Mean curvature flow

1. Intro/Motivation

1.1 Context: Historically geometers studied:

until 80s mostly curved spaces/shapes

in which time did not play a role

Since 80s: great progress in understanding shapes that evolve in time.

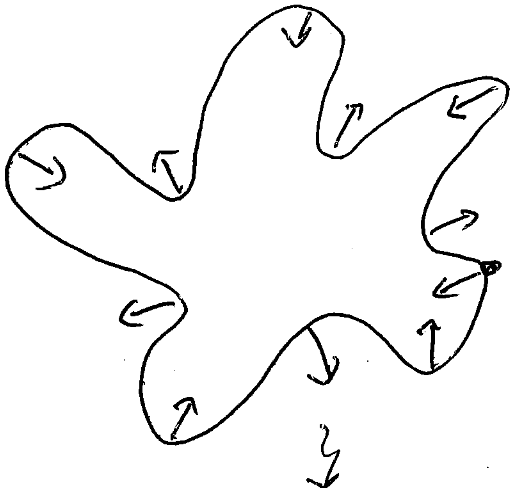
Intrinsic geometry:

Evolve Riemannian manifolds by Ricci flow $\partial_t g = -2Rc(g)$

Extrinsic geometry:

Evolve ~~surfaces~~ submanifolds by mean curvature flow $\partial_t x = \vec{H}(x)$

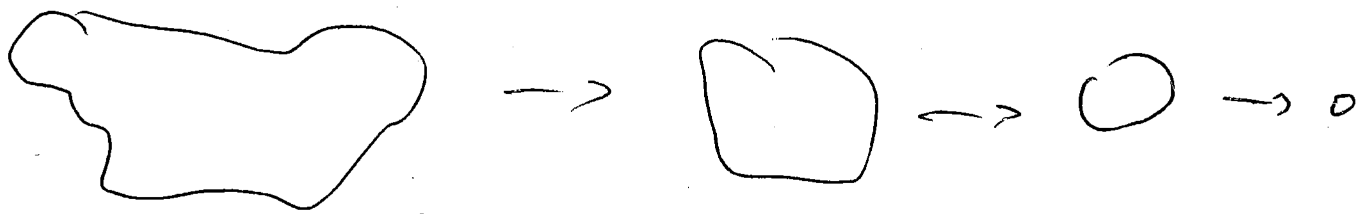
1.2 Curve shortening flow (1-dim mean curv. flow)



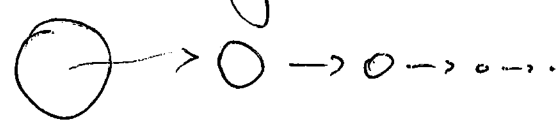
$\gamma_0 \subset \mathbb{R}^2$ curve in the plane

$\rightsquigarrow \{\gamma_t\}_{t\in[0,T)}$ 1-par. family of curves

$$\boxed{\partial_t x = \vec{k}(x)}, \quad x \in \gamma_t$$



Ex (shrinking circle) γ_0 = round circle of radius r_0

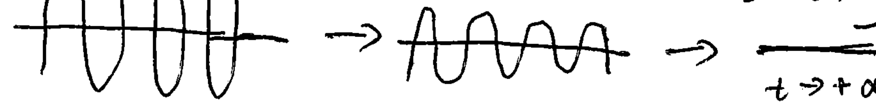


$\partial_t r = -\frac{1}{r} \Rightarrow \gamma_t$ = round circle of radius $r(t) = \sqrt{r_0^2 - 2t}$

$t \in [0, r_0^2/2)$

Ex ("hairclip")

$$\gamma_t = \{ (x(t), y(t)) : \sinh y(t) = e^{-t} \cos x(t) \}$$



$t \to +\infty$

General properties

- short time smoothing:

  γ_0 say C^2 $\Rightarrow$ γ_t is C^∞ (in fact analytic) for $t > 0$.
  (or finite length Jordan curve)

- decreases arclength:

