Δ-Definability of Sequence-Coding Operations

Let $p_1 = 2$, $p_2 = 3$, …, $p_i = i^{\text{th}}$ prime number.

$\mathbb{N}^{<\mathbb{N}}$ is the set of finite sequences of natural numbers.

Sequence-coding function $\langle\rangle : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$ is defined by

$$\langle a_1, \ldots, a_k \rangle := \prod_{i=1}^{k} p_i^{a_i+1}.$$

If $a = \langle a_1, \ldots, a_k \rangle$, then $|a| = k$ and $(a)_i = a_i$.

All of the sequence-coding operations are Δ*-definable (hence, representable).*

We have Δ-formulas:

$$Divides(y,x) :\equiv (\exists z \leq x)[x = y \cdot z],$$

$$Prime(x) :\equiv S0 < x \wedge (\forall y \leq x)[Divides(y,x) \rightarrow (y = 1 \vee y = x)]$$

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$$Divides(y, x) :\equiv (\exists z \leq x)[x = y \cdot z],$$

$$Prime(x) :\equiv S0 < x \wedge (\forall y \leq x)[\underbrace{Divides(y, x) \rightarrow (y = 1 \vee y = x)}_{\neg Divides(y,x) \vee (y = 1 \vee y = x)}]$$

Remark: Technically, $\neg Divides(y, x)$ is not a Δ-formula. However, it is equivalent to a Δ-formula.

The set

$$\textsc{PrimePair} := \{(p_i, p_{i+1}) : i \geq 1\} \subseteq \mathbb{N}^2$$

is defined by the Δ-formula

$$\begin{aligned} PrimePair(x, y) :\equiv\ & Prime(x) \wedge Prime(y) \wedge x < y \\ & \wedge (\forall z < y)[Prime(z) \rightarrow z \leq x]. \end{aligned}$$

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The set

$$\textsc{Codenumber} = \{ \underbrace{\langle a_1, \ldots, a_k \rangle}_{p_1^{a_1+1} p_2^{a_2+1} \cdots p_k^{a_k+1}} : (a_1, \ldots, a_k) \in \mathbb{N}^{<\mathbb{N}} \} \subseteq \mathbb{N}$$

is defined by the Δ-formula

$$\begin{aligned} \mathit{Codenumber}(c) :\equiv\ & \mathit{Divides}(SS0, c) \wedge (\forall z < c)(\forall y < z) \\ & \big[\big(\mathit{PrimePair}(y,z) \wedge \mathit{Divides}(z,c)\big) \rightarrow \mathit{Divides}(y,c)\big]. \end{aligned}$$

This formula expresses: "c is divisible by 2 and for every prime pair (p_i, p_{i+1}), if p_{i+1} divides c, so p_i divides c."

Continuing, we define the set

$$\textsc{Yardstick} := \{\underbrace{\langle 0, 1, 2, \ldots, k-1\rangle}_{2^1 3^2 5^3 \ldots (p_k)^k} : k \in \mathbb{N}\}$$

by the Δ-formula

$$\begin{aligned} \mathit{Yardstick}(x) :\equiv \mathit{Divides}(\overline{2}, x) \wedge \neg \mathit{Divides}(\overline{4}, x) \\ \wedge (\forall y \leq x)(\forall z \leq x)(\forall i < x) \\ \Big[\big(\mathit{PrimePair}(y, z) \wedge \mathit{Divides}(z, x)\big) \\ \rightarrow \big(\mathit{Divides}(\underbrace{y\, E\, i}_{y^i}, x) \leftrightarrow \mathit{Divides}(\underbrace{z\, E\, Si}_{z^{i+1}}, x)\big)\Big]. \end{aligned}$$

We next define the set

$$\textsc{IthPrime} := \{(i, p_i) : i \in \mathbb{N}_{\geq 1}\}$$

by the Δ-formula

$$\mathit{IthPrime}(i, y) :\equiv \mathit{Prime}(y) \wedge (\exists x \leq \text{some term}) \begin{bmatrix} \mathit{Yardstick}(x) \wedge \\ \mathit{Divides}(y^i, x) \wedge \\ \neg \mathit{Divides}(y^{i+1}, x) \end{bmatrix}.$$

Question: What term suffices for this bounded quantifier?

