Peano Arithmetic

Definition. The axioms of ***Peano Arithmetic*** (1889), denoted PA, consist of the eleven axioms of Robinson arithmetic together with axioms

$$Induction_{\varphi} :\equiv \Big[\varphi(0) \wedge (\forall x)[\varphi(x) \rightarrow \varphi(Sx)]\Big] \rightarrow (\forall x)\varphi(x)$$

for each $\mathcal{L}_{NT}$-formula $\varphi(x)$ with one free variable.

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- PA is easily seen to be recursive: there is a simple algorithm to decide membership in $\{\ulcorner\alpha\urcorner : \alpha \in PA\}$. By 1st Incompleteness Theorem, there exists a sentence θ such that $\mathfrak{N} \models \theta$ but $PA \not\vdash \theta$. (In particular, PA is not complete.)

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- Whereas Robinson arithmetic N is very weak (it doesn't prove $(\forall x)(\forall y)(x+y = y+x)$), Peano arithmetic PA is quite powerful – it proves any result you have seen in MAT315. (It is even claimed that $PA \vdash$ Fermat's Last Theorem.)

2nd Incompleteness Theorem

The sentence Con_A:

Let A be a recursive set of $\mathcal{L}_{NT}$-sentences.

Recall that the set $\textsc{Thm}_A := \{\ulcorner\varphi\urcorner : A \vdash \varphi\}$ is Σ-definable. Fix a Σ-formula $Thm_A(x)$ which defines $\textsc{Thm}_A$.

Let Con_A be the sentence

$$Con_A :\equiv \neg Thm_A(\overline{\ulcorner\bot\urcorner}).$$

This sentence expresses "A is consistent": note that A is consistent if, and only if, $\mathfrak{N} \models Con_A$.

Theorem 6.6.3 (Godel's 2nd Incompleteness Theorem)
If A is any consistent, recursive set of $\mathcal{L}_{NT}$-sentences which extends PA, then $A \nvdash Con_A$.

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- PA itself is consistent and recursive. Therefore, $PA \nvdash Con_{PA}$.
- How do you and I know that PA is consistent? We can prove $\mathfrak{N}$ is a model of Con_{PA} using the usual axioms of ZFC (Zermelo-Frankl set theory with choice). Therefore, $ZFC \vdash Con_{PA}$ (interpreting the sentence Con_{PA} in the language of set theory).

  However, $ZFC \nvdash Con_{ZFC}$.

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  However, $ZFC \nvdash Con_{ZFC}$.
- 2nd Incompleteness Theorem answered a question asked by David Hilbert in 1900 by showing that no "sufficiently powerful formal system" (including set theory ZFC) can prove its own consistency.

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- Alternative phrasing of 2nd Incompleteness Theorem: *If* A *is recursive extension of* PA*, then* A *is consistent* $\Leftrightarrow A \nvdash Con_A$*.*

  (If A is inconsistent, then $A \vdash Con_A$ since A proves everything.)

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- Since $PA \nvdash Con_{PA}$, it follows that $PA \cup \{\neg Con_{PA}\}$ is consistent. (This is because, if we assume that $PA \cup \{\neg Con_{PA}\} \vdash \bot$, then $PA \vdash \neg Con_{PA} \to \bot$ by the Deduction Theorem; it would then follow that $PA \vdash Con_{PA}$ by the (PC) rule, but this contradictions the fact that $PA \nvdash Con_{PA}$.)

  Therefore, there exists a model $\mathfrak{M}$ of $PA \cup \{\neg Con_{PA}\}$.

  (Note: This model looks similar to $\mathfrak{N}$ — for example, addition $+^{\mathfrak{M}}$ is commutative. However, $Th(\mathfrak{M}) \neq Th(\mathfrak{N})$.)

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- QUESTION: Since PA is consistent, why not take Con_{PA} as an additional axiom?

  Let $PA' := PA \cup \{Con_{PA}\}$. Then $PA' \vdash Con_{PA}$, but $PA' \nvdash Con_{PA'}$. So we are left with the same problem.

Hilbert-Bernays Derivability Conditions

Lemma. PA satisfies the following "derivability conditions" for all formulas α and β:

(D1) If $PA \vdash \alpha$, then $PA \vdash Thm_{PA}(\overline{\ulcorner \alpha \urcorner})$.

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PA proves "if PA proves α and PA proves $\alpha \rightarrow \beta$, then PA proves β".

Moreover, if A is a recursive extension of PA, then A satisfies derivability conditions (D1)–(D3) with respect to $Thm_A(x)$.

Proof of 2nd Incompleteness Theorem. Let A be a consistent, recursive extension of PA. Let θ be a sentence such that

$$(*) \qquad N \vdash \theta \leftrightarrow \neg Thm_A(\overline{\ulcorner \theta \urcorner}).$$

By proof of 1st Incompleteness Theorem, we know that $A \nvdash \theta$.

CLAIM: $A \vdash Con_A \rightarrow \theta$. (It follows that $A \nvdash Con_A$.)

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By (PC) rule, $A \vdash Thm_A(\overline{\ulcorner \theta \urcorner}) \to Thm_A(\overline{\ulcorner \neg\theta \urcorner})$.

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Next step: $A \vdash Thm_A(\overline{\ulcorner \theta \urcorner}) \to \underbrace{Thm_A(\overline{\ulcorner \bot \urcorner})}_{\neg Con_{PA}}$.

Taking the contrapositive, we have $A \vdash Con_A \to \neg Thm_A(\overline{\ulcorner \theta \urcorner})$.

Finally, by $(*)$ and (PC) rule: $A \vdash Con_A \to \theta$. Q.E.D.

Complete, Consistent, Recursive Theories

The 1st Incompleteness Theorem implies that $Th(\mathfrak{N})$ has no complete, consistent, recursive axiomatization.

The same is true of any theory that "interprets" $Th(\mathfrak{N})$, such as models of ZFC (set theory).

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In contrast, there are beautiful examples of complete, consistent, recursive theories:

- $Th(\mathbb{N}, 0, 1, +)$ (Presburger Arithmetic)
- $Th(\mathbb{R}, 0, 1, +, \cdot, <)$ (the theory of real closed fields)
- Euclidean Geometry: $Th(\mathbb{R}^2, \text{Between}, \text{Congruent})$ where

$$\text{Between} := \{(a, b, c) \in (\mathbb{R}^2)^3 : b \in ac\},$$
$$\text{Congruent} := \{(a, b, c, d) \in (\mathbb{R}^2)^4 : |ab| = |cd|\}.$$

  Here ab denotes the line segment between points $a, b \in \mathbb{R}^2$, and $|ab|$ is the length of ab. (See Tarski's axioms.)