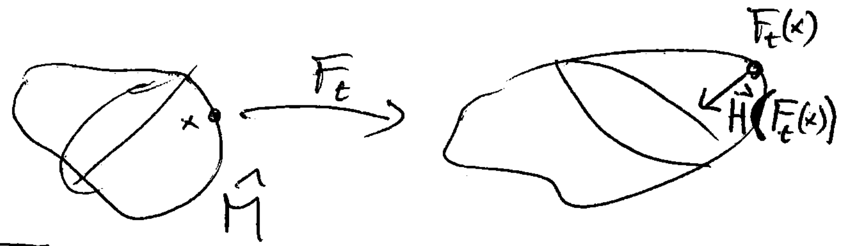
Evolution equations under MCF

$F_t : \hat{M}^n \longrightarrow \mathbb{R}^{n+1}$ 1-parameter family of embeddings

$F_t(x) \equiv F(x,t)$

$F : \hat{M}^n \times [0,T) \longrightarrow \mathbb{R}^{n+1}$



$$\boxed{\partial_t F(x,t) = \vec{H}(F(x,t))} \qquad \vec{H} = H\nu \qquad M_t = F_t(\hat{M}) \subset \mathbb{R}^{n+1}$$

<u>Q:</u> If F evolves by MCF, what evolution equations do g, $d\mu$, ν, A, H, etc satisfy?

$\cdot)$ $g_{ij} = \langle \partial_i F, \partial_j F \rangle$

$$\Rightarrow \partial_t g_{ij} = \langle \partial_i \partial_t F, \partial_j F \rangle + \langle \partial_i F, \partial_j \partial_t F \rangle$$

$$= \langle \partial_i (H\nu), \partial_j F \rangle + (i \leftrightarrow j)$$

$$= \partial_i H \underbrace{\langle \nu, \partial_j F \rangle}_{=0} + H \underbrace{\langle \partial_i \nu, \partial_j F \rangle}_{=-A_{ij}} + (i \leftrightarrow j)$$

$$= -HA_{ij} + (i \leftrightarrow j)$$

$$\Rightarrow \boxed{\partial_t g_{ij} = -2HA_{ij}}$$

•) $d\mu = \sqrt{\det g_{ij}}\, d^n x$

$$\partial_t \sqrt{\det g_{ij}} = \frac{1}{2\sqrt{\det g_{ij}}}\partial_t \det g_{ij} = \frac{1}{2\sqrt{\det g_{ij}}} \det g_{ij} \operatorname{tr}_g(\partial_t g_{ij})$$

Exer (Hint: Show first that $\frac{d}{dt}\det(1+tM) = \operatorname{tr} M$)

$$= -H^2 \sqrt{\det g_{ij}}$$

$$\Rightarrow \boxed{\partial_t\, d\mu = -H^2\, d\mu}$$

Cor $\underline{\frac{d}{dt}\text{Area}(M_t)} = \frac{d}{dt}\int_{M_t} 1\, d\mu = \underline{-\int_{M_t} H^2\, d\mu}$

•) $\partial_t g^{ij} = ?$

$$g^{ik} g_{kl} = \delta^i_l$$

$$\Rightarrow \partial_t g^{ik} g_{kl} + g^{ik}\partial_t g_{kl} = 0$$

$$\Rightarrow \partial_t g^{ij} = -g^{ik}\partial_t g_{kl}\, g^{jl}$$

$$\Rightarrow \partial_t g^{ij} = -g^{ik}(-2HA_{kl})\, g^{jl}$$

$$\Rightarrow \boxed{\partial_t g^{ij} = 2HA^{ij}}$$

•) $\partial_t \nu = ?$

$$\langle \nu, \partial_i F \rangle = 0$$

$$\Rightarrow \langle \partial_t \nu, \partial_i F \rangle = -\langle \nu, \partial_i \partial_t F \rangle$$

$$= -\langle \nu, \partial_i (H\nu) \rangle = -\partial_i H \underbrace{\langle \nu, \nu \rangle}_{=1} - H \underbrace{\langle \nu, \partial_i \nu \rangle}_{=0}$$

$$\underset{\partial_t \nu \perp \nu}{\Longrightarrow} \boxed{\partial_t \nu = -\nabla H}$$

