CSC2429 / MAT1304: Circuit Complexity March 14, 2019

Lecture 9: Monotone Lower Bounds for CLIQUE (continued)

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Note: In these notes I consider the (usual) k-clique function on n-vertex graphs (rather than k-partite graphs on kn vertices, as in lecture). This changes nothing essential, but simplifies notation in some places.

I have also revised the definition of *approximator* to avoid an issue that came up in lecture. (In lecture I defined "approximator" as function that is both closed and trimmed; here approximators are monotone functions of the form $\mathrm{trim}(\mathrm{cl}(f))$.) This fixes a small issue, but otherwise the proof is the same.

The k-clique function

Graphs are simple graphs with no isolated vertices: $G = (V(G), E(G))$ where $E(G) \subseteq \binom{V(G)}{2}$ and $V(G) = \bigcup_{e \in E(G)} e$.

Let n be a growing parameter. Let $\mathcal{G}$ be the set of graphs G with $V(G) \subseteq [n]$. Since graphs (with no isolated vertices) are determined by their edge sets, we may identify with graphs with $\mathcal{G}$ with subsets of $\binom{[n]}{2}$. This allows us to identify $\mathcal{G}$ with the hypercube $\{0,1\}^{\binom{n}{2}}$. In this sense, we view functions $\mathcal{G} \to \{0,1\}$ as boolean functions.

f, g, h shall always be monotone functions from $\mathcal{G}$ to $\{0,1\}$. (Here *monotone* means that $f(G_1) \leq f(G_2)$ whenever $G_1 \subseteq G_2$.) A graph $\mathcal{G}$ is a *minterm* of f if $f(X) = 1$ and $f(X') = 0$ for all $X' \subsetneq X$. Let $\mathrm{Minterms}(f)$ denote the set of minterms of f.

For $X \in \mathcal{G}$, let $\mathrm{Ind}_X : \mathcal{G} \to \{0,1\}$ be the X-subgraph indicator function

$$\mathrm{Ind}_X(G) = 1 \overset{\text{def}}{\iff} X \subseteq G.$$

Every monotone function $f : \mathcal{G} \to \{0,1\}$ is equivalent to the disjunction of subgraph indicators over its minterms, that is, $f = \bigvee_{X \in \mathrm{Minterms}(f)} \mathrm{Ind}_X$.

For a fixed graph H with $|V(H)| \leq k$, let $\mathcal{G}_H := \{G \in \mathcal{G} : G \text{ is isomorphic to } H\}$. For example, letting K_k be the complete graph of order k (a.k.a. the k-clique graph), $\mathcal{G}_{K_k}$ is the set of k-cliques among vertices $[n]$. Let $\mathrm{Sub}_H : \mathcal{G} \to \mathbb{N}$ be defined by

$$\mathrm{Sub}_H(G) := \sum_{X \in \mathcal{G}_H} \mathrm{Ind}_X.$$

That is, $\mathrm{Sub}_H(G)$ is the number of (not necessarily induced) subgraphs of G that are isomorphic to H. Finally, for a monotone function f, let $\mathrm{Minterms}_H(f) := \mathrm{Minterms}(f) \cap \mathcal{G}_H$; that is, $\mathrm{Minterms}_H(f)$ is the set of minterms of f that are isomorphic to H.

For a fixed constant $k \geq 3$, let k-CLIQUE $: \mathcal{G} \to \{0,1\}$ be the monotone function (i.e., sequence of monotone functions $k\text{-CLIQUE}_n : \mathcal{G}_n \to \{0,1\}$) defined by $\mathrm{Minterms}(k\text{-CLIQUE}) = \mathcal{G}_{K_k}$. That is, $k\text{-CLIQUE}(G) = 1$ iff G has a complete subgraph of order k.

Let $\mathcal{Y}$ be a uniform random graph in $\mathcal{G}_{K_k}$. We call $\mathcal{Y}$ a "positive test instance".

