Reminder of Num, Sub, $\textsc{Thm}_N$

Functions $\mathrm{Num} : \mathbb{N} \to \mathbb{N}$ and $\mathrm{Sub} : \mathbb{N}^3 \to \mathbb{N}$,

$$\mathrm{Num}(a) := \ulcorner \overline{a} \urcorner,$$
$$\mathrm{Sub}(\ulcorner \varphi \urcorner, \ulcorner x \urcorner, \ulcorner t \urcorner) := \ulcorner \varphi^x_t \urcorner,$$

defined by Δ-formulas $\mathit{Num}(x, y)$ and $\mathit{Sub}(x_1, x_2, x_3, y)$.

The set $\textsc{Thm}_N \subseteq \mathbb{N}$,

$$\textsc{Thm}_N := \{\ulcorner \varphi \urcorner : N \vdash \varphi\},$$

is defined by a Σ-formula $\mathit{Thm}_N(x)$:

$$\mathit{Thm}_N(x) := \exists d \; \mathit{Deduction}_N(d, x)$$

where $\mathit{Deduction}_N(d, x)$ is a Δ-formula expressing "$d = \langle \ulcorner \delta_1 \urcorner, \ldots, \ulcorner \delta_n \urcorner \rangle$ where $(\delta_1, \ldots, \delta_n)$ is a deduction from N and $x = \ulcorner \delta_n \urcorner$".

Representable $\Rightarrow$ Σ-Definable

Previously, we showed that ***every Δ-definable set is representable***.

(This is a straightforward corollary of Proposition 5.3.13: N *proves every* Σ-*sentence which is true in* $\mathfrak{N}$.)

Next, we show that ***every representable set is Σ-definable***.

Proposition 6.3.3. *If $A \subseteq \mathbb{N}^k$ is representable, then A is Σ-definable.*

Proof. Let $A \subseteq \mathbb{N}$ for simplicity and assume that $\varphi(v_1)$ represents A.

(Without loss of generality, we take v_1 to be the free variable of φ.)

Proposition 6.3.3.

If $A \subseteq \mathbb{N}^k$ is representable, then A is Σ-definable.

Proof. Let $A \subseteq \mathbb{N}$ for simplicity and assume that $\varphi(v_1)$ represents A.

Let $\beta(x)$ be the following Σ-formula:

$$\beta(x) :\equiv \exists x \exists y \big[Num(x, y) \wedge Sub(\overline{\ulcorner \varphi \urcorner}, \overline{\ulcorner v_1 \urcorner}, y, z) \wedge Thm_N(z) \big].$$

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CLAIM: $\beta(x)$ defines A. That is, for every $n \in \mathbb{N}$, we have

$$n \in A \iff \mathfrak{N} \models \beta(\overline{n}).$$

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CLAIM: $\beta(x)$ defines A. That is, for every $n \in \mathbb{N}$, we have

$$n \in A \iff \mathfrak{N} \models \beta(\overline{n}).$$

First, assume $n \in A$. Since φ represents A, we have $N \vdash \varphi(\overline{n})$. Therefore, $\ulcorner \varphi(\overline{n}) \urcorner \in \text{THM}_N$ (by definition of the set THM_N).

Since the formula $Thm_N(z)$ defines THM_N, it follows that $\mathfrak{N} \models Thm_N(\overline{\ulcorner \varphi(\overline{n}) \urcorner})$.

We have $\mathfrak{N} \models (Num(x, y) \wedge Sub(\overline{\ulcorner \varphi \urcorner}, \overline{\ulcorner v_1 \urcorner}, y, z) \wedge Thm_N(z))[s]$ under a variable assignment function $s : Vars \to \mathbb{N}$ with

$$s(x) = n, \quad s(y) = \text{Num}(n) = \ulcorner \overline{n} \urcorner, \quad s(z) = \ulcorner \text{Sub}(\ulcorner \varphi \urcorner, \ulcorner v_1 \urcorner, \ulcorner \overline{n} \urcorner) \urcorner = \ulcorner \varphi(\overline{n}) \urcorner.$$

Therefore, $\mathfrak{N} \models \beta(x)[s] \equiv \beta(\overline{n})$.

Proposition 6.3.3. *If $A \subseteq \mathbb{N}^k$ is representable, then A is Σ-definable.*

Proof. Let $A \subseteq \mathbb{N}$ for simplicity and assume that $\varphi(v_1)$ represents A.

Let $\beta(x)$ be the following Σ-formula:

$$\beta(x) :\equiv \exists x \exists y \big[\mathit{Num}(x, y) \wedge \mathit{Sub}(\overline{\ulcorner\varphi\urcorner}, \overline{\ulcorner v_1 \urcorner}, y, z) \wedge \mathit{Thm}_N(z) \big].$$

CLAIM: $\beta(x)$ defines A. That is, for every $n \in \mathbb{N}$, we have

$$n \in A \iff \mathfrak{N} \models \beta(\overline{n}).$$

For the other direction, assume $\mathfrak{N} \models \beta(\overline{n})$. Then there exists $s : \mathit{Vars} \to \mathbb{N}$ with $s(x) = n$ such that $\mathfrak{N} \models (\mathit{Num}(x, y) \wedge \mathit{Sub}(\overline{\ulcorner\varphi\urcorner}, \overline{\ulcorner v_1 \urcorner}, y, z) \wedge \mathit{Thm}_N(z))[s]$.

