ON BRENDLE'S ESTIMATE FOR THE INSCRIBED RADIUS UNDER MEAN CURVATURE FLOW

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ABSTRACT. In a recent paper [Bre13], Brendle proved that the inscribed radius of closed embedded mean convex hypersurfaces moving by mean curvature flow is at least $\frac{1}{(1+\delta)H}$ at all points with $H \geq C(\delta, M_0)$. In this note, we give a shorter proof of Brendle's estimate, and of a more general result for α-Andrews flows, based on our recent estimates from Haslhofer-Kleiner [HK13].

1. INTRODUCTION

Let $\{M_t \subset \mathbb{R}^{n+1}\}$ be a family of closed embedded mean convex hypersurfaces that evolve by mean curvature flow. In a recent paper [Bre13], Brendle considered the inscribed radius $r_{\text{in}}(p,t)$ and the outer radius $r_{\text{out}}(p,t)$, i.e. the maximal radius of interior respectively exterior balls tangent at $p \in M_t$, and proved sharp estimates for them:

Theorem 1.1 (Brendle [Bre13, Thm. 1, Thm. 2]). *Let $\{M_t \subset \mathbb{R}^{n+1}\}$ be a smooth family of closed embedded mean convex hypersurfaces which evolve by the mean curvature flow. Then, given any constant $\delta > 0$ there exists a constant $C = C(\delta, M_0) < \infty$ such that*

$$r_{\text{in}}(p,t) \geq \frac{1}{(1+\delta)H(p,t)} \qquad \text{and} \qquad r_{\text{out}}(p,t) \geq \frac{1}{\delta H(p,t)}, \tag{1.2}$$

whenever $H(p,t) \geq C$.

Brendle's proof is based on some quite sophisticated computations. The purpose of this note is to give a shorter proof of Brendle's estimate.

In fact, we prove a more general theorem (Theorem 1.3), that also applies to certain local and weak solutions. To describe our setting, recall that a mean curvature flow of mean convex hypersurfaces $M_t = \partial K_t$ satisfies the α-Andrews condition [HK13, Def. 1.1], if each boundary point $p \in \partial K_t$ admits interior and exterior balls tangent at p of radius at least $\frac{\alpha}{H(p,t)}$; in other words if $\inf\{Hr_{\text{in}}, Hr_{\text{out}}\} \geq \alpha$. By the main

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theorem of Andrews [And13] (which we extended to weak solutions in [HK13, Thm. 1.5]), the Andrews condition is preserved under the flow, i.e. if K_0 satisfies the α-Andrews condition, then so does K_t for all $t \geq 0$. The class of α-Andrews flows [HK13, Def. 1.6] is then defined as the smallest class of set flows which contains all compact α-Andrews level set flows with smooth initial condition and all smooth α-Andrews flows $\{K_t \subseteq U\}_{t\in I}$ in any open set $U \subseteq \mathbb{R}^{n+1}$, and which is closed under the operations of restriction, parabolic rescaling, and passage to Hausdorff limits.

Theorem 1.3. *For all $\delta > 0$, $\alpha > 0$, there exists $\eta = \eta(\delta, \alpha) < \infty$ with the following property. If $\{K_t\}$ is an α-Andrews flow in a parabolic ball $P(p, t, \eta r)$ centered at a boundary point $p \in \partial K_t$ with $H(p, t) \leq r^{-1}$, then*

$$r_{\text{in}}(p,t) \geq \frac{r}{1+\delta} \qquad \text{and} \qquad r_{\text{out}}(p,t) \geq \frac{r}{\delta}. \tag{1.4}$$

Note that Theorem 1.3 immediately implies Theorem 1.1. More generally, Theorem 1.3 shows that the scale invariant quantities Hr_{in} and Hr_{out} approach the optimal values 1 respectively ∞, under the mere assumption that the flow is defined on a parabolic ball that is large enough compared to the curvature scale; the optimal value 1 is attained for the shrinking cylinder $S^1 \times \mathbb{R}^{n-1}$. Philosophically, the assumption on the domain gives the maximum principle enough room to trigger in, and thus to improve the quantities Hr_{in} and Hr_{out} as much as allowed by the example of the cylinder. In particular, as another interesting consequence of Theorem 1.3, we obtain:

Corollary 1.5. *Any ancient α-Andrews flow $\{K_t \subset \mathbb{R}^{n+1}\}$ is in fact a 1-Andrews flow.*

Corollary 1.5 shows that $\alpha = 1$ is the universal noncollapsing constant for ancient mean convex mean curvature flows. It shares some similarities with Perelman's theorem [Per03, 1.5] that any ancient κ-solution for three-dimensional Ricci flow is either a κ_0-solution or a quotient of the round sphere; our constant is the only sharp one, though.

