Gödel Numbers of Terms and Formulas

We assign a unique number to each symbol in $\mathcal{L}_{NT}$ as follows:

$\neg$	1	$\cdot$	15
$\vee$	3	E	17
$\forall$	5	$<$	19
$=$	7	$($	21
0	9	$)$	23
S	11	v_i	$2i$
$+$	13		

Suppose $s :\equiv s_1 \dots s_n$ is a string of symbols, which constituting a well-formed term or formula of $\mathcal{L}_{NT}$.

Naively, we could encode s by the number $\langle \#(s_1), \dots, \#(s_n) \rangle$ where $\#(s_i)$ is the number corresponding to the symbol s_i.

However, it much better to encode s according to the inductive type of terms and formulas.

Def 5.7.1. For each term t and formula φ, the Gödel numbers $\ulcorner t \urcorner$ and $\ulcorner \varphi \urcorner$ are defined as follows:

$$
\begin{aligned}
\ulcorner \neg\alpha \urcorner &= \langle 1, \ulcorner \alpha \urcorner \rangle & \ulcorner {+}t_1t_2 \urcorner &= \langle 13, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \\
\ulcorner (\alpha \vee \beta) \urcorner &= \langle 3, \ulcorner \alpha \urcorner, \ulcorner \beta \urcorner \rangle & \ulcorner \cdot\, t_1t_2 \urcorner &= \langle 15, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \\
\ulcorner (\forall v_i)(\alpha) \urcorner &= \langle 5, \ulcorner v_i \urcorner, \ulcorner \alpha \urcorner \rangle & \ulcorner Et_1t_2 \urcorner &= \langle 17, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \\
\ulcorner {=}t_1t_2 \urcorner &= \langle 7, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle & \ulcorner {<}t_1t_2 \urcorner &= \langle 19, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \\
\ulcorner 0 \urcorner &= \langle 9 \rangle & \ulcorner v_i \urcorner &= \langle 2i \rangle. \\
\ulcorner St \urcorner &= \langle 11, \ulcorner t \urcorner \rangle
\end{aligned}
$$

Def 5.7.1. For each term t and formula φ, the Gödel numbers $\ulcorner t \urcorner$ and $\ulcorner \varphi \urcorner$ are defined as follows:

$$\begin{aligned}
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\ulcorner =t_1t_2 \urcorner &= \langle 7, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle & \ulcorner <t_1t_2 \urcorner &= \langle 19, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \\
\ulcorner 0 \urcorner &= \langle 9 \rangle & \ulcorner v_i \urcorner &= \langle 2i \rangle. \\
\ulcorner St \urcorner &= \langle 11, \ulcorner t \urcorner \rangle
\end{aligned}$$

Obs. $\ulcorner t \urcorner$ and $\ulcorner \varphi \urcorner$ are never divisible by 7. (Why?)

Def 5.7.1. For each term t and formula φ, the Gödel numbers $\ulcorner t \urcorner$ and $\ulcorner \varphi \urcorner$ are defined as follows:

$$\begin{aligned}
\ulcorner \neg\alpha \urcorner &= \langle 1, \ulcorner \alpha \urcorner \rangle \\
\ulcorner (\alpha \vee \beta) \urcorner &= \langle 3, \ulcorner \alpha \urcorner, \ulcorner \beta \urcorner \rangle \\
\ulcorner (\forall v_i)(\alpha) \urcorner &= \langle 5, \ulcorner v_i \urcorner, \ulcorner \alpha \urcorner \rangle \\
\ulcorner = t_1 t_2 \urcorner &= \langle 7, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \\
\ulcorner 0 \urcorner &= \langle 9 \rangle \\
\ulcorner S t \urcorner &= \langle 11, \ulcorner t \urcorner \rangle
\end{aligned}$$

$$\begin{aligned}
\ulcorner + t_1 t_2 \urcorner &= \langle 13, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \\
\ulcorner \cdot\, t_1 t_2 \urcorner &= \langle 15, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \\
\ulcorner E t_1 t_2 \urcorner &= \langle 17, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \\
\ulcorner < t_1 t_2 \urcorner &= \langle 19, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \\
\ulcorner v_i \urcorner &= \langle 2i \rangle.
\end{aligned}$$

