set  $\mathcal{H}$  of subgraphs of  $P$  such that  $\mathcal{H}$  contains at least one element from every union sequence.*

DEFINITION 2.12 (parameters  $\kappa(P)$ ,  $\kappa_{\alpha,\beta}(P)$  and  $\kappa_{\text{col}}(P)$ ).

(i) *If  $P$  is balanced, then  $\kappa(P)$  is defined by*

$$\kappa(P) := \min_{\text{union seq. } Q_1, \dots, Q_t} \max_{i \in [t]} v(Q_i) - \theta(P)e(Q_i).$$

*For  $P$  which is not balanced, we define  $\kappa(P) := \kappa(\text{Bal}(P))$ .*

(ii) *For  $(\alpha, \beta) \in \theta_{\text{col}}(P)$ , let*

$$\kappa_{\alpha,\beta}(P) := \min_{\text{union seq. } Q_1, \dots, Q_t} \max_{i \in [t]} \alpha(Q_i) - \beta(Q_i).$$

(iii) *Let  $\kappa_{\text{col}}(P) := \max_{(\alpha,\beta) \in \theta_{\text{col}}(P)} \kappa_{\alpha,\beta}(P)$  (the maximum exists since  $\kappa_{\alpha,\beta}(P)$ , viewed as a function of  $\alpha, \beta$  for a fixed  $P$ , is continuous).*

Remark 2.13. Later on we will see that in this definition we could restrict ourselves to threshold pairs with  $\alpha \equiv 1$  (Corollary 4.2). But since arbitrary threshold pairs appear quite naturally in our lower bound proofs in section 4.2, we prefer to give this more general definition at once.

The next lemma is key to linking our upper and lower bounds on the average-case AC<sup>0</sup> complexity of SUBGRAPH( $P$ ).

LEMMA 2.14 (minimax principle for  $\kappa(P)$  and  $\kappa_{\alpha,\beta}(P)$ ).

1. *If  $P$  is balanced, then*

$$\kappa(P) = \max_{\mathcal{H}} \min_{Q \in \mathcal{H}} v(Q) - \theta(P)e(Q),$$

*where  $\mathcal{H}$  ranges over hitting sets for  $P$ .*

2. *Similarly,  $\kappa_{\alpha,\beta}(P) = \max_{\mathcal{H}} \min_{Q \in \mathcal{H}} \alpha(Q) - \beta(Q)$  for all  $(\alpha, \beta) \in \theta_{\text{col}}(P)$ .*

*Proof.* The argument is the same for 1 and 2. Let  $f(Q) := v(Q) - \theta(P)e(Q)$  (the proof works for any real-valued objective function). First, we will prove that  $\max_{\mathcal{H}} \min_{Q \in \mathcal{H}} f(Q) \leq \kappa(P)$ . Since  $\mathcal{H}$  is a hitting set, for any union sequence  $\{Q_i\}$ , there exists some  $Q_i \in \mathcal{H}$ . It follows that  $\min_{Q \in \mathcal{H}} f(Q) \leq \max_i f(Q_i)$ , and thus  $\min_{Q \in \mathcal{H}} f(Q) \leq \kappa(P)$  as  $\{Q_i\}$  is taken arbitrarily.

On the other hand, let us prove  $\kappa(P) \leq \max_{\mathcal{H}} \min_{Q \in \mathcal{H}} f(Q)$ . Enumerate all union sequences  $\{Q_i^{(j)}\}$ ,  $j = 1, 2, \dots$  (each  $\{Q_i^{(j)}\}$  is a finite sequence). For each  $j$ , take a subgraph  $S^{(j)}$  in  $\{Q_i^{(j)}\}$  with maximal  $f(Q_i^{(j)})$ . Let  $\mathcal{S} = \{S^{(1)}, S^{(2)}, \dots\}$ . It is easily seen that  $\mathcal{S}$  is a hitting set, as every union sequence has some element in it. By

definition,

$$\max_{\mathcal{H}} \min_{Q \in \mathcal{H}} f(Q) \geq \min_{S^{(j)} \in \mathcal{S}} f(S^{(j)}) = \min_j \max_i f(Q_i^{(j)}) = \kappa(P),$$

which completes the proof. □

**2.6. Comparison with previous work.** The dual (max-min) expression for  $\kappa(P)$  given by Lemma 2.14.1 is naturally suited to *lower bounds*. It is this dual version of  $\kappa(P)$  which we use to prove  $C_{\text{ave}}(P) \geq \kappa(P)$  in the next section. This dual expression—which maximizes over hitting sets for a pattern  $P$ —generalizes Rossman’s proof of  $C_{\text{ave}}(K_k) \geq k/4$  in [27], which considers a specific hitting set for  $K_k$ .<sup>6</sup>

Previous work of Amano [4] also generalizes the technique of [27] to obtain a lower bound  $C_{\text{ave}}(P) \geq \ell(P)$  for general patterns  $P$ . The function  $\ell(P)$  defined by Amano (which is denoted  $Z_P^*$  in [4]) is similar to the dual expression for  $\kappa(P)$ , except it restricts attention to hitting sets of a particular form:

$$\ell(P) := \max_{s: 2 \leq s \leq v(P)} \min_{Q \subseteq P: s/2 < v(Q) \leq s} v(Q) - \theta(P)e(Q).$$

Clearly,  $\ell(P) \leq \kappa(P)$  for all patterns  $P$ . In some cases of interest, such as grid  $G_{k,k}$ , Amano shows that  $\ell(G_{k,k}) = \Omega(k)$ . However,  $\ell(P)$  is slack in general (for example,  $\ell(K_k) = 2k/9 + O(1)$  while  $\kappa(K_k) = k/4 + O(1)$ ). A key insight of the present paper is that the stronger parameter  $\kappa(P)$ —in its primal form, which minimizes over union sequences—leads to *upper bounds* on  $C_{\text{ave}}(P)$  which are tight within a multiplicative constant.

Another result of Amano [4] is a construction of nearly optimal  $\text{AC}^0$  circuits for the average-case  $k$ -clique problem, which match the lower bound of [27] by showing  $C_{\text{ave}}(K_k) \leq k/4 + O(1)$ . Nakagawa and Watanabe [23] observed that Amano’s construction generalizes to an upper bound  $C_{\text{ave}}(P) \leq u(P) + O(1)$ , where  $u(P)$  is defined by

$$u(P) := \min_{\text{linear orderings } v_1 < \dots < v_k \text{ of } V(P)} \max_{j \in [k]} j - \theta(P)e(\{v_1, \dots, v_j\})$$

and  $e(\{v_1, \dots, v_i\})$  is the number of edges in  $P$  among vertices  $v_1, \dots, v_i$ . This parameter  $u(P)$  is similar to the (primal) definition of  $\kappa(P)$ , except that  $u(P)$  is restricted to union sequences  $Q_1, \dots, Q_t$  where  $|V(Q_{i+1}) \setminus V(Q_i)| \leq 1$ . Thus,  $u(P) \geq \kappa(P)$ . However, in contrast to  $\kappa(P)$ , Nakagawa and Watanabe showed that  $u(P)$  is not bounded by any function of  $C_{\text{ave}}(P)$ : there is a sequence of patterns  $P_1, P_2, \dots$  with  $C_{\text{ave}}(P_i) = O(1)$  while  $\lim_i u(P_i) = \infty$ .

In summary, our bounds  $\kappa(P) \leq C_{\text{ave}}(P) \leq 2\kappa(P) + O(1)$  (Theorem 1.2) both achieve a tighter generalization of [27] and close the (arbitrarily large) gap between the previous bounds  $\ell(P) \leq C_{\text{ave}}(P) \leq u(P) + O(1)$  of [4, 23]. Our results on  $C_{\text{col}}(P)$ , including the definitions of  $\theta_{\text{col}}(P)$  and  $\kappa_{\text{col}}(P)$ , are completely new to this paper (the colored setting was not considered in [4, 23, 27]).

**3. Average-case  $\text{AC}^0$  complexity.** In this section, we prove Theorem 1.2 ( $\kappa(P) \leq C_{\text{ave}}(P) \leq 2\kappa(P) + O(1)$ ), which gives a combinatorial characterization of the  $\text{AC}^0$ -complexity of  $\text{SUBGRAPH}_{\text{ave}}(P)$  up to a quadratic factor. More generally, we prove a family of average-case lower and upper bounds for the average-case colorful  $P$ -subgraph isomorphism problem:

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<sup>6</sup>This hitting set consists of the “medium” subgraphs of  $K_k$ , defined as subgraphs  $Q$  such that (i)  $Q$  has  $> k/2$  vertices and (ii)  $Q$  is a union of two subgraphs with  $\leq k/2$  vertices.

**THEOREM 3.1.** *For every pattern  $P$  and  $(\alpha, \beta) \in \theta_{\text{col}}(P)$ , the average-case AC<sup>0</sup>-complexity of  $\text{SUBGRAPH}_{\text{col}}(P)$  on the  $P$ -colored random graph  $G_{\alpha, \beta}(n)$  is between  $n^{\kappa_{\alpha, \beta}(P) - o(1)}$  and  $n^{2\kappa_{\alpha, \beta}(P) + O(1)}$ .*

Rather than proving Theorems 1.2 and 3.1 separately, to avoid redundancy we present a proof of the latter only. For balanced  $P$  the proof of Theorem 1.2 looks exactly like the proof of Theorem 3.1 in the special case where  $\alpha \equiv 1$  and  $\beta \equiv \theta(P)$  (see Remark 2.9). The general case is reduced to the balanced one since for an arbitrary pattern  $P$  we have  $\kappa(P) = \kappa(\text{Bal}(P))$  (by definition of  $\kappa(P)$ ) and  $C_{\text{ave}}(P) = C_{\text{ave}}(\text{Bal}(P))$  (by Lemma 2.6.2).

Theorem 3.1 also plays a key role in our other main result, Theorem 1.1 (the worst-case lower bound  $C_{\text{col}}(P) \geq \Omega(\text{tw}(P)/\log \text{tw}(P))$ ). Since the worst-case AC<sup>0</sup>-complexity of  $\text{SUBGRAPH}_{\text{col}}(P)$  is lower-bounded by the average-case AC<sup>0</sup>-complexity of  $\text{SUBGRAPH}_{\text{col}}(P)$  on  $G_{\alpha, \beta}(n)$  for every  $(\alpha, \beta) \in \theta_{\text{col}}(P)$  (and  $G_{\alpha, \beta}(n)$  is supported on graphs with  $n^{1+o(1)}$  vertices), Theorem 3.1 directly implies the following.

**COROLLARY 3.2.**  $C_{\text{col}}(P) \geq \kappa_{\text{col}}(P)$ .

In section 4, we will show that  $\kappa_{\text{col}}(P) \geq \Omega(\text{tw}(P)/\log \text{tw}(P))$ ; together with Corollary 3.2, this proves Theorem 1.1.

The remainder of this section contains the proof of Theorem 3.1. The  $n^{2\kappa_{\alpha, \beta}(P) + O(1)}$  upper bound is proved in section 3.1, followed by the  $n^{\kappa_{\alpha, \beta}(P) - o(1)}$  lower bound in section 3.2.

**3.1. Proof of Theorem 3.1 (upper bound).** Fix a pattern  $P$  and a threshold pair  $(\alpha, \beta) \in \theta_{\text{col}}(P)$ . For a  $P$ -colored graph  $G$  and  $Q \subseteq P$ , let  $\text{sub}(Q, G)$  denote the number of (colored)  $Q$ -subgraphs of  $G$ . We write  $\mathbf{G}$  for the  $P$ -colored random graph  $G_{\alpha, \beta}(n)$ . Note that  $\mathbb{E}[\text{sub}(Q, \mathbf{G})] \leq n^{\alpha(Q) - \beta(Q)}$ .

We begin with a sketch of the algorithm that determines, correctly with high probability, whether  $\mathbf{G}$  contains a  $P$ -subgraph. We first establish that, in typical instances of  $\mathbf{G}$ , the number  $\text{sub}(Q, \mathbf{G})$  is highly concentrated around its mean (in particular, we have  $\text{sub}(Q, \mathbf{G}) \leq n^{\alpha(Q) - \beta(Q) + 1}$  for all  $Q \subseteq P$  with very high probability). For such typical instances of  $\mathbf{G}$ , the basic idea is “search” for a  $P$ -subgraph by exhaustively computing the lists  $L_i$  of all  $Q_i$ -subgraphs in  $\mathbf{G}$  for all elements  $Q_i$  of an optimal (that is, with  $\max_i \alpha(Q_i) - \beta(Q_i) = \kappa_{\alpha, \beta}(P)$ ) union sequence  $Q_1, \dots, Q_t = P$ . Each list  $L_i$  will have size at most  $n^{\alpha(Q_i) - \beta(Q_i) + 1}$ ; for  $Q_k = Q_i \cup Q_j$  in the union sequence, the list  $L_k$  is obtained by merging lists  $L_i$  and  $L_j$  at a cost of roughly  $|L_i| \cdot |L_j|$ , which is at most  $n^{2\kappa_{\alpha, \beta}(P) + 2}$ . Nonemptiness of the final list  $L_t$  is equivalent to  $\mathbf{G}$  containing a  $P$ -subgraph. To implement this algorithm by AC<sup>0</sup> circuits, we first invert the role of randomness by designing a random AC<sup>0</sup> circuit which solves the problem in the worst case over the set of “typical instances”; lists  $L_i$  are constructed by random hashing.

We now turn to the details of the construction. Let  $\mathcal{G}_{\alpha, \beta}(n)$  denote the support of  $\mathbf{G}$ , that is, the set of  $P$ -colored graphs with vertex set  $\{(v, i) : v \in V(P), 1 \leq i \leq \lfloor n^{\alpha(v)} \rfloor\}$  and the vertex-coloring  $(v, i) \mapsto v$ . Let also

$$\mathcal{G}'_{\alpha, \beta}(n) := \{G \in \mathcal{G}_{\alpha, \beta}(n) : \text{sub}(Q, G) \leq n^{\alpha(Q) - \beta(Q) + 1} \text{ for all } Q \subseteq P\}.$$

The next lemma says that  $\mathbf{G}$  is extremely unlikely to contain significantly more than  $n^{\alpha(Q) - \beta(Q)}$   $Q$ -subgraphs for any  $Q \subseteq P$ . It is proved by a straightforward application of Markov’s inequality.

**LEMMA 3.3.**  $\Pr[\mathbf{G} \notin \mathcal{G}'_{\alpha, \beta}(n)] = o(1)$ .

We wish to construct a *deterministic* AC<sup>0</sup>-circuit  $C$  which solves  $\text{SUBGRAPH}_{\text{col}}(P)$

correctly on  $\mathbf{G}$  with probability  $1 - o(1)$ . We will invert the role of randomness and instead construct a *random*  $\text{AC}^0$ -circuit  $\mathbf{C}$  which solves  $\text{SUBGRAPH}_{\text{col}}(P)$  correctly with probability  $1 - o(1)$  on every  $G \in \mathcal{G}'_{\alpha,\beta}(n)$ . That is, we will show the following.

LEMMA 3.4. *There exists a random  $\text{AC}^0$  circuit  $\mathbf{C}$  of size  $n^{2\kappa_{\alpha,\beta}(P)+O(1)}$  and depth<sup>7</sup>  $O(e(P))$  such that for every  $G \in \mathcal{G}'_{\alpha,\beta}(n)$ ,*

$$\Pr[\mathbf{C}(G) = 1 \Leftrightarrow \text{sub}(P, G) \geq 1] = 1 - o(1).$$

The upper bound of Theorem 3.1 follows as a corollary of Lemmas 3.3 and 3.4.

PROPOSITION 3.5. *There exists a  $\text{AC}^0$  circuit  $\mathbf{C}$  of size  $n^{2\kappa_{\alpha,\beta}(P)+O(1)}$  such that*

$$\Pr[\mathbf{C}(\mathbf{G}) = 1 \Leftrightarrow \text{sub}(P, \mathbf{G}) \geq 1] = 1 - o(1).$$

*Proof.* Lemmas 3.3 and 3.4 imply that  $\Pr[\mathbf{C}(\mathbf{G}) = 1 \Leftrightarrow \text{sub}(P, \mathbf{G}) \geq 1] = 1 - o(1)$ . Now Proposition 3.5 follows by a straightforward application of Yao’s principle [29].  $\square$

**The random circuit  $\mathbf{C}$ .** It remains to define the randomized  $\text{AC}^0$ -algorithm solving  $\text{SUBGRAPH}_{\text{col}}(P)$  with high probability on every  $G \in \mathcal{G}'_{\alpha,\beta}(n)$ . We first describe the algorithm informally. We then check that this algorithm can be implemented by circuits of size  $n^{2\kappa_{\alpha,\beta}(P)+O(1)}$  and depth  $O(e(P))$ .