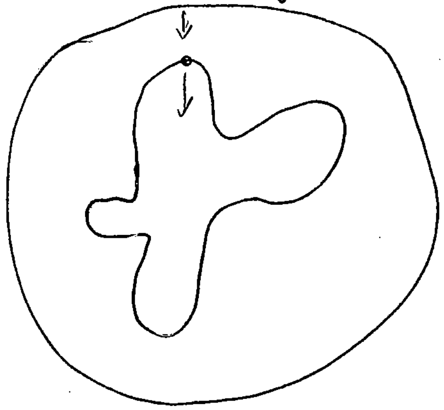
$$\frac{d}{dt} L(\gamma_t) = -\int \vec{k} \cdot \vec{v} \, ds = -\int |\vec{k}|^2 ds$$

  ⇝ "curveshortening flow"

- avoidance principle:

  $\{\gamma_t\}$, $\{\tilde{\gamma}_t\}$ two curveshortening flows

$$\gamma_{t_0} \cap \tilde{\gamma}_{t_0} = \phi \Rightarrow \gamma_t \cap \tilde{\gamma}_t = \phi \quad \forall t \geq t_0$$



<u>also:</u> embeddedness preserved.

<u>Cor</u> $\gamma_0 \subset B_r$

$\Rightarrow$ lifetime $< r^2/2$.

- $A(t) :=$ area enclosed by γ_t

$$\Rightarrow \frac{d}{dt} A(t) = -\int_{\gamma_t} k \, ds = -2\pi$$

$$\Rightarrow A(t) = A(0) - 2\pi t$$

<u>Cor</u> lifetime $\leq A(0)/2\pi$.

- Evolution eqn. for curvature:

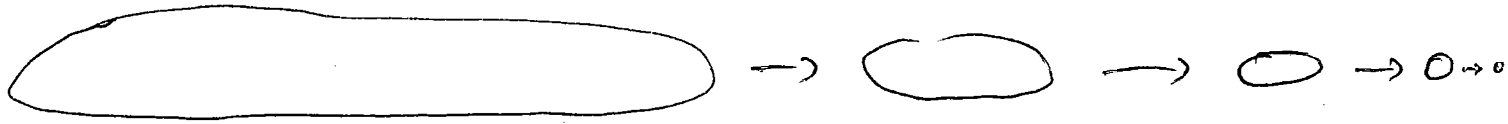
$$\partial_t x = \vec{k}(x) \Rightarrow \underline{\partial_t k = \partial_s \partial_s k + k^3}$$

<u>Cor</u> (convexity preserved) $k > 0$ at $t = 0 \Rightarrow k > 0 \ \forall t \geq 0$

in fact $\min k \nearrow$ in t

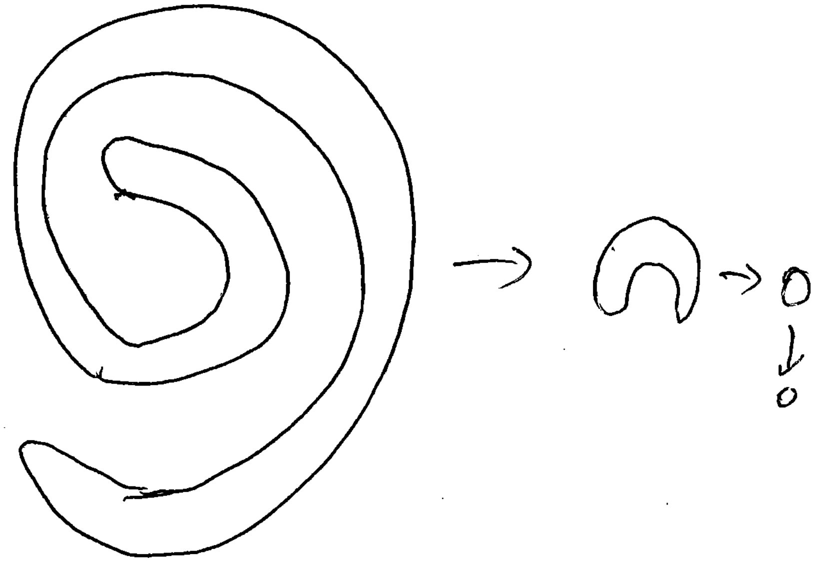
<u>Thm</u> (Gage-Hamilton 86):

Under curve shortening flow, any convex embedded curve shrinks to a round point.