Example.
$$\begin{aligned}
\ulcorner = 0 S 0 \urcorner &= \langle 7, \ulcorner 0 \urcorner, \ulcorner S0 \urcorner \rangle \\
&= \langle 7, \langle 9 \rangle, \langle 11, \langle 9 \rangle \rangle \rangle \\
&= \langle 7, 2^{10}, \langle 11, 2^{10} \rangle \rangle = 2^8 3^{1025} 5^{(2^{12} 3^{1025} + 1)}.
\end{aligned}$$

Notice how fast $\ulcorner SSSS0 \urcorner$ grows:

$$\ulcorner SSSS0 \urcorner = \langle 11, \langle 11, \langle 11, \langle 11, \langle 9 \rangle \rangle \rangle \rangle \rangle = 2^{12} 3^{2^{12} 3^{2^{12} 3^{2^{12} 3^{2^{10}}}}}$$

NEXT STEPS (Section 5.8)

Δ-definability of sets

$$\begin{aligned} \text{TERMS} := \{\ulcorner t \urcorner : \text{terms } t\} &= \{a \in \mathbb{N} : a = \ulcorner t \urcorner \text{ for some term } t\}, \\ \text{FORMULAS} := \{\ulcorner \varphi \urcorner : \text{formulas } \varphi\} &= \{a \in \mathbb{N} : a = \ulcorner \varphi \urcorner \text{ for some formula } \varphi\}. \end{aligned}$$

Δ-Definition of Terms $= \{\ulcorner t \urcorner : t \text{ is a term}\}$

$\ulcorner \neg\alpha \urcorner = \langle 1, \ulcorner \alpha \urcorner \rangle$	$\ulcorner {=} t_1 t_2 \urcorner = \langle 7, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$	$\ulcorner + t_1 t_2 \urcorner = \langle 13, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$	$\ulcorner < t_1 t_2 \urcorner = \langle 19, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$
$\ulcorner (\alpha \vee \beta) \urcorner = \langle 3, \ulcorner \alpha \urcorner, \ulcorner \beta \urcorner \rangle$	$\ulcorner 0 \urcorner = \langle 9 \rangle$	$\ulcorner \cdot\, t_1 t_2 \urcorner = \langle 15, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$	$\ulcorner v_i \urcorner = \langle 2i \rangle$
$\ulcorner (\forall v_i)(\alpha) \urcorner = \langle 5, \ulcorner v_i \urcorner, \ulcorner \alpha \urcorner \rangle$	$\ulcorner S t \urcorner = \langle 11, \ulcorner t \urcorner \rangle$	$\ulcorner E t_1 t_2 \urcorner = \langle 17, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$	

Δ-Definition of Terms $= \{\ulcorner t \urcorner : t \text{ is a term}\}$

$\ulcorner \neg\alpha \urcorner = \langle 1, \ulcorner \alpha \urcorner \rangle$	$\ulcorner = t_1 t_2 \urcorner = \langle 7, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$	$\ulcorner + t_1 t_2 \urcorner = \langle 13, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$	$\ulcorner < t_1 t_2 \urcorner = \langle 19, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$
$\ulcorner (\alpha \vee \beta) \urcorner = \langle 3, \ulcorner \alpha \urcorner, \ulcorner \beta \urcorner \rangle$	$\ulcorner 0 \urcorner = \langle 9 \rangle$	$\ulcorner \cdot\, t_1 t_2 \urcorner = \langle 15, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$	$\ulcorner v_i \urcorner = \langle 2i \rangle$
$\ulcorner (\forall v_i)(\alpha) \urcorner = \langle 5, \ulcorner v_i \urcorner, \ulcorner \alpha \urcorner \rangle$	$\ulcorner S t \urcorner = \langle 11, \ulcorner t \urcorner \rangle$	$\ulcorner E t_1 t_2 \urcorner = \langle 17, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$	

Recall the inductive definition of an $\mathcal{L}_{NT}$**-term** t: it is either

- a variable symbol v_i,
- the constant symbol 0,
- St_1 where t_1 is term,
- $+t_1t_2$ or $\cdot\, t_1t_2$ or Et_1t_2 where t_1, t_2 are terms.