Let $\mathcal{N}$ be the Erdos-Renyi random graph which includes each potential edge in $\binom{[n]}{2}$ independently with probability $p = n^{-2/(k-1)}$. With this choice of p, the probability that $\mathcal{N}$ contains a k-clique is bounded away from 1. This justifies calling $\mathcal{N}$ a "negative test instance".

Lemma 1. $\mathbb{P}[\mathcal{N}$ *contains a k-clique*$] < 3/4$

Proof. By a union bound,

$$\mathbb{P}[\mathcal{N} \text{ contains a } k\text{-clique}] \leq \tbinom{n}{k} p^{\binom{k}{2}} \leq (en/k)^k (n^{-2/(k-1)})^{\binom{k}{2}} = (e/k)^k \leq (e/3)^3 < 3/4. \qquad \square$$

Corollary 2. $\mathbb{P}[k\text{-CLIQUE}(\mathcal{Y}) = 1] + \mathbb{P}[k\text{-CLIQUE}(\mathcal{N}) = 0] > 5/4.$

We will show (Theorem 15) that every monotone C satisfies

$$\mathbb{P}[C(\mathcal{Y}) = 1] + \mathbb{P}[C(\mathcal{N}) = 0] \leq 1 + o(1) + \frac{\text{size}(C)}{\Omega(n^{k/4})}.$$

This implies an $\Omega(n^{k/4})$ lower bound on the monotone circuit size of k-CLIQUE. Moreover, this is *tight* in the average-case (for monotone circuit which solve k-CLIQUE with probability 0.9 on the distribution that is half the time $\mathcal{Y}$ and half the time $\mathcal{N}$).

Small and medium graphs

Definition 3. A graph H is:

- *small* if $|V(H)| < k/2$,
- *medium* if $|V(H)| \geq k/2$ and there exist small graphs H_1, H_2 such that $H = H_1 \cup H_2$.

Medium graphs will be responsible for the "bottlenecks" in our lower bound. Note that the union of two small graphs is either small or medium.

It will be useful to understand the expected value of $\text{Sub}_H(\mathcal{N})$ when H is small or medium. By linearity of expectations,

$$\mathbb{E}[\text{Sub}_H(\mathcal{N})] = |\mathcal{G}_H| \cdot p^{|E(H)|} = \tfrac{|V(H)|!}{|\text{Aut}(H)|} \tbinom{n}{|V(H)|} p^{|E(H)|} = \Theta(n^{|V(H)|} p^{|E(H)|}) = \Theta(n^{|V(H)| - \frac{2}{k-1}|E(H)|}).$$

(The constants here depend on k.) If $|V(H)| = \lambda k$ where $\lambda \in [0,1]$, then we have

$$|V(H)| - \tfrac{2}{k-1}|E(H)| \geq \lambda k - \tfrac{2}{k-1}\tbinom{\lambda k}{2} = \lambda k - \tfrac{\lambda k(\lambda k - 1)}{k-1} \geq \lambda(1-\lambda)k.$$

It follows that $\text{Sub}_H(\mathcal{N})$ is close to $\Omega(n^{k/4})$ when $|V(H)|$ is close to $k/2$. It turns out that $\Omega(n^{k/4})$ is a lower bound on $\text{Sub}_H(\mathcal{N})$ for all medium graphs H (this is a simple calculation to work out, which we omit).

Lemma 4. *For every medium graph H,*

$$\mathbb{E}[\text{Sub}_H(\mathcal{N})] = \Omega(n^{k/4 + \Omega(1/k)}).$$

Closed monotone functions

Definition 5. Set $\varepsilon := n^{-3k}$. A monotone function $f : \mathcal{G} \to \{0,1\}$ is *closed* if, for every small-or-medium $X \in \mathcal{G}$,

$$\mathbb{P}[f(\mathcal{N} \cup X) = 1] \geq 1 - \varepsilon \implies f(X) = 1,$$

Note that if f and g are closed, then so is $f \wedge g$. It follows that every monotone function f has a unique *closure*, denoted $\mathrm{cl}(f)$, defined as the minimum closed function such that $f \leq \mathrm{cl}(f)$.

