Classification of ancient noncollapsed flows in $\mathbb{R}^4$

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(joint work with Kyeongsu Choi)

In this note, we discuss our recent classification of all noncollapsed singularities for the mean curvature flow in $\mathbb{R}^4$. In stark contrast to $\mathbb{R}^3$, a classification in $\mathbb{R}^4$ until recently seemed out of reach. Fundamentally, this is because of the existence of examples with reduced symmetry. More precisely, Hoffman-Ilmanen-Martin-White constructed a 1-parameter family of 3d-translators that interpolate between the round 3d-bowl and $\mathbb{R}\times$2d-bowl, and are only $\mathbb{Z}_2 \times \mathrm{O}_2$-symmetric [13]. Similarly, in joint work with Du we constructed a 1-parameter family of 3d-ovals that interpolate between the $\mathrm{O}_2 \times \mathrm{O}_2$-symmetric 3d-oval and $\mathbb{R}\times$2d-oval, and are only $\mathbb{Z}_2^2 \times \mathrm{O}_2$-symmetric [10]. Recently, in joint work with K. Choi, building also on our earlier collaborations with B. Choi, Daskalopoulos, Du, Hershkovits, and Sesum, we obtained a complete classification of all noncollapsed singularities:

Theorem (classification [6, 7, 9, 8, 4, 5])**.** *Any ancient noncollapsed flow in* $\mathbb{R}^4$ *is, up to scaling and rigid motion,*

- *either one of the standard shrinkers* S^3, $\mathbb{R} \times S^2$, $\mathbb{R}^2 \times S^1$ *or* $\mathbb{R}^3$,
- *or the 3d-bowl, or* $\mathbb{R}\times$*2d-bowl, or belongs to the 1-parameter family of* $\mathbb{Z}_2 \times \mathrm{O}_2$*-symmetric translators from* [13]*,*
- *or the* $\mathbb{Z}_2 \times \mathrm{O}_3$*-symmetric 3d-oval, or the* $\mathrm{O}_2 \times \mathrm{O}_2$*-symmetric 3d-oval, or* $\mathbb{R}\times$*2d-oval, or belongs to the 1-parameter family of* $\mathbb{Z}_2^2 \times \mathrm{O}_2$*-symmetric 3d-ovals from* [10]*.*

In addition to the 1-parameter families of translators and ovals discussed above, our list of course also contains all classical historical examples, in particular the two examples of cohomogeneity-one 3d-ovals from [14] and [12], respectively. As an immediate consequence we obtain a classification of all potential blowup limits (and thus a canonical neighborhood theorem) for mean-convex flows in $\mathbb{R}^4$:

Corollary (canonical neighborhoods)**.** *For the mean curvature flow of mean-convex hypersurfaces in* $\mathbb{R}^4$ *(or in a 4-manifold) every blowup limit is given by one of the solutions from the above list. In particular, for every* $\varepsilon > 0$ *there is an* $H_\varepsilon = H_\varepsilon(M_0) < \infty$*, such that around any space-time point* (p,t) *with* $H(p,t) \geq H_\varepsilon$ *the flow is* ε*-close (after rescaling) to one of the solutions from the above list.*

More generally, in light of Ilmanen's multiplicity-one and mean-convex neighborhood conjecture, the conclusion of the corollary is also expected to hold for blowup limits near any generic singularity.

To outline our approach, let $\mathcal{M} = \{M_t\}$ be an ancient noncollapsed mean curvature flow in $\mathbb{R}^4$ that is neither a static plane nor a round shrinking sphere. By general theory the tangent flow at $-\infty$ is either a neck or a bubble-sheet. Since

the neck-case has already been dealt with in the fundamental work by Brendle-Choi [2, 3] and Angenent-Daskalopoulos-Sesum [1], we can from now assume that

$$\lim_{\lambda\to 0} \mathcal{D}_\lambda \mathcal{M} = \{\mathbb{R}^2 \times S^1(\sqrt{2|t|})\}_{t<0}\,. \tag{1}$$

The analysis of such ancient solutions starts by considering the bubble-sheet function $u = u(y, \vartheta, \tau)$, which captures the deviation of the renormalized flow $\bar{M}_\tau = e^{\tau/2} M_{-e^{-\tau}}$ from the round bubble-sheet $\mathbb{R}^2 \times S^1(\sqrt{2})$. The evolution of u is governed by the Ornstein-Uhlenbeck type operator

$$\mathcal{L} = \partial_{y_1}^2 + \partial_{y_2}^2 - \tfrac{y_1}{2}\partial_{y_1} - \tfrac{y_2}{2}\partial_{y_2} + \tfrac{1}{2}\partial_\vartheta^2 + 1, \tag{2}$$

which has the unstable eigenfunctions $1, y_1, y_2, \cos\vartheta, \sin\vartheta$, and the neutral eigenfunctions $y_1^2-2, y_2^2-2, y_1y_2, y_1\cos\vartheta, y_1\sin\vartheta, y_2\cos\vartheta, y_2\sin\vartheta$. Based on these spectral properties, and taking also into account that the ϑ-dependence is tiny thanks to Zhu's symmetry improvement result [15], in joint work with Du we proved:

