**Applications of mean curvature flow**

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(joint work with Reto Buzano, Or Hershkovits, Dan Ketover, Mohammad Ivaki)

We discussed some recent topological, geometric and analytic applications of mean curvature flow.

To put the topological application into context, consider the moduli space of embedded $n$-spheres in $\mathbb{R}^{n+1}$, i.e. the space $\mathcal{M}(S^n) = \mathrm{Emb}(S^n, \mathbb{R}^{n+1})/\mathrm{Diff}(S^n)$ equipped with the smooth topology. By a theorem of Smale [15] the space $\mathcal{M}(S^1)$ is contractible, and by Hatcher's solution of the Smale conjecture $\mathcal{M}(S^2)$ is also contractible. For $n \geq 3$, there are many non-vanishing homotopy groups, see e.g. [3]. In the view of the topological complexity of $\mathcal{M}(S^n)$ for general $n$, it is an interesting question whether one can still derive some positive results on the space of embedded $n$-spheres under some curvature conditions. Motivated by the topological classification result from [10], we consider the subspace $\mathcal{M}_{2\text{-conv}}(S^n) \subset \mathcal{M}(S^n)$ of 2-convex embedded $n$-spheres in $\mathbb{R}^{n+1}$, i.e. we impose the condition that the sum of the two smallest principal curvatures is positive. We proved:

**Theorem 1** (Buzano-Haslhofer-Hershkovits [2])**.** *The moduli space $\mathcal{M}_{2\text{-conv}}(S^n)$ is path-connected in every dimension $n$.*

Our proof uses mean curvature flow with surgery [8, 10], a connected sum operation for 2-convex hypersurfaces, and a scheme inspired by the work of Marques on the moduli space of positive scalar curvature metrics on the three-sphere [13].

For the geometric application, recall first that by a classical theorem of Lusternik-Schnirelmann [5, 12] every $(S^2, g)$ contains at least 3 simple closed geodesics.

Moving up one dimension, one might hope to prove that any $(S^3, g)$ contains at least 4 embedded minimal two-spheres. The existence of at least 1 embedded minimal two-sphere was established by Simon-Smith [14]. While there are indeed 4 cohomology classes $\alpha, \ldots, \alpha^4$ in the space of embedded two-spheres, the the major difficulty is the phenomenon of *multiplicity* in min-max theory. Namely, it could happen that the min-max spheres associated with the second, third and fourth family, just give the Simon-Smith sphere counted with higher integer multiplicities.

Using combined efforts from min-max theory and mean curvature flow we proved:

**Theorem 2** (Haslhofer-Ketover [7])**.** *Any $(S^3, g)$ equipped with a bumpy metric contains at least 2 embedded minimal two-spheres. More precisely, exactly one of the following alternatives holds:*

(1) *$(S^3, g)$ contains at least 1 stable embedded minimal two-sphere, and at least 2 embedded minimal two-spheres of index one.*
(2) *$(S^3, g)$ contains no stable embedded minimal two-sphere, at least 1 embedded minimal two-sphere $\Gamma_1$ of index one, and at least 1 embedded minimal two-sphere $\Gamma_2$ of index two. In this case, $|\Gamma_2| < 2|\Gamma_1|$.*

We note that White [16] previously proved the existence of at least 2 minimal two-spheres in the special case that $(S^3, g)$ has positive Ricci curvature.

Illustrative examples for Theorem 2 are dumbbells for case (1) and ellipsoids for case (2). The main way how mean curvature flow enters the proof is via the following theorem (of independent interest) which establishes the existence of smooth mean convex foliations in three-manifolds:

**Theorem 3** (Haslhofer-Ketover [7])**.** *Let $D \subset (M^3, g)$ be a smooth three-disc with mean convex boundary. Then exactly one of the following alternatives holds true:*

(1) *There exists an embedded stable minimal two-sphere $\Gamma \subset Int(D)$.*
(2) *There exists a smooth foliation $\{\Sigma_t\}_{t\in[0,1]}$ of $D$ by mean convex embedded two-spheres.*

Namely, if $(S^3, g)$ contains no stable embedded minimal two-sphere then we can use Theorem 3 to produce a foliation $\{\Sigma_t\}_{t\in[-1,1]}$ of $(S^3, g)$ such that the Simon-Smith sphere sits in the middle as $\Sigma_0$ and such that $|\Sigma_t| < |\Sigma_0|$ for all $t \neq 0$. We can then construct a suitable 2-parameter sweepout $\{\Sigma_{s,t}\}$ detecting $\alpha^2$ with

$$\sup_{s,t} |\Sigma_{s,t}| < 2|\Gamma_1|. \tag{1}$$

Roughly speaking, $\Sigma_{s,t}$ looks like $\Sigma_s$ connected to $\Sigma_t$ along a small neck, which we open up near $(s, t) \approx (0, 0)$, using the catenoid estimate from [11]. The estimate (1) ensures that min-max for $\Sigma_{s,t}$ doesn't give $\Gamma_1$ with multiplicity two.

The proof of Theorem 3 is again based on mean curvature flow with surgery, refining the methodology from the proof of Theorem 1.

Finally, for the analytic application we consider the Allen-Cahn equation

$$\Delta u = \frac{1}{\epsilon^2} u(1 - u^2) \tag{2}$$

on any bumpy $(S^3, g)$. The recent work of Gaspar-Guaraco [4] establishes the existence of solutions of arbitrarily large index as $\epsilon$ becomes smaller. In a different direction, we examine solutions of low index / with a simple interface, and prove:

**Theorem 4** (Haslhofer-Ivaki [6])**.** *The Allen-Cahn equation* (2) *on any bumpy $(S^3, g)$ has at least 4 solutions with spherical interface and index at most two.*

## References


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