Final Exam

- Tuesday December 10, 9-12am
- Closed book exam
- Cover Chapters 1-6 of the textbook. Might also include a question on quantifier rank or the Ehrenfeucht-Fraïssé game (see Oct 29 slides).
- Greater focus on Chapter 3.3 through 6.
- Won't ask about material that we didn't cover (or only barely) cover in lecture. For example, no question on the Lowenheim-Skolem Theorems.

Final Exam

- Expect at least one question involving the Compactness Theorem.
- Expect a question on writing down ($\Sigma/\Pi/\Delta$-)formulas to express specific properties.
- You don't need to memorize the axioms of Robinson Arithmetic.
- You won't be asked to write a deduction. However, by now you should know the logical axioms and rules of inference.
- Don't need to memorize the proof of completeness theorem, but good to understand the principle behind Henkin axioms, etc.
- You should know the sequence-coding function $\langle \cdot \rangle$. However, you don't need to memorize the specific definition of Gödel numbers (e.g., $\ulcorner t_1 + t_2 \urcorner = \langle 13, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$).

Final Exam

- Know the definitions of isomorphism (page 27) and elementary equivalence.
- Know what is a term vs. formula vs. number: $\ulcorner t \urcorner$, $\ulcorner \varphi \urcorner$, $\langle a_1, \ldots, a_k \rangle$ are numbers: $\overline{a}$ and $\overline{\ulcorner \varphi \urcorner}$ are terms, $Deduction_N(\overline{a}, \overline{b})$ is a Δ-sentence, etc.
- Know the definition of $\Sigma/\Pi/\Delta$-definable sets, (weakly) representable sets, and recursive/complete/consistent sets of $\mathcal{L}_{NT}$-formulas (today's lecture).
- Understand the concept of "construction sequences" as a means of building Δ-formulas.
- Know that $\textsc{Deduction}_N$ is Δ-definable, $\textsc{Thm}_N$ is Σ-definable, etc.
- Know the key definitions from Chapter 1-3: free variables, substitution, $\models$ and $\vdash$, statements of Soundness/Deduction/Model Existence/Completeness/Compactness Theorems.

Key Results

1. Rosser's Lemma: $N \vdash (\forall x < \overline{a})(x = \overline{0} \vee x = \overline{1} \vee \cdots \vee x = \overline{a-1})$,
$N \vdash [(\forall x < \overline{a})\varphi(x)] \leftrightarrow [\varphi(\overline{0}) \wedge \cdots \wedge \varphi(\overline{a-1})]$

2. Every Σ-sentence which is true in $\mathfrak{N}$ is provable from N.

3. Every Δ-definable set is representable.

4. Church's Thesis: A set $A \subseteq \mathbb{N}$ is (weakly) representable if, and only if, the question $n \in ?\ A$ is (semi-)decidable.

5. Num, Sub, $\textsc{Deduction}_N$ are Δ-definable, via the notion of construction sequences.

6. $\textsc{Thm}_N$ is Σ-definable.

7. Every representable set is Σ-definable.

Self-Reference Lemma: For every formula $\beta(x)$, there exists a sentence θ such that $N \vdash \theta \leftrightarrow \beta(\overline{\ulcorner \theta \urcorner})$.

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Summary of Proof:

(1) Let $\gamma(v_1) :\equiv (\exists y)(\exists z)[Num(v_1, y) \wedge Sub(v_1, \overline{\ulcorner v_1 \urcorner}, y, z) \wedge \beta(z)]$.

This has the property that, for every formula $\varphi(v_1)$,

$$\mathfrak{N} \models \gamma(\overline{\ulcorner \varphi \urcorner}) \leftrightarrow \beta(\overline{\ulcorner \varphi(\overline{\ulcorner \varphi \urcorner}) \urcorner}).$$

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(2) Letting $\theta :\equiv \gamma(\overline{\ulcorner \gamma \urcorner})$, we have $\mathfrak{N} \models \theta \leftrightarrow \beta(\overline{\ulcorner \theta \urcorner})$.

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(2) Letting $\theta :\equiv \gamma(\overline{\ulcorner\gamma\urcorner})$, we have $\mathfrak{N} \models \theta \leftrightarrow \beta(\overline{\ulcorner\theta\urcorner})$.

