Grayson's theorem

Thm (Grayson 87)

If $\Gamma \subset \mathbb{R}^2$ is a closed embedded curve, then the CSF $\{\Gamma_t\}_{t\in[0,T)}$ with $\Gamma_0 = \Gamma$ exists until $T = A_\Gamma / 2\pi$ and converges for $t \to T$ to a round point, i.e. $\exists! \, x_0 \in \mathbb{R}^2$ s.t.

$$\Gamma_t^\lambda := \lambda \cdot \left(\Gamma_{T+\lambda^{-2}t} - x_0 \right)$$

converges for $\lambda \to \infty$ to $\{\partial B_{\sqrt{-2t}}\}_{t\in(-\infty,0)}$.

Rmk $\exists$ many proofs.
We will give the one by Huisken.

Lemma (Grayson) Along CSF we have

$$\frac{d}{dt}\int_{\Gamma_t} |\kappa|\, ds = -2 \sum_{x:\,\kappa(x,t)=0} |\kappa_s|(x,t)$$

Proof Since solutions of CSF are analytic, there are only finitely many inflection points. Compute:

$$\frac{d}{dt}\left(\int_{\{\kappa\geq 0\}} \kappa\, ds - \int_{\{\kappa \leq 0\}} \kappa\, ds \right) = \int_{\{\kappa\geq 0\}} \kappa_{ss}\, ds - \int_{\{\kappa\leq 0\}} \kappa_{ss}\, ds$$

integration by parts $\Rightarrow$ assertion $\square$

Proof of Grayson's thm

Let $T<\infty$ be the maximal existence time of CSF starting at Γ.

Recall from last week:

If $\limsup_{t\to T} (T-t) \max_{\Gamma_t} \kappa^2 < \infty$, then $T = A_\Gamma / 2\pi$ and for $t \to T$ we have convergence to a round point (we proved this using Huisken's monotonicity formula)

Now suppose towards a contradiction that
$$\limsup_{t \to T} (T-t) \max_{\Gamma_t} \kappa^2 = \infty \quad (\mathrm{II})$$

We will perform a type II blowup:

For all $j \gg 1$, let $t_j \in [0, T - 1/j]$, $x_j \in S^1$ be st.
$$\kappa^2(x_j, t_j)(T - 1/j - t_j) = \max_{\substack{t \le T - 1/j \\ x \in S^1}} \kappa^2(x,t)(T - \tfrac{1}{j} - t)$$

Set:
$$\lambda_j := \kappa(x_j, t_j), \quad t_j^{(0)} := -\lambda_j^2 t_j, \quad t_j^{(1)} := \lambda_j^2 (T - \tfrac{1}{j} - t_j).$$

By (II), given any $M < \infty$, $\exists \bar{t} < T, \bar{x} \in S^1$

s.t. $\kappa^2(\bar{x}, \bar{t})(T - \bar{t}) > 2M$.

For $j \gg 1$, we have

$$\bar{t} < T - 1/j, \quad \kappa^2(\bar{x}, \bar{t})(T - \bar{t} - \tfrac{1}{j}) > M.$$

This yields

$$t_j^{(1)} = \kappa^2(x_j, t_j)(T - \tfrac{1}{j} - t_j)$$

$$\geq \kappa^2(\bar{x}, \bar{t})(T - \tfrac{1}{j} - \bar{t}) > M$$

Hence, $t_j^{(1)} \to \infty$.

Thus, $\lambda_j \to \infty$, $t_j \to T$, $t_j^{(0)} \to -\infty$.

Now consider the sequence

$$\Gamma_t^j := \lambda_j \cdot (\Gamma_{t_j + \lambda_j^{-2} t} - x_j),$$

where $t \in [t_j^{(0)}, t_j^{(1)})$.

By construction Γ_t^j has $\varkappa(0,0) = 1$.

Furthermore

$$\varkappa_j^2(x,t) \leq \frac{T - 1/j - t_j}{T - 1/j - t_j - \lambda_j^2 t} = \frac{t_j^{(1)}}{t_j^{(1)} - t}$$

for $t \in [t_j^{(0)}, t_j^{(1)})$.

Hence, along a subsequence we can pass to a smooth limit $\{\Gamma_t^\infty\}_{t\in(-\infty,\infty)}$.

Limit satisfies:

- $\varkappa(0,0) = 1$
- $\varkappa^2 \leq 1$ everywhere.
- $\displaystyle\int_{-\infty}^{\infty} \sum_{x:\,\varkappa(x,t)=0} |\varkappa_s|(x,t)\,dt = 0$ (by Lemma)

If $\varkappa(x,t) = 0 \Rightarrow \varkappa_s(x,t) = 0$

$\Rightarrow \{\Gamma_t^\infty\}$ is a straigth line $\lightning$ $\varkappa(0,0)=1$

$\Rightarrow \kappa > 0$.

equality case of Hamilton's Harnack inequality $\Rightarrow \{\Gamma_t^\infty\}$ is a translating soliton

HW $\Rightarrow \{\Gamma_t^\infty\}$ is the grim reaper

↳ Huisken's bound for ratio between intrinsic & extrinsic distance. $\square$