Δ-Definition of Terms $= \{\ulcorner t \urcorner : t \text{ is a term}\}$

$\ulcorner \neg\alpha \urcorner = \langle 1, \ulcorner \alpha \urcorner \rangle$	$\ulcorner {=} t_1 t_2 \urcorner = \langle 7, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$	$\ulcorner + t_1 t_2 \urcorner = \langle 13, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$	$\ulcorner < t_1 t_2 \urcorner = \langle 19, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle$
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Recall the inductive definition of an $\mathcal{L}_{NT}$-**term** t: it is either

- a variable symbol v_i,
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Let's start with Δ-definition of

$$\text{Variables} := \{\ulcorner v_i \urcorner : i = 1, 2, \ldots\} \quad (= \{2^{2i+1} : i = 1, 2, \ldots\}).$$

by the formula

$$Variable(x) :\equiv (\exists y < x)[Even(y) \wedge (0 < y) \wedge (x = 2^{Sy})].$$

Δ-Definition of Terms $= \{\ulcorner t \urcorner : t \text{ is a term}\}$

$\ulcorner \neg\alpha \urcorner = \langle 1, \ulcorner\alpha\urcorner\rangle$	$\ulcorner {=}t_1t_2 \urcorner = \langle 7, \ulcorner t_1\urcorner, \ulcorner t_2\urcorner\rangle$	$\ulcorner {+}t_1t_2 \urcorner = \langle 13, \ulcorner t_1\urcorner, \ulcorner t_2\urcorner\rangle$	$\ulcorner {<}t_1t_2 \urcorner = \langle 19, \ulcorner t_1\urcorner, \ulcorner t_2\urcorner\rangle$
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$\ulcorner (\forall v_i)(\alpha) \urcorner = \langle 5, \ulcorner v_i\urcorner, \ulcorner\alpha\urcorner\rangle$	$\ulcorner St \urcorner = \langle 11, \ulcorner t\urcorner\rangle$	$\ulcorner Et_1t_2 \urcorner = \langle 17, \ulcorner t_1\urcorner, \ulcorner t_2\urcorner\rangle$	

Recall the inductive definition of an $\mathcal{L}_{NT}$-**term** t: it is either

- a variable symbol v_i,
- St_1 where t_1 is term,
- the constant symbol 0,
- $+t_1t_2$ or $\cdot\, t_1t_2$ or Et_1t_2 where t_1, t_2 are terms.

We would like to write:

$$\mathit{Term}(x) :\equiv \mathit{Variable}(x) \vee \overbrace{x = \overline{2^{10}}}^{\text{"}x\text{ is }\ulcorner 0\urcorner\text{"}} \vee \overbrace{(\exists y < x)[\mathit{Term}(y) \wedge x = \underbrace{\overline{2^{12}}\cdot\overline{3}^{Sy}}_{\langle 11,y\rangle}]}^{\text{"}x\text{ is }\ulcorner St_1\urcorner\text{ for some term }t_1\text{"}}$$
$$\vee \underbrace{\cdots}_{\text{"}x\text{ is }+t_1t_2\text{ or }\cdot\, t_1t_2\text{ or }Et_1t_2\text{"}}$$

However, there is a problem with this "Δ-formula": *It is a not legitimate formula of first-order logic! Note the circular use of the subformula* $\mathit{Term}(y)$.

Δ-Definition of $\text{Terms} = \{\ulcorner t \urcorner : t \text{ is a term}\}$

Definition. A *term construction sequence* for a term t is a finite sequence of terms $(t_1, \ldots, t_\ell)$ such that $t_\ell :\equiv t$ and, for each $k \in \{1, \ldots, \ell\}$, the term t_k is either

- a variable symbol,
- the constant symbol 0,
- St_j for some $j < k$, or
- $+t_i t_j$ or $\cdot t_i t_j$ or $Et_i t_j$ for some $i, j < k$.

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Example. $(0, v_1, Sv_1, +0Sv_1)$ is term construction sequence for the $+0Sv_1$.

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- $+t_it_j$ or $\cdot t_it_j$ or Et_it_j for some $i, j < k$.

Example. $(0, v_1, Sv_1, +0Sv_1)$ is term construction sequence for the $+0Sv_1$.

Lemma. Every term t has a term construction sequence of length at most the number of symbols in t.

(Easy proof by induction.)

Δ-Definition of $\textsc{Terms} = \{\ulcorner t \urcorner : t \text{ is a term}\}$

Definition. A *term construction sequence* for a term t is a finite sequence of terms $(t_1, \ldots, t_\ell)$ such that $t_\ell :\equiv t$ and, for each $k \in \{1, \ldots, \ell\}$, the term t_k is either

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- St_j for some $j < k$, or
- $+t_it_j$ or $\cdot t_it_j$ or Et_it_j for some $i, j < k$.

