Differential forms

Idea: A k-form is what one integrates over k-dim oriented mfds.

Def: A k-form ω on M is a smooth section of $\Lambda^k T^*M$.

Notation: $\Omega^k(M) = \{\omega \mid \omega \text{ is a } k\text{-form on } M\}$.

So to each $p \in M$ we associate $\omega(p) \in \Lambda^k T_p^*M$ depending smoothly on p.

Note: $\cdot)$ $\Lambda^0 T_p^*M = \mathbb{R}$, so a 0-form is simply a smooth function.

$\cdot)$ $\Lambda^1 T_p^*M = T_p^*M$, so a 1-form is a covector field.

<u>Ex</u> $f: M \to \mathbb{R}$ smooth

$\rightsquigarrow$ $df_p : T_pM \to \mathbb{R}$ linear, so $df_p \in T_p^*M$,

i.e. df is a 1-form.

In particular $x^i : U \to \mathbb{R}$ local coords,

gives 1-forms $dx^i \in \Omega^1(U)$.

Recalling the formula $df_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \frac{\partial}{\partial x^j}\Big|_p f$,

we see that $dx^i_p\left(\frac{\partial}{\partial x^j}\Big|_p\right) = \delta^i_j$,

so $T_pM = \operatorname{span}\left\{\frac{\partial}{\partial x^1}\Big|_p, \dots, \frac{\partial}{\partial x^n}\Big|_p\right\}$

$T_p^*M = \operatorname{span}\{dx^1_p, \dots, dx^n_p\}$

and $df_p = \frac{\partial f}{\partial x^i}(p)\, dx^i_p$,

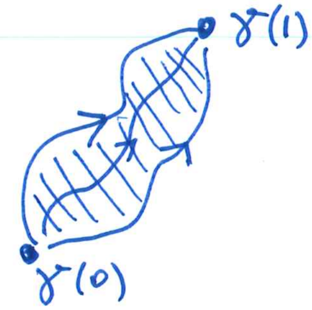
namely $\boxed{df = \frac{\partial f}{\partial x^i} dx^i}$

Ex $\omega \in \Omega^1(M)$, $\gamma : [0,1] \to M$ smooth curve.

Can define the line integral

$$\int_\gamma \omega := \int_0^1 \omega_{\gamma(t)}(\dot\gamma(t))\, dt$$

In particular $\int_\gamma df = \int_0^1 \underbrace{df_{\gamma(t)}(\dot\gamma(t))}_{=(f\circ\gamma)^\bullet(t)}\, dt = f(\gamma(1)) - f(\gamma(0))$



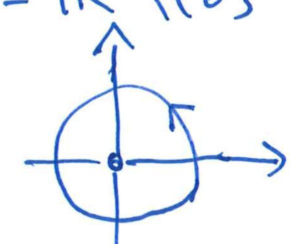
So $\int_\gamma df$ only depends on $[\gamma]$.

Also note that for $\tilde\gamma(t) = \gamma(1-t)$ get $\int_{\tilde\gamma} \omega = -\int_\gamma \omega$,

so $\int_\gamma \omega$ depends on orientation.

Eg. for $\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2}$ on $M = \mathbb{R}^2 \setminus \{0\}$

$\gamma(t) = (\cos t, \sin t)$



$$\int_\gamma \omega = \int_0^{2\pi} \left(\frac{\cos t\, dy - \sin t\, dx}{\sin^2 t + \cos^2 t}\right)\left(-\sin t \frac{\partial}{\partial x} + \cos t \frac{\partial}{\partial y}\right) dt$$

$= \int_0^{2\pi} dt = 2\pi$, cf. "winding number".

