Completeness Theorem. *If* $\Sigma \models \varphi$, *then* $\Sigma \vdash \varphi$.

Follows from:

Model Existence Theorem. *Every consistent set of sentences has a model. (If* $\Sigma \nvdash \bot$, *then there exists a model* $\mathfrak{A} \models \Sigma$.*)*

Completeness Theorem. *If* $\Sigma \models \varphi$, *then* $\Sigma \vdash \varphi$.

Model Existence Theorem. *If* $\Sigma \nvdash \bot$, *then* $\exists$ *a model* $\mathfrak{A} \models \Sigma$.

Proof of Completeness Thm from Model Existence Thm.

1. Suffices to prove Completeness Theorem in the case where Σ is a set of *sentences* and φ is a *sentence*. (Follows from the fact that $\{\alpha\} \vdash \forall\alpha$ and $\{\forall x\alpha\} \vdash \alpha$ for all formulas α.)

2. Assume $\Sigma \models \varphi$.

   Then $\mathfrak{A} \models \Sigma \Rightarrow \mathfrak{A} \models \varphi$ for all structures $\mathfrak{A}$.

   Therefore, $\Sigma \cup \{\neg\varphi\}$ has no models.

   By Model Existence Theorem, $\Sigma \cup \{\neg\varphi\} \vdash \bot$.

   By Deduction Theorem, $\Sigma \vdash (\neg\varphi \rightarrow \bot)$.

   By (PC) rule, $\Sigma \vdash \varphi$ (via the tautology $(\neg P \rightarrow \mathit{False}) \rightarrow P)$). Q.E.D.

Proof of the Model Existence Theorem

Let $\mathcal{L}$ be a countable language. (The case of uncountable languages uses Zorn's Lemma, as explained in tutorial.)

Let Σ be a *consistent* set of $\mathcal{L}$-sentences. We will construct a model $\mathfrak{A} \models \Sigma$.

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We first construct a language $\mathcal{L}' = \mathcal{L} \cup \{\text{countably many new constant symbols}\}$ and a set of $\mathcal{L}'$-sentences $\widehat{\Sigma}$ such that

(1) $\Sigma \subseteq \widehat{\Sigma}$,

(2) $\widehat{\Sigma}$ is consistent,

(3) $\widehat{\Sigma}$ contains a Henkin axiom $(\exists x\theta) \to \theta^x_c$ for each $\mathcal{L}'$-sentence $\exists x\theta$.

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We then complete $\widehat{\Sigma}$ to a maximal consistent set of $\mathcal{L}'$-sentences Σ' (by adding either φ or $\neg\varphi$ one-at-a-time for each $\mathcal{L}'$-sentence φ). In addition to properties (1)–(3), Σ' has the property:

(4) For every $\mathcal{L}'$-sentence φ,

$$\varphi \in \Sigma' \iff \Sigma' \vdash \varphi \iff \neg\varphi \notin \Sigma' \iff \Sigma' \nvdash \neg\varphi.$$

The $\mathcal{L}'$-Structure $\mathfrak{A}$:

We define an equivalence relation $\sim$ on $\{$variable-free $\mathcal{L}'$-terms$\}$ by

$$t \sim u \overset{\text{def}}{\iff} (t = u) \in \Sigma' \iff \Sigma' \vdash (t = u).$$

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CLAIM: $\sim$ is an equivalence relation.

PROOF: Here is the argument that $\sim$ is symmetric:

Assume $t \sim u$. Then $\Sigma' \vdash (t = u)$.

We know $\vdash (\forall x)(\forall y)[(x = y) \to (y = x)]$ (from Chapter 2).

Therefore, $\vdash (t = u) \to (u = t)$ by (Q1) axiom and (PC) rule.

Therefore, $\Sigma' \vdash (u = t)$ by (PC) rule; hence $u \sim t$.

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Let $\mathfrak{A}$ be the following structure:

- $A := \{[t]_\sim : t \text{ is a variable-free } \mathcal{L}'\text{-term}\}$ (universe of $\mathfrak{A}$)
- $c^{\mathfrak{A}} := [c]_\sim$ for each constant symbol c
- $f^{\mathfrak{A}}([t_1]_\sim, \ldots, [t_n]_\sim) := [f t_1 \ldots t_n]_\sim$ for each n-ary f
- $([t_1]_\sim, \ldots, [t_n]_\sim) \in R^{\mathfrak{A}} \overset{\text{def}}{\iff} R t_1 \ldots t_n \in \Sigma'$ for each n-ary R

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- $f^{\mathfrak{A}}([t_1]_\sim, \ldots, [t_n]_\sim) := [ft_1 \ldots t_n]_\sim$ for each n-ary f
- $([t_1]_\sim, \ldots, [t_n]_\sim) \in R^{\mathfrak{A}} \overset{\text{def}}{\iff} Rt_1 \ldots t_n \in \Sigma'$ for each n-ary R

NOTE: To show that $f^{\mathfrak{A}}$ and $R^{\mathfrak{A}}$ are well-defined, we must show that if $t_i \sim u_i$ for $i = 1, \ldots, n$, then

- $[ft_1 \ldots t_n]_\sim = [fu_1 \ldots u_n]_\sim$ (that is, $(ft_1 \ldots t_n = fu_1 \ldots u_n) \in \Sigma'$) and
- $Rt_1 \ldots t_n \in \Sigma' \iff Ru_1 \ldots u_n \in \Sigma'$.

