A mass-decreasing flow in dimension three

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Let (M, g_{ij}) be an asymptotically flat three-manifold with nonnegative integrable scalar curvature. The ADM-mass [1] from general relativity is defined as

$$m(g) := \lim_{r\to\infty} \int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj})\, dA^i. \tag{1}$$

By the positve mass theorem, the mass is always nonnegative and vanishes only for flat space. Beautiful proofs employing a variety of techniques have been discovered by Schoen-Yau, Witten and Huisken-Ilmanen [2, 3, 4].
The purpose here is to introduce a geometric flow that decreases the mass, for full details please see [5]. Our mass-decreasing flow is defined by iterating a suitable Ricci flow with surgery and conformal rescalings.

Definition 1. *Let (M, g_0) be an orientable asymptotically flat three manifold with nonnegative integrable scalar curvature, and fix a parameter $\varepsilon > 0$.*

- *Let $(M(t), g(t))_{t\in[0,\varepsilon]}$ be the surgical Ricci flow solution of* [6] *starting at g_0, with all connected components except the one containing the asymptotically flat end thrown away.*
- *Solve the elliptic equation $(-8\Delta_{g(\varepsilon)} + R_{g(\varepsilon)})w_1 = 0$, $w_1 \to 1$ at ∞, and conformally rescale to the scalar flat metric $g_1 := w_1^4 g(\varepsilon)$.*
- *Let $(M(\varepsilon), g_1)$ be the new initial condition and iterate the above procedure.*

The concatenation 'flow, conformal rescaling, flow, conformal rescaling, ...' gives an evolution $(M(t), g(t))_{t\in[0,\infty)}$ which we call the mass-decreasing flow.

The point is, that conformal rescalings to scalar flat metrics squeeze out of the manifold as much mass as possible. However, unless the manifold is flat, the scalar curvature becomes strictly positive again under the Ricci flow and thus the mass can be decreased even more by another conformal rescaling. This process can be iterated forever.

Theorem 2. *The mass-decreasing flow exists for all times, and preserves asymptotic flatness and nonnegative integrable scalar curvature. The mass is constant in the time intervals $t \in ((k-1)\varepsilon, k\varepsilon)$ and jumps down by*

$$\delta m_k = -\int_M (8|\nabla w_k|^2 + R w_k^2) dV \tag{2}$$

at the conformal rescaling times $t_k = k\varepsilon$, where w_k is the solution of

$$\left(-8\Delta_{g(t_k)} + R_{g(t_k)}\right) w_k = 0\,, \qquad w_k \to 1 \quad at \quad \infty. \tag{3}$$

The monotonicity of the mass is strict as long as the metric is nonflat.

We remark that the formal limiting equations for $\varepsilon \to 0$ are

$$\partial_t g = -2\mathrm{Ric} + \Delta^{-1}|\mathrm{Ric}|^2 g\,, \qquad \partial_t m = -2\int_M |\mathrm{Ric}|^2 dV. \tag{4}$$

The equations (4) have been discovered independently by Hubert Bray and Lars Andersson. Short-time existence for this nonlocal flow has been proved very recently by Lu-Qing-Zheng [7]. However, we will actually work with the discrete ε-iteration ($\varepsilon > 0$). The point is that our long-time existence result (Theorem 2) relies heavily on the theory of Ricci flow with surgery due to Perelman [8, 9], and the nice variant for noncompact manifolds due to Bessières-Besson-Maillot [6]. Regarding the topological aspects of the long-time behavior, recall that the Ricci flow with surgery on a closed 3-manifold that admits a metric with positive scalar curvature becomes extinct in finite time [10, 11]. In a similar spirit, along the mass-decreasing flow wormholes pinch off and nontrivial spherical space forms bubble off in finite time.

Theorem 3. *There exists a $T < \infty$, such that $M(t) \cong \mathbb{R}^3$ for $t > T$. In particular, the initial manifold had the diffeomorphism type*

$$M \cong \mathbb{R}^3 \# S^3/\Gamma_1 \# \ldots \# S^3/\Gamma_k \# (S^1 \times S^2) \# \ldots \# (S^1 \times S^2). \tag{5}$$

Moreover, one can in fact take $T = \frac{A_0}{4\pi}$, where A_0 is the area of the largest outermost minimal two-sphere in (M, g_0).

To investigate the geometric-analytic aspects of the long-time behavior we will follow the general principle that monotonicity formulas are a very useful tool. However, Perelman's λ-energy vanishes for all asymptotically flat manifolds with nonnegative scalar curvature. To overcome this difficulty, we consider instead the following variant of Perelman's λ-functional,

$$\lambda_{\mathrm{AF}}(g) := \inf_{w : w \to 1} \int_M \left(4|\nabla w|^2 + R w^2\right) dV, \tag{6}$$

where the infimum is now taken over all $w \in C^\infty(M)$ such that $w = 1 + O(r^{-1})$ at infinity.

Theorem 4. *Away from the conformal rescaling and surgery times, we have the monotonicity formula*

$$\tfrac{d}{dt} \lambda_{\mathrm{AF}}(g(t)) = 2 \int_M |Ric + \nabla^2 f|^2 e^{-f} dV \geq 0, \tag{7}$$

where $(-4\Delta + R)e^{-f/2} = 0$, $f \to 0$ at ∞. At the conformal rescaling times, λ_{AF} jumps down, but the mass jumps down more, i.e. $m - \lambda_{AF}$ is (almost) monotone decreasing at all times (the cumulative error term from the surgeries can be made arbitrarily small by choosing the surgery parameters suitably).

In fact, we had already introduced the energy-functional λ_{AF} in our previous note [12], where we also observed it gives a lower bound for the mass, i.e.

$$m(g) \geq \lambda_{AF}(g). \tag{8}$$

In passing, we remark that the renormalized Perelman-functional also motivates a stability inequality for Ricci-flat cones that we investigated thoroughly in a joint work with Hall and Siepmann [13]. Coming back to the long-time behavior of the mass-decreasing flow we have:

Theorem 5. *Let $(M(t), g(t))_{t\in[0,\infty)}$ be a solution of the mass-decreasing flow and assume a-priori there exist a constant $c > 0$, such that $\lambda_{AF}(g(t_k)) \geq cm(g(t_k))^2$ for all positive integers k. Then there exists a constant $C < \infty$ such that $m(g(t)) \leq C/t$. In particular, the mass-decreasing flow squeezes out all the initial mass, i.e.* $\lim_{t\to\infty} m(g(t)) = 0$.

The a-priori assumption is (partly) motivated by considering the flow on an end close to Schwarzschild. However, we actually have:

Conjecture 6. *The mass-decreasing flow squeezes out all the initial mass even without a-priori assumptions, i.e.* $\lim_{t\to\infty} m(g(t)) = 0$.

Going one step further, one might ask:

Question 7. *Can the mass-decreasing flow be used to give an independent proof of the positive mass theorem?*

The idea is to show that the mass-decreasing flow converges (for $t \to \infty$) to flat space in a sense strong enough to conclude that the mass limits to zero (and hence that the mass was nonnegative at the initial time). Understanding the geometric long time behavior is already very difficul in the case of the Ricci flow with surgery on closed 3-manifolds, see however the recent progress by Bamler [14].

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