•) $\partial_t A_{ij} = ?$

$$A_{ij} = \langle \partial_i \partial_j F, \nu \rangle$$

$$\partial_t A_{ij} = \langle \partial_i \partial_j (H\nu), \nu \rangle + \langle \underbrace{\partial_i \partial_j F}, -\nabla H \rangle$$

$$= \partial_i \partial_j H + H \langle \partial_i \partial_j \nu, \nu \rangle + \langle \overbrace{\Gamma_{ij}^k \partial_k F + A_{ij} \nu}^{\text{Gauss-Weingarten}}, -\nabla H \rangle$$

$$\underset{(*)}{=} \partial_i \partial_j H - \Gamma_{ij}^k \langle \partial_k F, \nabla H \rangle - H \langle \partial_i (A_{jk} g^{kl} \partial_l F), \nu \rangle$$

$$= \underbrace{\partial_i \partial_j H - \Gamma_{ij}^k \partial_k H} - H A_{jk} g^{kl} \langle \partial_i \partial_l F, \nu \rangle$$

$$= \nabla_i \nabla_j H - H A_{jk} g^{kl} A_{li}$$

$$\Rightarrow \boxed{\partial_t A_{ij} = \Delta A_{ij} - 2H A_{jk} g^{kl} A_{li} + |A|^2 A_{ij}}$$

Simon's identity

for (*) we used: $\partial_j \nu = -A_{jk} g^{kl} \partial_l F$

indeed: $\partial_j \nu = \lambda_j^l \partial_l F$, $\lambda_j^l = ?$

$$-A_{ij} = \langle \partial_j \nu, \partial_i F \rangle = \lambda_j^l \langle \partial_l F, \partial_i F \rangle = \lambda_j^l g_{li} \quad | \, g^{ik}$$

$$\Rightarrow -A_{ij} g^{ik} = \lambda_j^l \delta_l^k = \lambda_j^k \Rightarrow \partial_j \nu = -A_{ij} g^{il} \partial_l F .$$

Exer

Cor: $\boxed{\partial_t A^i_j = \Delta A^i_j + |A|^2 A^i_j}$

Proof $\partial_t A^i_j = \partial_t \left(g^{ik} A_{kj} \right) = \left(\partial_t g^{ik} \right) A_{kj} + g^{ik} \partial_t A_{kj} = \dots$ $\square$

•) $\partial_t H = ?$

$$\partial_t H = \partial_t \left(g^{ij} A_{ij} \right) = \partial_t g^{ij} A_{ij} + g^{ij} \partial_t A_{ij}$$

$$= 2 H A^{ij} A_{ij} + \Delta H - 2 H A^{ij} A_{ij} + |A|^2 H$$

$$\Rightarrow \boxed{\partial_t H = \Delta H + |A|^2 H}$$

Maximum principle & consequences

Intuition: $\partial_t H = \Delta H + |A|^2 H$

$$H_{\min}(t) := \min_{M_t} H$$

$$\Rightarrow \frac{d}{dt} H_{\min} \geq |A|^2 H_{\min} \geq \frac{1}{n} H_{\min}^3 \quad (\text{if } H_{\min} \geq 0)$$

If $H_{\min}(0) \geq 0$, then:

$t \mapsto H_{\min}(t)$ is increasing in time

more precisely $H_{\min}(t) \geq \dfrac{H_{\min}(0)}{\sqrt{1 - \frac{2}{n} t H_{\min}^2(0)}}$

solution of ODE $\begin{cases} \frac{d\varphi}{dt} = \frac{1}{n}\varphi^3 \\ \varphi(0) = H_{\min}(0) \end{cases}$

Make precise:

Let $v : M^n \times [0,T) \longrightarrow \mathbb{R}$

closed mfd

loc. Lipschitz

Consider $\partial_t v = \Delta_{g_t} v + \langle X, \nabla v \rangle_{g_t} + F(v)$ (*)

time dep. metrick vectorfield (say C^1)