By definition of  $\kappa_{\alpha,\beta}(P)$ , there exists a union sequence  $Q_1, \dots, Q_t$  with  $Q_t = P$  such that  $\kappa_{\alpha,\beta}(P) = \max_{i \in [t]} \alpha(Q_i) - \beta(Q_i)$ . On a high level, the idea behind the algorithm was already sketched above: given a graph  $G \in \mathcal{G}'_{\alpha,\beta}(n)$  (the input), we will compute a sequence  $L_1, \dots, L_t$  of lists, where  $L_k$  contains all of the  $Q_k$ -subgraphs of  $G$  (with high probability). Many entries in  $L_k$  will be *blank* (signified by  $\emptyset$ ); by construction, every nonblank entry of  $L_k$  will contain the description of a  $Q_k$ -subgraph of  $G$  (as a string of length  $\alpha(Q_k) \log n$ ). Blank and nonblank entries will in general be interleaved.

We introduce some notation. We write  $\ell_k$  for the number of entries in the list  $L_k$ . For  $a \in [\ell_k]$ , we write  $L_k(a)$  for the contents of the  $a$ th entry in  $L_k$  (either  $\emptyset$  or a  $Q_k$ -subgraph of  $G$ ). We say that  $L_k$  is *good* (with respect to  $G$  and the randomness of the algorithm) if  $L_k$  contains all  $Q_k$ -subgraphs of  $G$  exactly once.

Lists  $L_1, \dots, L_t$  are computed, in order, as follows. For  $k \in [t]$ , assume that  $L_1, \dots, L_{k-1}$  have been computed and are good.

If  $Q_k$  is a single vertex  $v$ , the construction is trivial:  $L_k$  simply lists all  $(v, i)$ ,  $1 \leq i \leq \lfloor n^{\alpha(v)} \rfloor$ .

In the case that  $Q_k$  is a single edge of  $P$ , let  $L_k$  have  $\ell_k := n^{\alpha(Q_k)}$  entries, indexed by the potential  $Q_k$ -subgraphs of  $G$ . For  $a \in [\ell_k]$ , the  $a$ th entry  $L_k(a)$  will contain the  $a$ th potential  $Q_k$ -subgraph iff it is a  $Q_k$  subgraph of  $G$ ; otherwise  $L_k(a)$  is blank. Clearly  $L_k$  is good.

If  $Q_k$  is not a single edge, then by the definition of union sequence,  $Q_k = Q_i \cup Q_j$  for some  $1 \leq i < j < k$ . We compute  $L_k$  in three steps as follows.

*Step 1.* Let  $M_k$  be the  $\ell_i \times \ell_j$  array where, for  $a \in [\ell_i]$  and  $b \in [\ell_j]$ ,

$$M_k(a, b) := \begin{cases} L_i(a) \cup L_j(b) & \text{if } L_i(a) \text{ and } L_j(b) \text{ are both nonblank} \\ & \text{and consistent on } V(Q_i) \cap V(Q_j), \\ \emptyset & \text{otherwise.} \end{cases}$$

<sup>7</sup>In fact, the depth is linear in the *height* of the optimal union sequence, where the height is defined as the length of the longest path from the root to a leaf in the directed acyclic graph induced by a union sequence.

(Note that since  $L_i$  and  $L_j$  are good,  $M_k$  contains each  $Q_k$ -subgraph of  $G$  exactly once. That is,  $M_k$  satisfies the “good” condition that we want for  $L_k$ .)

*Step 2.* We hash  $M_k$  down to a smaller number of entries to obtain the list  $L_k$ . Let  $\text{Supp}(M_k) \subseteq [\ell_i] \times [\ell_j]$  denote the set of nonempty entries of  $M_k$ . Let  $m_k := \lceil n^{\alpha(Q_k) - \beta(Q_k) + 1} \rceil$  and note that  $m_k \geq \#\{Q_k\text{-subgraphs of } G\} = |\text{Supp}(M_k)|$ . Let  $\mathbf{h}_k$  be a uniform random function

$$\mathbf{h}_k : [\ell_i] \times [\ell_j] \rightarrow [m_k].$$

(Restricted to the  $\leq m_k$  nonempty entries of  $M_k$ , this gives a uniform random packing of  $\leq m_k$  balls into  $m_k$  bins.)

*Step 3.* Let  $\ell_k := m_k \ln m_k$ . Indexing entries of  $L_k$  by pairs  $(p, q) \in [m_k] \times [\ln m_k]$  (rather than elements of  $[\ell_k]$ ), let

$$L_k(p, q) := \begin{cases} \text{the } q\text{th element of } \mathbf{h}_k^{-1}(p) \cap \text{Supp}(M_k) & \text{if } |\mathbf{h}_k^{-1}(p) \cap \text{Supp}(M_k)| \geq q, \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that  $L_k$  is good iff

$$(3) \quad \bigwedge_{p \in [m_k]} |\mathbf{h}_k^{-1}(p) \cap \text{Supp}(M_k)| \leq \ln m_k.$$

After computing the final list  $L_t$ , the algorithm outputs 1 iff  $L_t$  has nonblank entries. Note that the output of the algorithm will be correct provided  $L_t$  is good.

To analyze the success probability of the algorithm, note the following elementary fact about balls-into-bins, established by a simple union bound:<sup>8</sup>

$\left\{ \begin{array}{l} \text{For any } \tilde{m} \leq m, \text{ the maximum load of a random function of } \tilde{m} \text{ balls to } m \text{ bins is} \\ \leq \ln m \text{ with probability } \geq 1 - 1/m. \end{array} \right.$

From this fact, we have

$$\begin{aligned} & \Pr_{\mathbf{h}_k} [L_k \text{ is not good} \mid L_1, \dots, L_{k-1} \text{ are good}] \\ & \leq \Pr_{\mathbf{h}_k} \left[ \bigvee_{p \in [m_k]} |\mathbf{h}_k^{-1}(p) \cap \text{Supp}(M_k)| > \ln m_k \right] \\ & \leq \frac{1}{m_k} \leq \frac{1}{n}. \end{aligned}$$

It follows that

$$\begin{aligned} \Pr_{\mathbf{h}_1, \dots, \mathbf{h}_t} [\text{erroneous output}] &= \sum_{k \in [t]} \Pr_{\mathbf{h}_1, \dots, \mathbf{h}_k} [L_k \text{ is not good, } L_1, \dots, L_{k-1} \text{ are good}] \\ &\leq \sum_{k \in [t]} \Pr_{\mathbf{h}_1, \dots, \mathbf{h}_k} [L_k \text{ is not good} \mid L_1, \dots, L_{k-1} \text{ are good}] \\ &\leq tn^{-1} \leq o(1). \end{aligned}$$

<sup>8</sup>A tighter analysis than we require shows that the maximum load is  $\leq 3 \ln m / \ln \ln m$  with probability  $\geq 1 - 1/m$ .

Therefore, the algorithm correctly solves  $\text{SUBGRAPH}_{\text{col}}(P)$  with high probability for every  $G \in \mathcal{G}'_{\alpha,\beta}(P)$ .

It remains to show that this algorithm can be implemented by a random circuit  $\mathbf{C}$  of size  $n^{2\kappa_{\alpha,\beta}(P)+O(1)}$  and depth  $O(e(P))$ . We will make an additional assumption about the random functions  $\mathbf{h}_1, \dots, \mathbf{h}_t$ :

$$(4) \quad |\mathbf{h}_k^{-1}(p)| \leq \frac{2\ell_i\ell_j}{m_k} \text{ for all } k \in [t] \text{ and } p \in [m_k].$$

That is,  $|\mathbf{h}_k^{-1}(p)|$  is at most twice its expectation for all  $k$  and  $p$ . By Chernoff and union bounds, (4) holds with probability  $1 - \exp(-n^{\Omega(1)})$ . So even with this assumption, the error probability of the circuits we describe remains  $o(1)$ .

Let us now fix<sup>9</sup> any particular hash functions  $h_1, \dots, h_t$  such that (4) holds. We will design constant-depth circuits of size  $n^{2\kappa_{\alpha,\beta}(P)+O(1)}$  computing the lists that correspond to our particular choice of  $h_1, \dots, h_t$  whenever all these lists are good, that is, satisfy (3). (If (3) fails for at least one  $k$ , this is the error case and we do not care what the algorithm does.)

We describe the subcircuit which computes the list  $L_k$  given lists  $L_1, \dots, L_{k-1}$ . If  $Q_k$  is a single vertex, this is trivial, and in the case when  $Q_k$  is a single edge, the list  $L_k$  is clearly computable by a depth-2 circuit of size  $\tilde{O}(n^{\alpha(Q_k)})$  (the  $\tilde{O}()$  coming from the fact that it takes  $\alpha(Q_k) \log n$  gates to encode each entry of  $L_k$ ). In the case that  $Q_k = Q_i \cup Q_j$ , first note that we can compute the array  $M_k$  by a circuit of size  $\tilde{O}(n\ell_i\ell_j)$  and depth  $O(1)$  (sitting on top of the subcircuits which compute lists  $L_i$  and  $L_j$ ); this is because checking that  $L_i(a)$  and  $L_j(b)$  agree on all vertices of  $V(Q_i) \cap V(Q_j)$  requires only  $O(n)$  size and depth 2. Having computed  $M_k$ , computing the entries  $L_k(p, q)$  requires finding the  $q$ th element in the 0-1 string that represents the characteristic function of  $h_k^{-1}(p) \cap \text{Supp}(M_k)$  within the known set  $h_k^{-1}(p)$ . This string has length  $|h_k^{-1}(p)| \leq O(\frac{\ell_i\ell_j}{m_k})$  and, by (3), has at most  $\ln m_k = O(\log n)$  1 entries. Hence, by [16, Theorem 6] applied with  $n := |h_k^{-1}(p)|$ ,  $k := \ln m_k$ ,  $\gamma := 2$  and  $m := 3$ , this can be done by a constant-depth circuit of size  $n^{o(1)} \frac{\ell_i\ell_j}{m_k}$ . As there are altogether  $\ell_k \leq \tilde{O}(m_k)$  pairs  $(p, q)$ , we get a constant-depth circuit which computes  $L_k$  (given  $L_i$  and  $L_j$ ) with total size  $n^{o(1)} \ell_i\ell_j \leq n^{\alpha(Q_i)-\beta(Q_i)+\alpha(Q_j)-\beta(Q_j)+O(1)} \leq n^{2\kappa_{\alpha,\beta}(P)+O(1)}$ .

After computing all lists  $L_1, \dots, L_t$ , we have a circuit of size  $n^{2\kappa_{\alpha,\beta}(P)+O(1)}$ . Finally, note that the depth of this circuit will be  $O(d)$  where  $d$  is the height of the poset where  $i, j \prec k$  iff  $Q_k = Q_i \cup Q_j$  for  $i, j < k$ . Clearly,  $d \leq e(P)$  as long as all graphs in the sequence are pairwise distinct.

**3.2. Proof of Theorem 3.1 (lower bound).** This subsection gives the proof of the lower bound in Theorem 3.1 (the average-case  $\text{AC}^0$ -complexity of  $\text{SUBGRAPH}_{\text{col}}(P)$  on  $G_{\alpha,\beta}(n)$  is at least  $n^{\kappa_{\alpha,\beta}(P)-o(1)}$ ). The argument closely follows the technique of [27, 28].

It will be convenient to work with an alternative characterization of  $\text{AC}^0$  as boolean circuits with fan-in 2. We distinguish between “type-I” and “type-II”  $\text{AC}^0$  circuits as follows:

- type-I: polynomial-size constant-depth  $\{\text{AND}_\infty, \text{OR}_\infty, \text{NOT}\}$ -circuits with unbounded fan-in (this is the standard definition of  $\text{AC}^0$ ).
- type-II: polynomial-size  $\{\text{AND}_2, \text{OR}_2, \text{NOT}\}$ -circuits with fan-in 2 and arbitrary depth, but  $O(1)$  alternations between AND and OR gates (where w.l.o.g. NOT gates are on the bottom level).

<sup>9</sup>It is important for our argument that the random functions  $\mathbf{h}_1, \dots, \mathbf{h}_t$  are generated in advance and that we do not attempt to make our construction uniform in  $\mathbf{h}_1, \dots, \mathbf{h}_t$ .

The conversion from type-I to type-II replaces each AND<sub>∞</sub> (resp., OR<sub>∞</sub>) gate with a binary tree of AND<sub>2</sub> (resp., OR<sub>2</sub>) gates. This conversion can result in a quadratic blow-up in size (= number of gates), as a type-I circuit with  $g$  gates and  $w \leq O(g^2)$  wires becomes a type-II circuit with  $O(w)$  gates. Therefore, a lower bound of  $S$  on the size of type-II circuits implies a lower bound of  $\Omega(\sqrt{S})$  on the size of type-I circuits.

We will first prove an  $n^{\kappa_{\alpha,\beta}(P)-o(1)}$  lower bound on the size of type-II circuits solving SUBGRAPH<sub>col</sub>( $P$ ) in the average case on  $G_{\alpha,\beta}(n)$ . This implies a weaker  $n^{\kappa_{\alpha,\beta}(P)/2-o(1)}$  lower bound for type-I circuits. The stronger  $n^{\kappa_{\alpha,\beta}(P)-o(1)}$  lower bound for type-I circuits is shown by an additional argument in section 3.3.

Let  $\mathcal{G} := \mathcal{G}_{\alpha,\beta}(n)$  denote the support of the random  $P$ -colored graph  $G_{\alpha,\beta}(n)$ , that is, the set of  $P$ -colored graphs with vertex set  $\{(v, i) : v \in V(P), 1 \leq i \leq \lfloor n^{\alpha(v)} \rfloor\}$  and the vertex-coloring  $(v, i) \mapsto v$ . (The following definitions are also used in the uncolored setting, where  $\mathcal{G}$  is the set of graphs with vertex  $\{1, \dots, n\}$ .) We identify  $\mathcal{G}$  with the hypercube  $\{0, 1\}^{\mathcal{E}}$ , where  $\mathcal{E}$  is the set of potential edges  $\{(v, i), (w, j)\}$  with  $\{v, w\} \in V(P)$ .

DEFINITION 3.6. *Let  $f$  be any function with domain  $\mathcal{G}$  (and arbitrary range), and let  $H$  be any graph in  $\mathcal{G}$ . The  $f$ -sensitive subgraph of  $H$ , denoted  $\text{Sens}(f, H)$ , is defined as the unique minimal subgraph  $S \subseteq H$  such that  $f(H') = f(H' \cap S)$  for every  $H' \subseteq H$ . We say that  $f$  is sensitive over  $H$  if  $\text{Sens}(f, H) = H$ .*

For all  $f$  and  $H$ , observe that

$$(5) \quad f \text{ is sensitive over } \text{Sens}(f, H) \text{ (i.e., } \text{Sens}(f, \text{Sens}(f, H)) = \text{Sens}(f, H)),$$

$$(6) \quad \text{if } f : \mathcal{G} \rightarrow \{0, 1\} \text{ is the AND or OR of } f_1, f_2, \text{ then } \text{Sens}(f, H)$$

$$\subseteq \text{Sens}(f_1, H) \cup \text{Sens}(f_2, H).$$

We say that a single-output boolean circuit whose variables encode potential edges in a graph is *sensitive over  $H$*  if its output function is so.

LEMMA 3.7. *Let  $C$  be a boolean circuit with fan-in 2, computing a function  $\mathcal{G} \rightarrow \{0, 1\}$ , such that  $C$  is sensitive over some nonempty graph  $H$ . Then there exists a union sequence  $H_1, \dots, H_t = H$  and a sequence  $C_1, \dots, C_t$  of subcircuits of  $C$  such that  $C_i$  is sensitive over  $H_i$  for all  $i \in \{1, \dots, t\}$ .*

*Proof.* We argue by induction on boolean circuits with fan-in 2. In the base case,  $C$  is a variable (corresponding to a possible edge). The assumption that  $C$  is sensitive over  $H$  implies that  $H$  is a single edge. Therefore,  $H$  itself is a union sequence of length 1 which satisfies the condition of the lemma.

For the induction step, note that if  $C = \text{NOT}(C')$ , then  $C'$  is sensitive over  $H$ ; therefore, the lemma holds by the induction hypothesis for  $C'$ . Finally, suppose  $C$  is the AND or OR of subcircuits  $C_1$  and  $C_2$ . If  $C_1$  or  $C_2$  is sensitive over  $H$ , then appealing to the induction hypothesis, we are done. So we will assume that neither  $C_1$  nor  $C_2$  is sensitive over  $H$ . Let  $H_i := \text{Sens}(C_i, H)$  for  $i = 1, 2$ . Then  $C_i$  is sensitive over  $H_i$  by observation (5). By observation (6),

$$H = \text{Sens}(C, H) \subseteq \text{Sens}(C_1, H) \cup \text{Sens}(C_2, H) = H_1 \cup H_2.$$

Hence  $H = H_1 \cup H_2$ . By the induction hypothesis, there exist union sequence  $S_1, \dots, S_s = H_1$  and  $T_1, \dots, T_t = H_2$  which satisfy the condition in the lemma with respect to  $C_1, H_1$  and  $C_2, H_2$ , respectively. Then  $S_1, \dots, S_s, T_1, \dots, T_t, H$  is a union sequence which satisfies the condition in the lemma with respect to  $C$  and  $H$ .  $\square$

DEFINITION 3.8. *If  $f$  is a function with domain  $\mathcal{G}$  (and arbitrary range) and  $G$  is a graph in  $\mathcal{G}$ , then let  $f^{\cup G}$  denote the function  $f^{\cup G}(H) := f(G \cup H)$ .*

Note that if a boolean circuit  $C$  computes a function  $f$  on  $\mathcal{G}$ , then the circuit  $C^{\cup G}$  that substitutes 1 for variables corresponding to edges in  $G$  computes  $f^{\cup G}$ .

