Reminder of Notation

Language is always $\mathcal{L}_{NT} = (0, S, +, \cdot, E, <)$.

$\mathfrak{N}$ is the natural numbers as $\mathcal{L}_{NT}$-structure $\mathfrak{N} = (\mathbb{N}, 0, S, +, \cdot, E, <)$.

$N = \{N_1, \ldots, N_{11}\}$ is the set of axioms of Robinson Arithmetic.

For $a \in \mathbb{N}$, we let $\overline{a}$ stand for the variable-free term $\underbrace{SS\ldots S}_{a \text{ times}}0$.

For a variable-free term t, we let $t^{\mathfrak{N}} \in \mathbb{N}$ stand for the interpretation of t in $\mathfrak{N}$. (For example, $(SSS0 \cdot SS0)^{\mathfrak{N}}$ equals 6.)

The Power of Robinson Arithmetic

Robinson Arithmetic. The eleven axioms of N are:

(N1) $(\forall x)\neg[Sx = 0]$

(N2) $(\forall x)(\forall y)[Sx = Sy \rightarrow x = y]$

(N3) $(\forall x)[x + 0 = x]$

(N4) $(\forall x)(\forall y)[x + Sy = S(x + y)]$

(N5) $(\forall x)[x \cdot 0 = 0]$

(N6) $(\forall x)(\forall y)[(x \cdot Sy) = (x \cdot y) + x]$

(N7) $(\forall x)[xE0 = S0]$

(N8) $(\forall x)(\forall y)[xE(Sy) = (xEy) \cdot x]$

(N9) $(\forall x)\neg[x < 0]$

(N10) $(\forall x)(\forall y)[x < Sy \leftrightarrow (x < y \vee x = y)]$

(N11) $(\forall x)(\forall y)[x < y \vee x = y \vee y < x]$.

The Power of Robinson Arithmetic

Lemma 2.8.4. For all natural numbers a and b:

1. If $a = b$, then $N \vdash \overline{a} = \overline{b}$.
2. If $a \neq b$, then $N \vdash \overline{a} \neq \overline{b}$.
3. If $a < b$, then $N \vdash \overline{a} < \overline{b}$.
4. If $a \not< b$, then $N \vdash \neg(\overline{a} < \overline{b})$.
5. $N \vdash \overline{a} + \overline{b} = \overline{a + b}$.
6. $N \vdash \overline{a} \cdot \overline{b} = \overline{a \cdot b}$.
7. $N \vdash \overline{a} E \overline{b} = \overline{a^b}$.

Lemma 5.3.10. $N \vdash (t = \overline{t^{\mathfrak{N}}})$ for every variable-free term t. (Proof by induction on t, on blackboard.)

For example, if $t :\equiv (SS0 + S0) \cdot SS0$, then this lemma tells us $N \vdash ((SS0 + S0) \cdot SS0 = SSSSSS0)$.

The Power of Robinson Arithmetic

Lemma 5.3.11 (Rosser's Lemma). For every $a \in \mathbb{N}$,

$$N \vdash (\forall x < \overline{a})\big[x = \overline{0} \vee x = \overline{1} \vee \cdots \vee x = \overline{a-1}\big].$$

Proof by induction on a (on blackboard)

Lemma 5.3.11 (Rosser's Lemma). For every $a \in \mathbb{N}$,

$$N \vdash (\forall x < \overline{a})\big[x = \overline{0} \vee x = \overline{1} \vee \cdots \vee x = \overline{a-1}\big].$$

Proof by induction on a (on blackboard)

Corollary 5.3.12. For every $a \in \mathbb{N}$ and formula $\varphi(x)$,

$$N \vdash \underbrace{\big[(\forall x < \overline{a})\varphi(x)\big] \leftrightarrow \big[\varphi(\overline{0}) \wedge \varphi(\overline{1}) \wedge \cdots \wedge \varphi(\overline{a-1})\big]}_{\text{that is, } \big[(\forall x < \overline{a})\varphi\big] \leftrightarrow \big[\varphi^x_{\overline{0}} \wedge \varphi^x_{\overline{1}} \wedge \cdots \wedge \varphi^x_{\overline{a-1}}\big]}$$

(Proof given as Exercise 11 in Section 5.3; solution on page 319. *Good exercise to try on your own!*)

The Power of Robinson Arithmetic

Proposition 5.3.13. *If* φ *is a* Σ*-sentence such that* $\mathfrak{N} \models \varphi$*, then* $N \vdash \varphi$*.*

In other words, N proves every Σ-sentence which is true in $\mathfrak{N}$.