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Question: What term suffices for this bounded quantifier?

Answer: We want x to be the i-th yardstick number $(p_1)^1(p_2)^2 \ldots (p_i)^i$. This is at most

$$(p_i)(p_i)^2(p_i)^3 \cdots (p_i)^i = (p_i)^{\binom{i+1}{2}} \leq (p_i)^{i^2}.$$

Therefore, if $y = p_i$, it suffices to take $x \leq y^{i^2}$.

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Therefore, if $y = p_i$, it suffices to take $x \leq y^{i^2}$.

Alternatively, since $p_i \leq (i+1)^i$ (easy fact), we could instead use $x \leq (i+1)^{i^2}$.

We next define the set

$$\textsc{IthPrime} := \{(i, p_i) : i \in \mathbb{N}_{\geq 1}\}$$

by the Δ-formula

$$\mathit{IthPrime}(i, y) :\equiv \mathit{Prime}(y) \wedge (\exists x \leq y^{i \cdot i}) \begin{bmatrix} \mathit{Yardstick}(x) \wedge \\ \mathit{Divides}(y^i, x) \wedge \\ \neg \mathit{Divides}(y^{i+1}, x) \end{bmatrix}.$$

Exercise. Convince yourself that $\mathit{IthPrime}(i, y)$ indeed defines $\textsc{IthPrime}$. That is, show that

- $\mathfrak{N} \models \mathit{IthPrime}(\overline{k}, \overline{p_k})$ for every $k \geq 1$,
- $\mathfrak{N} \models \neg \mathit{IthPrime}(\overline{a}, \overline{b})$ whenever $a = 0$ or $b \neq p_a$.

We next define the set

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by the Δ-formula

$$\mathit{IthPrime}(i, y) :\equiv \mathit{Prime}(y) \wedge (\exists x \leq y^{i \cdot i}) \begin{bmatrix} \mathit{Yardstick}(x) \wedge \\ \mathit{Divides}(y^i, x) \wedge \\ \neg \mathit{Divides}(y^{i+1}, x) \end{bmatrix}.$$

Remark. The set $\textsc{IthPrime} \subseteq \mathbb{N}^2$ corresponds to the function $\mathbb{N} \to \mathbb{N}$ defined by $i \mapsto p_i$. Therefore, we say that the formula $\mathit{IthPrime}(i, y)$ defines the function $i \mapsto p_i$.

Continuing, the set

$$\textsc{Length} := \{(\langle a_1, \ldots, a_k \rangle, k) : k \geq 1 \text{ and } (a_1, \ldots, a_k) \in \mathbb{N}^k\}$$

is defined by the Δ-formula

$$\begin{aligned} \mathit{Length}(c, \ell) :\equiv\ & \mathit{Codenumber}(c) \\ & \wedge (\exists y \leq c) \left[\begin{aligned} & \mathit{IthPrime}(\ell, y) \wedge \mathit{Divides}(y, c) \\ & \wedge (\forall z \leq c)[\mathit{PrimePair}(y, z) \rightarrow \neg \mathit{Divides}(z, c)] \end{aligned} \right]. \end{aligned}$$

Continuing, the set

$$\text{Length} := \{(\langle a_1, \ldots, a_k \rangle, k) : k \geq 1 \text{ and } (a_1, \ldots, a_k) \in \mathbb{N}^k\}$$

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The set

$$\text{IthElement} := \{(a_j, j, \langle a_1, \ldots, a_k \rangle) : 1 \leq j \leq k \text{ and } (a_1, \ldots, a_k) \in \mathbb{N}^k\}$$

is defined by the Δ-formula

$$\begin{aligned} \mathit{IthElement}(e, i, c) :\equiv\ & \mathit{Codenumber}(c) \wedge (\exists y \leq c) \\ & \left[\mathit{IthPrime}(i, y) \wedge \mathit{Divides}(y^{Se}, c) \wedge \neg \mathit{Divides}(y^{SSe}, c) \right]. \end{aligned}$$

Δ-Definability of Sequence-Coding Operations

For practice, try writing a Δ-formula that defines the set $\textsc{Concatenation} \subseteq \mathbb{N}^3$ of triples of the form $(\langle a_1, \ldots, a_k \rangle, \langle b_1, \ldots, b_\ell \rangle, \langle a_1, \ldots, a_k, b_1, \ldots, b_\ell \rangle)$.