We now fix a pattern  $P$  and a threshold pair  $(\alpha, \beta) \in \theta_{\text{col}}(P)$ . Without loss of generality, we assume that  $\beta(e) > 0$  for all  $e \in E(P)$  (otherwise we replace  $P$  with the subgraph with edge set  $\{e \in E(P) : \beta(e) > 0\}$ ). We continue to write  $\mathbf{G}$  for the  $P$ -colored random graph  $G_{\alpha, \beta}(n)$ . Independently, let  $\mathbf{P} \in \mathcal{G}$  be a uniform random “planted”  $P$ -subgraph (viewed as element of  $\mathcal{G}$ ). That is, after ignoring isolated vertices,  $\mathbf{P}$  is the  $P$ -subgraph with vertex set  $\{(v, i_v) : v \in V(P)\}$ , where  $i_v$  is uniform random in  $\{1, \dots, \lfloor n^{\alpha(v)} \rfloor\}$ . For a subgraph  $Q \subseteq P$ , let  $\mathbf{Q} \in \mathcal{G}$  denote the corresponding subgraph of  $\mathbf{P}$ . (While  $\mathbf{P}$  and  $\mathbf{G}$  are independent,  $\mathbf{Q}$  is completely determined by  $\mathbf{P}$  and hence is also independent of  $\mathbf{G}$ .)

We next state two technical lemmas.

LEMMA 3.9. *Suppose  $f : \mathcal{G} \rightarrow \{0, 1\}$  solves  $\text{SUBGRAPH}_{\text{col}}(P)$  in the average case on  $\mathbf{G}$ , that is,*

$$(7) \quad \Pr_{\mathbf{G}}[f(\mathbf{G}) = 1 \Leftrightarrow \mathbf{G} \text{ has a } P\text{-subgraph}] = 1 - o(1).$$

Then

$$(8) \quad \liminf_{n \rightarrow \infty} \Pr_{\mathbf{G}, \mathbf{P}}[f^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{P}] > 0.$$

The full proof of Lemma 3.9 is included in Appendix B. We give a brief outline here. First, we show that the event “ $E(\mathbf{P}) \cap E(\mathbf{G}) = \emptyset$  and  $\mathbf{P}$  is the unique  $P$ -subgraph in  $\mathbf{G} \cup \mathbf{P}$ ” holds with probability bounded away from 0. (In particular, we have  $\Pr[E(\mathbf{P}) \cap E(\mathbf{G}) = \emptyset] = 1 - o(1)$  and  $\Pr[\mathbf{P} \text{ is the unique } P\text{-subgraph in } \mathbf{G} \cup \mathbf{P}] = \Omega(1)$ , as shown in Lemma B.1.3.) Note that if  $f$  computes  $\text{SUBGRAPH}_{\text{col}}(P)$  exactly, then this event implies that  $f^{\cup \mathbf{G}}$  is the AND function over the edges of  $\mathbf{P}$  (i.e., for all  $Q \subseteq P$ ,  $f(\mathbf{G} \cup \mathbf{Q}) = 1$  iff  $Q = P$ ) and is therefore sensitive over  $\mathbf{P}$ . However, if  $f$  merely agrees with  $\text{SUBGRAPH}_{\text{col}}(P)$  on  $\mathbf{G}$  a.a.s. (as in the hypothesis of Lemma 3.9), then we require an additional argument bounding the total variation distance between  $\mathbf{G}$  and  $\mathbf{G} \cup \mathbf{Q}$  for subgraphs  $Q \subseteq P$  (see Appendix B for details).<sup>10</sup>

The second technical lemma relies on Håstad’s switching lemma [15] and its proof closely follows Proposition 3.11 of [28]. (The reader who wishes to skip the technical details is encouraged to jump ahead to Theorem 3.11.)

LEMMA 3.10. *Suppose  $f : \mathcal{G} \rightarrow \{0, 1\}$  is  $\text{AC}^0$ -computable. Then for every  $Q \subseteq P$ ,*

$$\Pr_{\mathbf{G}, \mathbf{Q}}[f^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{Q}] \leq n^{-\alpha(Q) + \beta(Q) + o(1)}.$$

*Proof.* Let  $\mathcal{E}$  be the set of potential edges of graphs in  $\mathcal{G}$ . We identify  $\mathcal{G}$  with the hypercube  $\{0, 1\}^{\mathcal{E}}$  and we view  $f$  as a boolean function  $\{0, 1\}^{\mathcal{E}} \rightarrow \{0, 1\}$ . For  $e \in \mathcal{E}$ ,

<sup>10</sup>If we are content with *worst-case* lower bounds for  $\text{SUBGRAPH}_{\text{col}}(P)$  (as opposed to *average-case* lower bounds with respect to  $\mathbf{G}$ ), then it suffices to prove Lemma 3.9 in the special case where  $f$  computes  $\text{SUBGRAPH}_{\text{col}}(P)$  exactly. For this, we merely require the bound  $\Pr[E(\mathbf{P}) \cap E(\mathbf{G}) = \emptyset \text{ and } \mathbf{P} \text{ is the unique } P\text{-subgraph in } \mathbf{G} \cup \mathbf{P}] = \Omega(1)$  (Lemma B.1.3), thus avoiding most of the work in Appendix B that deals with total variation distance. An even easier way to establish the worst-case lower bound is to prove (8) for  $\mathbf{G}$  in  $G_{\alpha, \beta'}(G)$ , where  $(\alpha, \beta')$  is a subthreshold pair, that is,  $\beta'(e) = \beta(e) + \epsilon$  for an arbitrarily small  $\epsilon > 0$ , which follows from a union bound.

let  $\hat{e} \in E(P)$  be the corresponding edge of  $P$  (under the  $V(P)$ -coloring  $(v, i) \mapsto v$  of graphs in  $\mathcal{G}$ ).

Let  $\delta > 0$  be an arbitrarily small constant, which is independent of  $n$  but may depend on  $P, \alpha, \beta$ . We generate a random restriction  $\rho : \mathcal{E} \rightarrow \{0, 1, \star\}$  where, independently for all  $e \in \mathcal{E}$ ,

$$(9) \quad \Pr[\rho(e) = \star] = n^{-\beta(\hat{e})-\delta}, \quad \Pr[\rho(e) = 1 \mid \rho(e) \neq \star] = n^{-\beta(\hat{e})}.$$

Let  $H_\rho$  denote the  $P$ -colored graph with edge set  $E(H_\rho) = \rho^{-1}(\star)$ . Note that  $H_\rho$  has distribution  $G_{\alpha, \beta+\delta}(n)$ . In particular,  $\mathbb{E}_\rho[\text{sub}(Q, H_\rho)] = n^{\alpha(Q)-\beta(Q)-\delta|E(Q)|}$ . We assume  $\delta$  is sufficiently small so that  $\alpha(Q) - \beta(Q) - \delta|E(Q)| > 0$  (here we assume  $\alpha(Q) - \beta(Q) > 0$  since otherwise the lemma is trivial). Using the lower-tail version of Janson's inequality [19], it can be shown that

$$(10) \quad \Pr_\rho[\text{sub}(Q, H_\rho) < \frac{1}{2}n^{\alpha(Q)-\beta(Q)-\delta|E(Q)|}] = n^{-\omega(1)}.$$

That is, with very high probability,  $H_\rho$  contains at least half the expected number of  $Q$ -subgraphs. (Since  $P$  and  $\delta > 0$  are fixed,  $n^{-\omega(1)}$  is  $O(n^{-c})$  for every constant  $c = c(P, \delta)$  which may depend on  $P$  and  $\delta$ .)

Let  $L_\rho$  denote the subgraph of  $H_\rho$  with edge set

$$E(L_\rho) = \{e \in \rho^{-1}(\star) : \text{restricted function } f \upharpoonright_\rho : \{0, 1\}^{\rho^{-1}(\star)} \rightarrow \{0, 1\} \text{ depends}^{11} \text{ on coordinate } e\}.$$

Note that in the language of graphs predominantly used in this proof, we have  $E(L_\rho) = \text{Sens}(f \upharpoonright_\rho, \rho^{-1}(\{1, \star\}))$ .

We now use the fact that  $f$  is computed by an AC<sup>0</sup>-circuit (in particular, a type-I AC<sup>0</sup>-circuit). A bottom-up depth-reduction argument using Håstad's switching lemma [15] shows that

$$(11) \quad \Pr_\rho[|L_\rho| > n^\delta] \leq S \cdot (5n^{-\delta/d} \delta \log n)^{\delta \log n} = n^{-\omega(1)},$$

where  $S$  and  $d$  are the size and depth of the circuit defining  $f$ . (This bound is  $n^{-\omega(1)}$  since  $S = n^{O(1)}$  and  $d = O(1)$  and  $\delta = \Omega(1)$ .) The proof of (11) is included in Appendix C.

Let  $\mathcal{A} = \mathcal{A}(\rho)$  denote the event that  $\text{sub}(Q, H_\rho) \geq \frac{1}{2}n^{\alpha(Q)-\beta(Q)-\delta|E(Q)|}$  and  $|L_\rho| \leq n^\delta$ . Note that  $\Pr[\neg \mathcal{A}] = n^{-\omega(1)}$  (by (10) and (11)) and

$$\mathcal{A} \implies \frac{\text{sub}(Q, L_\rho)}{\text{sub}(Q, H_\rho)} \leq 2n^{-\alpha(Q)+\beta(Q)+\delta|E(Q)|+\delta|V(Q)|}.$$

We now generate a pair  $(\mathbf{G}', \mathbf{Q}')$  of random variables by the following two-step process:

- Independently, for all  $e \in \mathcal{E}$ , let

$$(12) \quad \Pr[e \in E(\mathbf{G}') \mid \rho] = \begin{cases} \rho(e) & \text{if } \rho(e) \in \{0, 1\}, \\ n^{-\beta(\hat{e})} & \text{if } \rho(e) = \star. \end{cases}$$

<sup>11</sup>A function  $g : \{0, 1\}^I \rightarrow \{0, 1\}$  depends on a coordinate  $i \in I$  if there exists  $x \in \{0, 1\}^I$  such that  $g(x) \neq g(x^{(i)})$ , where  $x^{(i)}$  is  $x$  with its  $i$ th coordinate flipped.

- If  $\mathcal{A}(\rho)$  holds (in particular,  $\text{sub}(Q, H_\rho) \neq \emptyset$ ), we let  $\mathbf{Q}'$  be a uniform random  $Q$ -subgraph of  $H_\rho$ . Otherwise, we let  $\mathbf{Q}' := \perp$  ( $\perp$  stands for "undefined").

We claim that  $(\mathbf{G}', \mathbf{Q}')$  under the condition  $\mathbf{Q}' \neq \perp$  is distributed identically with  $(\mathbf{G}, \mathbf{Q})$ .

Indeed, by inspecting the definitions (9) and (12), we see that  $\Pr_{\mathbf{G}'}[e \in \mathbf{G}'] = n^{-\beta(\textcircled{2})}$ . This implies that, first,  $\mathbf{G}' \sim G_{\alpha,\beta}(n)$  and, second,  $\mathbf{G}'$  and  $H_\rho$  are independent. As  $\mathbf{Q}'$  is a function of  $H_\rho$ , it is independent of  $\mathbf{G}'$  as well. Finally,  $\mathbf{Q}'$  conditioned by the event  $\mathbf{Q}' \neq \perp$  is distributed identically with  $\mathbf{Q}$  simply by symmetry.

This observation, along with the crucial fact  $\Pr[\neg \mathcal{A}] = n^{-\omega(1)}$  mentioned above, implies that we can rephrase the inequality we are proving as

$$\Pr_{\rho, \mathbf{G}', \mathbf{Q}'} [f^{\cup \mathbf{G}'} \text{ is sensitive over } \mathbf{Q}' \mid \mathcal{A}(\rho)] \leq n^{-\alpha(Q)+\beta(Q)+o(1)}.$$

We now fix an arbitrary  $\rho$  such that  $\mathcal{A}$  holds.

Since  $\mathbf{Q}' \subseteq H_\rho = \{e : \rho(e) = \star\}$ , it follows from definitions that if  $f^{\cup \mathbf{G}'}$  is sensitive over  $\mathbf{Q}'$ , then  $\mathbf{Q}' \subseteq L_\rho$ . We now have

$$\Pr_{\mathbf{G}', \mathbf{Q}'} [f^{\cup \mathbf{G}'} \text{ is sensitive over } \mathbf{Q}'] \leq 2n^{-\alpha(Q)+\beta(Q)+\delta|E(Q)|+\delta|V(Q)|}.$$

The lemma follows, since we can choose  $\delta > 0$  arbitrarily small relative to  $|E(Q)|/\beta(Q)$  and  $|V(Q)|/\alpha(Q)$ . □

Finally, the main result of this subsection (the lower bound of Theorem 3.1) is as follows.

**THEOREM 3.11.** *Suppose  $\mathbf{C}$  is a type-II  $\text{AC}^0$  circuit that solves  $\text{SUBGRAPH}_{\text{col}}(P)$  in the average case on  $G_{\alpha,\beta}(P)$ . Then  $\mathbf{C}$  has size at least  $n^{\kappa_{\alpha,\beta}(P)-o(1)}$ .*

*Proof.* For contradiction, assume  $\mathbf{C}$  has size  $\leq n^{\kappa_{\alpha,\beta}(P)-\varepsilon}$  and  $d$  alternations for constants  $\varepsilon, d > 0$  (independent of  $n$ ). By Lemma 2.14, there exists a hitting set  $\mathcal{H}$  for  $P$  such that  $\kappa_{\alpha,\beta}(P) = \min_{Q \in \mathcal{H}} \alpha(Q) - \beta(Q)$ . Note that every subcircuit  $\mathbf{C}'$  of  $\mathbf{C}$  is computable by a type-I  $\text{AC}^0$  circuit of depth  $d$  (by combining all adjacent AND and OR gates in  $\mathbf{C}'$ ). Therefore, by Lemma 3.10,

$$\begin{aligned} \Pr_{\mathbf{G}, \mathbf{P}} \left[ \bigvee_{Q \in \mathcal{H}} \mathbf{C}'^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{Q} \right] &\leq \sum_{Q \in \mathcal{H}} \Pr_{\mathbf{G}, \mathbf{Q}} [\mathbf{C}'^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{Q}] \\ &\leq \sum_{Q \in \mathcal{H}} n^{-\alpha(Q)+\beta(Q)+o(1)} \\ &\leq |\mathcal{H}| \cdot \max_{Q \in \mathcal{H}} n^{-\alpha(Q)+\beta(Q)+o(1)} \\ &= n^{-\kappa_{\alpha,\beta}(P)+o(1)} \quad (\text{since } |\mathcal{H}| \leq 2^{|E(P)|} = n^{o(1)}). \end{aligned}$$

Taking a union bound over the  $\leq n^{\kappa_{\alpha,\beta}(P)-\varepsilon}$  subcircuits of  $\mathbf{C}$ , we have

$$(13) \quad \Pr_{\mathbf{G}, \mathbf{P}} \left[ \bigvee_{\text{subcircuits } \mathbf{C}'} \bigvee_{Q \in \mathcal{H}} \mathbf{C}'^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{Q} \right] \leq n^{-\varepsilon+o(1)} = o(1).$$

We now derive a contradiction to (13). By Lemma 3.7 (with respect to the circuit  $\mathbf{C}^{\cup \mathbf{G}}$ ), if  $\mathbf{C}^{\cup \mathbf{G}}$  is sensitive over  $\mathbf{P}$ , then there exist  $Q \in \mathcal{H}$  and a subcircuit  $\mathbf{C}'$  of  $\mathbf{C}$  such that  $\mathbf{C}'^{\cup \mathbf{G}}$  is sensitive over  $\mathbf{Q}$ . It follows that

$$\Pr_{\mathbf{G}, \mathbf{P}} \left[ \bigvee_{\text{subcircuits } C'} \bigvee_{Q \in \mathcal{H}} C'^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{Q} \right] \geq \Pr_{\mathbf{G}, \mathbf{P}} \left[ C^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{P} \right]$$

(14) = Ω(1) (by Lemma 3.9).

Inequalities (13) and (14) give a contradiction, which completes the proof. □

**3.3. Unbounded fan-in.** As discussed at the beginning of this section, our lower bound for type-II circuits (Theorem 3.11) implies an  $n^{\frac{1}{2}\kappa_{\alpha, \beta}(P) - o(1)}$  lower bound on the size of type-I (that is, bounded-depth unbounded-fan-in AC<sup>0</sup>) circuits. We now modify the proof of Theorem 3.11 to obtain the stronger  $n^{\kappa_{\alpha, \beta}(P) - o(1)}$  lower bound for type-I circuits. (The argument presented below follows section 3.4 of [28].)