The Power of Robinson Arithmetic

Proposition 5.3.13. *If φ is a Σ-sentence such that $\mathfrak{N} \models \varphi$, then $N \vdash \varphi$.*

In other words, N proves every Σ-sentence which is true in $\mathfrak{N}$.

RECALL: As we have discussed before, N does *not* prove every sentence which is true in $\mathfrak{N}$. In particular, $N \nvdash (\forall x)\neg[x < x]$ and $N \nvdash (\forall x)(\forall y)[x+y = y+x]$.

Proposition 5.3.13. *If φ is a Σ-sentence such that $\mathfrak{N} \models \varphi$, then $N \vdash \varphi$.*

PROOF. Let φ be a Σ-sentence such that $\mathfrak{N} \models \varphi$. We argue by induction on the complexity of φ.

Proposition 5.3.13. *If* φ *is a* Σ*-sentence such that* $\mathfrak{N} \models \varphi$*, then* $N \vdash \varphi$*.*

PROOF. Let φ be a Σ-sentence such that $\mathfrak{N} \models \varphi$. We argue by induction on the complexity of φ.

Base case φ is atomic or $\neg$(atomic).

Suppose (for example) φ is $t < u$.

Then $\mathfrak{N} \models \varphi$ means that $t^{\mathfrak{N}} < u^{\mathfrak{N}}$.

So by Lemma 2.8.4, $N \vdash \overline{t^{\mathfrak{N}}} < \overline{u^{\mathfrak{N}}}$.

By Lemma 5.3.10 (which we just proved), $\mathfrak{N} \models t = \overline{t^{\mathfrak{N}}}$ and $\mathfrak{N} \models u = \overline{u^{\mathfrak{N}}}$.

Therefore, $N \vdash t < u$ (using the (E3) axiom).

Proposition 5.3.13. *If φ is a Σ-sentence such that $\mathfrak{N} \models \varphi$, then $N \vdash \varphi$.*

PROOF. Let φ be a Σ-sentence such that $\mathfrak{N} \models \varphi$. We argue by induction on the complexity of φ.

Suppose $\varphi :\equiv (\alpha \vee \beta)$.

Without loss of generality, assume $\mathfrak{N} \models \alpha$.

By induction hypothesis, $N \vdash \alpha$.

Therefore, $N \vdash \varphi$ by (PC) rule.

Proposition 5.3.13. *If φ is a Σ-sentence such that $\mathfrak{N} \models \varphi$, then $N \vdash \varphi$.*

PROOF. Let φ be a Σ-sentence such that $\mathfrak{N} \models \varphi$. We argue by induction on the complexity of φ.

NOTE: We do not need to consider the case $\varphi :\equiv \neg\alpha$, since Σ-sentences are not closed under negation.

Proposition 5.3.13. *If φ is a Σ-sentence such that $\mathfrak{N} \models \varphi$, then $N \vdash \varphi$.*

PROOF. Let φ be a Σ-sentence such that $\mathfrak{N} \models \varphi$. We argue by induction on the complexity of φ.

Suppose $\varphi :\equiv (\exists y)\alpha$.

Since $\mathfrak{N} \models \varphi$, there exists $a \in \mathbb{N}$ such that $\mathfrak{N} \models \alpha_{\overline{a}}^{y}$.

Note that $\alpha_{\overline{a}}^{y}$ is a Σ-sentence with lower complexity than φ (that is, fewer $\vee$ and $\forall$ symbols). (NOTE: $\alpha_{\overline{a}}^{y}$ possibly has greater length as a string.)

By induction hypothesis, $N \vdash \alpha_{\overline{a}}^{y}$.

By (Q2) axiom: $\vdash \alpha_{\overline{a}}^{y} \to (\exists y)\alpha$. (Since $\overline{a}$ is variable-free, it is substitutable for y in α.)

Therefore, $N \vdash \varphi$ by (PC) rule.