(3) To get the stronger property

$$N \vdash \theta \leftrightarrow \beta(\overline{\ulcorner\theta\urcorner}),$$

we replace $Num(x, y)$ and $Sub(x_1, x_2, x_3, y)$ in θ with formulas

$$Num^*(x, y) :\equiv Num(x, y) \wedge (\forall z < y)[\neg Num(x, z)]$$
$$Sub^*(x_1, x_2, x_3, y) :\equiv Sub(x_1, x_2, x_3, y) \wedge (\forall z < y)[\neg Sub(x_1, x_2, x_3, z)]$$

which *strongly represent* functions Num and Sub (see book).

Theorem 6.3.10 (Tarski's Undefinability Theorem, 1936). *The set* $\{\ulcorner\varphi\urcorner : \mathfrak{N} \models \varphi\}$ *of Gödel numbers of formulas true in* $\mathfrak{N}$ *is not definable.*

In other words, truth is undefinable!

Proof. Toward a contradiction, assume there is a formula $\alpha(x)$ which defines $\{\ulcorner\varphi\urcorner : \mathfrak{N} \models \varphi\}$. This mean that, for every formula φ,

$$\mathfrak{N} \models \alpha(\overline{\ulcorner\varphi\urcorner}) \iff \mathfrak{N} \models \varphi.$$

By the Self-Reference Lemma applied to the formula $\beta(x) :\equiv \neg\alpha(x)$, there is a sentence θ such that

$$N \vdash \theta \leftrightarrow \beta(\overline{\ulcorner\theta\urcorner}) \quad \text{(and therefore } \mathfrak{N} \models \theta \leftrightarrow \beta(\overline{\ulcorner\theta\urcorner})\text{).}$$

(Think of θ as saying "I am true in $\mathfrak{N}$ if, and only if, I am false in $\mathfrak{N}$.")

This immediately yields a contradiction, as we have

$$\mathfrak{N} \models \theta \iff \mathfrak{N} \models \beta(\overline{\ulcorner\theta\urcorner}) \iff \mathfrak{N} \not\models \alpha(\overline{\ulcorner\theta\urcorner}) \iff \mathfrak{N} \not\models \theta.$$

Q.E.D.

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Corollary. $\{\ulcorner\varphi\urcorner : \mathfrak{N} \models \varphi\}$ *is not representable. Therefore (assuming Church's Thesis), there is no computer program which, given an* $\mathcal{L}_{NT}$*-formula* φ *as input, outputs "true" if* $\mathfrak{N} \models \varphi$ *and outputs "false" if* $\mathfrak{N} \not\models \varphi$*.*

Theories (in an arbitrary language $\mathcal{L}$)

A **theory** is a collection of formulas T that is closed under deduction: for every formula φ, if $T \vdash \varphi$, then $\varphi \in T$.

A theory T is **consistent** if $T \nvdash \bot$ (equivalently: if T has a model).

A theory T is **complete** if, for every sentence σ, either $\sigma \in T$ or $\neg\sigma \in T$.

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Recall: If $\mathfrak{A}$ is a structure, then the **theory of** $\mathfrak{A}$, written $Th(\mathfrak{A})$, is the set of formulas that are true in $\mathfrak{A}$:

$$Th(\mathfrak{A}) = \{\text{formulas } \varphi : \mathfrak{A} \models \varphi\}.$$

Exercise: (Try on your own!)

1. For every structure $\mathfrak{A}$, the theory $Th(\mathfrak{A})$ is complete and consistent.
2. Every complete and consistent theory equals $Th(\mathfrak{A})$ for some structure $\mathfrak{A}$.

Some Interesting Complete Theories

- $Th(\mathfrak{N}) = Th(\mathbb{N}, 0, 1, S, +, \cdot, E, <)$ (True Arithmetic, a.k.a., the Theory of Arithmetic)
- $Th(\mathbb{N}, 0, 1, +)$ (Presburger Arithmetic)
- $Th(\mathbb{R}, 0, 1, +, \cdot, <)$ (the Theory of Real Closed Fields)
- Euclidean Geometry: $Th(\mathbb{R}^2, \text{Between}, \text{Congruent})$ where

$$\text{Between} := \{(a, b, c) \in (\mathbb{R}^2)^3 : b \in ac\},$$
$$\text{Congruent} := \{(a, b, c, d) \in (\mathbb{R}^2)^4 : |ab| = |cd|\}.$$

  Here ab denotes the line segment between points $a, b \in \mathbb{R}^2$, and $|ab|$ is the length of ab.