Proposition 3.2.6. For every $\mathcal{L}'$-sentence φ, we have

$$\varphi \in \Sigma' \iff \mathfrak{A} \models \varphi.$$

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The Model Existence Theorem follows from Prop 3.2.5 as follows:

- As a consequence of Prop 3.2.6, $\mathfrak{A} \models \Sigma'$.
- Since $\Sigma \subset \widehat{\Sigma} \subset \Sigma'$, it follows that $\mathfrak{A} \models \Sigma$.
- If we ignore that interpretation of constant symbols in $\mathcal{L}' \setminus \mathcal{L}$, then the "reduct" $\mathcal{A}|_{\mathcal{L}}$ is a model of Σ in the original language $\mathcal{L}$.

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PROOF. We argue by induction on the complexity of φ.

As usual, we consider five cases:

1. $\varphi :\equiv Rt_1 \ldots t_n$ (base case)
2. $\varphi :\equiv t_1 = t_2$ (base case)
3. $\varphi :\equiv \neg\alpha$ (easy induction step)
4. $\varphi :\equiv \alpha \vee \beta$ (easy induction step)
5. $\varphi :\equiv \forall x \alpha$ (non-trivial induction step)

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CASE 1. $\varphi :\equiv Rt_1 \dots t_n$

We have

$$\varphi \in \Sigma' \iff ([t_1]_\sim, \dots, [t_n]_\sim) \in R^{\mathfrak{A}} \iff \mathfrak{A} \models \varphi.$$

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CASE 2. $\varphi :\equiv t_1 = t_2$

We use the fact (shown by induction) that $\overline{s}(t) = [t]_\sim$ for every variable-free term t and $s : \mathit{Vars} \to A$.

It follows:

$$\begin{aligned}
\mathfrak{A} \models \varphi &\iff \mathfrak{A} \models \varphi[s] \text{ for every } s : \mathit{Vars} \to A \\
&\iff \overline{s}(t_1) = \overline{s}(t_2) \text{ for every } s : \mathit{Vars} \to A \\
&\iff [t_1]_\sim = [t_2]_\sim \text{ (since } \overline{s}(t) = [t]_\sim) \\
&\iff (t_1 = t_2) \in \Sigma' \\
&\iff \varphi \in \Sigma'.
\end{aligned}$$

Proposition 3.2.6. For every $\mathcal{L}'$-sentence φ, we have

$$\varphi \in \Sigma' \iff \mathfrak{A} \models \varphi.$$

CASE 3. $\varphi :\equiv \neg\alpha$

$$\begin{aligned} \mathfrak{A} \models \varphi &\iff \mathfrak{A} \not\models \alpha \\ &\iff \alpha \notin \Sigma' \text{ (induction hypothesis applied to } \alpha) \\ &\iff \neg\alpha \in \Sigma' \text{ (since } \alpha \in \Sigma' \iff \neg\alpha \notin \Sigma') \\ &\iff \varphi \in \Sigma'. \end{aligned}$$

Proposition 3.2.6. For every $\mathcal{L}'$-sentence φ, we have

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CASE 4. $\varphi :\equiv \alpha \vee \beta$

Another trivial induction step.

Proposition 3.2.6. For every $\mathcal{L}'$-sentence φ, we have

$$\varphi \in \Sigma' \iff \mathfrak{A} \models \varphi.$$

CASE 5. $\varphi :\equiv \forall x \alpha$ $\boxed{\varphi \in \Sigma' \Longrightarrow \mathfrak{A} \models \varphi}$

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CASE 5. $\varphi :\equiv \forall x \alpha$ $\boxed{\varphi \in \Sigma' \implies \mathfrak{A} \models \varphi}$

Assume $\varphi \in \Sigma'$. To show $\mathfrak{A} \models \varphi$, we must show

$$\begin{array}{rll} & \mathfrak{A} \models \varphi[s] & \text{for every } s : \mathit{Vars} \to A, \\ \text{that is,} & \mathfrak{A} \models \alpha[s[x|a]] & \text{for every } s : \mathit{Vars} \to A \text{ and } a \in A. \end{array}$$

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Fix any $s : \mathit{Vars} \to A$ and $a = [t]_\sim \in A$. We have

$$\begin{aligned} \mathfrak{A} \models \alpha[s[x|a]] &\iff \mathfrak{A} \models \alpha[s[x|\overline{s}(t)]] && (\text{since } \overline{s}(t) = [t]_\sim = a) \\ &\iff \mathfrak{A} \models \alpha^x_t[s] && (\text{by Theorem 2.6.2}) \\ &\iff \mathfrak{A} \models \alpha^x_t && (\text{since } \alpha^x_t \text{ is a sentence}). \end{aligned}$$

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Since $\varphi \in \Sigma'$, we have $\Sigma' \vdash \forall x \alpha$, hence $\Sigma \vdash \alpha^x_t$ by (Q1) axiom and (PC) rule. (OBS: Since t is variable-free, it is substitutable for x.)