Review multilinear algebra

V an n-dim. real vector space

$V^* = \{ L : V \to \mathbb{R} \mid L \text{ linear} \}$ dual space

$\{e_i\}_{i=1,\dots,n}$ basis for V

$\rightsquigarrow$ dual basis e^i for V^* defined by $e^i(e_j) = \delta^i_j$

(in part. $\dim V^* = \dim V = n$)

$\alpha : \underbrace{V \times \dots \times V}_{k\text{-times}} \longrightarrow \mathbb{R}$ multilinear,

i.e. $\alpha(v_1, \dots, \lambda v_i + \tilde{v}_i, \dots, v_k)$

$= \lambda \alpha(v_1, \dots, v_i, \dots, v_k) + \alpha(v_1, \dots, \tilde{v}_i, \dots, v_k)$

$\bigotimes^k V^* := \{ \alpha : \underbrace{V \times \dots \times V}_{k\text{-times}} \longrightarrow \mathbb{R} \text{ multilinear} \}$

has basis $\{e^{i_1} \otimes \dots \otimes e^{i_k}\}_{i_1, \dots, i_k = 1, \dots, n}$

Indeed, can write any $\alpha \in \bigotimes^k V^*$ uniquely as

$\alpha = \alpha_{i_1 \dots i_k} e^{i_1} \otimes \dots \otimes e^{i_k}$, where $\alpha_{i_1 \dots i_k} = \alpha(e_{i_1}, \dots, e_{i_k})$

In particular, $\dim \bigotimes^k V^* = n^k$.

Def: $\Lambda^k V^* := \{\alpha \in \bigotimes^k V^* \mid \alpha \text{ is alternating}\}$.

Here α <u>alternating</u> means

$$\alpha(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -\alpha(v_1, \dots, v_j, \dots, v_i, \dots, v_k).$$

Define projection $\bigotimes^k V^* \xrightarrow{\mathrm{Alt}} \Lambda^k V^*$ by

$$(\mathrm{Alt}\,\alpha)(v_1, \dots, v_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma)\, \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})$$

<u>Note:</u> α alternating $\Leftrightarrow$ $\mathrm{Alt}\,\alpha = \alpha$.

The space $\Lambda^k V^*$ has the basis $\{e^{i_1} \wedge \dots \wedge e^{i_k}\}_{1 \le i_1 < \dots \le i_k \le n}$,

where $e^{i_1} \wedge \dots \wedge e^{i_k} := k!\,\mathrm{Alt}(e^{i_1} \otimes \dots \otimes e^{i_k})$

In particular, $\dim \Lambda^k V^* = \binom{n}{k}$.

<u>Ex</u> $\Lambda^2(\mathbb{R}^3)^*$ has the basis

$$e^1 \wedge e^2 = \cancel{\tfrac{1}{2}}(e^1 \otimes e^2 - e^2 \otimes e^1)$$
$$e^1 \wedge e^3 = \cancel{\tfrac{1}{2}}(e^1 \otimes e^3 - e^3 \otimes e^1)$$
$$e^2 \wedge e^3 = \cancel{\tfrac{1}{2}}(e^2 \otimes e^3 - e^3 \otimes e^2).$$

More generally, can define <u>wedge product</u>

$$\wedge : \Lambda^k V^* \times \Lambda^\ell V^* \longrightarrow \Lambda^{k+\ell} V^*$$

$$\alpha \wedge \beta := \frac{(k+\ell)!}{k!\,\ell!} \mathrm{Alt}(\alpha \otimes \beta) .$$

<u>Note:</u> Here, the prefactor $\frac{(k+\ell)!}{k!\,\ell!}$ is to ensure that if $\alpha = e^1 \wedge \ldots \wedge e^k$ and $\beta = e^{k+1} \wedge \ldots \wedge e^{k+\ell}$, then $\alpha \wedge \beta = e^1 \wedge \ldots \wedge e^{k+\ell}$.