Lemma 6. *If f is closed and $H \subseteq K_k$ is small-or-medium, then*

$$|\mathrm{Minterms}_H(f)| \leq \frac{O(k^2 \log(1/\varepsilon))^{k^2}}{p^{|E(H)|}} = \frac{n^{o(1)}}{p^{|E(H)|}}.$$

Proof Sketch. Let $\mathcal{M} = \mathrm{Minterms}_H(f)$ and let $t = |E(H)|$. View elements of $\mathcal{M}$ as t-element subsets of E_0.

Erdos-Rado Sunflower Lemma: *If $|\mathcal{M}| \geq t!s^t$, then $\mathcal{M}$ contains a sunflower of size s, that is, there exist $X_1, \dots, X_s \in \mathcal{M}$ such that sets $X_1 \setminus Z, \dots, X_s \setminus Z$ are disjoint where $Z := (X_1 \cap \dots \cap X_s)$.*

We use the Sunflower Lemma to prove an upper bound on $|\mathcal{M}|$ which is weaker than Lemma 6, but conveys the basic idea. Set $s := \ln(1/\varepsilon)/p^t$. We claim that $|\mathcal{M}| < t!s^t$. For contradiction, assume $|\mathcal{M}| \geq t!s^t$. Fix a sunflower $X_1, \dots, X_s \in \mathcal{M}$ with core Z.

Note that $f(Z) = 0$, since Z is a proper subset of a minterm of f. However,

$$\begin{aligned} \mathbb{P}[f(\mathcal{N} \cup Z) = 0] &= \mathbb{P}[\textstyle\bigwedge_{i=1}^s (X_i \setminus Z) \not\subseteq \mathcal{N}] \\ &= \textstyle\prod_{i=1}^s \mathbb{P}[(X_i \setminus Z) \not\subseteq \mathcal{N}] \\ &= (1 - p^{|X_i \setminus Z|})^s \leq (1 - p^t)^s < e^{-sp^t} = \varepsilon. \end{aligned}$$

This contradicts the fact that f is closed.

This proves the bound

$$|\mathrm{Minterms}_H(f)| \leq t!s^t \leq \left(\frac{t \ln(1/\varepsilon)}{p^t}\right)^t \leq \frac{O(k^2 \ln(1/\varepsilon))^{k^2}}{p^{|E(H)|^2}}$$

(using $t = |E(H)| \leq \binom{k}{2}$). To prove the stronger bound stated in the lemma (with $p^{|E(H)|}$ in place of $p^{|E(H)|^2}$, we use a generalization of the Sunflower Lemma called the "Approximate Sunflower Lemma" (see `http://www.math.toronto.edu/rossman/approx-sunflowers.pdf`). □

Lemma 7. *For every monotone function $f : \mathcal{G} \to \{0,1\}$,*

$$\mathbb{P}[f(\mathcal{N}) = 0 \text{ and } \mathrm{cl}(f)(\mathcal{N}) = 1] \leq O(\varepsilon n^k) = O(n^{-2k}).$$

Proof. We form an increasing sequence of monotone functions $f = h_0 < h_1 < \dots < h_t = \mathrm{cl}(f)$ as follows. If h_{i-1} is not closed, then let X_i be any graph in $\bigcup_{H \subseteq K_k} \mathcal{G}_H$ such that

$$\mathbb{P}[h_{i-1}(\mathcal{N} \cup X_i) = 1] \in [1 - \varepsilon, 1).$$

(Such X_i exists by definition of *closed*.) Let $h_i := h_{i-1} \vee \mathrm{Ind}_{X_i}$. Note that $h_{i-1} < h_i$ (since $h_{i-1}(X_i) = 0$ as a consequence of $\mathbb{P}[h_{i-1}(\mathcal{N} \cup X_i) = 1] < 1$). Note that graphs $X_1, X_2, \dots$ must be distinct, therefore this process terminates with a closed function h_t after $t \leq |\bigcup_{H \subseteq K_k} \mathcal{G}_H| = O(n^k)$ steps. Finally, by induction we have $h_i \leq \mathrm{cl}(f)$ for all i. Therefore, $h_t = \mathrm{cl}(f)$ (since h_t is closed and $h_t \leq \mathrm{cl}(f)$).