We say u is a supersolution of (*) if

$$\partial_t u \geq \Delta_{g_t} u + \langle X, \nabla u \rangle + F(u)$$

Subsolution $\partial_t u \leq \Delta_{g_t} u + \langle X, \nabla u \rangle + F(u)$

3

Thm (Maximum principle)

i.e. C^2 on $M^n \times (0,T)$, C^0 on $M^n \times [0,T)$

Suppose $u : M^n \times [0,T) \to \mathbb{R}$ is a C^2-supersolution of $(*)$

Suppose $\min_{x \in M} u(x,0) \geq m$

Let φ be the solution of the ODE $\begin{cases} \frac{d\varphi}{dt} = F(\varphi) \\ \varphi(0) = m \end{cases}$

Then $u(x,t) \geq \varphi(t)$ $\forall x \in M$ $\forall t \in [0,T)$ st $\varphi(t)$ exists.

Similarly: u subsolution, $\max_{x \in M} u(x,0) \leq M \Rightarrow u(x,t) \leq \varphi(t)$.

<u>Proof</u> i) case $F \equiv 0$

<u>Claim:</u> $\min u(\cdot,0) \geq 0 \Rightarrow \min u(\cdot,t) \geq 0 \ \forall t \in [0,T)$

<u>Proof</u> Let $u_\varepsilon(x,t) = u(x,t) + \varepsilon t + \varepsilon$ $(\varepsilon > 0)$

$$\Rightarrow \begin{cases} \partial_t u_\varepsilon \geq \Delta u_\varepsilon + \langle X, \nabla u_\varepsilon \rangle + \varepsilon \\ u_\varepsilon(\cdot,0) \geq \varepsilon \end{cases}$$

Suppose $u_\varepsilon(x_0,t_0) = 0$, and t_0 is the first time when this happens.

$$\Rightarrow 0 \geq \underbrace{\partial_t u_\varepsilon(x_0,t_0)}_{\leq 0} \geq \underbrace{\Delta u_\varepsilon(x_0,t_0)}_{\geq 0} + \langle X, \underbrace{\nabla u_\varepsilon(x_0,t_0)}_{=0} \rangle + \varepsilon \geq \varepsilon > 0 \ \text{↯}$$

$$\Rightarrow u_\varepsilon > 0 \ \forall t \in [0,T) \underset{\forall \varepsilon > 0}{\Rightarrow} u(\cdot,t) \geq 0 \ \forall t \in [0,T) \ \square$$

ii) $\partial_t u \geq \Delta u + \langle X, \nabla u \rangle - Cu$

<u>Claim:</u> $u \geq 0$ at $t = 0 \Rightarrow u \geq 0 \ \forall t$

<u>Proof</u> $\tilde{u}(x,t) := e^{Ct} u(x,t)$

$$\Rightarrow \partial_t \tilde{u} \geq \Delta \tilde{u} + \langle X, \nabla \tilde{u} \rangle \underset{i)}{\Rightarrow} \tilde{u} \geq 0 \ \forall t \quad \square$$

iii) general case:

$$\partial_t (u - \varphi) \geq \Delta (u - \varphi) + \langle X, \nabla (u - \varphi) \rangle + \underbrace{F(u) - F(\varphi)}_{\geq -C_{T'} \cdot (u - \varphi) \text{ on } M \times [0,T']}$$

$$\underset{ii)}{\Rightarrow} u - \varphi \geq 0 \text{ on } M \times [0,T') \underset{\forall T' < T}{\Rightarrow} \text{Thm} \quad \square$$

Rmk If $H \geq 0$ at $t = 0$

then either $H > 0$ $\forall t > 0$.

or $H \equiv 0$.

(look up "strong maximum principle")

Short time existence

Thm Given $F_0 : \hat{M}^n \hookrightarrow \mathbb{R}^{n+1}$ smooth closed embedded hypersurface, there exists a unique smooth solution of

$$\begin{cases} \partial_t F(x,t) = H(x,t)\, \nu(x,t) \\ F(x,0) = F_0(x) \end{cases} \qquad (MCF)$$

defined in some positive time interval.