<u>Thm</u> (Grayson 87)

Under CSF, embedded curves become convex and thus (by Gage-Hamilton) shrink to round point.



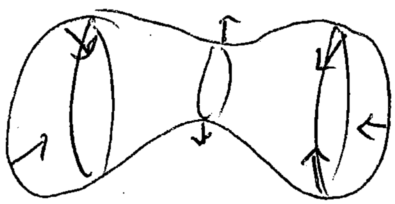
eg. curve spiraling around 10^{1000} times.

<u>Cor</u> lifespan $= A(0)/2\pi$.

1.3 Overview MCF

$$\partial_t x = \vec{H}(x), \quad x \in M_t \qquad M^2 \subset \mathbb{R}^3 \text{ surface}$$

(or $M^n \subset \mathbb{R}^{n+1}$)



(first appeared in material science Mullins 56)

similar basic properties:

- short time smoothing:

$$M_0 \ C^2 \Rightarrow M_t \text{ analytic for } t > 0$$

(in fact enough: M_0 Lipschitz: Ecker–Huisken or even certain fractals: Hershkovits)

- area decreases. In fact:

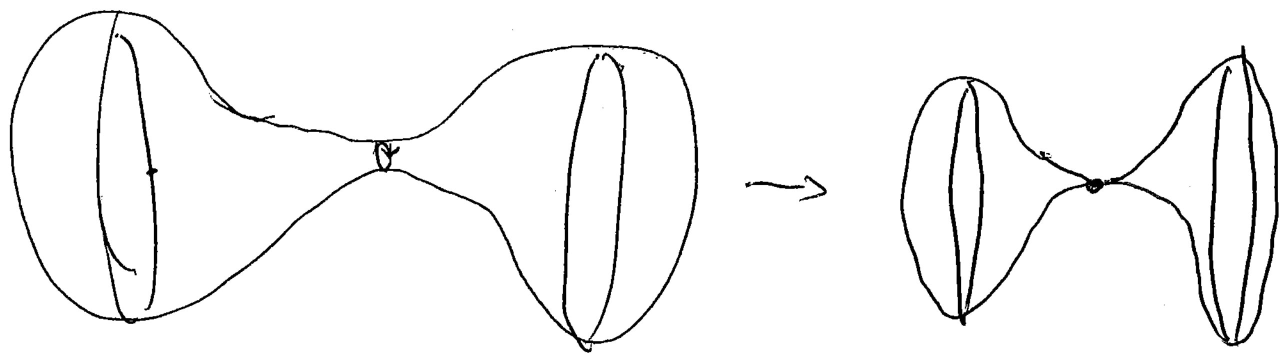
$$\frac{d}{dt}\mathrm{Area}(M_t) = -\int_{M_t} \vec{H}\cdot\vec{v}\, dA = -\int_{M_t} H^2 dA$$

- disjoint surfaces stay disjoint & embeddedness preserved

<u>Thm</u> (Huisken 84) Under MCF, ~~for~~ $n \geq 2$, any n-dim. compact convex surface in $\mathbb{R}^{n+1}$ shrinks to a round point.

⚠ However, analog of Grayson's thm. is false for surfaces!

Ex (neckpinch singularity)



M_0 = dumbbell

Qs: i) Continue flow through singularities?

ii) Size of the singular set?

iii) Nature of singularities?

i) approaches: generalized solutions

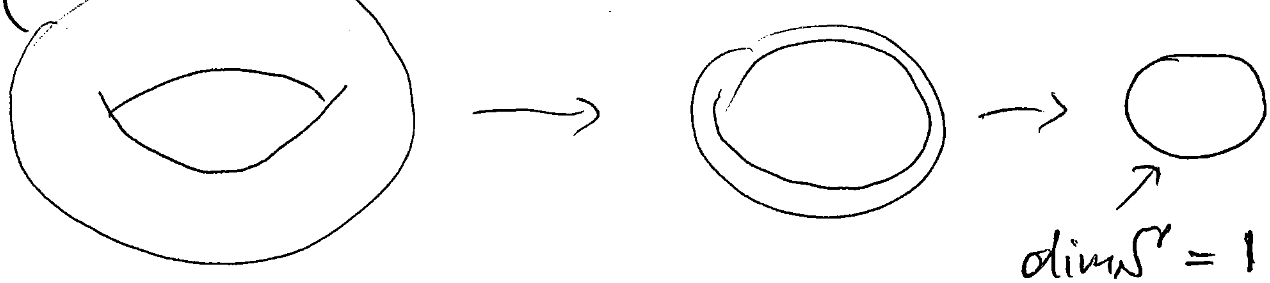
(Brakke flow, level set flow, ...)

or surgery (cut along necks & glue in caps)

ii) Thm (Ilmanen) For a.e. initial hypersurface M_0

M_t is smooth a.e. for a.e. t.