Observe that $\boxed{\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha}$

("graded commutative")

Finally, can package everything together into

$$\boxed{\Lambda V^* := \bigoplus_{k=0}^{n} \Lambda^k V^*}$$ (<u>exterior algebra</u>)

<u>Note:</u> $\dim \Lambda V^* = \sum_{k=0}^{n} \binom{n}{k} = 2^n$

So back to mfds, any $\omega \in \Omega^k(M) = \Gamma(\Lambda^k T^*M)$

can be locally written as $\omega = \sum_{1 \le i_1 < \ldots < i_k \le n} \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$,

where $\omega_{i_1 \ldots i_k} = \omega\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right)$.

Moreover, we get that $\Omega^*(M) := \bigoplus_{k=0}^{n} \Omega^k(M)$

has the structure of a graded commutative algebra,

i.e. we can add & multiply differential forms,

and $\omega \wedge \rho = (-1)^{k\ell} \rho \wedge \omega$ if $\omega \in \Omega^k(M)$, $\rho \in \Omega^\ell(M)$.

<u>Note:</u> Only have to remember that

$$dx^i \wedge dx^j = -dx^j \wedge dx^i$$

Then can compute everything via multilinearity.

<u>Ex</u> $f^1, \dots, f^n : M^n \longrightarrow \mathbb{R}$

$$\underline{\underline{df^1 \wedge \dots \wedge df^n}} = \sum_{1 \le i_1, \dots, i_n \le n} \frac{\partial f^1}{\partial x^{i_1}} dx^{i_1} \wedge \dots \wedge \frac{\partial f^n}{\partial x^{i_n}} dx^{i_n}$$

$$= \sum_{i_1, \dots, i_n \text{ all different}} \frac{\partial f^1}{\partial x^{i_1}} \cdots \frac{\partial f^n}{\partial x^{i_n}} dx^{i_1} \wedge \dots \wedge dx^{i_n}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \frac{\partial f^1}{\partial x^{\sigma(1)}} \cdots \frac{\partial f^n}{\partial x^{\sigma(n)}} dx^1 \wedge \dots \wedge dx^n$$

$$= \underline{\underline{\det\left(\frac{\partial f^j}{\partial x^i}\right) dx^1 \wedge \dots \wedge dx^n}}$$

In particular, if (x^i), $(\tilde{x}^j)$ are two local coordinates, then

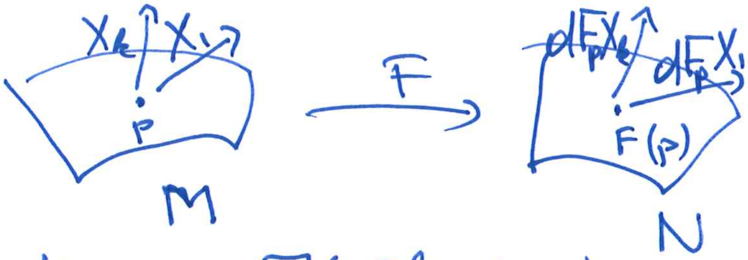
$$\boxed{d\tilde{x}^1 \wedge \dots \wedge d\tilde{x}^n = \det\left(\frac{\partial \tilde{x}^j}{\partial x^i}\right) dx^1 \wedge \dots \wedge dx^n}$$

<u>Note:</u> ·) M orientable $\Longleftrightarrow$ M admits nowhere vanishing n-form

$\Longleftrightarrow \Lambda^n T^* M \cong M \times \mathbb{R}$

·) choice of orientation $\Longleftrightarrow$ choice of positive ~~volume~~ n-form (volume form).