**THEOREM 3.12.** *Suppose C is a type-I AC<sup>0</sup> circuit that solves SUBGRAPH<sub>col</sub>(P) in the average case on G<sub>α,β</sub>(P). Then C has size at least n<sup>κ<sub>α,β</sub>(P) - o(1)</sup>.*

Our proof requires a slightly stronger version of Lemma 3.10.

**LEMMA 3.13.** *Suppose f : G → {0, 1}<sup>m</sup> is AC<sup>0</sup>-computable where m = n<sup>o(1)</sup> (that is, f is computed by an O(1)-depth, n<sup>O(1)</sup>-size circuit with n<sup>o(1)</sup> output gates). Then for every Q ⊆ P,*

$$\Pr_{\mathbf{G}, \mathbf{Q}} [f^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{Q}] \leq n^{-\alpha(Q) + \beta(Q) + o(1)}.$$

*Proof.* We modify the proof of Lemma 3.10 as follows. Let boolean functions  $f_1, \dots, f_m : \mathcal{G} \rightarrow \{0, 1\}$  be the coordinates of the output of  $f$ . Consider the random restriction  $\rho$  and the graph  $H_\rho$  as before (that is, as defined in the proof of Lemma 3.10 with  $E(H_\rho) = \rho^{-1}(\star)$ ). Also as before, let  $L_\rho$  denote the subgraph of  $H_\rho$  with edge set

$$E(L_\rho) = \{e \in \rho^{-1}(\star) : f|_\rho : \{0, 1\}^{\rho^{-1}(\star)} \rightarrow \{0, 1\}^m \text{ depends on coordinate } e\}.$$

Let us now also consider graphs  $L_{1, \rho}, \dots, L_{m, \rho} \subseteq H_\rho$  defined by

$$E(L_{i, \rho}) = \{e \in \rho^{-1}(\star) : f_i|_\rho : \{0, 1\}^{\rho^{-1}(\star)} \rightarrow \{0, 1\} \text{ depends on coordinate } e\}.$$

Clearly, we have  $L_\rho = L_{1, \rho} \cup \dots \cup L_{m, \rho}$ .

Recall that  $\delta > 0$  is an arbitrary constant (which we allow to depend on  $P$  and  $(\alpha, \beta)$  but will be independent of  $n$ ). For each  $i \in \{1, \dots, m\}$ , the argument in Appendix C gives the bound

$$\Pr_\rho[|L_{i, \rho}| > n^\delta] \leq S \cdot (5n^{-\delta/d} \delta \log n)^{\delta \log n} = n^{-\omega(1)},$$

where  $S$  and  $d$  are the size and depth of the circuit defining  $f$ . (This bound is  $n^{-\omega(1)}$  since  $S = n^{O(1)}$  and  $d = O(1)$  and  $\delta = \Omega(1)$ .) Similarly, we have  $\Pr_\rho[|L_{i, \rho}| > n^{\delta/2}] \leq n^{-\omega(1)}$  (replacing  $\delta$  with  $\delta/2$  above). Since  $m = n^{o(1)}$ , for all  $n$  sufficiently large, we get the bound

(15)

$$\Pr_\rho[|L_\rho| > n^\delta] \leq \Pr_\rho \left[ \bigvee_{i=1}^m |L_{i, \rho}| > n^{\delta/2} \right] \leq \sum_{i=1}^m \Pr_\rho[|L_{i, \rho}| > n^{\delta/2}] = m \cdot n^{-\omega(1)} = n^{-\omega(1)}.$$

With (15) in place of (11), the rest of the proof is identical to that of Lemma 3.10. □

To prove Theorem 3.12, we shall once again convert type-I circuits into type-II circuits. However, this time we replace each unbounded fan-in gate with a *unbalanced* tree of fan-in 2 gates.

DEFINITION 3.14. *For every type-I circuit  $C$  computing a boolean function  $\mathcal{G} \rightarrow \{0, 1\}$ , we shall define an equivalent type-II circuit  $\tilde{C}$ . In addition, if the output of  $C$  is an AND/OR gate of fan-in  $m$ , then we shall also define auxiliary functions  $g^C : \mathcal{G} \rightarrow \{0, 1\}^m$  and  $f^C : \mathcal{G} \rightarrow \{0, 1\}^{\lceil \log(m+1) \rceil}$ . The definition is inductive:*

- If  $C$  is an input (variable or constant), then  $\tilde{C} := C$ .
- If  $C = \text{NOT}(C')$ , then  $\tilde{C} := \text{NOT}(\tilde{C}')$ .
- If  $C = \text{OR}(C_1, \dots, C_m)$ , then

$$\tilde{C} := \tilde{C}^{(m)}, \text{ where } \tilde{C}^{(i)} := \begin{cases} \tilde{C}_1 & \text{if } i = 1, \\ \text{OR}(\tilde{C}^{(i-1)}, \tilde{C}_i) & \text{if } i \in \{2, \dots, m\} \end{cases}$$

(that is,  $\tilde{C} = \text{OR}(\dots \text{OR}(\text{OR}(\tilde{C}_1, \tilde{C}_2), \tilde{C}_3) \dots, \tilde{C}_m)$  and, for all  $i \in [m]$ ,

$$g_i^C(G) := C_1(G) \vee \dots \vee C_i(G) \text{ (= the boolean function computed by } \tilde{C}^{(i)}),$$

and

$$f^C(G) := \begin{cases} 0 & \text{if } C(G) = 0, \\ \text{binary}(i) & \text{if } C_i(G) = 1 \text{ and } C_1(G) = \dots = C_{i-1}(G) = 0, \end{cases}$$

where  $\text{binary}(i) \in \{0, 1\}^{\lceil \log(m+1) \rceil}$  is the binary representation of  $i$ . (That is,  $f^C(G)$  reports that  $C(G) = 0$  or points to the first 1-value among  $C_1(G), \dots, C_m(G)$ .)

- If  $C = \text{AND}(C_1, \dots, C_m)$ , then the definition is dual to the case of OR.

We make some straightforward observations.

- (i) Note that  $|\{\text{gates in } \tilde{C}\}| = |\{\text{wires in } C\}| = O(|\{\text{gates in } C\}|^2)$ . (Indeed, the conversion  $C \mapsto \tilde{C}$  incurs a quadratic blow-up in size in the worst case. Our proof of Theorem 3.12 will avoid taking a union bound over gates in  $\tilde{C}$ .)
- (ii) The operation  $C \mapsto \tilde{C}$  commutes with the operation  $C \mapsto C^{\cup G}$ . That is, we have  $\tilde{C}^{\cup G} = \tilde{C^{\cup G}}$ .
- (iii) If  $C = \text{OR}(C_1, \dots, C_m)$ , then  $g^C$  reports the vector of values of subcircuits  $\tilde{C}^{(1)}, \dots, \tilde{C}^{(m)}$ . Note that  $g^C(G)$  can achieve only  $m + 1$  possible values:

$$\underbrace{(0, \dots, 0)}_m, \underbrace{(0, \dots, 0, 1)}_{m-1}, \underbrace{(0, \dots, 0, 1, 1)}_{m-2}, \dots, \text{ or } \underbrace{(1, \dots, 1)}_m.$$

- (iv) Function  $g^C : \mathcal{G} \rightarrow \{0, 1\}^m$  are  $f^C : \mathcal{G} \rightarrow \{0, 1\}^{\lceil \log(m+1) \rceil}$  information-theoretically equivalent (that is, there is a bijection  $\varphi : \text{Range}(g^C) \xrightarrow{\cong} \text{Range}(f^C)$  such that  $f^C(H) = \varphi(g^C(H))$  for all  $H \in \mathcal{G}$ ). Consequently, for every graph  $H \in \mathcal{G}$ , we have  $\text{Sens}(g^C, H) = \text{Sens}(f^C, H)$  and therefore

$$g^C \text{ is sensitive over } H \iff f^C \text{ is sensitive over } H.$$

- (v) We have  $\text{Sens}(g^C, H) = \bigcup_{i \in [m]} \text{Sens}(\tilde{C}^{(i)}, H)$ . (In general,  $\text{Sens}(h, H) = \text{Sens}(h_1, H) \cup \dots \cup \text{Sens}(h_t, H)$  for any multioutput function  $h : \mathcal{G} \rightarrow \{0, 1\}^t$ .) In particular,

$$\bigvee_{i \in [m]} (\tilde{C}^{(i)} \text{ is sensitive over } H) \implies g^C \text{ is sensitive over } H.$$

(vi) If  $C$  has depth  $d = O(1)$  and size  $\text{poly}(n)$ , then  $f^C$  is computable by a circuit of depth  $d + 2 = O(1)$  and size  $\text{poly}(n)$  with  $\lceil \log(m + 1) \rceil = O(\log n) = n^{o(1)}$  output gates.

Observations (iii)–(vi) apply equally if  $C = \text{AND}(C_1, \dots, C_m)$  (with only the roles of 0 and 1 exchanged in observation (iii)).

LEMMA 3.15. *Suppose  $C$  is a type-I AC<sup>0</sup> circuit with top fan-in  $m \geq 2$  computing a boolean function  $\mathcal{G} \rightarrow \{0, 1\}$ . Then for every  $Q \subseteq P$ , we have*

$$\Pr_{\mathbf{G}, \mathbf{Q}} \left[ \bigvee_{i \in [m]} ((\tilde{C}^{(i)})^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{Q}) \right] \leq n^{-\alpha(Q) + \beta(Q) + o(1)}.$$

*Proof.* By observations (ii), (iv), and (v), we have the implication

$$\bigvee_{i \in [m]} ((\tilde{C}^{(i)})^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{Q}) \implies (f^C)^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{Q}.$$

By observation (vi) and Lemma 3.13, we have

$$\Pr_{\mathbf{G}, \mathbf{Q}} [(f^C)^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{Q}] \leq n^{-\alpha(Q) + \beta(Q) + o(1)}.$$

The lemma follows from these two facts. □

We now present the proof of Theorem 3.12.

*Proof of Theorem 3.12.* Let  $C$  be a type-I circuit of depth  $d = O(1)$  that solves  $\text{SUBGRAPH}_{\text{col}}(P)$  in the average case on  $G_{\alpha, \beta}(P)$ . Toward a contradiction, assume  $C$  has size  $\leq n^{\kappa_{\alpha, \beta}(P) - \varepsilon}$  for some constant  $\varepsilon > 0$  (independent of  $n$ ). By Lemma 2.14, we may fix a hitting set  $\mathcal{H}$  for  $P$  such that  $\kappa_{\alpha, \beta}(P) = \min_{Q \in \mathcal{H}} \alpha(Q) - \beta(Q)$ .

Let  $\mathcal{S}$  (resp.,  $\tilde{\mathcal{S}}$ ) be the set of subcircuits of  $C$  (resp.,  $\tilde{C}$ ). Note that, for each  $D \in \tilde{\mathcal{S}}$ , either

- $D = \tilde{C}^{(i)}$  for some  $C' \in \mathcal{S}$  with top fan-in 1 (i.e.,  $C'$  is an input or has a NOT gate on top), or
- $D = \tilde{C}'^{(i)}$  for some  $C' \in \mathcal{S}$  with top fan-in  $m \geq 2$  and some  $i \in [m]$ .

Applying Lemma 3.7 to the type-II circuit  $\tilde{C}^{\cup \mathbf{G}}$ , we have that if  $\tilde{C}^{\cup \mathbf{G}}$  is sensitive over  $\mathbf{P}$ , then there exist  $Q \in \mathcal{H}$  and a subcircuit  $D \in \tilde{\mathcal{S}}$  such that  $D^{\cup \mathbf{G}}$  is sensitive over  $\mathbf{Q}$ . Therefore,

$$\begin{aligned} \Omega(1) &\leq \Pr_{\mathbf{G}, \mathbf{P}} [C^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{P}] \quad (\text{by Lemma 3.9}) \\ &= \Pr_{\mathbf{G}, \mathbf{P}} [\tilde{C}^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{P}] \\ &\leq \Pr_{\mathbf{G}, \mathbf{P}} \left[ \bigvee_{Q \in \mathcal{H}} \bigvee_{D \in \tilde{\mathcal{S}}} D^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{Q} \right] \\ &\leq \sum_{Q \in \mathcal{H}} \Pr_{\mathbf{G}, \mathbf{Q}} \left[ \bigvee_{D \in \tilde{\mathcal{S}}} D^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{Q} \right] \\ &\leq |\mathcal{H}| \cdot \max_{Q \in \mathcal{H}} \Pr_{\mathbf{G}, \mathbf{Q}} \left[ \bigvee_{D \in \tilde{\mathcal{S}}} D^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{Q} \right]. \end{aligned}$$

For each  $Q \in \mathcal{H}$ , by Lemmas 3.10 and 3.15, we have

$$\begin{aligned} & \Pr_{\mathbf{G}, \mathbf{Q}} \left[ \bigvee_{D \in \tilde{\mathcal{S}}} D^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{Q} \right] \\ & \leq \sum_{C' \in \mathcal{S}} \begin{cases} \Pr_{\mathbf{G}, \mathbf{Q}} [(C')^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{Q}] & \text{if } C' \text{ has top fan-in } 1, \\ \Pr_{\mathbf{G}, \mathbf{Q}} [\bigvee_{i \in [m]} (\tilde{C}^{(i)})^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{Q}] & \text{if } C' \text{ has top fan-in } m \geq 2, \end{cases} \\ & \leq \text{size}(\mathbf{C}) \cdot n^{-\alpha(Q) + \beta(Q) + o(1)}. \end{aligned}$$

It follows that

$$\begin{aligned} \Omega(1) & \leq \text{size}(\mathbf{C}) \cdot |\mathcal{H}| \cdot \max_{Q \in \mathcal{H}} n^{-\alpha(Q) + \beta(Q) + o(1)} \\ & = \text{size}(\mathbf{C}) \cdot |\mathcal{H}| \cdot n^{-\kappa_{\alpha, \beta}(P) + o(1)} \\ & = n^{-\varepsilon + o(1)} \end{aligned}$$

(since  $|\mathcal{H}| = O(1)$  and using our assumption that  $\text{size}(\mathbf{C}) = n^{\kappa_{\alpha, \beta}(P) - \varepsilon}$ ). This is the desired contradiction, which completes the proof of the theorem.  $\square$

**4. Bounds on  $\kappa_{\text{col}}(P)$ .** In the previous section, we proved that  $C_{\text{col}}(P) \geq \kappa_{\text{col}}(P)$ , that is,  $n^{\kappa_{\text{col}}(P) - o(1)}$  is a lower bound on the  $\text{AC}^0$  complexity of  $\text{SUBGRAPH}_{\text{col}}(P)$ . In section 4.2 below we will complete the proof of Theorem 1.1 by showing that  $\kappa_{\text{col}}(P) \geq \Omega(\text{tw}(P) / \log \text{tw}(P))$ . But, as a warm-up, let us do a simple combinatorial upper bound on  $\kappa_{\text{col}}(P)$  (and also present a useful construction in Lemma 4.1 that we will need for lower bound proofs).

**4.1. Upper bound on  $\kappa_{\text{col}}(P)$ .** We have already established that  $\kappa_{\text{col}}(P) \leq C_{\text{col}}(P)$  (Corollary 3.2) and  $C_{\text{col}}(P) \leq \text{tw}(P) + 1$  (Lemma 2.3(1)). By these lower and upper bounds in circuit complexity, it follows that  $\kappa_{\text{col}}(P) \leq \text{tw}(P) + 1$ . In this subsection, we give a direct proof that  $\kappa_{\text{col}}(P) \leq \text{tw}(P) + 1$ .

We need the following fact, which shows that the max in the definition  $\kappa_{\text{col}}(P) := \max_{(\alpha, \beta) \in \theta_{\text{col}}(P)} \kappa_{\alpha, \beta}(P)$  is always achieved by some  $(\alpha, \beta) \in \theta_{\text{col}}(P)$  with  $\alpha \equiv 1$ .

LEMMA 4.1. *Assume that  $(\alpha, \beta) \in \theta_{\text{col}}(P)$  and define  $\beta'$  by the formula*

$$\beta'(\{v, w\}) := \beta(\{v, w\}) + \frac{1 - \alpha(v)}{d_P(v)} + \frac{1 - \alpha(w)}{d_P(w)},$$

where  $d_P(v)$  is the degree of the vertex  $v$ . Then  $(1, \beta') \in \theta_{\text{col}}(P)$  and, moreover,  $v(Q) - \beta'(Q) \geq \alpha(Q) - \beta(Q)$  for any  $Q \subseteq P$ .

