MEAN CURVATURE FLOW THROUGH NECK-SINGULARITIES

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A family of surfaces moves by mean curvature flow if the velocity at each point is given by the mean curvature vector. Mean curvature flow first arose in material science as a model of evolving interfaces and has been extensively studied over the last 40 years.

Curve shortening flow. To gain intuition, we first consider the case of evolving curves. Recall that if γ is a curve in the plane, then its second derivative with respect to arc length s gives the curvature vector $\mathbf{k}$. Now, given any closed embedded curve as initial condition we can evolve it in time by curve shortening flow,

$$\partial_t \gamma(p,t) = \mathbf{k}(p,t). \tag{1}$$

A short computation yields that the total length of the curve satisfies

$$\dot{L}(t) = -\int_{\gamma_t} |\mathbf{k}|^2 \, ds\,. \tag{2}$$

This shows that the curve shortening flow is the gradient flow of the length functional, i.e. the most efficient way to decrease the length of curves. From the PDE perspective, equation (1) can be viewed as a geometric version of the heat equation. Motivated by the physical intuition of heat diffusion one thus hopes that the curve shortening flow deforms any initial curve towards an optimal homogeneous limit. This process indeed works amazingly well:

Theorem 1 (Grayson [1]). *The curve shortening flow starting at any closed embedded curve in the plane converges to a round point.*

Here, the conclusion means that the curve shrinks to a point, and after dilating to unit size converges to a round circle. In particular, Grayson's theorem shows that a curve spiralling around many times always unwinds itself quicker than it shrinks; for a nice illustration look at
https://www.youtube.com/watch?v=wHfpacPLHIA

Mean curvature flow. We now consider surfaces M in $\mathbb{R}^3$. Recall that at any point $x \in M$ the mean curvature $\mathbf{H}$ is a vector normal to the surface whose length is given by the sum of the principal curvatures. Now, given any closed embedded initial surface we can evolve it in time by mean curvature flow,

$$\partial_t x = \mathbf{H}(x,t). \tag{3}$$

Generalizing (2), we have

$$\dot{A}(t) = -\int_{M_t} |\mathbf{H}|^2 \, dA\,, \tag{4}$$

i.e. the mean curvature flow is the gradient flow of the area functional. Moreover, since the mean curvature can be expressed as a nonlinear Laplacian of the position vector, the mean curvature flow can be viewed as geometric version of the heat equation. The subject was kicked off by the following classical result:

Theorem 2 (Huisken [2]). *The mean curvature flow starting at any convex closed embedded surface $M \subset \mathbb{R}^3$ converges to a round point.*

On the other hand, if the initial surface is not convex then the flow typically encounters singularities:

Example 3 (neck-pinch singularity). *If the initial surface looks like a dumbbell, then the neck pinches off, and the surface gets broken up into two components.*

A related example is the degenerate neck-pinch, where one starts with an asymmetric dumbbell whose geometry is fine-tuned such that the smaller sphere pulls itself through the neck exactly at the same moment as the neck pinches off. Most of the research since Huisken's classical result has focussed on analyzing the formation of singularities and developing methods to continue the flow through them. Indeed, getting a hold of singularities is crucial for most striking applications in topology, geometry, and physics, see e.g. [3, 4, 5, 6].

Blowup analysis. To study singularities one looks at them under the microscope. To describe this properly, let $\mathcal{M} = \{M_t\}$ be a mean curvature flow of surfaces. Given a space-time point $X_0 = (x_0, t_0)$ and scaling factors $\lambda_i \to \infty$, one considers the sequence of flows that is obtained from $\mathcal{M}$ by shifting X_0 to the origin and parabolically dilating by λ_i. This gives rise to the notion of a tangent flow at X_0:

$$\hat{\mathcal{M}}_{X_0} := \lim_{i\to\infty} \mathcal{D}_{\lambda_i}(\mathcal{M} - X_0). \tag{5}$$

Roughly speaking, every tangent flow is either (i) a round shrinking cylinder

$$\hat{\mathcal{M}}_{X_0} = \{S^1(\sqrt{-2t}) \times \mathbb{R}\}_{t<0}, \tag{6}$$

or (ii) a self-similarly shrinking asymptotically conical surface. In case (i) we say the flow has a neck-singularity, and in case (ii) a conical singularity. It has been known since the 90s that mean curvature flow through conical singularities is nonunique:

Example 4 (non-uniqeness). *Mean curvature flow of surfaces can form conical singularities, and evolve out of them in many different ways.*

This has been shown theoretically [7] and experimentally [8]. On the other hand, it has been conjectured in the 90s [9], that mean curvature flow through cylindrical singularities is unique, and that every cylindrical singularity has a mean-convex neighborhood. We will describe our solution of these conjectures in the following subsection.

Flow through neck singularities. In [11, 12], joint with Choi-Hershkovits and Choi-Hershkovits-White, we proved the mean-convex neighborhood conjecture:

Theorem 5 (mean-convex neighborhoods)**.** *If $\mathcal{M} = \{M_t\}$ has a neck-singularity at X_0, then there exists a space-time neighborhood of X_0 in which the flow moves in one direction.*

Combined with an earlier result by Hershkovits-White [10], this allowed us to confirm the uniqueness conjecture for mean curvature flow through neck-singularities:

Theorem 6 (uniqueness)**.** *Mean curvature flow through neck-singularities is unique.*

Together with a result of Brendle [13], we also made progress towards the two-sphere conjecture:

Theorem 7 (flow of two-spheres)**.** *Assuming Ilmanen's multiplicity one conjecture, mean curvature flow of embedded two-spheres is well-posed.*

The major difficulty in tackling the mean-convex neighborhood conjecture is that tangent flows only partially capture singularities. To fully capture singularities, one needs to understand all limit flows

$$\mathcal{M}^\infty = \lim_{i\to\infty} \mathcal{D}_{\lambda_i}(\mathcal{M} - X_i), \tag{7}$$

where now X_i is allowed to vary (e.g. to capture a degenerate neck pinch, one wants to choose X_i that follow the tip). A prior one only knows that any suitable limit flow $\mathcal{M}^\infty$ near a neck singularity is an ancient asymptotically cylindrical flow, namely an ancient, unit-regular, cyclic, integral Brakke flow whose tangent flow at $-\infty$ is a round shrinking cylinder. We classified all such flows:

Theorem 8 (classification)**.** *Any ancient asymptotically cylindrical flow is either a round shrinking cylinder, a translating bowl soliton, or an ancient oval.*

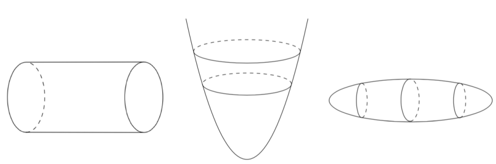

FIGURE 1. Cylinder, bowl and ancient oval.

We note that the classification theorem quickly implies the mean-convex neighborhood theorem via a short argument by contradiction. Our classification builds on recent breakthroughs by Angenent-Daskalopoulos-Sesum [14] and Brendle-Choi [15]. However, we assume neither convexity nor self-similarity, something which has never been accomplished before for any geometric flow. This is crucial for our proof of the mean-convex neighborhood conjecture and uniqueness conjecture. Let us conclude with a sentence about potential topological and geometric applications. A full resolution of the two-sphere conjecture would yield a mean curvature proof of the Smale conjecture that $\text{Diff}(S^3) \sim O(4)$, and would also be very helpful to attack the Lusternik-Schnirelman conjecture that the three-sphere equipped with any Riemannian metric contains at least four minimal embedded two-spheres.

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