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Therefore, $\alpha^x_t \in \Sigma'$. By induction hypothesis applied to α^x_t, it follows that $\mathfrak{A} \models \alpha^x_t$. Hence, $\mathfrak{A} \models \varphi$.

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CASE 5. $\varphi :\equiv \forall x \alpha$

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Assume $\varphi \notin \Sigma'$. Then $\neg\varphi \in \Sigma'$, hence $\Sigma' \vdash \neg\varphi :\equiv \neg\forall x\alpha$, hence $\Sigma' \vdash \exists x \neg\alpha$.

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The set $\widehat{\Sigma}$ (and hence also Σ') contains a Henkin axiom

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Hence $\Sigma' \vdash (\exists x \neg\alpha) \to (\neg\alpha)^x_c$. Therefore, by (PC) rule (modus ponens),

$$\Sigma' \vdash (\neg\alpha)^x_c.$$

By the induction hypothesis applied to the sentence $(\neg\alpha)^x_c$, we have $\mathfrak{A} \models (\neg\alpha)^x_c$.

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By the induction hypothesis applied to the sentence $(\neg\alpha)^x_c$, we have $\mathfrak{A} \models (\neg\alpha)^x_c$.

It follows that $\mathfrak{A} \models \exists x \neg\alpha$, hence $\mathfrak{A} \not\models \neg\exists x \neg\alpha$, hence $\mathfrak{A} \not\models \varphi$ as required.

Q.E.D.

✓ **Model Existence Theorem.** Every consistent set of sentences has a model.

✓ **Completeness Theorem.** If $\Sigma \models \varphi$, then $\Sigma \vdash \varphi$.

☐ **Compactness Theorem.** $\Sigma \models \varphi$ iff $\Sigma_0 \models \varphi$ for some finite $\Sigma_0 \subseteq \Sigma$.

The Compactness Theorem

COMPACTNESS THEOREM (Theorem 3.3.1) Let Σ be any set of formulas. Then Σ has a model (a.k.a. it is "satisfiable") if, and only if, every finite subset of Σ has a model.

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($\Leftarrow$): Assume that Σ has no model. We will show that some finite $\Sigma_0 \subseteq \Sigma$ has no model.

Since Σ has no model, this means $\Sigma \models \bot$. By the Completeness Theorem (or the contrapositive of the Model Existence Theorem), it follows that $\Sigma \vdash \bot$.

That is, there exists a deduction $(\delta_1, \ldots, \delta_n)$ of $\bot$ from Σ.

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That is, there exists a deduction $(\delta_1, \ldots, \delta_n)$ of $\bot$ from Σ.

Let $\Sigma_0 := \Sigma \cap \{\delta_1, \ldots, \delta_n\}$. Observe that Σ_0 is a finite set and $\Sigma_0 \vdash \bot$ (since the same sequence $(\delta_1, \ldots, \delta_n)$ is a deduction of $\bot$ from Σ_0).

Therefore, $\Sigma_0 \models \bot$ by the Soundness Theorem. This means that Σ_0 has no model, as required.

COMPACTNESS THEOREM (Theorem 3.3.1) Let Σ be any set of formulas. Then Σ has a model (a.k.a. it is "satisfiable") if, and only if, every finite subset of Σ has a model.

Corollary 3.2.2. $\Sigma \models \varphi$ iff $\Sigma_0 \models \varphi$ for some finite subset Σ_0 of Σ.

In lectures and tutorial, we discussed several applications of the Compactness Theorem:

1. If Σ has finite models of size $\geq n$ for every $n \in \mathbb{N}$, then Σ has an infinite model.
2. Every set S admits a linear order $<$.
3. König's Infinity Lemma (Example 3.3.8): Every infinite tree contains a vertex of infinite degree or an infinite simple path.
4. Wang tiles (wikipedia): A set of Wang tiles can tile the plane iff it can tile any $n \times n$ region.
5. The 4-Color Theorem for infinite planar graphs (Examples 3.3.6)
6. Non-standard models of arithmetic (Example 3.3.3)
7. Non-standard analysis (Example 3.3.5)
8. We briefly mentioned (a version of) the Löwenheim-Skolem Theorems (§3.4)
9. Infinite Ramsey Theorem $\Rightarrow$ Finite Ramsey Theorem (homework problem)