*Proof.* First,  $\beta(\{v, w\}) \leq \alpha(v) + \alpha(w)$  (by Definition 2.8(i) applied to  $Q := \{u, v\}$ ) and  $d_P(v), d_P(w) \geq 1$ . Hence  $\beta'(\{v, w\}) \leq 2$ . Now, for all  $Q \subseteq P$  we have

$$\begin{aligned} v(Q) - \beta'(Q) & = v(Q) - \sum_{\{v, w\} \in E(Q)} \left( \beta(\{v, w\}) + \frac{1 - \alpha(v)}{d_P(v)} + \frac{1 - \alpha(w)}{d_P(w)} \right) \\ & = \sum_{v \in V(Q)} \underbrace{\left( 1 - \frac{d_Q(v)}{d_P(v)} (1 - \alpha(v)) \right)}_{\geq \alpha(v)} - \sum_{\{v, w\} \in E(Q)} \beta(\{v, w\}) \\ & \geq \alpha(Q) - \beta(Q) \end{aligned}$$

with equality when  $Q = P$ .  $\square$

COROLLARY 4.2. *For all  $P$ , there exists  $\beta : E(P) \rightarrow [0, 2]$  such that  $(1, \beta) \in \theta_{\text{col}}(P)$  and  $\kappa_{\text{col}}(P) = \kappa_{1, \beta}(P)$ .*

*Proof.* Let  $(\alpha, \beta) \in \theta_{\text{col}}(P)$  be such that  $\kappa_{\text{col}}(P) = \kappa_{\alpha, \beta}(P)$ . Then the element  $(1, \beta') \in \theta_{\text{col}}(P)$  constructed from  $(\alpha, \beta)$  as in Lemma 4.1 has the desired property.  $\square$

PROPOSITION 4.3.  $\kappa_{\text{col}}(P) \leq tw(P) + 1$ .

*Proof.* In fact, we will prove  $\kappa_{\text{col}}(P) \leq bw(P)$ , where  $bw(P)$  is the branch-width of  $P$  (it is known that  $bw(P) \leq tw(P) + 1$  by [26]). Recall that a *branch decomposition* of  $P$  is a pair  $(T, b)$ , where  $T$  is a ternary tree and  $b$  is a bijection from  $\text{Leaves}(T)$  to  $E$ . Each edge in  $T$  determines a partition of  $\text{Leaves}(T)$  (and hence of  $E$ ) into two sets. The *width* of  $(T, b)$  is the maximum of  $|V(E_1) \cap V(E_2)|$  over partitions  $E = E_1 \uplus E_2$  determined by the edges of  $T$ , and the *branch-width*  $bw(P)$  is the minimum possible width of a branch decomposition of  $P$ .

Suppose  $bw(P) = k$ . This means that there exists a branch decomposition  $(T, b)$  of width  $k$ . Fix an arbitrary root of  $T$  and let  $x_1, \dots, x_t$  be a postorder traversal of nodes in  $T$  (in particular,  $x_t$  is the root). For every  $i \in [t]$ , let  $Q_i$  and  $\overline{Q_i}$  be the subgraphs of  $P$  with  $E(Q_i) := \{b(y) : y \text{ is a leaf of } T \text{ lying below } x_i\}$  and  $E(\overline{Q_i}) := E(P) \setminus E(Q_i)$  and  $V(Q_i) := \bigcup_{e \in E(Q_i)} e$  and  $V(\overline{Q_i}) := \bigcup_{e \in E(\overline{Q_i})} e$ . Note that  $Q_1, \dots, Q_t$  is a union sequence for  $P$ . Since  $(T, b)$  has width  $\leq k$ , we have  $|V(Q_i) \cap V(\overline{Q_i})| \leq k$  for all  $i \in [t]$ .

By Corollary 4.2, there exists  $\beta : E(P) \rightarrow [0, 2]$  such that  $(1, \beta) \in \theta_{\text{col}}(P)$  and  $\kappa_{\text{col}}(P) = \kappa_{1, \beta}(P)$ . For all  $i \in [t]$ , we have

$$\begin{aligned} v(Q_i) - \beta(Q_i) &\leq v(Q_i) - \beta(Q_i) + v(\overline{Q_i}) - \beta(\overline{Q_i}) && \text{(since } v(\overline{Q_i}) \geq \beta(\overline{Q_i})\text{)} \\ &= v(Q_i) + v(\overline{Q_i}) - v(P) && \text{(since } \beta(Q_i) + \beta(\overline{Q_i}) = \beta(P) = v(P)\text{)} \\ &= |V(Q_i) \cap V(\overline{Q_i})| \leq k. \end{aligned}$$

Therefore,  $\kappa_{\text{col}}(P) = \kappa_{1, \beta}(P) \leq \max_{i \in [t]} v(Q_i) - \beta(Q_i) \leq k \leq bw(P) \leq tw(P) + 1$ .  $\square$

**4.2. Lower bounds on  $\kappa_{\text{col}}(P)$ .** We give two lower bounds on  $\kappa_{\text{col}}(P)$ . The first applies to all patterns  $P$ .

THEOREM 4.4.  $\kappa_{\text{col}}(P) \geq \Omega(tw(P) / \log tw(P))$ .

Together with the fact that  $C_{\text{col}}(P) \geq \kappa_{\text{col}}(P)$  (Corollary 3.2), this completes the proof of our main theorem (Theorem 1.1).

Our proof of Theorem 4.4 uses a characterization of treewidth from Marx [21] (based on results of Feige, Hajiaghai, and Lee [12]): for every  $P$  with  $tw(P) = k$ , there is a subset  $W \subseteq V(P)$  of size  $|W| = \Omega(k)$  and a concurrent flow on  $P$  which routes  $\Omega(1/k \log k)$  flow between every pair of distinct vertices in  $W$  (Lemma 4.8). Given such a concurrent flow on  $P$ , we construct a corresponding threshold pair  $(\alpha, \beta) \in \theta_{\text{col}}(P)$  and show that  $\kappa_{\alpha, \beta}(P)$  gives the desired bound.

We also include a lower bound on  $\kappa_{\text{col}}(P)$  in terms of the expansion of  $P$  (Theorem 4.9), which improves Theorem 4.4 in the case where  $P$  is a constant-degree expander.

DEFINITION 4.5.

- (i) Let  $\text{Paths}(P)$  denote the set of paths in  $P$  (i.e., subgraphs of  $P$  isomorphic to an (undirected, simple) path of length  $\geq 1$ ).
- (ii) Let  $\text{Flows}(P)$  denote the set of concurrent flows on  $P$  with node-capacity 1, that is, functions  $f : \text{Paths}(P) \rightarrow [0, 1]$  such that for all  $v \in V(P)$ ,  $\sum_{\substack{\pi \in \text{Paths}(P) \\ v \in V(\pi)}} f(\pi) \leq 1$ .

- (iii) For  $f \in \text{Flows}(P)$  and disjoint  $S, T \subseteq V(P)$ , let  $f(S, T)$  denote the total flow that  $f$  sends between  $S$  and  $T$ , that is,

$$f(S, T) := \sum_{\substack{\pi \in \text{Paths}(P) \\ \pi \text{ has endpoints in } S \text{ and } T}} f(\pi).$$

For two distinct vertices  $v, w$ , we let  $f(v, w) := f(\{v\}, \{w\})$ .

- (iv) For  $\pi \in \text{Paths}(P)$ , define  $\alpha_\pi : V(P) \rightarrow [0, 1]$  and  $\beta_\pi : E(P) \rightarrow [0, 2]$  by

$$\alpha_\pi(v) := \begin{cases} 1/2 & \text{if } v \text{ is an endpoint of } \pi, \\ 1 & \text{if } v \text{ is an interior vertex of } \pi, \\ 0 & \text{if } v \notin V(\pi), \end{cases} \quad \beta_\pi(e) := \begin{cases} 1 & \text{if } e \in E(\pi), \\ 0 & \text{if } e \notin E(\pi). \end{cases}$$

- (v) For  $f \in \text{Flows}(P)$ , define  $\alpha_f : V(P) \rightarrow [0, 1]$  and  $\beta_f : E(P) \rightarrow [0, 2]$  by

$$\alpha_f(v) := \sum_{\pi \in \text{Paths}(P)} f(\pi) \cdot \alpha_\pi(v), \quad \beta_f(e) := \sum_{\pi \in \text{Paths}(P)} f(\pi) \cdot \beta_\pi(e).$$

LEMMA 4.6.  $(\alpha_f, \beta_f) \in \theta_{\text{col}}(P)$  for all  $f \in \text{Flows}(P)$ .

*Proof.* Clearly,  $\alpha_\pi(P) = \beta_\pi(P)$  ( $= |E(\pi)|$ ) and  $\alpha_\pi(Q) \geq \beta_\pi(Q)$  for all  $Q \subseteq P$  and  $\pi \in \text{Paths}(P)$ .  $(\alpha_f, \beta_f) \in \theta_{\text{col}}(P)$  follows by convexity.  $\square$

LEMMA 4.7. For all  $Q \subseteq P$  and  $f \in \text{Flows}(P)$ ,

$$\alpha_f(Q) - \beta_f(Q) \geq \frac{1}{2}f(V(Q), \overline{V(Q)}).$$

*Proof.* Note that  $f(S, T) = \sum_{\pi \in \text{Paths}(P)} f(\pi) \cdot \chi_\pi(S, T)$ , where

$$\chi_\pi(S, T) := \begin{cases} 1 & \text{if } \pi \text{ has one endpoint in } S \text{ and another in } T, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, it suffices to show, for all  $\pi \in \text{Paths}(P)$ , that

$$(16) \quad \alpha_\pi(Q) - \beta_\pi(Q) \geq \frac{1}{2}\chi_\pi(V(Q), \overline{V(Q)}).$$

If both endpoints of  $\pi$  belong to the same set among  $V(Q), \overline{V(Q)}$ , then  $\frac{1}{2}\chi_\pi(V(Q), \overline{V(Q)}) = 0$ , while  $\alpha_\pi(Q) - \beta_\pi(Q) \geq 0$  by Lemma 4.6 (since  $(\alpha_\pi, \beta_\pi) \in \theta_{\text{col}}(P)$ ); so (16) holds. On the other hand, if  $\pi$  has one endpoint in  $V(Q)$  and another in  $\overline{V(Q)}$ , then  $\frac{1}{2}\chi_\pi(V(Q), \overline{V(Q)}) = \frac{1}{2}$ , while

$$\alpha_\pi(Q) - \beta_\pi(Q) \geq \frac{1}{2}|\{\text{edges of } \pi \text{ that cross between } V(Q) \text{ and } \overline{V(Q)}\}| \geq \frac{1}{2},$$

so again (16) holds.  $\square$

Our lower bound on  $\kappa_{\text{col}}(P)$  relies on a characterization of treewidth in terms of concurrent flows.

LEMMA 4.8. If  $P$  has treewidth  $k$ , then there exists  $W \subseteq V(P)$  with  $|W| \geq 2k/3$  and  $f \in \text{Flows}(P)$  such that  $f(v, w) \geq 1/ck \log k$  for all distinct  $v, w \in W$ , where  $c > 0$  is a universal constant.

*Proof.* This lemma is implicit in [12, 21]. Utilizing the notation from the latter paper, [21, Lemma 3.2(a)] implies that there exists  $W \subseteq V(P)$  with  $|W| > 2k/3$  that has no balanced  $k/3$ -separator. By [21, Lemma 3.3],  $\alpha^w(G) \geq (\frac{4}{3}k + 1)^{-1}$ . By the contrapositive of [21, Theorem 3.5], all solutions to the linear program (LP2) are  $\Omega(1/k \log k)$ , hence its dual (LP1) has a solution  $\alpha \geq \Omega(1/k \log k)$ , and it gives us the required concurrent flow.  $\square$

*Proof of Theorem 4.4.* Suppose  $tw(P) = k$  and fix  $W \subseteq V(P)$  and  $f \in \text{Flows}(P)$  as in Lemma 4.8. Let  $\mathcal{H}$  be the set of subgraphs  $Q \subseteq P$  such that  $2k/9 \leq |W \cap V(Q)| \leq 4k/9$ . Clearly  $\mathcal{H}$  is a hitting set for  $P$  (i.e., every union sequence for  $P$  contains a graph in this set). For every  $Q \in \mathcal{H}$ , we have

$$\begin{aligned} \alpha_f(Q) - \beta_f(Q) &\geq \frac{f(V(Q), \overline{V(Q)})}{2} \geq \frac{f(W \cap V(Q), W \setminus V(Q))}{2} \\ &\geq \frac{|W \cap V(Q)| \cdot |W \setminus V(Q)|}{2ck \log k} \geq \frac{4k}{81c \log k} = \Omega\left(\frac{k}{\log k}\right). \end{aligned}$$

Therefore,  $\kappa_{\text{col}}(P) \geq \kappa_{\alpha_f, \beta_f}(P) = \Omega(k/\log k)$ .  $\square$

**Tight lower bound for expanders.** We conclude this section by giving a second lower bound on  $\kappa_{\text{col}}(P)$  in terms of edge expansion; this sometimes gives the optimal  $\Omega(tw(P))$  lower bound in the case that  $P$  is a good expander such as  $K_k$  or  $G_{k,k}$ . Let  $\Delta(P)$  denote the maximum degree of  $P$ . For  $S \subseteq V(P)$ , let  $e_P(S, \overline{S}) := |\{\{v, w\} \in E(P) : v \in S \text{ and } w \in V(P) \setminus S\}|$ . Recall that the *edge expansion* of  $P$  is defined by

$$h(P) := \min_{S: \emptyset \subset S \subset V(P)} \frac{e_P(S, \overline{S})}{\min\{|S|, |\overline{S}|\}}.$$

THEOREM 4.9.  $\kappa_{\text{col}}(P) \geq \frac{h(P)v(P)}{3\Delta(P)}$ .

*Proof.* Let us apply the construction from Lemma 4.1 to the pair  $(0, 0) \in \theta_{\text{col}}(P)$ . This gives us the function  $\beta : E(P) \rightarrow [0, 2]$  defined by

$$\beta(\{v, w\}) := \frac{1}{d_P(v)} + \frac{1}{d_P(w)}$$

such that  $(1, \beta) \in \theta_{\text{col}}(P)$ .

Consider the hitting set  $\mathcal{H}$  consisting of subgraphs  $Q \subseteq P$  such that  $\frac{1}{3}v(P) \leq v(Q) \leq \frac{2}{3}v(P)$ . For every  $Q \in \mathcal{H}$  the calculation in the proof of Lemma 4.1 gives us

$$\begin{aligned} v(Q) - \beta(Q) &= \sum_{v \in V(Q)} \left(1 - \frac{d_Q(v)}{d_P(v)}\right) \geq \frac{1}{\Delta(P)} \sum_{v \in V(Q)} (d_P(v) - d_Q(v)) \\ &= \frac{e_P(V(Q), \overline{V(Q)})}{\Delta(P)} \\ &\geq \frac{h(P) \min\{v(Q), v(P) - v(Q)\}}{\Delta(P)} \\ &\geq \frac{h(P)v(P)}{3\Delta(P)}. \end{aligned}$$

Completing the proof,

$$\kappa_{\text{col}}(P) \geq \kappa_{1, \beta}(P) \geq \min_{Q \in \mathcal{H}} (v(Q) - \beta(Q)) \geq \frac{h(P)v(P)}{3\Delta(P)}. \quad \square$$

**5. Minor-monotonicity and monotone projections.** In this section, we prove that  $\kappa_{\text{col}}(P)$  and  $C_{\text{col}}(P)$  are minor-monotone graph parameters. First, a few definitions.

Recall that a *minor* of  $G$  is any graph that can be obtained from  $G$  by a sequence of vertex deletions, edge deletions, and edge contractions. A real-valued graph parameter  $f$  is *minor-monotone* if  $f(G) \leq f(G')$  whenever  $G$  is a minor of  $G'$ .

**THEOREM 5.1.**  *$\kappa_{\text{col}}$  and  $C_{\text{col}}$  are minor-monotone.*

The algorithmic problem  $\text{SUBGRAPH}_{\text{col}}(P)$  was defined in section 2.3 in such a way that the coloring  $\chi : G \rightarrow P$  is a part of its input. We first observe that the parameter  $C_{\text{col}}(P)$  does not change if we consider instead its more structured version  $\text{SUBGRAPH}_{\text{col},n}(P)$  in which (cf. Definition 2.8) we demand that the target graph  $G$  has the vertex set  $V(P) \times [n]$ , and  $\chi$  is the projection onto the first coordinate. An easy  $\text{AC}^0$ -reduction from  $\text{SUBGRAPH}_{\text{col}}(P)$  to  $\text{SUBGRAPH}_{\text{col},n}(P)$  works as follows. Assume that we are given an input  $(G, \chi)$  to the problem  $\text{SUBGRAPH}_{\text{col}}(P)$ , and assume w.l.o.g. that  $V(G) = [n]$ . We map it to the pair  $(G', \chi')$ , where  $V(G') := V(P) \times [n]$ ,  $\chi'$  is the projection onto the first coordinate, and  $E(G')$  is defined as follows:  $E(G') := \{(\chi(i), i), (\chi(j), j)\} : \{i, j\} \in E(G)\}$ . (Thus, all vertices  $(v, i)$  with  $v \neq \chi(i)$  remain isolated.) Hence Theorem 5.1 readily follows from the following lemma.<sup>12</sup>

**LEMMA 5.2.** *Suppose  $P$  is a minor of  $P'$ . Then*

1. *for every  $(\alpha, \beta) \in \theta_{\text{col}}(P)$ , there exists  $(\alpha', \beta') \in \theta_{\text{col}}(P')$  such that  $\kappa_{\alpha, \beta}(P) \leq \kappa_{\alpha', \beta'}(P')$ ,*
2.  $\text{SUBGRAPH}_{\text{col},n}(P) \leq_{\text{mp}} \text{SUBGRAPH}_{\text{col},n}(P')$ .