Axioms

Notational switch: A will be represent a set of formulas (rather than the universe of a structure, or a subset of $\mathbb{N}^k$)

We will regard formulas of A as "axioms" describing a particular theory of interest, such as $Th(\mathfrak{N})$ or the theory of linear orders.

Axioms

If A is a set of formulas, then the **theory of** A, written $Th(A)$, is defined by

$$Th(A) = \{\text{formulas } \varphi : A \vdash \varphi\}.$$

Note that $Th(A)$ is the smallest theory that includes A.

A theory T is **axiomatized** by a set of formulas A if $T = Th(A)$.

Example: Let

$$A := \{\ x = y \lor x < y \lor y < x, \quad (x < y \land y < z) \rightarrow x < z, \quad \neg(x < x)\ \}.$$

Then A axiomatizes the theory

$$T = \{\mathcal{L}_{\{<\}}\text{-formulas that are true in all linear orders}\}.$$

Non-example: Robinson Arithmetic N does *not* axiomatize $Th(\mathfrak{N})$

Robinson Arithmetic

The eleven axioms of N are:

$$
\begin{array}{ll}
N_1 & (\forall x)\neg[Sx = 0] \\
N_2 & (\forall x)(\forall y)[Sx = Sy \rightarrow x = y] \\
N_3 & (\forall x)[x + 0 = x] \\
N_4 & (\forall x)(\forall y)[x + Sy = S(x + y)] \\
N_5 & (\forall x)[x \cdot 0 = 0] \\
N_6 & (\forall x)(\forall y)[(x \cdot Sy) = (x \cdot y) + x] \\
N_7 & (\forall x)[xE0 = S0] \\
N_8 & (\forall x)(\forall y)[xE(Sy) = (xEy) \cdot x] \\
N_9 & (\forall x)\neg[x < 0] \\
N_{10} & (\forall x)(\forall y)[x < Sy \leftrightarrow (x < y \vee x = y)] \\
N_{11} & (\forall x)(\forall y)[x < y \vee x = y \vee y < x].
\end{array}
$$

Robinson Arithmetic

N is a very nice set of axioms. It is finite! And axioms $N_1, \dots, N_{11}$ are evidently true in $\mathfrak{N}$.

N is reasonably powerful: It proves every Σ-sentence in $Th(\mathfrak{N})$.

However, N does not axiomatize $Th(\mathfrak{N})$, since (for example)

$$\forall x \forall y (x + y = y + x) \notin Th(\mathfrak{N}) \setminus Th(N).$$

"Useful" Axiomatization of $Th(\mathfrak{N})$

We would like to strengthen N by including more axioms, which allow us prove more theorems in $Th(\mathfrak{N})$.

Ideally, we would like a set of axioms A which axiomatizes $Th(\mathfrak{N})$ (that is, for any formula φ, we want $\mathfrak{N} \models \varphi \iff A \vdash \varphi$).

We could take $A = Th(\mathfrak{N})$, but this is not a useful axiomatization: How do you recognize if a formula φ belongs to A? (It is unclear how to check whether $\mathfrak{N} \models \varphi$ in finite time...)

Question: *Is there a "useful" axiomatization of $Th(\mathfrak{N})$?*

"Useful" could mean finite. But this seems too restrictive; after all, there are infinitely many logical axioms (e.g., $x_1 = x_1$, $x_2 = x_2$, $\ldots$), which admits a nice, uniform description.

A more reasonable interpretation of "useful": A should be decidable by an algorithm.

Recursive sets of axioms

Definition. *Let* A *be a set of axioms of* $\mathcal{L}_{NT}$. *We say that* A *is* ***recursive*** *if the set* $\{\ulcorner\alpha\urcorner : \alpha \in A\}$ *is representable.*

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Recall Church's Thesis: A set $B \subseteq \mathbb{N}$ is representable if, and only if, there exists a computer program which, given $n \in \mathbb{N}$ as input, eventually outputs "yes" if $n \in B$ and eventually outputs "no" if $n \notin B$.