We now have the following bound

$$\begin{aligned}
\mathbb{P}[f(\mathcal{N}) = 0 \text{ and } \mathrm{cl}(f)(\mathcal{N}) = 1] &= \mathbb{P}[h_0(\mathcal{N}) = 0 \text{ and } h_t(\mathcal{N}) = 1] \\
&= \textstyle\sum_{i=1}^t \mathbb{P}[h_{i-1}(\mathcal{N}) = 0 \text{ and } (h_{i-1} \vee \mathrm{Ind}_{X_i})(\mathcal{N}) = 1] \\
&= \textstyle\sum_{i=1}^t \mathbb{P}[h_{i-1}(\mathcal{N}) = 0 \text{ and } (\mathcal{X}_i \subseteq \mathcal{N})] \\
&\leq \textstyle\sum_{i=1}^t \mathbb{P}[h_{i-1}(\mathcal{N} \cup \mathcal{X}_i) = 0] \\
&\leq t\varepsilon \quad (\text{since } \mathbb{P}[h_{i-1}(\mathcal{N} \cup \mathcal{X}_i) = 1] \geq 1 - \varepsilon) \\
&= O(\varepsilon n^k) \\
&\leq O(n^{-2k}).
\end{aligned}$$

□

Trimmed monotone functions

Definition 8. A monotone function $f : \mathcal{G} \to \{0,1\}$ is *trimmed* if every minterm of f is small. For any f, let $\mathrm{trim}(f) : \mathcal{G} \to \{0,1\}$ be the monotone function

$$\mathrm{trim}(f)(G) = 1 \overset{\text{def}}{\iff} (\exists X \subseteq G)(X \text{ is small and } f(X) = 1).$$

Note that $\mathrm{trim}(f) \leq f$, with equality if and only f is trimmed.

Lemma 9. *If f and g are trimmed monotone functions, then all minterms of* $\mathrm{cl}(f \wedge g)$ *and* $\mathrm{cl}(f \vee g)$ *are small-or-medium.*

Proof. For any monotone functions f and g, every minterm of $f \wedge g$ is the union of a minterm of f and a minterm of g. If f are g are trimmed, this means that every minterm of $f \wedge g$ is the union of two small graphs and is therefore small or medium. Finally, as the proof of Lemma 7 shows, $\mathrm{cl}(f \wedge g) = (f \wedge g) \vee \mathrm{Ind}_{X_1} \vee \dots \vee \mathrm{Ind}_{X_t}$ for some sequence of small-or-medium graphs $X_1, \dots, X_t \in \mathcal{G}$. Therefore, every minterm of $\mathrm{cl}(f \wedge g)$ is small or medium. Similarly, since all minterms of $f \vee g$ are small, the minterms of $\mathrm{cl}(f \vee g)$ are small-or-medium. □

Lemma 10. *Suppose f is closed and every minterm of f is small or medium. Then*

$$\mathbb{P}[f(\mathcal{Y}) = 1 \textit{ and } \mathrm{trim}(f)(\mathcal{Y}) = 0] \leq \frac{1}{\Omega(n^{k/4})}.$$