Proposition 5.3.13. *If φ is a Σ-sentence such that $\mathfrak{N} \models \varphi$, then $N \vdash \varphi$.*

PROOF. Let φ be a Σ-sentence such that $\mathfrak{N} \models \varphi$. We argue by induction on the complexity of φ.

Suppose $\varphi :\equiv (\forall y < u)\alpha$ where u is a variable-free term.

Since $\mathfrak{N} \models \varphi$, it follows that $\mathfrak{N} \models \alpha^y_{\overline{a}}$ for every $a < u^{\mathfrak{N}}$.

By the induction hypothesis, $N \vdash \alpha^y_{\overline{a}}$ for every $a < u^{\mathfrak{N}}$.

By Corollary 4.3.8 (the corollary of Rosser's Lemma), we have

$$N \vdash \left[(\forall y < \overline{u^{\mathfrak{N}}})\alpha\right] \leftrightarrow \left[\alpha^y_{\overline{0}} \land \alpha^y_{\overline{1}} \land \cdots \land \alpha^y_{\overline{u^{\mathfrak{N}}-1}}\right].$$

By (PC) rule, $N \vdash (\forall y < \overline{u^{\mathfrak{N}}})\alpha$.

By Lemma 4.3.6, $N \vdash u = \overline{u^{\mathfrak{N}}}$. This lets us derive $N \vdash (\forall y < u)\alpha]$ as required.

Q.E.D.

Definable and Representable Sets

A set $A \subseteq \mathbb{N}^k$ is $\Sigma/\Pi/\Delta$-***definable*** if there exists a $\Sigma/\Pi/\Delta$-formula $\varphi(x_1, \ldots, x_k)$ such that

- $\mathfrak{N} \models \varphi(\overline{a_1}, \ldots, \overline{a_k})$ for every $(a_1, \ldots, a_k) \in A$
- $\mathfrak{N} \models \neg\varphi(\overline{b_1}, \ldots, \overline{b_k})$ for every $(b_1, \ldots, b_k) \in \mathbb{N}^k \setminus A$.

Definable and Representable Sets

A set $A \subseteq \mathbb{N}^k$ is $\Sigma/\Pi/\Delta$-***definable*** if there exists a $\Sigma/\Pi/\Delta$-formula $\varphi(x_1, \ldots, x_k)$ such that

- $\mathfrak{N} \models \varphi(\overline{a_1}, \ldots, \overline{a_k})$ for every $(a_1, \ldots, a_k) \in A$
- $\mathfrak{N} \models \neg\varphi(\overline{b_1}, \ldots, \overline{b_k})$ for every $(b_1, \ldots, b_k) \in \mathbb{N}^k \setminus A$.

A set $A \subseteq \mathbb{N}^k$ is ***representable*** if there exists a formula $\varphi(x_1, \ldots, x_k)$ such that

- $N \vdash \varphi(\overline{a_1}, \ldots, \overline{a_k})$ for every $(a_1, \ldots, a_k) \in A$
- $N \vdash \neg\varphi(\overline{b_1}, \ldots, \overline{b_k})$ for every $(b_1, \ldots, b_k) \in \mathbb{N}^k \setminus A$.

Definable and Representable Sets

A set $A \subseteq \mathbb{N}^k$ is $\Sigma/\Pi/\Delta$***-definable*** if there exists a $\Sigma/\Pi/\Delta$-formula $\varphi(x_1, \ldots, x_k)$ such that

- $\mathfrak{N} \models \varphi(\overline{a_1}, \ldots, \overline{a_k})$ for every $(a_1, \ldots, a_k) \in A$
- $\mathfrak{N} \models \neg\varphi(\overline{b_1}, \ldots, \overline{b_k})$ for every $(b_1, \ldots, b_k) \in \mathbb{N}^k \setminus A$.

A set $A \subseteq \mathbb{N}^k$ is ***representable*** if there exists a formula $\varphi(x_1, \ldots, x_k)$ such that

- $N \vdash \varphi(\overline{a_1}, \ldots, \overline{a_k})$ for every $(a_1, \ldots, a_k) \in A$
- $N \vdash \neg\varphi(\overline{b_1}, \ldots, \overline{b_k})$ for every $(b_1, \ldots, b_k) \in \mathbb{N}^k \setminus A$.