Note: If $\hat{M}^n \xrightarrow{F_t} \mathbb{R}^{n+1}$ evolves by MCF, and $\varphi \in \mathrm{Diff}(\hat{M}^n)$, then $F_t \circ \varphi$ also solves the MCF eqn.

$\rightsquigarrow$ MCF is only <u>degenerate parabolic</u>.

Indeed, e.g. if $\gamma : S^1 \times [0,T) \to \mathbb{R}^2$, $\partial_t \gamma = \partial_s^2 \gamma$

then $\partial_t \gamma = \partial_s^2 \gamma \underset{\substack{\nearrow \\ ds = |\partial_x \gamma| dx}}{=} \frac{1}{|\partial_x \gamma|} \partial_x \left(\frac{1}{|\partial_x \gamma|} \partial_x \gamma \right)$

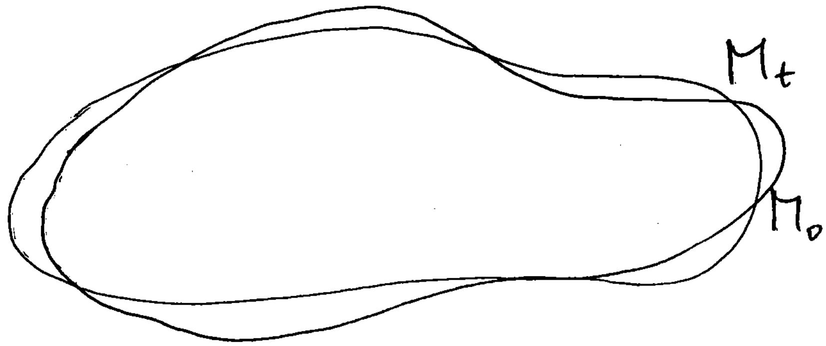
$$= \frac{1}{|\partial_x \gamma|^2} \left(\partial_x^2 \gamma - \left\langle \frac{\partial_x \gamma}{|\partial_x \gamma|}, \partial_x^2 \gamma \right\rangle \frac{\partial_x \gamma}{|\partial_x \gamma|} \right)$$

In components $\gamma = (\gamma^1, \gamma^2)$ this reads

$$\begin{pmatrix} \partial_t \gamma^1 \\ \partial_t \gamma^2 \end{pmatrix} = |\partial_x \gamma|^{-4} \underbrace{\begin{pmatrix} (\partial_x \gamma^2)^2 & \partial_x \gamma^1 \partial_x \gamma^2 \\ \partial_x \gamma^1 \partial_x \gamma^2 & (\partial_x \gamma^1)^2 \end{pmatrix}}_{\substack{\nearrow \\ \text{not strictly positive definite} \\ (\mathrm{tr} > 0,\ \det = 0)}} \begin{pmatrix} \partial_x^2 \gamma^1 \\ \partial_x^2 \gamma^2 \end{pmatrix}$$

$\Rightarrow$ Std. theory for strictly parabolic systems cannot be applied.

To prove the thm we break the Diff-invariance ("fix a gauge"):



Write M_t as graph over M_0, i.e.

$$F(x,t) = F_0(x) + f(x,t)\,\nu_0(x), \quad f(x,0) = 0.$$

$$\begin{cases} \langle \partial_t F(x,t), \nu(x,t) \rangle = H(x,t) \\ F(x,0) = F_0(x) \end{cases} \qquad (\mathrm{MCF}^*)$$

$*$ = up to tangential motion.