Since $\mathfrak{N} \models \mathit{Num}(x, y)[s]$, it must be that $s(y) = \mathrm{Num}(s(x)) = \mathrm{Num}(n) = \ulcorner \overline{n} \urcorner$. Since $\mathfrak{N} \models \mathit{Sub}(\overline{\ulcorner\varphi\urcorner}, \overline{\ulcorner v_1 \urcorner}, y, z)[s]$, it must be that $s(z) = \ulcorner \mathrm{Sub}(\ulcorner\varphi\urcorner, \ulcorner v_1 \urcorner, s(y)) \urcorner = \ulcorner \varphi(\overline{n}) \urcorner$. Since $\mathfrak{N} \models \mathit{Thm}_N(z)[s]$, we have $\mathfrak{N} \models \mathit{Thm}_N(\overline{s(z)}) \equiv \mathit{Thm}_N(\overline{\ulcorner \varphi(\overline{n}) \urcorner})$.

Since $\mathit{Thm}_N(z)$ defines $\mathrm{T{\scriptsize HM}}_N$, it follows that $\ulcorner \varphi(\overline{n}) \urcorner \in \mathrm{T{\scriptsize HM}}_N$, hence $N \vdash \varphi(\overline{n})$. Finally, since φ represents A, we conclude that $n \in A$. Q.E.D.

Proposition 6.3.3. *If $A \subseteq \mathbb{N}^k$ is representable, then A is Σ-definable.*

Proof. Let $A \subseteq \mathbb{N}$ for simplicity and assume that $\varphi(v_1)$ represents A.

Let $\beta(x)$ be the following Σ-formula:

$$\beta(x) :\equiv \exists d \text{ ``}\mathit{Deduction}_N(d, \mathrm{Sub}(\overline{\ulcorner\varphi\urcorner}, \overline{\ulcorner v_1 \urcorner}, \mathrm{Num}(x)))\text{''}.$$

CLAIM: $\beta(x)$ defines A. That is, $n \in A \iff \mathfrak{N} \models \beta(\overline{n})$ for all $n \in \mathbb{N}$.

First, suppose $n \in A$. Then $N \vdash \varphi(\overline{n})$ (since φ represents A). So there exists a deduction $(\delta_1, \ldots, \delta_k)$ of $\varphi(\overline{n})$ from N.

Let $d := \langle \ulcorner\delta_1\urcorner, \ldots, \ulcorner\delta_k\urcorner \rangle$. Then

$$\begin{aligned} \mathfrak{N} \models \text{``}&\mathit{Deduction}_N(d, \ulcorner \varphi^{v_1}_{\overline{n}} \urcorner)\text{''} \\ &\equiv \text{``}\mathit{Deduction}_N(d, \mathrm{Sub}(\overline{\ulcorner\varphi\urcorner}, \overline{\ulcorner v_1 \urcorner}, \mathrm{Num}(a))\text{''} \\ &\equiv \exists y \exists z \big[\mathit{Num}(\overline{n}, y) \wedge \mathit{Sub}(\overline{\ulcorner\varphi\urcorner}, \overline{\ulcorner v_1 \urcorner}, y, z) \wedge \mathit{Deduction}_N(d, z)\big]. \end{aligned}$$

Therefore, $\mathfrak{N} \models \beta(\overline{n})$.

Proposition 6.3.3. *If $A \subseteq \mathbb{N}^k$ is representable, then A is Σ-definable.*

Proof. Let $A \subseteq \mathbb{N}$ for simplicity and assume that $\varphi(v_1)$ represents A.

Let $\beta(x)$ be the following Σ-formula:

$$\beta(x) :\equiv \exists d \text{ ``}\mathit{Deduction}_N(d, \mathrm{Sub}(\overline{\ulcorner \varphi \urcorner}, \overline{\ulcorner v_1 \urcorner}, \mathrm{Num}(x)))\text{''}.$$

CLAIM: $\beta(x)$ defines A. That is, $n \in A \iff \mathfrak{N} \models \beta(\overline{n})$ for all $n \in \mathbb{N}$.

Conversely, suppose $\mathfrak{N} \models \beta(\overline{n})$, that is,

$$\mathfrak{N} \models \text{``}\mathit{Deduction}_N(d, \ulcorner \varphi(v_1) \urcorner)\text{''}.$$

Then there exists a deduction of $\varphi(v_1)$ from N, hence $N \vdash \varphi(\overline{n})$.

Since φ represents A, it follows that $n \in A$. Q.E.D.