Proof. If $f(\mathcal{Y}) = 1$ and $\mathrm{trim}(f)(\mathcal{Y}) = 0$, then it must be the case that some medium minterm of f

is a subgraph of $\mathcal{Y}$. Therefore,

$$\begin{aligned}
&\mathbb{P}[f(\mathcal{Y}) = 0 \text{ and } \operatorname{trim}(f)(\mathcal{Y}) = 1] \\
&\qquad \leq \mathbb{P}\Big[\bigvee_{\substack{\text{medium graphs } H \\ \text{(up to isomorphism)}}} \ \bigvee_{X \in \mathrm{Minterms}_H(f \wedge g)} (X \subseteq \mathcal{Y})\Big] \\
&\qquad \leq \sum_{\substack{\text{medium graphs } H \\ \text{(up to isomorphism)}}} \ \sum_{X \in \mathrm{Minterms}_H(f \wedge g)} \mathbb{P}[X \subseteq \mathcal{Y}] \\
&\qquad = \sum_{\text{medium } H \subseteq K_k} \ \sum_{X \in \mathrm{Minterms}_H(f \wedge g)} \frac{\binom{n-|V(H)|}{k-|V(H)|}}{\binom{n}{k}} \\
&\qquad \leq \sum_{\text{medium } H \subseteq K_k} |\mathrm{Minterms}_H(f \wedge g)| \cdot \frac{1}{\Omega(n^{|V(H)|})} \\
&\qquad \leq \sum_{\text{medium } H \subseteq K_k} \frac{n^{o(1)}}{p^{|E(H)|}} \cdot \frac{1}{\Omega(n^{|V(H)|})} \quad \text{(by Lemma 6)} \\
&\qquad = \sum_{\text{medium } H \subseteq K_k} \frac{n^{o(1)}}{\Omega(\mathbb{E}[\mathrm{Sub}_H(\mathcal{N})])} \\
&\qquad \leq \sum_{\text{medium } H \subseteq K_k} \frac{n^{o(1)}}{\Omega(n^{k/4 + \Omega(1/k)})} \quad \text{(by Lemma 4).}
\end{aligned}$$

Since there are at most $2^{\binom{k}{2}} = O(1)$ medium graphs up to isomorphism (each has at most k vertices), and since $n^{\Omega(1/k)}$ dominates $n^{o(1)}$, we conclude that this bound is at most $1/\Omega(n^{k/4})$. □

Approximators

Definition 11. Let $\mathcal{A} := \{\operatorname{trim}(\operatorname{cl}(h)) : h \text{ is a monotone function}\}$. Functions in $\mathcal{A}$ are called *approximators*. We define operations $\sqcup, \sqcap : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ as follows: for approximators $f, g \in \mathcal{A}$, let

$$\begin{aligned}
f \sqcup g &:= \operatorname{trim}(\operatorname{cl}(f \vee g)), \\
f \sqcap g &:= \operatorname{trim}(\operatorname{cl}(f \wedge g)).
\end{aligned}$$

Note that every edge-indicator function Ind_e (where $e \in E_0$) is an approximator.

Definition 12. If C is a monotone $\{\vee, \wedge\}$-circuit (with inputs labeled by edge-indicators Ind_e), let $C^{\mathcal{A}}$ be the corresponding $\{\sqcup, \sqcap\}$-circuit, which computes an approximator.

We now show that $C^{\mathcal{A}}$ closely approximates C (with one-sided error) on both distributions $\mathcal{Y}$ and $\mathcal{N}$.

First, we bound the error on $\mathcal{Y}$:

$$\begin{aligned}
&\mathbb{P}[(f \wedge g)(\mathcal{Y}) = 1 \text{ and } (f \sqcap g)(\mathcal{Y}) = 0] \\
&\qquad = \mathbb{P}[(f \wedge g)(\mathcal{Y}) = 1 \text{ and } \operatorname{trim}(\operatorname{cl}(f \wedge g))(\mathcal{Y}) = 0] \\
&\qquad \leq \mathbb{P}[\operatorname{cl}(f \wedge g)(\mathcal{Y}) = 1 \text{ and } \operatorname{trim}(\operatorname{cl}(f \wedge g))(\mathcal{Y}) = 0] && \text{(since } f \wedge g \leq \operatorname{cl}(f \wedge g)\text{)} \\
&\qquad \leq 1/\Omega(n^{k/4}) && \text{(by Lemmas 9 and 10),}
\end{aligned}$$