Therefore, a set of axioms A is recursive if, and only if, membership in A can be decided algorithmically in finite time (by a computer or, in principle, a human mathematician).

Recursive sets of axioms

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Proof. Since A is recursive, the set

$$\textsc{Axiom}_A := \{\ulcorner\alpha\urcorner : \alpha \in A\}$$

is representable and therefore definable by a Σ-formula $Axiom_A(x)$.

Recall that $Deduction_N(y, z)$ is a Δ-formula, which has $Axiom_N(x)$ as a Δ-subformula. By replacing $Axiom_N(x)$ with $Axiom_A(x)$, the result is a Σ-formula $Deduction_A(y, z)$ which defines the set

$$\textsc{Deduction}_A := \Big\{(c, a) : c = \langle\ulcorner\delta_1\urcorner, \ldots, \ulcorner\delta_n\urcorner\rangle \text{ and } a = \ulcorner\varphi\urcorner \text{ where } (\delta_1, \ldots, \delta_n) \text{ is a deduction from } A \text{ of } \varphi\Big\}.$$

It follows that the set $\textsc{Thm}_A$ is defined by Σ-formula

$$Thm_A(x) :\equiv (\exists y) Deduction_A(y, x).$$

Theorem 6.3.6 (Gödel's First Incompleteness Theorem, 1931). *Suppose that* A *is a consistent and recursive set of axioms in the language* $\mathcal{L}_{NT}$*. Then there is a sentence* θ *such that* $\mathfrak{N} \models \theta$ *but* $A \nvdash \theta$*.*

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Proof. If $A \nvdash N_i$ for any of the axioms $N_1, \ldots, N_{11}$ of Robinson arithmetic, then we are done (simply let $\theta :\equiv N_i$). So we will assume that $A \vdash N$.

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Applying the Self-Reference Lemma to the formula $\beta(x) :\equiv \neg Thm_A(x)$, we get a sentence θ such that

$$(*) \qquad N \vdash \theta \leftrightarrow \neg Thm_A(\overline{\ulcorner \theta \urcorner}).$$

Think of θ as saying "I am true if, and only if, I am not provable from A."

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Since $\mathfrak{N} \models N$, it follows from $(*)$ that $\mathfrak{N} \models \theta \leftrightarrow \neg Thm_A(\overline{\ulcorner \theta \urcorner})$. Therefore,

$$(**) \qquad \begin{aligned} \mathfrak{N} \models \theta &\iff \mathfrak{N} \models \neg Thm_A(\overline{\ulcorner \theta \urcorner}) \\ &\iff \ulcorner \theta \urcorner \notin \textsc{Thm}_A \\ &\iff A \nvdash \theta. \end{aligned}$$

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Proof, cont'd. We have: $A \vdash N$,

(∗) $$N \vdash \theta \leftrightarrow \neg Thm_A(\overline{\ulcorner \theta \urcorner}),$$

(∗∗) $$\mathfrak{N} \models \theta \iff \mathfrak{N} \models \neg Thm_A(\overline{\ulcorner \theta \urcorner}) \iff A \nvdash \theta.$$

Claim. It must be the case $\mathfrak{N} \models \theta$ (hence $A \nvdash \theta$ by (∗∗), finishing the proof).

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Proof of Claim. Toward a contradiction, assume that $\mathfrak{N} \not\models \theta$. Then (∗∗) implies $A \vdash \theta$.

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(∗∗) also implies $\mathfrak{N} \models Thm_A(\overline{\ulcorner \theta \urcorner})$. Since $Thm_A(\overline{\ulcorner \theta \urcorner})$ is Σ-sentence which is true in $\mathfrak{N}$, we have $N \vdash Thm_A(\overline{\ulcorner \theta \urcorner})$ (Proposition 5.3.13). By (∗) and the (PC) rule, it follows that $N \vdash \neg\theta$. Since $A \vdash N$, we have $A \vdash \neg\theta$.

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We have now shown that $A \vdash \theta$ and $A \vdash \neg\theta$. But this means that $A \vdash \bot$, contradicting our initial assumption that A is consistent. Q.E.D.