Express (MCF^*) in terms of f:

$$\begin{cases} \partial_t f(x,t) = \underbrace{\dfrac{1}{\langle \nu_0(x), \nu(x,t) \rangle} H(x,t)} \\ f(x,0) \equiv 0 \end{cases}$$

TODO: write in terms of f

$$\partial_i F(x,t) = \partial_i F_0(x) + \partial_i f(x,t)\,\nu_0(x) - f(x,t)\,\underset{\substack{\uparrow\\ \text{2nd fund. form of } M_0}}{A_i^k(x)}\,\partial_k F_0(x)$$

$$\Rightarrow \nu(x,t) = \frac{\nu_0(x) - \langle \nu_0(x), \partial_i F(x,t)\rangle g^{ij}(x,t)\,\partial_j F(x,t)}{|\nu_0(x) - \langle \nu_0(x), \partial_i F(x,t)\rangle g^{ij}(x,t)\,\partial_j F(x,t)|}$$

= small perturbation of $\nu_0(x)$,
provided $\|f\|_{C^1}$ small enough.

here $g_{ij}(x,t) = \langle \partial_i F(x,t), \partial_j F(x,t)\rangle$

$$= \langle \partial_i F_0 + \partial_i f\,\nu_0 - f A_i^k \partial_k F_0, \partial_j F_0 + \partial_j f\,\nu_0 - f A_j^l \partial_l F_0\rangle$$

$$= g_{ij}(x,0) - 2f A_{ij} + \partial_i f\,\partial_j f + f^2 A_{ik} A_j^k$$

$$= g_{ij}(x,0) + \text{small, provided } \|f\|_{C^1} \text{ small enough}$$

In particular, $\forall \varepsilon > 0\ \exists \delta > 0$:

$$|1 - \langle \nu_0(x), \nu(x,t)\rangle| < \varepsilon \quad \text{provided } \|f\|_{C^1} < \delta.$$

$$A_{ij}(x,t) = \langle \nu(x,t), \partial_i \partial_j F(x,t) \rangle$$

$$= \langle \nu, \partial_i \partial_j f \, \nu_0 + \partial_i \partial_j F_0 - \partial_i f A^k_j \partial_k F_0 - \partial_j f A^k_i \partial_k F_0 - f \partial_j A^k_i \partial_k F_0 - f A^k_i \partial_j \partial_k F_0 \rangle$$

$$= \langle \nu(x,t), \nu_0(x) \rangle \partial_i \partial_j f(x,t) + P_{ij}(x, f(x,t), \partial f(x,t))$$

$$H(x,t) = g^{ij}(x,t) A_{ij}(x,t)$$

$$= \langle \nu(x,t), \nu_0(x) \rangle g^{ij}(x,t) \partial_i \partial_j f(x,t) + P(x, f(x,t), \partial f(x,t))$$

$$\Rightarrow \underline{(MCF^*)}: \begin{cases} \partial_t f(x,t) = g^{ij}(x,t) \partial_i \partial_j f(x,t) + Q(x, f(x,t), \partial f(x,t)) \\ f(\cdot, 0) \equiv 0. \end{cases}$$

quasilinear strictly parabolic PDE

$\Rightarrow$ $\exists$ unique solution on short time interval.

$(MCF^*) \rightsquigarrow (MCF)$:

$$\partial_t F_t(x) = H(x,t)\,\nu(x,t) + \underbrace{X(x,t)}_{\text{(tangential)}} \qquad (MCF^*)$$

Ansatz: $\tilde{F}_t = F_t \circ \varphi_t$, $\quad \varphi_t \in \mathrm{Diff}(M^n)$.

$$\Rightarrow \partial_t \tilde{F}_t(\tilde{x}) = \partial_t F_t(\varphi_t(x)) + dF_t(\varphi_t(x))\frac{\partial \varphi_t}{\partial t}(x), \qquad \tilde{x} = \varphi_t(x)$$

$$= \tilde{H}(\tilde{x},t)\tilde{\nu}(\tilde{x},t) + X(\varphi_t(x),t) + dF_t(\varphi_t(x))\frac{\partial \varphi_t}{\partial t}(x)$$

$$= \underline{\tilde{H}(\tilde{x},t)\tilde{\nu}(\tilde{x},t)} \qquad (MCF)$$

provided that

$$\frac{\partial \varphi_t}{\partial t}(x) = -\,dF_t(\varphi_t(x))^{-1}X(\varphi_t(x),t).$$

(and vice versa)