$$\begin{aligned}
&\mathbb{P}[(f \vee g)(\mathcal{Y}) = 1 \text{ and } (f \sqcup g)(\mathcal{Y}) = 0] \\
&\qquad = \mathbb{P}[(f \vee g)(\mathcal{Y}) = 1 \text{ and } \operatorname{trim}(\operatorname{cl}(f \vee g))(\mathcal{Y}) = 0] \\
&\qquad \leq \mathbb{P}[\operatorname{cl}(f \vee g)(\mathcal{Y}) = 1 \text{ and } \operatorname{trim}(\operatorname{cl}(f \vee g))(\mathcal{Y}) = 0] && \text{(since } f \vee g \leq \operatorname{cl}(f \vee g)\text{)} \\
&\qquad \leq 1/\Omega(n^{k/4}) && \text{(by Lemmas 9 and 10).}
\end{aligned}$$

It follows that

$$\mathbb{P}[\mathcal{C}(\mathcal{Y}) = 1 \text{ and } \mathcal{C}^{\mathcal{A}}(\mathcal{Y}) = 0] \leq \frac{\operatorname{size}(C)}{\Omega(n^{k/4})}.$$

Next, we bound the error on $\mathcal{N}$:

$$\begin{aligned}
&\mathbb{P}[(f \wedge g)(\mathcal{N}) = 0 \text{ and } (f \sqcap g)(\mathcal{N}) = 1] \\
&\qquad = \mathbb{P}[(f \wedge g)(\mathcal{N}) = 0 \text{ and } \operatorname{trim}(\operatorname{cl}(f \wedge g))(\mathcal{N}) = 1] \\
&\qquad \leq \mathbb{P}[(f \wedge g)(\mathcal{N}) = 0 \text{ and } \operatorname{cl}(f \wedge g)(\mathcal{N}) = 1] && \text{(since } \operatorname{trim}(\operatorname{cl}(f \wedge g)) \leq \operatorname{cl}(f \wedge g)\text{)} \\
&\qquad \leq 1/\Omega(n^{2k}) && \text{(by Lemma 7),}
\end{aligned}$$

$$\begin{aligned}
&\mathbb{P}[(f \vee g)(\mathcal{N}) = 0 \text{ and } (f \sqcup g)(\mathcal{N}) = 1] \\
&\qquad = \mathbb{P}[(f \vee g)(\mathcal{N}) = 0 \text{ and } \operatorname{trim}(\operatorname{cl}(f \vee g))(\mathcal{N}) = 1] \\
&\qquad \leq \mathbb{P}[(f \vee g)(\mathcal{N}) = 0 \text{ and } \operatorname{cl}(f \vee g)(\mathcal{N}) = 1] && \text{(since } \operatorname{trim}(\operatorname{cl}(f \vee g)) \leq \operatorname{cl}(f \vee g)\text{)} \\
&\qquad \leq 1/\Omega(n^{2k}) && \text{(by Lemma 7).}
\end{aligned}$$

It follows that

$$\mathbb{P}[\mathcal{C}(\mathcal{N}) = 0 \text{ and } \mathcal{C}^{\mathcal{A}}(\mathcal{N}) = 1] \leq \frac{\operatorname{size}(C)}{\Omega(n^{2k})}.$$

Combining these bounds, the denominator $\Omega(n^{k/4})$ dominates and we get

Lemma 13. $\mathbb{P}[\mathcal{C}(\mathcal{Y}) = 1 \textit{ and } \mathcal{C}^{\mathcal{A}}(\mathcal{Y}) = 0] + \mathbb{P}[\mathcal{C}(\mathcal{N}) = 0 \textit{ and } \mathcal{C}^{\mathcal{A}}(\mathcal{N}) = 1] \leq \dfrac{\operatorname{size}(C)}{\Omega(n^{k/4})}$.