Self-Reference Lemma. *If* $\beta(x)$ *is an* $\mathcal{L}_{NT}$*-formula with only* x *free, then there is a sentence* θ *such that* $N \vdash \theta \leftrightarrow \beta(\overline{\ulcorner\theta\urcorner})$*.*

Tarski's Undefinability Theorem. *The set* $\{\ulcorner\varphi\urcorner : \mathfrak{N} \models \varphi\}$ *of Gödel numbers of formulas true in* $\mathfrak{N}$ *is not definable.*

<u>Idea.</u> Toward a contradiction, assume $\alpha(x)$ defines $\{\ulcorner\varphi\urcorner : \mathfrak{N} \models \varphi\}$. By Self-Reference Lemma applied to $\beta(x) :\equiv \neg\alpha(x)$, there is a sentence θ such that $\mathfrak{N} \models \theta \leftrightarrow \neg\alpha(\overline{\ulcorner\theta\urcorner})$. This yields a contradiction:

$$\mathfrak{N} \models \theta \iff \mathfrak{N} \models \neg\beta(\overline{\ulcorner\theta\urcorner}) \iff \mathfrak{N} \not\models \theta.$$

Gödel's 1st Incompleteness Theorem. *Suppose that* A *is a consistent and recursive set of axioms in the language* $\mathcal{L}_{NT}$*. Then there is a sentence* θ *such that* $\mathfrak{N} \models \theta$ *but* $A \nvdash \theta$*.*

<u>Idea.</u> If $A \nvdash N$, we're done. If $A \vdash N$, let $\beta(x) :\equiv \neg\, Thm_A(x)$. By Self-Reference Lemma, there is a sentence θ such that

$$N \vdash \theta \leftrightarrow \neg\, Thm_A(\overline{\ulcorner\theta\urcorner}).$$

This forces $\mathfrak{N} \models \theta$ and $A \nvdash \theta$ (otherwise we get a contradiction).

Theorem 6.4.5 (Rosser's Theorem). *If A is a set of $\mathcal{L}_{NT}$-axioms that is recursive, consistent, and extends N, then A is incomplete.*

In other words, Robinson arithmetic N is not "recursively completable".

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Proof Idea (Rosser's Trick). Use the Self-Reference Lemma to construction a sentence θ which expresses: "I am true if, and only if, for every deduction of $A \vdash \theta$, there is a shorter deduction of $A \vdash \neg\theta$."

Obs: If $a := \ulcorner\theta\urcorner$, then $\ulcorner\neg\theta\urcorner = \langle 1, \ulcorner\theta\urcorner\rangle = 2^2 3^{\ulcorner\theta\urcorner+1} = 4 \cdot 3^{a+1}$.

Let $\beta(x) :\equiv (\forall y)\big[Deduction_A(y, x) \to (\exists z < y) Deduction_A(z, \overline{4} \cdot \overline{3}^{Sx})\big]$.

By the Self-Reference Lemma, we get a sentence θ such that

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We get a contradiction if we assume $A \vdash \theta$, and we get a contradiction if we assume $A \vdash \neg\theta$. Therefore, $A \nvdash \theta$ and $A \nvdash \neg\theta$; hence A is incomplete. Q.E.D.

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Let $\beta(x) :\equiv (\forall y)\big[Deduction_A(y, x) \to (\exists z < y) Deduction_A(z, \overline{4} \cdot \overline{3}^{Sx})\big]$.

By the Self-Reference Lemma, we get a sentence θ such that

$$\begin{aligned} N \vdash \theta &\leftrightarrow \beta(\overline{\ulcorner\theta\urcorner}) \\ &\equiv \theta \leftrightarrow (\forall y)\big[Deduction_A(y, \overline{\ulcorner\theta\urcorner}) \to (\exists z < y) Deduction_A(z, \ulcorner\neg\theta\urcorner)\big]. \end{aligned}$$

We get a contradiction if we assume $A \vdash \theta$, and we get a contradiction if we assume $A \vdash \neg\theta$. Therefore, $A \nvdash \theta$ and $A \nvdash \neg\theta$; hence A is incomplete. Q.E.D.

Remark. Either $\mathfrak{N} \models \theta$ or $\mathfrak{N} \models \neg\theta$. Either way, we get a sentence true in $\mathfrak{N}$ but not provable from A. Thus, Rosser's Theorem $\Rightarrow$ 1st Incompleteness Theorem.