Theorem (normal form [8, 9]). *For $\tau \to -\infty$, in suitable coordinates, in Gaussian L^2-norm we have*

$$u = O(e^{\tau/2}) \quad \text{or} \quad u = \tfrac{4-y_1^2-y_2^2}{\sqrt{8}|\tau|} + o(\tfrac{1}{|\tau|}) \quad \text{or} \quad u = \tfrac{2-y_2^2}{\sqrt{8}|\tau|} + o(\tfrac{1}{|\tau|}). \tag{3}$$

Accordingly, the classification problem can be split up into 3 cases, which we call the case of fast convergence, slow convergence, and mixed convergence, respectively. In the case of fast convergence, which is easiest case, we have:

Theorem (no wings [6]). *There are no wing-like ancient noncollapsed flows in $\mathbb{R}^4$. In particular, if the convergence is fast, then $\mathcal{M}$ is either a round shrinking $\mathbb{R}^2 \times S^1$ or a translating $\mathbb{R}\times$2d-bowl.*

To prove this, we showed that

$$u^X = (a_1y_1 + a_2y_2)e^{\tau/2} + o(e^{\tau/2}) \tag{4}$$

for all τ negative enough depending only on the bubble-sheet scale. Analyzing this expansion along potential different edges, we concluded that $\mathcal{M}$ in fact splits off a line (hence is not wing-like) and is selfsimilar. The case of slow convergence has been settled in joint work with B. Choi, Daskalopoulos, Du and Sesum:

Theorem (bubble-sheet ovals [4]). *If the convergence is slow, then $\mathcal{M}$ is either the $\mathrm{O}_2\times\mathrm{O}_2$-symmetric 3d-oval, or belongs to the one-parameter family of $\mathbb{Z}_2^2\times\mathrm{O}_2$-symmetric 3d-ovals from* [10].

Regarding the proof, let us just mention that (3) in the case of slow convergence means inwards quadratic bending in all directions, which yields that M_t is compact with axes of length approximately $\sqrt{2|t|\log|t|}$. Hence, up to technical challenges, the problem turned out to be amenable to the techniques from [1]. Finally, in joint work with Choi, we settled the most difficult case of mixed convergence:

Theorem (mixed convergence [5]). *If the convergence is mixed, then $\mathcal{M}$ is either $\mathbb{R}\times$2d-oval or is selfsimilarly translating (and hence by* [7] *is either $\mathbb{R}\times$2d-bowl, or belongs to the 1-parameter family of $\mathbb{Z}_2\times\mathrm{O}_2$-symmetric translators from* [13]*).*

Loosely speaking, to capture the (dauntingly small) slope in y_1-direction, we consider the derivative $u_1^X = \partial u^X / \partial y_1$, which kills the leading order dependence on y_2, and prove that

$$u_1^X = a e^{\tau/2} + o(e^{\tau/2}). \tag{5}$$

This differential neck theorem, which goes vastly beyond (4), can then be used to conclude that $\mathcal{M}$ is noncompact (hence there are no exotic ovals) and either splits off a line or is selfsimilarly translating.

Finally, for the related problem for 4d Ricci flow we conjecture:

Conjecture ([11])**.** *Any κ-solution in 4d Ricci flow is, up to scaling and finite quotients, given by one of the following solutions.*

- *shrinkers: S^4, $\mathbb{C}P^2$, $S^2 \times S^2$, $\mathbb{R} \times S^3$ or $\mathbb{R}^2 \times S^2$.*
- *steadies: 4d Bryant soliton, the 3d Bryant soliton times a line, or belongs to the 1-parameter family of $\mathbb{Z}_2 \times \mathrm{O}_3$-symmetric steady solitons constructed by Lai.*
- *ovals: the $\mathbb{Z}_2 \times \mathrm{O}_4$-symmetric 4d oval from Perelman, the 3d oval times a line, the $\mathrm{O}_2 \times \mathrm{O}_3$-symmetric 4d oval constructed by Buttsworth, or belongs to the 1-parameter family of $\mathbb{Z}_2^2 \times \mathrm{O}_3$-symmetric ovals from* [11].

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