Ex (thin torus)



Thm (White) For the flow of mean convex surfaces $\dim S \leq 1$.

Open Qs: .) Remove mean convexity assumption?

.) Finitely many singular times?

.) Either curves or point singularities?

.) Nonround curve?

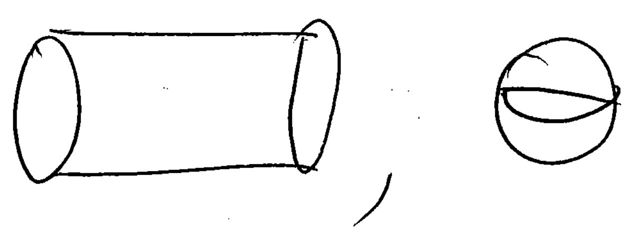
.) Generically only finitely many point singularities?

iii) neckpinch ⇝ singularity model = round shrinking cylinder

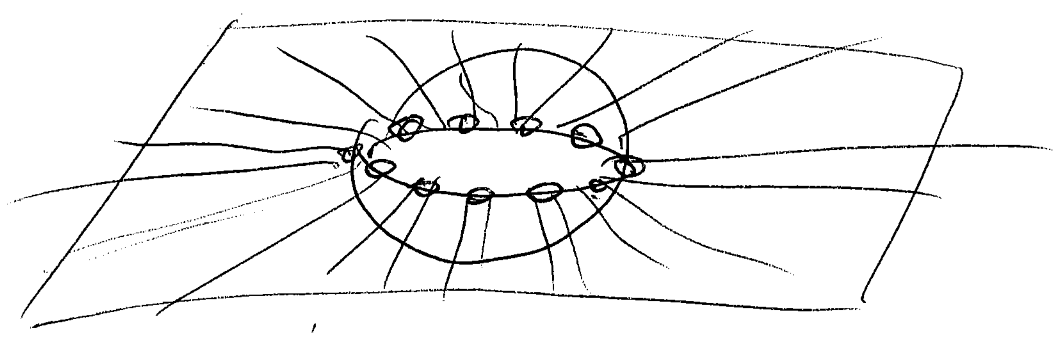
Principle (consequence of Huisken's monotonicity formula)

Singularities are modelled by selfsimilarly shrinking solutions.

Mean convex ones:



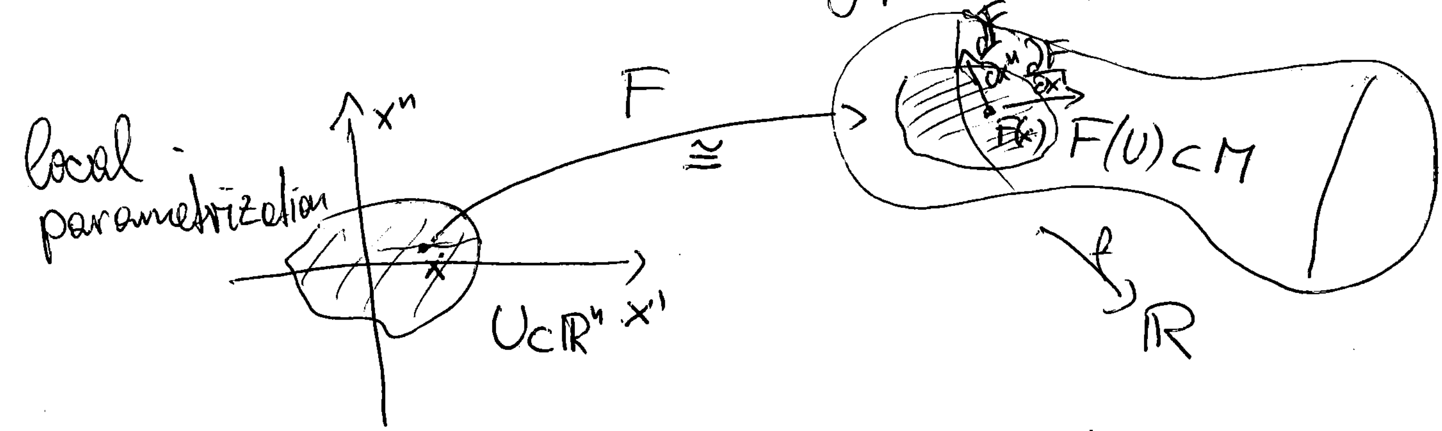
General ones can be very complicated:



(KKM)

Review: Geometry of hypersurfaces

$M^n \subset \mathbb{R}^{n+1}$ embedded hypersurface



tangent space $T_{F(x)}M$ is spanned by

$$\frac{\partial F}{\partial x^1}, \dots, \frac{\partial F}{\partial x^n}$$

induced metric $g_{ij} = \left\langle \frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \right\rangle$

$d\mu = \sqrt{\det g_{ij}}\, d^n x$ area element

$$\int_{F(U)} f \, d\mathcal{H}^n = \int_U f(F(x)) \sqrt{\det g_{ij}}\, d^n x$$

$\uparrow$ n-dim Hausdorff measure

Index notation:

X, Y tangent vectors

$$\Rightarrow X = X^i \frac{\partial F}{\partial x^i}, \; Y = Y^j \frac{\partial F}{\partial x^j} \quad \text{(Einstein summation convention)}$$

$$\Rightarrow \langle X, Y \rangle = \left\langle X^i \frac{\partial F}{\partial x^i}, Y^j \frac{\partial F}{\partial x^j} \right\rangle = g_{ij} X^i Y^j$$

Metric gives identification $TM \to T^*M$

$$X \longmapsto \langle X, \cdot \rangle$$

i.e. $\underset{\uparrow \text{ co-vector}}{X_i} = g_{ij} \underset{\uparrow \text{ vector}}{X^j}, \; X^i = \underset{\nwarrow \text{ inverse metric:}}{g^{ij}} X_j$

$$g^{ij} g_{jk} = \delta^i_k .$$

Extended Einstein convention

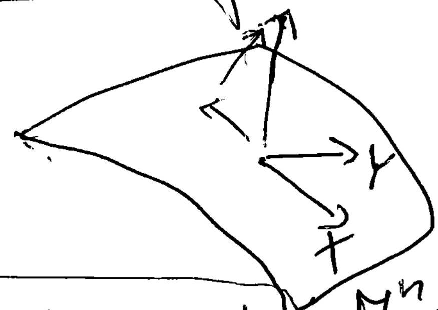
$$X_i Y_i := X_i Y^i = g^{ij} X_i Y_j$$

similarly for tensors T_{ij}

$$\text{eg } |T|^2 = T_{ij} T_{ij} = g^{ik} g^{jl} T_{ij} T_{kl}$$

$$= \lambda_1^2 + \dots + \lambda_n^2 \text{ if } T = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \; g_{ij} = \delta_{ij} .$$

Covariant derivative & 2nd fundamental form



Starting point:

$M^n \subset \mathbb{R}^{n+1}$

$$\boxed{D_X^{\mathbb{R}^{n+1}} Y = \underset{\substack{\uparrow \\ \text{tangential part}}}{\left(D_X^{\mathbb{R}^{n+1}} Y\right)^T} + \underset{\substack{\uparrow \\ \text{normal part}}}{\left(D_X^{\mathbb{R}^{n+1}} Y\right)^\perp}} \quad (*)$$

<u>Exer:</u> Check that $\nabla_X Y := \left(D_X^{\mathbb{R}^{n+1}} Y\right)^T$

is well-defined and satisfies the axioms of the Levi-Civita connection.

