Free boundary flow with surgery and applications

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Mean curvature flow with surgery for closed mean-convex surfaces has been constructed by Brendle-Huisken [4] and Kleiner and the author [12]. However, until recently the construction of a flow with surgery in the setting of mean-convex surfaces with free boundary seemed inaccessible, since both the approach from [4] and [12] crucially rely on the noncollapsing result of Andrews [1], which is only available in the setting without boundary. Recently, we solved this problem for mean-convex surfaces with free boundary in any strictly convex domain D:

Theorem ([10])**.** *There exists a free boundary flow with surgery starting at any smooth compact strictly mean-convex free boundary surface $M_0 \subset D$.*
Moreover, the flow either becomes extinct in finite time or for $t \to \infty$ converges to a finite collection of stable connected minimal surfaces with empty or free boundary.

Here, a free boundary flow with surgery is a free boundary $(\delta, \mathcal{H})$-flow. In particular, $\delta > 0$ is a small parameter that captures the quality of the surgery necks and half necks, and $\mathcal{H}$ is a triple of curvature scales $H_{\text{trigger}} \gg H_{\text{neck}} \gg H_{\text{thick}} \gg 1$, which is used to specify more precisely when and how surgeries are performed.

To prove the theorem we implemented our recent new approach from [9], which is based on weak solutions rather than a priori estimates for smooth solutions. Specifically, we study sequences $\mathcal{M}^j$ of free boundary $(\delta, \mathcal{H}^j)$-flows, with the same mean-convex initial condition $M_0 \subset D$, where the curvature scales $\mathcal{H}^j$ improve along the sequence. Given any rescaling factors $\lambda_j \to \infty$, we consider the blowup sequence $\widetilde{\mathcal{M}}^j = \mathcal{D}_{\lambda_j}(\mathcal{M}^j - X_j)$. We establish a hybrid compactness theorem, which allows us to pass to a limit of $\widetilde{\mathcal{M}}^j$, which is smooth near the surgery regions but potentially singular elsewhere. Moreover, using Edelen's monotonicty formula [6] we rule out microscopic surgeries. We then generalize the theory of mean-convex Brakke flows with free boundary from [7] to our setting of hybrid limits, and in particular establish multiplicity-one. As a consequence, taking also into account the recent classification of ancient solutions from [2, 3], we then establish a canonical neighborhood theorem, which allows us to conclude.

As an application, in joint work with Ketover we prove:

Theorem ([11])**.** *Every strictly convex 3-ball B with nonnegative Ricci-curvature contains at least 3 embedded free-boundary minimal disks in the generic case, and at least 2 solutions even without genericity assumption. Moreover, the area of our 2nd solution is always strictly less than twice the area of the Grüter-Jost solution.*

A natural family of examples of 3-balls to illustrate this are the ellipsoids

$$E(a,b,c) := \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\} \subset \mathbb{R}^3.$$

They contain at least 3 obvious 'planar' solutions, which are obtained by intersecting $E(a,b,c)$ with the coordinates planes. On the other hand, for $a \geq 2\max(b,c)$ our theorem produces a nonplanar embedded free-boundary minimal disk $\Sigma(a)$.

Moreover, for $a \to \infty$ our surfaces $\Sigma(a)$ converge in the sense of varifolds to the planar disk $\{x = 0\} \times E(b, c) \subset \mathbb{R} \times E(b, c)$ with multiplicity-two.

To outline our proof, recall that Grüter-Jost [8] already proved the existence of at least 1 solution. Moreover, by a beautiful degree theory argument of Maximo-Nunes-Smith [14] for generic metrics the number of solutions is always odd. Hence, our task is to produce a 2nd solution. To get started, sliding the Grüter-Jost disk a bit to both sides we can decompose $B = B^- \cup Z \cup B^+$, where Z is a short cylindrical region and $\partial B^\pm$ are smooth strictly mean-convex disks with free-boundary. Using the free boundary flow with surgery from above, and ideas from our earlier work with Buzano and Hershkovits [5], we produce an optimal free-boundary foliation of B, namely a foliation $\{\Sigma_t\}_{t\in[-1,1]}$ of B by free-boundary disks, such that the Grüter-Jost disk sits in the middle of the foliation as Σ_0 and all other slices have strictly less area. As an aside, we mention that these smooth foliations are of independent interest. Using our optimal foliation we can then form a certain two parameter family $\{\Sigma_{s,t}\}$. Loosely speaking, this family is constructed by joining the surfaces Σ_s and Σ_t by a thin half neck. Establishing a half version of the catenoid estimate from [13], we can suitably open up the half neck to arrange that

$$\sup_{s,t} |\Sigma_{s,t}| < 2|\Sigma_0|.$$

This guarantees that min-max for our two-parameter family does not simply produce the Grüter-Jost disk with multiplicity-two, and together with a standard Lusternik-Schnirelmann argument allows us to conclude.

References

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