Mean curvature flow of mean convex hypersurfaces

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(joint work with Bruce Kleiner)

In the last 15 years, White and Huisken-Sinestrari developed a far-reaching structure theory for the mean curvature flow of mean convex hypersurfaces [1, 2, 3, 4, 5, 6]. We recently gave a new treatment of this theory [7], based on the beautiful non-collapsing result of Andrews [8]. Our new proofs are both more elementary and substantially shorter than the original arguments.

Recall that for any mean convex hypersurface $M_0^n \subset \mathbb{R}^{n+1}$ (smooth, closed, embedded), there is a unique weak solution $\{M_t = \partial K_t\}_{t\geq 0}$ of the mean curvature flow starting at M_0. It is characterized by the condition that $\{K_t\}$ is the maximal family of closed sets satisfying the avoidance principle

$$K_{t_0} \cap L_{t_0} = \emptyset \quad \Rightarrow \quad K_t \cap L_t = \emptyset \quad (t \in [t_0, t_1]) \tag{1}$$

for every smooth mean curvature flow $\{L_t\}_{t\in[t_0,t_1]}$. By the main result of Andrews [8] (which we extended to the weak setting via elliptic regularization) we have

Theorem. *There exists a constant $\alpha = \alpha(K_0) > 0$ such that every point $p \in \partial K_t$ admits interior and exterior balls tangent at p of radius at least $\frac{\alpha}{H(p)}$.*

The Andrews condition immediately rules out higher multiplicity planes as potential blowup limits. However, taking a much broader perspective, it turned out that one can actually develop the entire theory based on the Andrews condition. The starting point was the following estimate, which says that curvature control at a single point implies curvature control in a whole parabolic ball.

Theorem (Curvature estimate)**.** *There exist constants $\delta = \delta(\alpha) > 0$ and $C = C(\alpha) < \infty$ such that for any α-Andrews flow K_t in a parabolic ball $P(p, t, r)$:*

$$H(p,t) \leq r^{-1} \qquad \Rightarrow \qquad \sup_{P(p,t,\rho r)} |A| \leq C r^{-1}. \tag{2}$$

To prove this we use comparison and the Andrews condition to show that the flow is, after suitable rescaling, Hausdorff close to a halfspace on a large time interval. Then the local regularity theorem implies the desired curvature bounds.

Theorem (Convexity estimate)**.** *For every $\varepsilon > 0$ there exists $\eta = \eta(\varepsilon, \alpha) < \infty$ such that if K_t is an α-Andrews flow defined in $P(p, t, \eta H(p, t)^{-1})$ then*

$$\frac{\lambda_1}{H}(p,t) \geq -\varepsilon. \tag{3}$$

In particular, blowup limits of α-Andrews flows are convex.

The proof is very short proof (one page as opposed to a couple of sophisticated papers): Take a sequence of counterexamples where the infimum of $\frac{\lambda_1}{H}$ is negative. By the local curvature estimate the infimum is actually a minimum. However, by the strict maximum principle, $\frac{\lambda_1}{H}$ can never attain a negative minimum.

Our treatment of the global theory is based on the following global convergence result, which says that after normalizing the mean curvature at a single point we can pass smoothly and globally to a limit.

Theorem (Global convergence)**.** *Every sequence of α-Andrews flows in $P(0,0,\eta_j)$ with $H(0,0)=1$ and $\eta_j\to\infty$ has a smoothly and globally convergent subsequence.*

Roughly, the idea of the proof is as follows: If the global convergence theorem failed, by looking at the first radius where the curvature blows up we could find a nonflat convex cone; this however cannot happen under mean curvature flow.

Theorem (Structure of ancient solutions)**.** *Ancient α-Andrews flows are smooth and convex until they become extinct. In particular, the only self-similarly shrinking ancient α-Andrews flows are the sphere, the cylinders, and the plane.*

For more about ancient solutions we refer to Haslhofer-Hershkovits [9]. By standard stratification the structure theorem immediately implies:

Theorem (Partial regularity)**.** *For every α-Andrews flow $K_t\subset\mathbb{R}^N$, the singular set has parabolic Hausdorff dimension at most $N-2$.*

In fact, using quantitative stratification this can be improved to Minkowski and L^p-estimates, in particular in the k-convex case, see Cheeger-Haslhofer-Naber [10].

Theorem (Cylindrical estimate)**.** *For every $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon,\alpha,\beta)>0$ such that if K_t is a uniformly k-convex (i.e. $\lambda_1+\ldots+\lambda_k\geq\beta H$ for some $\beta>0$) α-Andrews flow and $\frac{\lambda_1+\ldots+\lambda_{k-1}}{H}(p,t)<\delta$, then K_t is ε-close to a round shrinking cylinder $\mathbb{R}^{k-1}\times S^{n-k+1}$ near (p,t).*

All our estimates are local and universal, i.e. they only depend on the value of the Andrews constant. In a forthcoming paper [11], we will give a new construction of mean curvature flow with surgery based on these estimates.

References

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