Gödel Numbers of Terms and Formulas

We assign a unique number to each symbol in $\mathcal{L}_{NT}$ as follows:

$\neg$	1	$\cdot$	15
$\vee$	3	E	17
$\forall$	5	$<$	19
$=$	7	$($	21
0	9	$)$	23
S	11	v_i	$2i$
$+$	13		

Suppose $s :\equiv s_1 \ldots s_n$ is a string of symbols, which constituting a well-formed term or formula of $\mathcal{L}_{NT}$.

Naively, we could encode s by the number $\langle \#(s_1), \ldots, \#(s_n) \rangle$ where $\#(s_i)$ is the number corresponding to the symbol s_i.

However, it much better to encode s according to the inductive type of terms and formulas.

Def 5.7.1. For each term t and formula φ, the Gödel numbers $\ulcorner t \urcorner$ and $\ulcorner \varphi \urcorner$ are defined as follows:

$$\begin{aligned}
\ulcorner \neg\alpha \urcorner &= \langle 1, \ulcorner \alpha \urcorner \rangle & \ulcorner +t_1t_2 \urcorner &= \langle 13, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \\
\ulcorner (\alpha \vee \beta) \urcorner &= \langle 3, \ulcorner \alpha \urcorner, \ulcorner \beta \urcorner \rangle & \ulcorner \cdot\, t_1t_2 \urcorner &= \langle 15, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \\
\ulcorner (\forall v_i)(\alpha) \urcorner &= \langle 5, \ulcorner v_i \urcorner, \ulcorner \alpha \urcorner \rangle & \ulcorner Et_1t_2 \urcorner &= \langle 17, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \\
\ulcorner {=}t_1t_2 \urcorner &= \langle 7, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle & \ulcorner {<}t_1t_2 \urcorner &= \langle 19, \ulcorner t_1 \urcorner, \ulcorner t_2 \urcorner \rangle \\
\ulcorner 0 \urcorner &= \langle 9 \rangle & \ulcorner v_i \urcorner &= \langle 2i \rangle. \\
\ulcorner St \urcorner &= \langle 11, \ulcorner t \urcorner \rangle
\end{aligned}$$

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\end{aligned}$$

Obs. $\ulcorner t \urcorner$ and $\ulcorner \varphi \urcorner$ are never divisible by 7. (Why?)

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$$
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\end{aligned}
$$

Example.
$$
\begin{aligned}
\ulcorner {=}0S0 \urcorner &= \langle 7, \ulcorner 0 \urcorner, \ulcorner S0 \urcorner \rangle \\
&= \langle 7, \langle 9 \rangle, \langle 11, \langle 9 \rangle \rangle \rangle \\
&= \langle 7, 2^{10}, \langle 11, 2^{10} \rangle \rangle = 2^8 3^{1025} 5^{(2^{12}3^{1025}+1)}.
\end{aligned}
$$

Notice how fast $\ulcorner SSSS0 \urcorner$ grows:

$$\ulcorner SSSS0 \urcorner = \langle 11, \langle 11, \langle 11, \langle 11, \langle 9 \rangle \rangle \rangle \rangle \rangle = 2^{12}3^{2^{12}3^{2^{12}3^{2^{12}3^{2^{10}}}}}$$

Next Steps (Section 5.8)

Δ-definability of sets

$$\begin{aligned}
\textsc{Terms} &:= \{\ulcorner t \urcorner : \text{terms } t\} &&= \{a \in \mathbb{N} : a = \ulcorner t \urcorner \text{ for some term } t\},\\
\textsc{Formulas} &:= \{\ulcorner \varphi \urcorner : \text{formulas } \varphi\} &&= \{a \in \mathbb{N} : a = \ulcorner \varphi \urcorner \text{ for some formula } \varphi\}.
\end{aligned}$$