Gödel's Self-Reference Lemma

Lemma 6.2.2. *Let* $\beta(x)$ *be an* $\mathcal{L}_{NT}$*-formula with only* x *free. Then there is a sentence* θ *such that*

$$N \vdash \theta \leftrightarrow \beta(\overline{\ulcorner \theta \urcorner}).$$

Gödel's Self-Reference Lemma

Lemma 6.2.2. *Let* $\beta(x)$ *be an* $\mathcal{L}_{NT}$*-formula with only* x *free. Then there is a sentence* θ *such that*

$$N \vdash \theta \leftrightarrow \beta(\overline{\ulcorner\theta\urcorner}).$$

Proof Idea. Let's first define a formula θ with the *weaker* property that

$$\mathfrak{N} \models \theta \leftrightarrow \beta(\overline{\ulcorner\theta\urcorner}).$$

We obtain θ by first defining a formula $\gamma(v_1)$ such that, for every formula $\varphi(v_1)$,

$$\mathfrak{N} \models \gamma(\overline{\ulcorner\varphi\urcorner}) \leftrightarrow \beta(\overline{\ulcorner\varphi(\overline{\ulcorner\varphi\urcorner})\urcorner}).$$

Once we have such a formula $\gamma(v_1)$, we simply let $\theta :\equiv \gamma(\overline{\ulcorner\gamma\urcorner})$.

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Definition:

$$\gamma(v_1) :\equiv (\exists y)(\exists z)\big[Num(v_1, y) \land Sub(v_1, \overline{8}, y, z) \land \beta(z)\big]$$

(Note that $\ulcorner v_1 \urcorner = \langle 2 \rangle = 2^3 = 8$, so $\overline{8} \equiv \overline{\ulcorner v_1 \urcorner} \equiv SSSSSSSS0$.)

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Proof Idea. Let

$$\gamma(v_1) :\equiv (\exists y)(\exists z)\left[Num(v_1, y) \wedge Sub(v_1, \overline{\ulcorner v_1 \urcorner}, y, z) \wedge \beta(z)\right]$$

$$\theta :\equiv \gamma(\overline{\ulcorner \gamma \urcorner}) :\equiv (\exists y)(\exists z)\left[Num(\overline{\ulcorner \gamma \urcorner}, y) \wedge Sub(\overline{\ulcorner \gamma \urcorner}, \overline{\ulcorner v_1 \urcorner}, y, z) \wedge \beta(z)\right].$$

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Proof Idea. So, we have

$$\gamma(v_1) :\equiv (\exists y)(\exists z)\big[Num(v_1, y) \wedge Sub(v_1, \overline{\ulcorner v_1 \urcorner}, y, z) \wedge \beta(z)\big]$$

$$\theta :\equiv \gamma(\overline{\ulcorner \gamma \urcorner}) :\equiv (\exists y)(\exists z)\big[\underbrace{Num(\overline{\ulcorner \gamma \urcorner}, y)}_{\text{forces variable assignment } y \mapsto \ulcorner \overline{\ulcorner \gamma \urcorner} \urcorner} \wedge Sub(\overline{\ulcorner \gamma \urcorner}, \overline{\ulcorner v_1 \urcorner}, y, z) \wedge \beta(z)\big].$$

We see that

$$\mathfrak{N} \models \theta \iff \mathfrak{N} \models (\exists z)\big[\overbrace{Sub(\overline{\ulcorner \gamma \urcorner}, \overline{\ulcorner v_1 \urcorner}, \overline{\ulcorner \overline{\ulcorner \gamma \urcorner} \urcorner}, z)}^{\text{forces variable assignment } z \mapsto \ulcorner \gamma(\overline{\ulcorner \gamma \urcorner}) \urcorner \ (= \ulcorner \theta \urcorner)} \wedge \beta(z)\big] \iff \mathfrak{N} \models \beta(\overline{\ulcorner \theta \urcorner}).$$

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Lemma 6.2.2. *Let $\beta(x)$ be an $\mathcal{L}_{NT}$-formula with only x free. Then there is a sentence θ such that*

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Proof Idea, continued. This formula θ has the property that

$$\mathfrak{N} \models \theta \leftrightarrow \beta(\overline{\ulcorner \theta \urcorner}).$$

In order to prove the stronger assertion

$$N \vdash \theta \leftrightarrow \beta(\overline{\ulcorner \theta \urcorner}),$$

we need to replace Δ-formulas $Num(x, y)$ and $Sub(x_1, x_2, x_3, y)$ in θ with:

$$Num^*(x, y) :\equiv Num(x, y) \wedge (\forall z < y)[\neg Num(x, z)]$$

$$Sub^*(x_1, x_2, x_3, y) :\equiv Sub(x_1, x_2, x_3, y) \wedge (\forall z < y)[\neg Sub(x_1, x_2, x_3, z)].$$

This lets us use Rosser's Lemma to eliminate bounded quantifiers in N.