*Proof.* It suffices to show that the lemma holds in the two cases where  $P$  is a subgraph of  $P'$  and where  $P$  is obtained from  $P'$  by contracting a single edge  $\{x, y\}$  where  $x, y$  have no common neighbors. (Otherwise, we perform necessary edge deletions before contraction.)

*Subgraph case.* Suppose  $P$  is a subgraph of  $P'$ .

For (1). Consider any  $(\alpha, \beta) \in \theta_{\text{col}}(P)$ . Define  $\alpha' : V(P') \rightarrow [0, 1]$  and  $\beta' : E(P') \rightarrow [0, 2]$  by

$$\alpha'(v) := \begin{cases} \alpha(v) & \text{if } v \in V(P), \\ 0 & \text{otherwise,} \end{cases} \quad \beta'(e) := \begin{cases} \beta(e) & \text{if } e \in E(P), \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that  $(\alpha', \beta') \in \theta_{\text{col}}(P')$  and  $\kappa_{\alpha, \beta}(P) = \kappa_{\alpha', \beta'}(P')$ .

For (2). The monotone projection  $p$  is defined as follows:

$$p(\{(v, i), (w, j)\}) := \begin{cases} \{(v, i), (w, j)\} & \text{if } \{v, w\} \in E(P), \\ 1 & \text{if } \{v, w\} \in E(P') \setminus E(P). \end{cases}$$

Thus,  $p^*$  takes an input  $G$  to the problem  $\text{SUBGRAPH}_{\text{col},n}(P)$  and converts it into an input  $G'$  to  $\text{SUBGRAPH}_{\text{col},n}(P')$  by filling in complete bipartite graphs between  $\{v\} \times [n]$  and  $\{w\} \times [n]$  for all new edges  $\{v, w\} \in E(P') \setminus E(P)$ .

*Contraction case.* Now suppose  $P$  is obtained from  $P'$  by contracting a single edge  $\{x, y\}$  where  $x, y$  have no common neighbors. Let  $z$  label the contracted vertex in  $P$ , so that  $V(P) \setminus V(P') = \{z\}$  and  $V(P') \setminus V(P) = \{x, y\}$ . Let  $\rho : V(P') \rightarrow V(P)$  be the function  $x, y \mapsto z$  and  $v \mapsto v$  for all  $v \in V(P') \setminus \{x, y\}$ . For  $e = \{v, w\} \in E(P') \setminus \{x, y\}$ , let  $\rho(e) := \{\rho(v), \rho(w)\} \in E(P)$  ( $\rho(\{x, y\})$  is undefined).

<sup>12</sup>A somewhat similar (de)construction recently appeared in [9].

For (1). Consider any  $(\alpha, \beta) \in \theta_{\text{col}}(P)$ . Define  $\alpha' : V(P') \rightarrow [0, 1]$  and  $\beta' : E(P') \rightarrow [0, 2]$  by

$$\alpha'(v) := \alpha(\rho(v)), \quad \beta'(e) := \begin{cases} \alpha(z) & \text{if } e = \{x, y\}, \\ \beta(\rho(e)) & \text{otherwise.} \end{cases}$$

We now check that  $(\alpha', \beta') \in \theta_{\text{col}}(P')$  and  $\kappa_{\alpha', \beta'}(P') \geq \kappa_{\alpha, \beta}(P)$ . For that consider the mapping  $\widehat{\rho} : Q' \mapsto \rho(Q' \setminus \{\{x, y\}\})$  that takes subgraphs of  $P'$  to subgraphs of  $P$ . It is easy to see that  $\alpha'(Q') - \beta'(Q') \geq \alpha(\widehat{\rho}(Q')) - \beta(\widehat{\rho}(Q'))$  and that this is tight for  $Q' = P'$ : in the only nontrivial case  $\{x, y\} \in E(Q')$  we have  $\alpha(\widehat{\rho}(Q')) = \alpha'(Q') - \alpha(z)$  and  $\beta(\widehat{\rho}(Q')) = \beta'(Q') - \alpha(z)$ . This proves the first claim  $(\alpha', \beta') \in \theta_{\text{col}}(P')$ . To see that  $\kappa_{\alpha, \beta}(P) \leq \kappa_{\alpha', \beta'}(P')$ , it suffices to observe that  $\widehat{\rho}$  takes union sequences for  $P'$  into union sequences for  $P$  and thus  $\widehat{\rho}^{-1}$  maps hitting sets for  $P$  into hitting sets for  $P'$ .

For (2). This time the monotone projection  $p$  is defined by

$$p(\{(v, i), (w, j)\}) := \begin{cases} 1 & \text{if } \{v, w\} = \{x, y\} \text{ and } i = j, \\ 0 & \text{if } \{v, w\} = \{x, y\} \text{ and } i \neq j, \\ \{(\rho(v), i), (\rho(w), j)\} & \text{otherwise.} \end{cases}$$

(That is,  $p^*$  duplicates  $\{z\} \times [n]$  into two sets  $\{x\} \times [n]$ ,  $\{y\} \times [n]$  and then plants a perfect matching between twins.) This  $p$  is clearly a monotone projection from  $\text{SUBGRAPH}_{\text{col}, n}(P)$  to  $\text{SUBGRAPH}_{\text{col}, n}(P')$ .  $\square$

**5.1. Negative results in the uncolored setting.** In the colored setting, we have seen that  $\text{SUBGRAPH}_{\text{col}}(P)$  is minor-monotone via linear-size monotone projections. Highlighting a difference between the uncolored and colored settings, we now show that there is *no* monotone projection whatsoever that reduces  $\text{SUBGRAPH}(M_3)$  to  $\text{SUBGRAPH}(P_3 + M_2)$  (where  $P_3$  is a path on three vertices and  $M_k$  is a matching with  $k$  edges). While it remains an open problem whether  $C(P)$  is (even approximately) minor-monotone under general AC<sup>0</sup> reductions, this result strongly suggests that the colorful version of the subgraph isomorphism problem is much better structured and well-behaved than the standard (uncolored) one.

We begin with some properties of  $P_3 + M_2$ -free graphs.

LEMMA 5.3. *Every  $P_3 + M_2$ -free graph  $G$  satisfies one of the following conditions:*

- (i)  $G$  has  $\leq C$  edges for some absolute constant  $C$ .
- (ii)  $G$  is a matching.
- (iii)  $G$  contains a vertex of degree  $\geq 6$ .

Note that Lemma 5.3 is true with any integer replacing 6 in (iii), for an appropriate constant  $C$  in (i). The choice of 6 is what we need in the proof of Theorem 5.6 later on.

*Proof.* Assume  $G$  is  $P_3 + M_2$ -free, not a matching, and has maximum degree  $\leq 5$ . We will show that  $G$  has  $O(1)$  edges. Since  $G$  is not a matching, it contains a vertex  $u$  of degree  $\geq 2$ . Since  $G$  has maximum degree  $\leq 5$ , there is a constant  $C'$  such that if  $G$  has  $> C'$  nonisolated vertices, then it contains nonisolated vertices  $v, w$  such that any two of  $u, v, w$  have distance  $\geq 3$ ; in that case, we would have  $P_3 + M_2$ -subgraph of  $G$  by taking any two edges containing  $u$  plus any two edges containing  $v$  and  $w$ , respectively. Therefore,  $G$  has  $\leq C'$  nonisolated vertices. It follows that  $G$  has  $\leq 5C'/2$  edges.  $\square$

LEMMA 5.4. *If  $G$  is  $P_3 + M_2$ -free and contains an  $M_4$ -subgraph, then  $G$  is a matching.*

*Proof.* Suppose  $G$  contains an  $M_4$ -subgraph  $H$ , but  $G$  is not a matching. We will show that  $G$  contains a  $P_3 + M_2$ -subgraph. Since  $G$  is not a matching, it contains a  $P_3$ -subgraph  $K$ . If  $K$  is vertex-disjoint from  $H$ , then  $K$  plus any two edges from  $H$  is a  $P_3 + M_2$ -subgraph of  $G$ . Now assume that  $K$  contains a vertex in  $H$ . Then there is a  $P_3$ -subgraph  $K'$  which contains an edge in  $H$ . This  $K'$  is vertex-disjoint from at least two edges in  $H$ ; then  $K'$  plus these two edges is a  $P_3 + M_2$ -subgraph of  $G$ .  $\square$

LEMMA 5.5. *Suppose  $G$  contains a  $P_3 + M_2$ -subgraph and a vertex  $u$  of degree  $\geq 6$ . Then  $G$  contains a  $P_3 + M_2$ -subgraph in which  $u$  is the degree-2 vertex.*

*Proof.* Let  $H$  be any  $P_3 + M_2$ -subgraph of  $G$ .  $H$  contains an  $M_2$ -subgraph  $H'$  which does not include the vertex  $u$ . Since  $u$  has degree  $\geq 6$ , it has two distinct neighbors  $v$  and  $w$  such that  $\{u, v, w\} \cap V(H') = \emptyset$ . Then  $H'$  plus edges  $\{u, v\}$  and  $\{u, w\}$  is a  $P_3 + M_2$ -subgraph of  $G$  in which  $u$  is the degree-2 vertex.  $\square$

Now we have the main result of this subsection.

THEOREM 5.6. *SUBGRAPH( $M_3$ ) is not a monotone projection of SUBGRAPH( $P_3 + M_2$ ).*

*Proof.* Toward a contradiction, assume there exists a monotone projection  $p : \binom{[N]}{2} \rightarrow \binom{[n]}{2} \cup \{0, 1\}$  from SUBGRAPH( $M_3$ ) on  $n$ -vertex graphs to SUBGRAPH( $P_3 + M_2$ ) on  $N$ -vertex graphs for some  $n, N \in \mathbb{N}$ , where  $n \geq C + 2$  with  $C$  the constant from Lemma 5.3. That is, for every graph  $G$  with vertex set  $[n]$ , we have

$$G \text{ contains an } M_3\text{-subgraph} \Leftrightarrow p^*(G) \text{ contains a } P_3 + M_2\text{-subgraph,}$$

where  $p^*(G)$  is the graph with edge set  $p^{-1}(E(G) \cup \{1\})$ . Note that since the predicate in the left-hand side essentially depends on all variables  $e \in \binom{[n]}{2}$ , so must the function  $p^*$  or, in other words,  $p^{-1}(e)$  is nonempty for any  $e \in \binom{[n]}{2}$ .

For  $a \in [n]$ , let  $S_a$  denote the  $n$ -vertex star centered at  $a$  (i.e., with edge set  $\{e \in \binom{[n]}{2} : a \in e\}$ ). Let  $F_a := p^{-1}(S_a)$  (so that  $p^*(S_a)$  is the disjoint union of  $F_a$  and  $p^{-1}(1)$ ). Over the next few claims, we will show that  $F_a$  are stars of degree  $\geq 6$  with distinct centers. Since all  $p^{-1}(e)$  are nonempty,  $F_a$  contains at least  $n - 1$  ( $> C$ ) edges.

Since  $S_a$  is  $M_3$ -free,  $p^*(S_a)$  is  $P_3 + M_2$ -free, hence  $F_a$  is  $P_3 + M_2$ -free. By Lemma 5.3, it follows that either  $F_a$  is a matching or  $F_a$  contains a vertex of degree  $\geq 6$ . The next claim eliminates the first possibility.

CLAIM 5.7. *For every  $a \in [n]$ ,  $F_a$  is not a matching.*

For contradiction, assume  $F_a$  is a matching for some  $a \in [n]$ . Consider any  $b \in [n]$ . Note that  $S_a \cup S_b$  is  $M_3$ -free, hence  $p^*(S_a \cup S_b)$  is  $P_3 + M_2$ -free. Since  $p^*(S_a \cup S_b) \supseteq F_a \cup F_b$  and  $F_a$  contains an  $M_4$ -subgraph, Lemma 5.4 implies that  $F_a \cup F_b$  is a matching. In particular,  $F_b$  is a matching for any  $b \in [n]$ , and we can repeat the above argument with  $a := b$  to conclude that  $F_b \cup F_c$  is a matching for all  $b, c \in [n]$ . Therefore, the entire preimage  $p^{-1}(K_n)$  is a matching, where  $K_n$  is the complete graph on vertices  $[n]$ .

Since  $K_n$  contains an  $M_3$ -subgraph,  $p^*(K_n)$  ( $= p^{-1}(K_n) \cup p^{-1}(1)$ ) contains a  $P_3 + M_2$ -subgraph. It follows that either  $p^{-1}(1)$  contains a path of length 2 or  $p^{-1}(1)$  contains an edge with an endpoint in  $V(p^{-1}(K_n))$ . In both cases we get a contradiction, as it follows that  $p^*(S_c)$  contains a  $P_3 + M_2$ -subgraph for some  $c \in [n]$ , even though  $S_c$  is  $M_3$ -free. (If  $p^{-1}(1)$  contains a  $P_3$ -subgraph, then any  $c \in [n]$  will do; if  $p^{-1}(1)$  contains an edge with an endpoint  $v \in V(p^{-1}(K_n))$ , then any  $c \in [n]$  with  $v \in V(F_c)$  will do.)

For all  $a \in [n]$ , we have established that  $F_a$  is  $P_3 + M_2$ -free, has  $> C$  edges, and is not a matching. By Lemma 5.3, we conclude that  $F_a$  contains at least one vertex of degree  $\geq 6$ . Let us now fix a function  $z : [n] \rightarrow [N]$  such that  $z(a)$  is a vertex of degree  $\geq 6$  in  $F_a$  for all  $a \in [n]$ .

CLAIM 5.8.  $z$  is  $(\leq 2)$ -to-1.

For contradiction, assume there exist distinct  $a, b, c \in [n]$  such that  $v := z(a) = z(b) = z(c)$ . By Lemma 5.5,  $p^*(S_a \cup S_b \cup S_c)$  contains a  $P_3 + M_2$ -subgraph in which  $v$  is the degree-2 vertex. Let  $e, f \in \binom{[N]}{2}$  be the two edges in this subgraph which are not adjacent to  $v$ . Without loss of generality,  $\{e, f\} \subseteq p^*(S_a \cup S_b)$ . Since  $v$  has degree  $\geq 6$  in  $p^*(S_a \cup S_b)$ , we can find a different path of length 2 through  $v$  which is vertex-disjoint from edges  $e$  and  $f$ . Therefore,  $p^*(S_a \cup S_b)$  contains a  $P_3 + M_2$ -subgraph. Since  $S_a \cup S_b$  is  $M_3$ -free, this contradicts our assumption about  $p$ .

CLAIM 5.9.  $F_a$  is a star with center  $z(a)$  for all  $a \in [n]$ .

For contradiction, assume  $F_a$  is not a star with center  $z(a)$ . Then  $F_a$  contains an edge  $e$  with  $z(a) \notin e$ . Since  $z$  is  $(\leq 2)$ -to-1, there exists  $b \in [n]$  such that  $z(b) \notin \{z(a)\} \cup e$ . We may find a  $P_3 + M_2$ -subgraph within  $F_a \cup F_b$  by taking  $e$  together with a disjoint path of length 2 through  $z(a)$  and a disjoint edge containing  $z(b)$ . This contradicts the fact that  $p^*(S_a \cup S_b)$  is  $P_3 + M_2$ -free.

CLAIM 5.10.  $z$  is 1-to-1.

For contradiction, assume  $v := z(a) = z(b)$  for some  $a \neq b$ . Let  $c \in [n] \setminus \{a, b\}$ . Then  $z(c) \neq v$  and  $p^*(S_a \cup S_b \cup S_c) = F_a \cup F_b \cup F_c \cup p^{-1}(1)$  contains a  $P_3 + M_2$ -subgraph  $H$ . We may assume that  $H$  contains edges  $\{v, u\} \in E(F_a) \setminus E(F_b)$  and  $\{v, w\} \in E(F_b) \setminus E(F_a)$  since otherwise  $H$  would be a subgraph of either  $p^*(S_a \cup S_c)$  or  $p^*(S_b \cup S_c)$  contradicting  $P_3 + M_2$ -freeness of these graphs. Note that  $u \neq w$ . Since  $v$  has degree  $\geq 6$  in  $F_a$ , we can find an edge  $\{v, w'\} \in E(F_a)$  such that  $w' \notin V(H)$ . Let  $H'$  be the graph obtained by substituting the edge  $\{v, w'\}$  for  $\{v, w\}$ . Then  $H'$  is a  $P_3 + M_2$ -subgraph of  $p^*(S_a \cup S_c)$ , which is again a contradiction.