Key to defining $\textsc{Terms}$**:** We will write a Δ-formula defining the set

$$\textsc{TermConSeq} = \{(c, a) : c = \langle \ulcorner t_1 \urcorner, \ldots, \ulcorner t_\ell \urcorner \rangle \text{ and } a = \ulcorner t_\ell \urcorner \text{ where } (t_1, \ldots, t_\ell) \text{ is a term construction sequence}\}.$$

Δ-Definition of Terms $= \{\ulcorner t \urcorner : t \text{ is a term}\}$

$$
\begin{aligned}
& \mathit{TermConSeq}(c, a) :\equiv \\
& \quad \mathit{Codenumber}(c) \wedge (\exists \ell < c)\Big[\mathit{Length}(c, \ell) \wedge \mathit{IthElement}(a, \ell, c) \wedge \\
& \qquad (\forall k \leq \ell)(\exists e_k < c)\Big[\mathit{IthElement}(e_k, k, c) \wedge \\
& \qquad\quad \left(\begin{array}{l}
\mathit{Variable}(e_k) \\
\vee\, e_k = \overline{2^{10}} \;\} \text{ “}e_k \text{ is } \ulcorner 0 \urcorner\text{”} \\
\vee\, (\exists j < k)(\exists e_j < c)[\mathit{IthElement}(e_j, j, c) \wedge \overbrace{e_k = \overline{2^{12}} \cdot \overline{3}^{S e_j}}^{\text{“}e_k \text{ is } \ulcorner S e_j \urcorner\text{”}}] \\
\vee \cdots
\end{array}\right)\Big]\Big]
\end{aligned}
$$

Key to defining Terms**:** We will write a Δ-formula defining the set

$$
\text{TermConSeq} = \{(c, a) : c = \langle \ulcorner t_1 \urcorner, \ldots, \ulcorner t_\ell \urcorner \rangle \text{ and } a = \ulcorner t_\ell \urcorner \text{ where } (t_1, \ldots, t_\ell) \text{ is a term construction sequence}\}.
$$

Δ-Definition of Terms $= \{\ulcorner t \urcorner : t \text{ is a term}\}$

Now there is an obvious way to define $Term(a)$:

$$Term(a) :\equiv (\exists c)\, TermConSeq(c, a).$$

To make this a Δ-formula, we need an upper bound on c as a function of a.

Δ-Definition of Terms $= \{\ulcorner t \urcorner : t \text{ is a term}\}$

Now there is an obvious way to define $\mathit{Term}(a)$:

$$\mathit{Term}(a) :\equiv (\exists c)\, \mathit{TermConSeq}(c, a).$$

To make this a Δ-formula, we need an upper bound on c as a function of a.

Suppose $a = \ulcorner t \urcorner$. Another easy lemma by induction: The number of symbols in t is at most a. Therefore, there exists a term construction sequence $(t_1, \ldots, t_\ell)$ for t with length $\leq a$. We may assume that each t_k is a subterm of t, so that $\ulcorner t_k \urcorner \leq \ulcorner t \urcorner = a$ for all $k \in \{1, \ldots, \ell\}$.

Let $c := \langle \ulcorner t_1 \urcorner, \ldots, \ulcorner t_\ell \urcorner \rangle$. We have

$$c = 2^{\ulcorner t_1 \urcorner + 1} 3^{\ulcorner t_2 \urcorner + 1} \cdots (p_\ell)^{\ulcorner t_\ell \urcorner + 1} \leq (p_\ell)^{\ulcorner t_1 \urcorner + \cdots + \ulcorner t_\ell \urcorner + \ell} \leq (p_\ell)^{\ell a + \ell} \leq (p_a)^{a^2 + a} \leq (a+1)^{a^3}$$

using the (easy) fact that the a^{th} prime number p_a is at most $(a+1)^a$.

Δ-Definition of Terms $= \{\ulcorner t \urcorner : t \text{ is a term}\}$

Now there is an obvious way to define $Term(a)$:

$$Term(a) :\equiv (\exists c)\, TermConSeq(c, a).$$

To make this a Δ-formula, we need an upper bound on c as a function of a.