A set $A \subseteq \mathbb{N}^k$ is ***weakly representable*** if there exists a formula $\varphi(x_1, \ldots, x_k)$ such that

- $N \vdash \varphi(\overline{a_1}, \ldots, \overline{a_k})$ for every $(a_1, \ldots, a_k) \in A$
- $N \nvdash \varphi(\overline{b_1}, \ldots, \overline{b_k})$ for every $(b_1, \ldots, b_k) \in \mathbb{N}^k \setminus A$.

Definable and Representable Sets

A function $f : A \to \mathbb{N}$ where $A \subseteq \mathbb{N}^k$ is *definable* or *representable* according to the corresponding set $\{(a_1, \ldots, a_k, b) : f(a_1, \ldots, a_k) = b\} \subseteq \mathbb{N}^{k+1}$.

Example. The function $a \mapsto a^2$ is Δ-definable, since it is defined by the Δ-formula $\varphi(x, y) :\equiv (y = x \cdot x)$ (or $(y = x\, E\, SS0)$).

The Power of Robinson Arithmetic

Proposition 5.3.13. *If* φ *is a* Σ*-sentence such that* $\mathfrak{N} \models \varphi$*, then* $N \vdash \varphi$*.*

Corollary 5.3.15. *Every* Δ*-definable set is representable.*

This fact is extremely useful: it lets us show that various sets and functions are representable!

The Power of Robinson Arithmetic

Proposition 5.3.13. *If φ is a Σ-sentence such that $\mathfrak{N} \models \varphi$, then $N \vdash \varphi$.*

Corollary 5.3.15. *Every Δ-definable set is representable.*

This fact is extremely useful: it lets us show that various sets and functions are representable!

Proof of Proposition $\Rightarrow$ Corollary:

Suppose $A \subseteq \mathbb{N}^k$ is defined by the Δ-formula $\varphi(x_1, \ldots, x_n)$. Both $\varphi(x_1, \ldots, x_n)$ and $\neg\varphi(x_1, \ldots, x_n)$ are logically equivalent to Σ-formulas. Therefore, Proposition 5.3.13 implies

- $N \vdash \varphi(\overline{a_1}, \ldots, \overline{a_k})$ (since $\mathfrak{N} \models \varphi(\overline{a_1}, \ldots, \overline{a_k})$) for every $(a_1, \ldots, a_k) \in A$
- $N \vdash \neg\varphi(\overline{b_1}, \ldots, \overline{b_k})$ (since $\mathfrak{N} \models \neg\varphi(\overline{b_1}, \ldots, \overline{b_k})$) for every $(b_1, \ldots, b_k) \in \mathbb{N}^k \setminus A$.

Representable Functions and Computer Programs (Section 5.4)

Various mathematical model of "computable" sets and functions were proposed in the 1930s:

- Turing machines (most intuitive model)
- Church's λ-calculus
- Gödel's recursive functions
- *representable functions*

Remarkably, all these models capture the same notion: a set $A \subseteq \mathbb{N}^k$ (or function $f : \mathbb{N}^k \to \mathbb{N}$) is *representable* iff it is λ-computable iff it is Turing computable iff it is recursive.

The equivalence of these various notions of "computable" is a mathematical theorem.

Representable Functions and Computer Programs (Section 5.4)

Theorem.

- If $A \subseteq \mathbb{N}^k$ is representable, then there is an algorithm (computer program) which, given $(a_1, \ldots, a_k) \in \mathbb{N}^k$ as input, determines in finite time whether or not $(a_1, \ldots, a_k) \in A$.
- Conversely, if there exists a computer program which determines membership in a set $A \subseteq \mathbb{N}^k$, then A is representable.

Representable Functions and Computer Programs (Section 5.4)

Theorem.

- If $A \subseteq \mathbb{N}^k$ is representable, then there is an algorithm (computer program) which, given $(a_1, \ldots, a_k) \in \mathbb{N}^k$ as input, determines in finite time whether or not $(a_1, \ldots, a_k) \in A$.
- Conversely, if there exists a computer program which determines membership in a set $A \subseteq \mathbb{N}^k$, then A is representable.

The Church-Turing Thesis. A function on the natural numbers is computable by a human being following an algorithm, ignoring resource limitations, if and only if it is computable by a Turing machine or any other equivalent notion (e.g., representability).