At this point, we have established that graphs  $F_a$  ( $a \in [n]$ ) are stars of degree  $\geq 6$  with distinct centers.

CLAIM 5.11.  $|p^{-1}(e)| = 1$  for all  $e \in \binom{[N]}{2}$ .

Suppose  $e = \{a, b\}$ . Since  $F_a$  and  $F_b$  are stars with different centers and  $p^{-1}(e) \subseteq F_a \cap F_b$ , we conclude  $|p^{-1}(e)| \leq 1$ . Since  $p^{-1}(e)$  is nonempty, it follows that  $|p^{-1}(e)| = 1$ .

CLAIM 5.12.  $p^{-1}(1)$  is nonempty.

Let  $G$  be any copy of  $M_3$  (i.e., any three disjoint edges) among  $n$ -vertices. Then  $p^{-1}(G)$  has only three edges by Claim 5.11. Since  $p^*(G) = p^{-1}(G) \cup p^{-1}(1)$  has  $P_3 + M_2$ -subgraph, it contains at least four edges. Therefore,  $p^{-1}(1)$  is nonempty.

Fix any edge  $e$  in  $p^{-1}(1)$  and any  $a \neq b \in [n]$  such that  $z(a), z(b) \notin e$ . Then  $p^*(S_a \cup S_b)$  contains a  $P_3 + M_2$ -subgraph, even though  $S_a \cup S_b$  is  $M_3$ -free. This, finally, is the contradiction which completes the proof of Theorem 5.6.  $\square$

**6. Conclusion.** With the results of this paper, the state of knowledge on the average-/worst-case AC<sup>0</sup> complexity of the uncolored-/colorful  $P$ -subgraph isomorphism problem now stands as

$$\begin{aligned} \Omega\left(\frac{tw(P)}{\log tw(P)}\right) &\leq \kappa_{\text{col}}(P) \leq C_{\text{col}}(P) \leq tw(P) + 1 \\ &\quad \vee \\ &\quad C(P) \\ &\quad \vee \\ \kappa(P) &\leq C_{\text{ave}}(P) \leq 2\kappa(P) + O(1). \end{aligned}$$

We have examples showing that the gap between  $C_{\text{ave}}(P)$  and  $C(P)$  (i.e., the average-case versus worst-case  $\text{AC}^0$  complexity of  $\text{SUBGRAPH}(P)$ ) can be arbitrarily large (see Remark 2.7). We do not know of any gap between  $C(P)$  and  $C_{\text{col}}(P)$ . Equivalently, we can ask whether  $C(P)$  is bounded from below by any function of  $tw(P)$ . We restate Question 1 from the introduction.

**Question 1.** Is it possible to give general lower bounds on the worst-case  $\text{AC}^0$  complexity of  $\text{SUBGRAPH}(P)$  (uncolored  $P$ -subgraph isomorphism) in terms of the treewidth of  $P$  only?

When  $P$  is a core, we know that  $C(P) = C_{\text{col}}(P) = \tilde{\Theta}(tw(P))$ . At the opposite end of the spectrum, Question 1 is wide open for bipartite patterns  $P$ .

The next two questions seek to improve the parameters in our main results.

**Question 2.** Can the upper bound  $C_{\text{ave}}(P) \leq 2\kappa(P) + O(1)$  of Theorem 1.2 be improved to  $\kappa(P) + O(1)$ ?

**Question 3.** Can the  $\log tw(P)$  factor be eliminated from our lower bounds on  $\kappa_{\text{col}}(P)$  (Theorem 1.1) or at least  $C_{\text{col}}(P)$ ?

We are able to answer Question 3 affirmatively in the special case where  $P$  is a constant-degree expander (Theorem 4.9).

Another question raised by this work is whether the  $\text{AC}^0$  complexity of  $\text{SUBGRAPH}(P)$  is monotone with respect to minors or subgraphs. In contrast to the colorful setting, we showed that monotone projections (the simplest form of reduction) fail to give any reduction whatsoever from  $\text{SUBGRAPH}(Q)$  to  $\text{SUBGRAPH}(P)$ , even when  $Q$  is only a subgraph of  $P$ .

**Question 4.** Is  $C(P)$  minor-monotone or at least monotone under subgraphs?

More modestly, if  $Q$  is a minor (or subgraph) of  $P$ , is there a reduction from  $\text{SUBGRAPH}(Q)$  to  $\text{SUBGRAPH}(P)$  by  $\text{AC}^0$ -circuits of size  $O(n^c)$  for a constant  $c$  independent of  $P$  and  $Q$ ? That would imply  $C(Q) \leq O(C(P))$ ; currently we do not know if  $C(Q)$  can be bounded by *any* function in  $C(P)$ .

Finally, it would be interesting to investigate the relationship between  $\kappa_{\text{col}}(P)$  and the complexity of  $\text{SUBGRAPH}_{\text{col}}(P)$  beyond  $\text{AC}^0$ . In particular, we recall the result of Marx [21] that  $\text{SUBGRAPH}_{\text{col}}(P)$  has no  $n^{o(tw(P)/\log tw(P))}$ -time algorithm unless the ETH fails. Follow-up work of Alon and Marx [1] looked at the question of removing the  $\log tw(P)$  factor loss in the exponent of this result (toward the goal of showing that  $n^{\Theta(tw(P))}$  is the true complexity of  $\text{SUBGRAPH}_{\text{col}}(P)$ , at least assuming the ETH). Alon and Marx specifically identified constant-degree expanders as a case where “substantially different methods” are needed to eliminate the  $\log tw(P)$  factor loss incurred by the reduction of [21]. In light of our lower bounds  $C_{\text{col}}(P) \geq \kappa_{\text{col}}(P) = \Omega(|V(P)|)$  when  $P$  is a constant-degree expander, it becomes interesting to ask the following.

**Question 5.** Can it be shown that  $\text{SUBGRAPH}_{\text{col}}(P)$  has no  $n^{o(\kappa_{\text{col}}(P))}$ -time algorithm unless the ETH fails?

**Appendix A. Proof of Lemma 2.10.** Fix a pattern  $P$  and a nontrivial threshold pair  $(\alpha, \beta) \in \theta_{\text{col}}(P)$ . We can assume w.l.o.g. that  $\beta(e) > 0$  for all  $e \in E(P)$

(as the edges with  $\beta(e) = 0$  can be removed). Following the approach of Bollobás and Wierman [7], we fix a chain of (necessarily induced) subgraphs

$$\emptyset = Q_0 \subset Q_1 \subset \dots \subset Q_{t-1} \subset Q_t = P$$

satisfying

- $\alpha(Q_i) = \beta(Q_i)$  for all  $0 \leq i \leq t$ , and
- $\alpha(R) > \beta(R)$  for all  $1 \leq i \leq t$  and  $Q_{i-1} \subset R \subset Q_i$ .

Call such a sequence  $(Q_0, \dots, Q_t)$  an  $(\alpha, \beta)$ -grading of  $P$ . Clearly, at least one  $(\alpha, \beta)$ -grading exists. Note that  $1 \leq t \leq |E(P)|$ , since  $(\alpha, \beta)$  is nontrivial. (It is known that  $t$  is the same for all  $(\alpha, \beta)$ -gradings; however, we will not use this fact.)

Let  $\mathbf{G} := G_{\alpha, \beta}(P)$ . For  $0 \leq i \leq t$ , define random variable  $\mathbf{X}_i$  as the number of  $Q_i$ -subgraphs in  $\mathbf{G}$ . Obviously,  $\mathbf{X}_0 = 1$  (with probability 1). For  $1 \leq i \leq t$ , let  $\mathcal{L}(\mathbf{X}_i \mid \mathbf{X}_{i-1} = 1)$  denote the distribution of  $\mathbf{X}_i$  conditioned on the event  $\mathbf{X}_{i-1} = 1$ . We prove Lemma 2.10 by showing the following.

LEMMA A.1. *For all  $1 \leq i \leq t$ ,  $\mathcal{L}(\mathbf{X}_i \mid \mathbf{X}_{i-1} = 1)$  is asymptotically the Poisson distribution  $\text{Po}(1)$ . In particular,*

$$\lim_{n \rightarrow \infty} \Pr[\mathbf{X}_i = 0 \mid \mathbf{X}_{i-1} = 1] = \lim_{n \rightarrow \infty} \Pr[\mathbf{X}_i = 1 \mid \mathbf{X}_{i-1} = 1] = \frac{1}{e}.$$

The first inequality of Lemma 2.10 follows immediately, as we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \Pr[\mathbf{G} \text{ has no } P\text{-subgraph}] &\geq \liminf_{n \rightarrow \infty} \Pr[\mathbf{G} \text{ has no } Q_1\text{-subgraph}] \\ &= \liminf_{n \rightarrow \infty} \Pr[\mathbf{X}_1 = 0 \mid \mathbf{X}_0 = 1] \\ &= \frac{1}{e}. \end{aligned}$$

For the second inequality, we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \Pr[\mathbf{G} \text{ has a unique } P\text{-subgraph}] &= \liminf_{n \rightarrow \infty} \Pr[\mathbf{X}_t = 1] \\ &\geq \liminf_{n \rightarrow \infty} \Pr[\mathbf{X}_0 = \dots = \mathbf{X}_t = 1] \\ &= \liminf_{n \rightarrow \infty} \prod_{1 \leq i \leq t} \Pr[\mathbf{X}_i = 1 \mid \mathbf{X}_{i-1} = 1] \\ &= \frac{1}{e^t} \geq \frac{1}{e^{|E(P)|}}. \end{aligned}$$

In the remainder of this appendix we give the proof of Lemma A.1. We will use the following result on Poisson approximation. Before stating it, recall that the *total variation distance*  $d_{TV}(\mathbf{X}, \mathbf{Y})$  between two random variables  $\mathbf{X}$  and  $\mathbf{Y}$  with values in the same set (in particular, real-valued variables) is given by

$$d_{TV}(\mathbf{X}, \mathbf{Y}) := \sup_A |\Pr[\mathbf{X} \in A] - \Pr[\mathbf{Y} \in A]|.$$

$\{0, 1\}$ -valued random variables  $\mathbf{I}_1, \dots, \mathbf{I}_m$  on the same probability space are *positively related* if for any given  $i \in [m]$  one can find  $\{0, 1\}$ -valued random variables  $\mathbf{J}_{j_i}$  ( $j \neq i$ ) such that  $\mathbf{J}_{j_i} \geq \mathbf{I}_j$  and this tuple is distributed identically with the tuple  $\mathbf{I}_j$  ( $j \neq i$ ) conditioned by the event  $\mathbf{I}_i = 1$ .

LEMMA A.2 (Theorem 6.24 in [19]). *Suppose  $\mathbf{I}_1, \dots, \mathbf{I}_m$  are positively related  $\{0, 1\}$ -valued random variables, and let  $\mathbf{k} := \sum_i \mathbf{I}_i$ . Then*

$$d_{TV}(\mathbf{k}, \text{Po}(\mathbb{E}[\mathbf{k}])) \leq \frac{\text{Var}[\mathbf{k}]}{\mathbb{E}[\mathbf{k}]} - 1 + 2 \max_i \mathbb{E}[\mathbf{I}_i].$$

*Proof of Lemma A.1.* Fix  $i \in \{1, \dots, t\}$  and let  $Q := Q_i$  and  $Q' := Q_{i-1}$  and  $\mathbf{X} := \mathbf{X}_i$  and  $\mathbf{X}' := \mathbf{X}_{i-1}$ . To show that  $\mathcal{L}(\mathbf{X} \mid \mathbf{X}' = 1)$  is asymptotically  $\text{Po}(1)$ , we would like to sample  $\mathbf{G}$  conditioned on  $\mathbf{X}' = 1$  (i.e., the event that  $\mathbf{G}$  contains a unique  $Q'$ -subgraph). However, it will be convenient to condition on the entire  $V(Q')$ -colored part of  $\mathbf{G}$  (i.e., the induced subgraph of  $G$  on the vertices which map to  $V(Q')$  under the vertex-coloring of  $\mathbf{G}$ ). We shall therefore fix an arbitrary  $V(Q')$ -colored graph  $G'$  such that

- $G'$  equals the  $Q'$ -colored part of  $G$  for some  $G$  in the support of  $\mathbf{G}$ , and
- $G'$  contains a unique  $Q'$ -subgraph, which we will denote by  $H'$ .

We denote by  $\mathbf{G}|G'$  the random graph  $\mathbf{G}$  conditioned on the event that the  $V(Q')$ -colored part of  $\mathbf{G}$  equals  $G'$ . Note that  $\mathbf{G}|G'$  is a product distribution on the unrestricted edges.

Let  $\mathcal{Q} (= \mathcal{Q}(H'))$  be the set of potential  $Q$ -subgraphs which extend  $H'$ . For each  $H \in \mathcal{Q}$ , let  $\mathbf{I}_H$  be the indicator variable for the event that  $\mathbf{G}|G'$  contains  $H$ . These random variables are positively related: just let  $\mathbf{J}_H$  be the characteristic function of the event that  $\mathbf{G}$  contains  $E(H') \setminus E(H)$ . Let  $\mathbf{k} := \sum_{H \in \mathcal{Q}} \mathbf{I}_H$ . We will show that  $\mathbf{k}$  is asymptotically  $\text{Po}(1)$  using Lemma A.2. Since the event  $\{\mathbf{X}' = 1\}$  is the disjoint union of events  $\{G' \text{ is the } Q'\text{-colored part of } \mathbf{G}\}$  over all  $G'$ , it follows that  $\mathcal{L}(\mathbf{X} \mid \mathbf{X}' = 1)$  is asymptotically  $\text{Po}(1)$  by the convexity of  $d_{TV}$ .

We will now calculate the expectation of  $\mathbf{k}$ . First, we have

$$|\mathcal{Q}| = \prod_{v \in V(Q) \setminus V(Q')} [n^{\alpha(v)}] = (1 - o(1))n^{\alpha(Q) - \alpha(Q')}.$$

For each  $H \in \mathcal{Q}$ , we have

$$\mathbb{E}[\mathbf{I}_H] = n^{-\beta(Q) + \beta(Q')}.$$

(Note for the record that this is  $o(1)$  since  $\beta(Q') < \beta(Q)$  by the fact that  $Q_0, \dots, Q_t$  is an  $(\alpha, \beta)$ -grading.) Therefore,

$$\mathbb{E}[\mathbf{k}] = (1 - o(1))n^{\alpha(Q) - \alpha(Q')}n^{-\beta(Q) + \beta(Q')} = 1 - o(1),$$

using the fact that  $\alpha(Q) = \beta(Q)$  and  $\alpha(Q') = \beta(Q')$ . In particular,  $|\mathbb{E}[\mathbf{k}] - \mathbb{E}[\mathbf{k}]^2| \leq o(1)$ .

We next calculate  $\text{Var}[\mathbf{k}]$ . For  $H, K \in \mathcal{Q}$ , let  $U := \chi(V(H) \cap V(K))$  be the set of  $P$ -colors of vertices in the intersection of  $V(H)$  and  $V(K)$ . Note that

$$V(Q') \subseteq U \subseteq V(Q).$$

Thus,

$$\mathbb{E}[\mathbf{I}_H \mathbf{I}_K] = n^{-2\beta(Q) + \beta(Q') + \beta(U)},$$

where  $\beta(U) := \sum_{e \in E(P) \cap \binom{U}{2}} \beta(e)$ . For all  $V(Q') \subseteq U \subseteq V(Q)$ , we have

$$\#\{(H, K) \in \mathcal{Q} \times \mathcal{Q} : \chi(V(H) \cap V(K)) = U\} = (1 - o(1))n^{2\alpha(Q) - \alpha(Q') - \alpha(U)}.$$

Therefore,

$$\begin{aligned} \text{Var}[\mathbf{k}] &= \sum_{H,K \in \mathcal{Q}: H \neq K} \mathbb{E}[\mathbf{I}_H \mathbf{I}_K] + (\mathbb{E}[\mathbf{k}] - \mathbb{E}[\mathbf{k}]^2) \\ &= \sum_{U: V(Q') \subseteq U \subseteq V(Q)} (1 - o(1)) n^{2\alpha(Q) - \alpha(Q') - \alpha(U)} n^{-2\beta(Q) + \beta(Q') + \beta(U)} \pm o(1) \\ &= \sum_{U: V(Q') \subseteq U \subseteq V(Q)} (1 - o(1)) n^{\beta(U) - \alpha(U)} \pm o(1). \end{aligned}$$

Note that  $\beta(U) < \alpha(U)$  for all  $V(Q') \subset U \subset V(Q)$  (otherwise, letting  $R$  denote the induced subgraph of  $Q$  on  $U$ , we would have  $\alpha(R) = \beta(R)$ , contradicting the fact that  $Q_0, \dots, Q_t$  is an  $(\alpha, \beta)$ -grading). It follows that

$$\text{Var}[\mathbf{k}] = 1 \pm o(1).$$

Plugging the bounds  $\mathbb{E}[\mathbf{k}] = 1 - o(1)$  and  $\text{Var}[\mathbf{k}] = 1 \pm o(1)$  and  $\mathbb{E}[\mathbf{I}_H] = o(1)$  into Lemma A.2, we have

$$d_{TV}(\mathcal{L}(\mathbf{k}), \text{Po}(\mu)) \leq \frac{\text{Var}[\mathbf{k}]}{\mathbb{E}[\mathbf{k}]} - 1 + 2 \max_{H \in \mathcal{Q}} \mathbb{E}[\mathbf{I}_H] = o(1).$$

Finally, since  $d_{TV}(\text{Po}(1), \text{Po}(1 - o(1))) = o(1)$ , we conclude that  $\mathbf{k}$  is asymptotically  $\text{Po}(1)$ , which completes the proof.  $\square$

**Appendix B. Proof of Lemma 3.9.** In this section we continue to assume that  $\beta(e) > 0$  for all  $e \in P$  (see the paragraph before the statement of Lemma 3.9). This assumption in particular implies that  $E(\mathbf{G}) \cap E(\mathbf{P}) = \emptyset$  almost surely. Thus we only have to prove that with constant probability  $\mathbf{G} \cup \mathbf{P}$  does not contain any  $P$ -subgraphs other than  $\mathbf{P}$  itself (a formal argument is included at the end of this section).