We may therefore take

$$Term(a) :\equiv (\exists c \leq (a+1)^{a^3})\, TermConSeq(c, a).$$

Construction Sequences for Other Recursive Definitions

In a similar way, using the notion of a *formula construction sequence*, we get a Δ-definition of the set

$$\textsc{Formulas} = \{\ulcorner\varphi\urcorner : \varphi \text{ is a formula}\}.$$

Definition. A *formula construction sequence* for a formula φ is a finite sequence of terms $(\varphi_1, \ldots, \varphi_\ell)$ such that $\varphi_\ell :\equiv \varphi$ and, for each $k \in \{1, \ldots, \ell\}$, the term φ_k is either

- $=t_1t_2$ for some terms t_1 and t_2
- $<t_1t_2$ for some terms t_1 and t_2
- $\neg\varphi_j$ for some $j < k$
- $(\varphi_i \vee \varphi_j)$ for some $i, j < k$
- $(\forall x)(\varphi_i)$ for some $i < k$ and $x \in \mathit{Vars}$

Construction Sequences for General Recursive Definitions

This idea is very general: using an appropriate notion of *construction sequence*, we get a Δ-definition of any recursively defined set or function.

Suppose want a Δ-formula $\mathit{Factorial}(x, y)$ defining the function $\textsc{Factorial} : \mathbb{N} \to \mathbb{N}$ (i.e., the $\{(a, b) \in \mathbb{N}^2 : b = a!\}$).

Key idea: Write a Δ-formula defining the set

$\textsc{FactorialConstSeq} := \{(a, b, c) \in \mathbb{N}^3 : b = a! \text{ and } c = \langle 0!, 1!, 2!, \ldots, a! \rangle\}$

using *Codenumber*, *Length*, *IthElement* as subformulas. (Details in tutorial.)

Homework Problem (PSET 4). Write down a Δ-formula defining the function $\textsc{Fibonacci} : \mathbb{N} \to \mathbb{N}$.

Next Steps (Sections 5.11–5.12)

The following are Δ-definable:

$$\textsc{LogicalAxiom} := \{\ulcorner\varphi\urcorner : \varphi \text{ is a logical axiom}\}$$

$$\textsc{RuleOfInference} := \{(\langle\ulcorner\gamma_1\urcorner, \ldots, \ulcorner\gamma_n\urcorner\rangle, \ulcorner\varphi\urcorner) : (\{\gamma_1, \ldots, \gamma_n\}, \varphi) \text{ is a rule of inference}\}$$

$$\textsc{Axiom}_N := \{\ulcorner N_1\urcorner, \ldots, \ulcorner N_{11}\urcorner\}$$

$$\textsc{Deduction}_N := \{(\langle\ulcorner\delta_1\urcorner, \ldots, \ulcorner\delta_1\urcorner\rangle, \ulcorner\varphi\urcorner) : (\delta_1, \ldots, \delta_n) \text{ is a deduction from } N \text{ of } \varphi\}.$$

Important Δ-definable functions:

$$\mathrm{Num}(a) := \ulcorner\overline{a}\urcorner,$$

$$\mathrm{TermSub}(\ulcorner u\urcorner, \ulcorner x\urcorner, \ulcorner t\urcorner) := \ulcorner u_t^x\urcorner,$$

$$\mathrm{Sub}(\ulcorner\varphi\urcorner, \ulcorner x\urcorner, \ulcorner t\urcorner) := \ulcorner\varphi_t^x\urcorner.$$

Δ-definable sets and functions

Using an appropriate notion of "construction sequence", we get a Δ-formula $Num(x,y)$ which defines the function

$$\mathrm{Num}(a) := \ulcorner \overline{a} \urcorner.$$

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$$\mathrm{Num}(a) := \ulcorner \overline{a} \urcorner.$$

This means:

- $\mathfrak{N} \models Num(\overline{a}, \overline{b})$ for all $(a, b) \in \mathbb{N}^2$ such that $b = \ulcorner \overline{a} \urcorner$
- $\mathfrak{N} \models \neg Num(\overline{a}, \overline{b})$ for all $(a, b) \in \mathbb{N}^2$ such that $b \neq \ulcorner \overline{a} \urcorner$

Δ-definable sets and functions

Similarly (by a more complicated "construction sequence"), there is a Δ-formula $Sub(x_1, x_2, x_3, y)$ which define the function

$$\mathrm{Sub}(\ulcorner \varphi \urcorner, \ulcorner x \urcorner, \ulcorner t \urcorner) := \ulcorner \varphi_t^x \urcorner.$$