Our proof of Theorem 1.3 is related to our proof of the convexity estimate in [HK13, Thm. 1.10], see also White [Whi03]. The point is, once the relevant compactness theorem is established (see [HK13, Thm. 1.8] and [HK13, Thm. 1.12] respectively) we can pass to a suitable limit and apply the strong maximum principle.

2. The proof

Proof of Theorem 1.3. As explained in [HK13, Sec. 4] it suffices to establish the estimate for smooth α-Andrews flows; the estimate for general α-Andrews flows then follows by approximation.

Fix $\alpha < 1$. The α-Andrews condition implies that the assertion for the inscribed radius holds for $\delta = \frac{1}{\alpha} - 1$. Let $\delta_0 \leq \frac{1}{\alpha} - 1$ be the infimum of the δ's for which it holds, and suppose $\delta_0 > 0$.

It follows that there is a sequence $\{K_t^j\}$ of α-Andrews flows, where for all j, $0 \in \partial K_0^j$, $H(0,0) \leq 1$ and $\{K_t^j\}$ is defined in $P(0,0,j)$, but $r_{\mathrm{in}}(0,0) \to \frac{1}{1+\delta_0}$ as $j \to \infty$. By the global convergence theorem [HK13, Thm. 1.12], after passing to a subsequence, $\{K_t^j\}$ converges smoothly and globally to an ancient convex α-Andrews flow $\{K_t^\infty \subset \mathbb{R}^{n+1}\}$. Then for $\{K_t^\infty\}$ we have $r_{\mathrm{in}}(0,0) = \frac{1}{1+\delta_0}$ and $H(0,0) = 1$.

By construction, the quantity $\frac{1}{Hr_{\mathrm{in}}}$ attains its maximum over ∂K_t^∞ for all $t \leq 0$ at $(0,0)$. By Andrews-Langford-McCoy [ALM13] we have the evolution inequality

$$\partial_t \frac{1}{Hr_{\mathrm{in}}} \leq \Delta \frac{1}{Hr_{\mathrm{in}}} + 2\langle \nabla \log H, \nabla \frac{1}{Hr_{\mathrm{in}}} \rangle, \tag{2.1}$$

in the viscosity sense. Thus, by the strict maximum principle, Hr_{in} is constant. On the one hand, this constant is equal to $\frac{1}{1+\delta_0}$. On the other hand, recalling that the asymptotic shrinkers of $\{K_t^\infty\}$ must be cylinders or spheres [HK13, Rem. 1.20], we see that this constant must be at least 1; this gives the desired contradiction and proves the estimate for the inscribed radius.

Finally, the estimate for the outer radius follows from the convexity of ancient α-Andrews flows [HK13, Cor. 2.13]. □

References

[And13] B. Andrews, *Noncollapsing in mean-convex mean curvature flow*, Geom. Topol. 16(3):1413–1418, 2012

[ALM13] B. Andrews, M. Langford, J. McCoy. *Noncollapsing in fully nonlinear curvature flows*, Ann. Inst. H. Poincare ANL, 30(1):23–32, 2013.

[Bre13] S. Brendle, *A sharp bound for the inscribed radius under mean curvature flow*, arXiv:1309.1459, 2013.

[HK13] R. Haslhofer, B. Kleiner, *Mean curvature flow of mean convex hypersurfaces*, arXiv:1304.0926, 2013.

[Per03] G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv:math/0303109, 2013.

[Whi03] B. White, *The nature of singularities in mean curvature flow of mean convex sets*, J. Amer. Math. Soc. 16(1):123–138, 2003.

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