LEMMA B.1.

1. For every  $P$ -colored graph  $G$  in the support of  $\mathbf{G}$  and every subgraph  $Q \subseteq P$ ,

$$\frac{\Pr[\mathbf{G} \cup \mathbf{Q} = G]}{\Pr[\mathbf{G} = G]} = (1 + o(1)) \frac{\text{sub}(Q, G)}{n^{\alpha(Q) - \beta(Q)}}.$$

2. If  $\mathcal{A}$  is a property of  $P$ -colored graphs which holds a.a.s. for  $\mathbf{G}$ , then  $\mathcal{A}$  holds a.a.s. for  $\mathbf{G} \cup \mathbf{Q}$  for every  $Q \subseteq P$ .

3.  $\liminf_{n \rightarrow \infty} \Pr[\text{sub}(P, \mathbf{G} \cup \mathbf{P}) = 1] > 0$ . That is,  $\mathbf{G} \cup \mathbf{P}$  has a unique  $P$ -subgraph (namely,  $\mathbf{P}$ ) with probability bounded away from 0.

*Proof.*

1. Noting that the number of possible  $Q$ -subgraphs in  $\mathbf{G}$  is  $\prod_{v \in V(Q)} [n^{\alpha(v)}] = (1 - o(1)) n^{\alpha(Q)}$ , we have

$$\Pr[\mathbf{G} \cup \mathbf{Q} = G] = (1 + o(1)) n^{-\alpha(Q)} \sum_{K \in \text{Sub}(Q, G)} \sum_{H: G \setminus K \subseteq H \subseteq G} \Pr[\mathbf{G} = H].$$

For every  $K \in \text{Sub}(Q, G)$  and  $H$  such that  $G \setminus K \subseteq H \subseteq G$ , we have

$$\begin{aligned} \Pr[\mathbf{G} = H] &= \Pr[\mathbf{G} = G] \cdot \prod_{e \in E(K \setminus H)} \frac{1 - n^{-\beta(\bar{e})}}{n^{-\beta(\bar{e})}} \\ &= (1 - o(1)) \Pr[\mathbf{G} = G] \cdot \begin{cases} n^{\beta(Q)} & \text{if } H = G \setminus K, \\ n^{\beta(Q) - \Omega(1)} & \text{otherwise} \end{cases} \\ &\quad \text{(since } \beta \text{ positive).} \end{aligned}$$

(Above,  $\hat{e}$  is the edge in  $P$  corresponding to  $e$  under the vertex-coloring of  $G$ .) Since  $n^{-\Omega(1)}$  dominates  $2^{|E(Q)|} - 1$  (i.e., the number of summands where  $H \neq G \setminus K$ ), statement 1 follows.

2. Suppose  $\mathcal{A}$  holds a.a.s. with respect to  $\mathbf{G}$  (i.e.,  $\limsup_{n \rightarrow \infty} \Pr[\mathbf{G} \notin \mathcal{A}] = 0$ ) and let  $Q \subseteq P$ . Let  $c > 0$  be an arbitrary (large) constant. We split up the event  $\{\mathbf{G} \cup \mathbf{Q} \notin \mathcal{A}\}$  as follows:

$$\Pr[\mathbf{G} \cup \mathbf{Q} \notin \mathcal{A}] \leq \Pr[\text{sub}(Q, \mathbf{G} \cup \mathbf{Q}) \geq cn^{\alpha(Q)-\beta(Q)}] + \Pr[\mathbf{G} \cup \mathbf{Q} \notin \mathcal{A} \text{ and } \text{sub}(Q, \mathbf{G} \cup \mathbf{Q}) \leq cn^{\alpha(Q)-\beta(Q)}].$$

We bound each of the right-hand terms separately. First, note that

$$\begin{aligned} \mathbb{E}[\text{sub}(Q, \mathbf{G} \cup \mathbf{Q})] &= \sum_{R \subseteq Q} \mathbb{E}[|\{H \in \text{Sub}(Q, \mathbf{G} \cup \mathbf{Q}) : \chi(H \cap \mathbf{Q}) = R\}|] \\ &\leq \sum_{R \subseteq Q} n^{\alpha(Q)-\alpha(R)} \cdot n^{-\beta(Q \setminus R)} \\ &\leq 2^{|E(Q)|} \cdot n^{\alpha(Q)-\beta(Q)} \end{aligned}$$

(the latter inequality holds since  $\alpha(R) \geq \beta(R)$  for all  $R$ ). So by Markov's inequality,

$$\Pr[\text{sub}(Q, \mathbf{G} \cup \mathbf{Q}) \geq cn^{\alpha(Q)-\beta(Q)}] \leq \frac{2^{|E(Q)|}}{c}.$$

Second, we have

$$\begin{aligned} &\Pr[\mathbf{G} \cup \mathbf{Q} \notin \mathcal{A} \text{ and } \text{sub}(Q, \mathbf{G} \cup \mathbf{Q}) \leq cn^{\alpha(Q)-\beta(Q)}] \\ &= \sum_{G: G \notin \mathcal{A} \text{ and } \text{sub}(Q, G) \leq cn^{\alpha(Q)-\beta(Q)}} \Pr[\mathbf{G} \cup \mathbf{Q} = G] \\ &= \sum_{G: G \notin \mathcal{A} \text{ and } \text{sub}(Q, G) \leq cn^{\alpha(Q)-\beta(Q)}} (1 + o(1)) \Pr[\mathbf{G} = G] \frac{\text{sub}(Q, G)}{n^{\alpha(Q)-\beta(Q)}} \text{ (by (1))} \\ &\leq \sum_{G: G \notin \mathcal{A} \text{ and } \text{sub}(Q, G) \leq cn^{\alpha(Q)-\beta(Q)}} (1 + o(1))c \Pr[\mathbf{G} = G] \\ &\leq (1 + o(1))c \Pr[\mathbf{G} \notin \mathcal{A}]. \end{aligned}$$

Since  $\liminf_{n \rightarrow \infty} \Pr[\mathbf{G} \notin \mathcal{A}] = 0$ , it follows that

$$\liminf_{n \rightarrow \infty} \Pr[\mathbf{G} \cup \mathbf{Q} \notin \mathcal{A}] \leq \frac{2^{|E(Q)|}}{c}.$$

Since  $c$  may be chosen arbitrarily large, we conclude that  $\mathcal{A}$  holds a.a.s. with respect to  $\mathbf{G} \cup \mathbf{Q}$ .

3. Note that for  $Q = P$  and  $\text{sub}(P, G) = 1$  Lemma B.1(1) simplifies to  $\Pr[\mathbf{G} \cup \mathbf{P} = G] = (1 + o(1)) \Pr[\mathbf{G} = G]$ . Thus, we have

$$\begin{aligned} \liminf_{n \rightarrow \infty} \Pr[\text{sub}(P, \mathbf{G} \cup \mathbf{P}) = 1] &= \liminf_{n \rightarrow \infty} \Pr[\text{sub}(P, \mathbf{G}) = 1] \\ &> 0 \text{ (by Lemma 2.10)}. \end{aligned} \quad \square$$

*Proof of Lemma 3.9.* Let  $h : \mathcal{G}_{\alpha,\beta}(n) \rightarrow \{0, 1\}$  denote the  $\text{SUBGRAPH}_{\text{col}}(P)$  function, that is,  $h(G) = 1 \Leftrightarrow G$  contains a  $P$ -subgraph. Assume  $f : \mathcal{G}_{\alpha,\beta}(n) \rightarrow \{0, 1\}$  solves  $\text{SUBGRAPH}_{\text{col}}(P)$  in the average-case on  $\mathbf{G}$ , that is,

$$\Pr[f(\mathbf{G}) = h(\mathbf{G})] = 1 - o(1).$$

By Lemma B.1.2,

$$\Pr[f(\mathbf{G} \cup \mathbf{Q}) = h(\mathbf{G} \cup \mathbf{Q}) \text{ for all } Q \subseteq P] = 1 - o(1).$$

Since the event “ $f^{\cup \mathbf{G}}$  is sensitive over  $\mathbf{P}$ ” depends only on the values of  $f(\mathbf{G} \cup \mathbf{Q})$  for  $Q \subseteq P$ , we have

$$(17) \quad \Pr[f^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{P} \Leftrightarrow h^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{P}] = 1 - o(1).$$

As we already indicated above,

$$(18) \quad h^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{P} \Leftrightarrow E(\mathbf{G}) \cap E(\mathbf{P}) = \emptyset \text{ and } \text{sub}(P, \mathbf{G} \cup \mathbf{P}) = 1$$

(with probability 1). To see why, first assume  $E(\mathbf{G}) \cap E(\mathbf{P}) = \emptyset$  and  $\text{sub}(P, \mathbf{G} \cup \mathbf{P}) = 1$ . It follows that  $h^{\cup \mathbf{G}}(\mathbf{P} - \{e\}) = 0$  for all  $e \in E(\mathbf{P})$ . Since  $h^{\cup \mathbf{G}}(\mathbf{P}) = 1$ , this shows that  $h^{\cup \mathbf{G}}$  is sensitive over  $\mathbf{P}$ . For the opposite direction, consider the case that there exists  $e \in E(\mathbf{G}) \cap E(\mathbf{P})$ . Note that  $e$  does appear in  $\text{Sens}(h^{\cup \mathbf{G}}, \mathbf{P})$ . Therefore,  $h^{\cup \mathbf{G}}$  is not sensitive over  $\mathbf{P}$ . Finally, consider the case that  $\text{sub}(P, \mathbf{G} \cup \mathbf{P}) > 1$ . Then  $\mathbf{G} \cup \mathbf{P}$  contains a  $P$ -subgraph other than  $\mathbf{P}$ ; this  $P$ -subgraph necessarily does not include some edge  $e \in E(\mathbf{P})$ . Note that  $\text{sub}(P, \mathbf{G} \cup (\mathbf{P} - \{e\})) \geq 1$ , which means that  $e$  does appear in  $\text{Sens}(h^{\cup \mathbf{G}}, \mathbf{P})$ . So again  $h^{\cup \mathbf{G}}$  is not sensitive over  $\mathbf{P}$ .

From (17) and (18), we have

$$\begin{aligned} \Pr[f^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{P}] &\geq \Pr[E(\mathbf{G}) \cap E(\mathbf{P}) = \emptyset \text{ and } \text{sub}(P, \mathbf{G} \cup \mathbf{P}) = 1] - o(1) \\ &\geq \Pr[\text{sub}(P, \mathbf{G} \cup \mathbf{P}) = 1] - \Pr[E(\mathbf{G}) \cap E(\mathbf{P}) \neq \emptyset] - o(1). \end{aligned}$$

Since  $\beta$  is positive,

$$\Pr[E(\mathbf{G}) \cap E(\mathbf{P}) \neq \emptyset] \leq \sum_{e \in E(P)} n^{-\beta(e)} = o(1).$$

Completing the proof, we have

$$\liminf_{n \rightarrow \infty} \Pr[f^{\cup \mathbf{G}} \text{ is sensitive over } \mathbf{P}] \geq \liminf_{n \rightarrow \infty} \Pr[\text{sub}(P, \mathbf{G} \cup \mathbf{P}) = 1] > 0$$

by Lemma B.1.3. □

*Remark B.2.* We get analogous versions of Lemmas 3.9 and B.1 in the uncolored setting where  $P$  is balanced and  $\mathbf{G} = G(n, n^{-\theta(P)})$ . The analysis is essentially the same as we get the colored setting with respect to the threshold pair  $(\alpha, \beta) = (1, \theta(P))$ , that is,  $\alpha(Q) = |V(Q)|$  and  $\beta(Q) = \theta(P)|E(Q)|$ . However, all instances of  $n^{\alpha(Q)}$  become  $\binom{n}{|V(Q)|}$  in the uncolored setting; for instance, the expected number of  $Q$ -subgraphs in  $\mathbf{G}$  is  $\binom{n}{|V(Q)|} n^{-\theta(P)|E(Q)|}$ . Modulo this change, the proofs in this section adapt straightforwardly to the uncolored setting.

**Appendix C. The switching lemma proof of (11).** Let  $\mathbf{C}$  be a type-I (constant-depth, unbounded fan-in)  $\text{AC}^0$  circuit of size  $S = S(n) = n^{O(1)}$  and depth

$d = d(n) = O(1)$  over the set of variables  $\mathcal{E}$  (corresponding to potential edges of graphs in  $\mathcal{G}_{\alpha,\beta}(n)$ ). Recall that  $\rho : \mathcal{E} \rightarrow \{0, 1, \star\}$  is the (coordinatewise independent) random restriction with

$$\Pr[\rho(e) = \star] = n^{-\beta(\widehat{e})-\delta} (=: q_e), \quad \Pr[\rho(e) = 1 \mid \rho(e) \neq \star] = n^{-\beta(\widehat{e})} (=: p_e),$$

where  $\delta > 0$  is a fixed constant. Restating (11), we must show that

$$\Pr[\mathbf{C}\upharpoonright_\rho \text{ depends on } > n^\delta \text{ variables}] = n^{-\omega(1)}.$$

The proof uses nothing more than Håstad’s switching lemma [15]. We generate  $\rho$  as a composition of  $d + 1$  random restrictions  $\rho_0, \dots, \rho_d$ , where

- $\rho_0 : \mathcal{E} \rightarrow \{0, 1, \star\}$  is the *unbalanced* random restriction with

$$\Pr[\rho_0(e) = \star] = p_e, \quad \Pr[\rho_0(e) = 1 \mid \rho_0(e) \neq \star] = \frac{\frac{1}{2}(p_e + q_e) - p_e q_e}{1 - p_e} (=: r_e)$$

(note that  $r_e = \frac{1}{2}p_e + o(p_e)$  is a well-defined value in  $(0, 1)$  since  $q_e \ll p_e \ll 1$ ),

- for  $i \in \{1, \dots, d\}$ ,  $\rho_i : \rho_{i-1}^{-1}(\star) \rightarrow \{0, 1, \star\}$  is the *balanced* random restriction with

$$\Pr[\rho_i(e) = \star] = n^{-\delta/d}, \quad \Pr[\rho_i(e) = 1 \mid \rho_i(e) \neq \star] = 1/2.$$

It is easy to check that  $\rho_d \circ \dots \circ \rho_0$  is indeed the same distribution as  $\rho$ .

Let  $D(\cdot)$  denote *decision tree depth* of a boolean function. Let  $g$  be the function computed by a gate of height  $i \in \{1, \dots, d\}$  in  $\mathbf{C}$ . Let  $g_1, \dots, g_m$  be its children. (That is,  $g$  is the AND or the OR of  $g_1, \dots, g_m$  and has distance  $d - i$  to the output of  $\mathbf{C}$ .) Håstad’s switching lemma tells us

$$\Pr \left[ D(g\upharpoonright_{\rho_i \circ \dots \circ \rho_0}) > \delta \log n \mid \bigwedge_{j \in [m]} D(g_j\upharpoonright_{\rho_{i-1} \circ \dots \circ \rho_0}) \leq \delta \log n \right] \leq (5n^{-\delta/d} \delta \log n)^{\delta \log n}.$$

By a union bound, it follows that

$$\Pr[\mathbf{C}\upharpoonright_\rho \text{ depends on } > n^\delta \text{ variables}] \leq \Pr[D(\mathbf{C}\upharpoonright_\rho) > \delta \log n] \leq S \cdot (5n^{-\delta/d} \delta \log n)^{\delta \log n}.$$

Since  $S = n^{O(1)}$  and  $d = O(1)$  and  $\delta = \Omega(1)$  (with these constants only depending on the fixed pattern  $P$ ), we have

$$S \cdot (5n^{-\delta/d} \delta \log n)^{\delta \log n} = n^{-\omega(1)}.$$

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