<u>Cor:</u> $\nabla_i Y^j = \partial_i Y^j + \Gamma^j_{ik} Y^k$, where

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right)$$

$$\vec{A}(X,Y) := \left(D_X^{\mathbb{R}^{n+1}} Y\right)^{\perp} \quad 2^{\text{nd}} \text{ fundamental form}$$

Note: $\left(D_X^{\mathbb{R}^{n+1}} Y\right)^{\perp} - \left(D_Y^{\mathbb{R}^{n+1}} X\right)^{\perp} = [X,Y]^{\perp} = 0$

$$\Rightarrow \vec{A}(X,Y) = \vec{A}(Y,X),$$
$$\vec{A}(fX,Y) = f\,\vec{A}(X,Y) = \vec{A}(X,fY).$$

Write: $\vec{A} = A\vec{\nu}$, where $\vec{\nu} = \nu$ is a unit normal (usually choose inwards)

Eqn $(*)$ for $X = \frac{\partial F}{\partial x^i}$, $Y = \frac{\partial F}{\partial x^j}$ becomes:

$$\underbrace{\frac{\partial^2 F}{\partial x^i \partial x^j} = \Gamma_{ij}^k \frac{\partial F}{\partial x^k} + A_{ij}\,\nu}$$

(Gauss-Weingarten)

$$\Rightarrow \underbrace{A_{ij} = \left\langle \frac{\partial^2 F}{\partial x^i \partial x^j}, \nu \right\rangle \underset{\uparrow}{=} -\left\langle \frac{\partial F}{\partial x^i}, \frac{\partial \nu}{\partial x^j} \right\rangle}$$

$\left\langle \frac{\partial F}{\partial x^i}, \nu \right\rangle = 0$

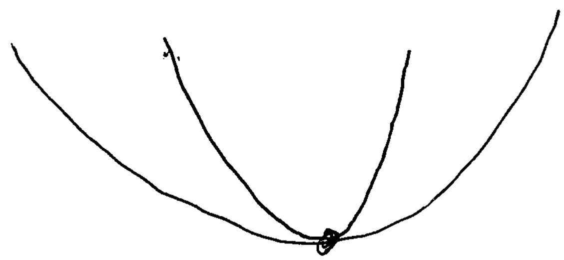
$\vec{H} = \operatorname{tr} \vec{A}$ mean curvature vector

$\vec{H} = H\nu$, $\quad H = \operatorname{tr} A = g^{ij} A_{ij}$

$$= \lambda_1 + \dots + \lambda_n$$

if $A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$, $g_{ij} = \delta_{ij}$

Geometric meaning: $\lambda_1 \le \dots \le \lambda_n$ principal curvatures



Exer Show that if $M \subset \mathbb{R}^{n+1}$ is locally the graph of a function $f : \mathbb{R}^n \to \mathbb{R}$, i.e. $F(x) = (x, f(x))$, then:

$$g_{ij} = \delta_{ij} + D_i f D_j f$$

$$A_{ij} = \frac{D_i D_j f}{\sqrt{1 + |Df|^2}}$$

$$H = \operatorname{div}\left(\frac{Df}{\sqrt{1+|Df|^2}}\right).$$

5

Riemann curvature tensor

$$\nabla_i \nabla_j X_k - \nabla_j \nabla_i X_k = R_{ijkl} X_l$$

Gauss eqn

$$R_{ijkl} = A_{ik} A_{jl} - A_{il} A_{jk}$$

(eg. $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$: $K = R_{1212} = \lambda_1 \cdot \lambda_2$)

Codazzi eqn

$$\nabla_i A_{jk} = \nabla_j A_{ik}$$

<u>Prop</u> (Simon's identity).

$$\Delta A_{ij} = \nabla_i \nabla_j H + H A_{ik} A_{kj} - |A|^2 A_{ij}$$

<u>Proof</u> $\Delta A_{ij} = \nabla_k \nabla_k A_{ij}$

$$\underset{\text{Codazzi}}{=} \nabla_k \nabla_i A_{kj}$$

$$\underset{\text{interchange derivatives}}{=} \nabla_i \nabla_k A_{kj} + R_{kikl} A_{lj} + R_{kijl} A_{kl}$$

$$= \nabla_i \nabla_j H + (H A_{il} - A_{kl} \cancel{A_{ki}}) A_{lj}$$
$$+ (\cancel{A_{kj}} A_{il} - A_{kl} A_{ij}) A_{kl}$$