Δ-definable sets and functions

Similarly (by a more complicated "construction sequence"), there is a Δ-formula $Sub(x_1, x_2, x_3, y)$ which define the function

$$\mathrm{Sub}(\ulcorner\varphi\urcorner, \ulcorner x\urcorner, \ulcorner t\urcorner) := \ulcorner\varphi^x_t\urcorner.$$

This means: for all $(a, b, c, d) \in \mathbb{N}^4$,

- $\mathfrak{N} \models Sub(\overline{a}, \overline{b}, \overline{c}, \overline{d})$ if $a = \ulcorner\varphi\urcorner$ and $b = \ulcorner x\urcorner$ and $c = \ulcorner t\urcorner$ and $d = \ulcorner\varphi^x_t\urcorner$ for some formula φ and variable symbol x and term t
- $\mathfrak{N} \models \neg Sub(\overline{a}, \overline{b}, \overline{c}, \overline{d})$ otherwise.

Using Δ-formulas $\mathit{Num}(x, y)$ and $\mathit{Sub}(x_1, x_2, x_3, y)$ (among other useful Δ-formulas such as *Free* and *Substitutable*)), we get Δ-formulas defining sets:

$$\textsc{LogicalAxiom} := \{\ulcorner\varphi\urcorner : \varphi \text{ is a logical axiom}\}$$

$$\textsc{Axiom}_N := \{\ulcorner N_1\urcorner, \ldots, \ulcorner N_{11}\urcorner\},$$

$$\textsc{RuleOfInf} := \{(c, a) : c = \langle\ulcorner\gamma_1\urcorner, \ldots, \ulcorner\gamma_n\urcorner\rangle \text{ and } a = \ulcorner\varphi\urcorner \text{ where } (\{\gamma_1, \ldots, \gamma_n\}, \varphi) \text{ is a rule of inference}\}.$$

Finally, we get a Δ-formula $\mathit{Deduction}_N(y, z)$ which defines the set

$$\textsc{Deduction}_N := \{(c, a) : c = \langle\ulcorner\delta_1\urcorner, \ldots, \ulcorner\delta_1\urcorner\rangle \text{ and } a = \ulcorner\varphi\urcorner \text{ where } (\delta_1, \ldots, \delta_n) \text{ is a deduction from } N \text{ of } \varphi\}.$$

The Σ-formula $Thm_N(x)$

The set

$$\text{Thm}_N := \{\ulcorner \varphi \urcorner : N \vdash \varphi\}$$

is defined by Σ-formula

$$Thm_N(x) :\equiv (\exists y) Deduction_N(y, x).$$

THE Σ-FORMULA $Thm_N(x)$

The set

$$\text{THM}_N := \{\ulcorner\varphi\urcorner : N \vdash \varphi\}$$

is defined by Σ-formula

$$Thm_N(x) :\equiv (\exists y) Deduction_N(y, x).$$

This means: for every $a \in \mathbb{N}$,

- $\mathfrak{N} \models Thm(\overline{a})$ if $a = \ulcorner\varphi\urcorner$ for some formula φ such that $N \vdash \varphi$,

  (In this case, $N \vdash Thm(\overline{a})$ since N proves every Σ-sentence which is true in $\mathfrak{N}$ by Proposition 5.3.13.)

- $\mathfrak{N} \models \neg Thm(\overline{a})$ otherwise.

  (We *cannot* conclude that $N \vdash \neg Thm(\overline{a})$ since $\neg Thm(\overline{a})$ is (equivalent to) a Π-sentence.)

The Σ-formula $Thm_N(x)$

The set

$$\text{Thm}_N := \{\ulcorner\varphi\urcorner : N \vdash \varphi\}$$

is defined by Σ-formula

$$Thm_N(x) :\equiv (\exists y) Deduction_N(y, x).$$

There is no obvious way to rewrite $Thm_N(x)$ as a Δ-sentence (in fact, this is impossible). For instance, we cannot replace $(\exists y)$ with $(\exists y < x^{x^x})$, since this would imply that every formula of length ℓ provable by φ has a deduction of length $< \ell^{\ell^\ell}$ (which is false).

Representable ⇒ Σ-Definable

Previously, we showed that ***every Δ-definable set is representable***.

(This is a straightforward corollary of Proposition 5.3.13: N *proves every* Σ-*sentence which is true in* $\mathfrak{N}$.)

Next, we show that ***every representable set is Σ-definable***.