It remains to show that every approximator makes many one-sided errors on at least one of $\mathcal{Y}$ and $\mathcal{N}$:

Lemma 14. *For every approximator $f \in \mathcal{A}$, we have*

$$\mathbb{P}[f(\mathcal{Y}) = 1] + \mathbb{P}[f(\mathcal{N}) = 0] \leq 1 + o(1).$$

Proof. If $f \equiv 1$, we are done. So we assume $f \not\equiv 1$ and will show that $\mathbb{P}[f(\mathcal{Y}) = 1] = o(1)$.

By definition of $\mathcal{A}$, there exists a closed monotone function h such that $f = \mathrm{trim}(h)$. Note that $h \not\equiv 1$, since $\mathrm{trim}(1) = 1$ (this is because 1 has the small minterm $\emptyset$). Therefore,

$$f = \mathrm{trim}(h) = \bigvee_{\substack{\text{nonempty small graphs } H \\ \text{(up to isomorphism)}}} \bigvee_{X \in \mathrm{Minterms}_H(h)} \mathrm{Ind}_X.$$

Therefore,

$$\begin{aligned}
\mathbb{P}[f(\mathcal{Y}) = 1] &\leq \sum_{\substack{\text{nonempty small graphs } H \\ \text{(up to isomorphism)}}} \sum_{X \in \mathrm{Minterms}_H(f \wedge g)} \mathbb{P}[X \subseteq \mathcal{Y}] \\
&\leq \sum_{\substack{\text{nonempty small graphs } H \\ \text{(up to isomorphism)}}} |\mathrm{Minterms}_H(f \wedge g)| \cdot \frac{\binom{n-|V(H)|}{k-|V(H)|}}{\binom{n}{k}} \\
&\leq \sum_{\substack{\text{nonempty small graphs } H \\ \text{(up to isomorphism)}}} \frac{n^{o(1)}}{p^{|E(H)|}} \cdot \frac{1}{\Omega(n^{|V(H)|})} \quad \text{(by Lemma 6)} \\
&= \sum_{\substack{\text{nonempty small graphs } H \\ \text{(up to isomorphism)}}} \frac{n^{o(1)}}{\Omega(\mathbb{E}[\mathrm{Sub}_H(\mathcal{N})])}.
\end{aligned}$$

This bound is $o(1)$, since $\mathbb{E}[\#$ of H-subgraphs of $\mathcal{N}] \geq n^{\Omega(1)}$ (in fact, $\geq \Omega(n^{2-2/(k-1)})$) for every nonempty small H. □

Combining Lemma 13 and 14, we get

Theorem 15. *For every monotone circuit C,*

$$\begin{aligned}
\mathbb{P}[C(\mathcal{Y}) = 1] + \mathbb{P}[C(\mathcal{N}) = 0] &\leq \mathbb{P}[C(\mathcal{Y}) = 1 \textit{ and } C^{\mathcal{A}}(\mathcal{Y}) = 0] + \mathbb{P}[C^{\mathcal{A}}(\mathcal{Y}) = 1] \\
&\quad + \mathbb{P}[C(\mathcal{N}) = 0 \textit{ and } C^{\mathcal{A}}(\mathcal{N}) = 1] + \mathbb{P}[C^{\mathcal{A}}(\mathcal{N}) = 0] \\
&\leq 1 + o(1) + \frac{\mathrm{size}(C)}{\Omega(n^{k/4})}.
\end{aligned}$$

Since $\mathbb{P}[k\text{-CLIQUE}(\mathcal{Y}) = 1] + \mathbb{P}[k\text{-CLIQUE}(\mathcal{N}) = 0] > 5/4$, it follows that:

Corollary 16. *Monotone circuits computing k-CLIQUE require size $\Omega(n^{k/4})$.*