Pullbacks



$F: M \to N$ smooth, $\alpha \in \Gamma(\bigotimes^k T^*N)$

Can define $F^*\alpha \in \Gamma(\bigotimes^k T^*M)$ by

$$(F^*\alpha)_p(X_1, \dots, X_k) := \alpha_{F(p)}(dF_pX_1, \dots, dF_pX_k)$$

Note:
- For $k=0$ simply have $F^*f = f \circ F$
- $F^*(\alpha \otimes \beta) = F^*\alpha \otimes F^*\beta$

  (in particular $F^*(f\alpha) = (f \circ F)F^*\alpha$)
- $(G \circ F)^* = F^* \circ G^*$, $Id^* = Id$

Ex: $M \xrightarrow{F} N$, $N \xrightarrow{f} \mathbb{R}$

$$(F^*df)_p(X) = df_{F(p)}(dF_pX)$$

$$\underset{\text{chain rule}}{=} d(f \circ F)_p(X) = d(F^*f)_p(X)$$

So $\underline{\underline{F^*df = dF^*f}}$

In particular, given $F: M \to N$ smooth, get

$$F^*: \Omega^*(N) \longrightarrow \Omega^*(M)$$

defined on k-forms by

$$(F^*\omega)_p(X_1, \dots, X_k) = \omega_{F(p)}(dF_p X_1, \dots, dF_p X_k).$$

<u>Note:</u> •) $F^*(\omega \wedge \rho) = F^*\omega \wedge F^*\rho$

•) $F^* f = f \circ F$, $F^* df = dF^* f$

•) $(G \circ F)^* = F^* \circ G^*$, $Id^* = Id$

So for $\omega = \sum \omega_{i_1 \dots i_k} dy^{i_1} \wedge \dots \wedge dy^{i_k}$ have

$$F^*\omega = \sum (\omega_{i_1 \dots i_k} \circ F)\, d(y^{i_1} \circ F) \wedge \dots \wedge d(y^{i_k} \circ F)$$

<u>Ex</u> $F: \mathbb{R}^2 \to \mathbb{R}^3$, $F(u,v) = (u, v, u^2 - v^2)$, $\omega = y\,dx \wedge dz + x\,dy \wedge dz$

$$\Rightarrow \underline{F^*(y\,dx \wedge dz + x\,dy \wedge dz)}$$

$$= v\,du \wedge d(u^2 - v^2) + u\,dv \wedge d(u^2 - v^2)$$

$$= -2v^2\,du \wedge dv + 2u^2\,dv \wedge du = \underline{-2(u^2+v^2)\,du \wedge dv}$$

Integration

goal: Integrate k-forms over oriented k-dim (sub)manifolds

recall: $\omega \in \Omega^1(M)$, $\gamma : [a,b] \to M$

$$\int_\gamma \omega = \int_a^b \omega_{\gamma(t)}(\dot\gamma(t))\,dt = \int_a^b (\gamma^*\omega)_t\left(\tfrac{\partial}{\partial t}\right) dt =: \int_{[a,b]} \gamma^*\omega$$

Idea: reduce to integration in $\mathbb{R}^n$ via pullback & partition of unity.

So consider first $\omega = f\,dx^1 \wedge \dots \wedge dx^n \in \Omega^n_c(U)$, where U is an open subset of $\mathbb{R}^n$ (or of $\mathbb{H}^n$).

Def: $\int_U \omega := \int_U f\,dx^1 \dots dx^n$ (the usual Riemann/ Lebesgue integral)

Note: If $\psi : V \to U$ is an orientation preserving diffeo, then setting $x^i = \psi^i(y)$ we have $dx^i = \frac{\partial \psi^i}{\partial y^j} dy^j$, hence $\psi^*\omega = f \circ \psi \det\left(\frac{\partial \psi^i}{\partial y^j}\right) dy^1 \wedge \dots \wedge dy^n$ (cf. last lecture)

Thus $$\underline{\int_V \psi^*\omega} \underset{\text{Def}}{=} \int_V f\circ\psi(y) \det\left(\frac{\partial \psi^i}{\partial y^j}\right) dy^1 \dots dy^n$$

$$\underset{\text{calculus change of variables}}{=} \int_U f(x)\,dx^1 \dots dx^n \underset{\text{Def}}{=} \underline{\int_U \omega} \qquad (*)$$

Now, let M^n be an oriented smooth mfd[*)], and consider

$$\Omega^n_c(M) := \{ \omega \in \Omega^n(M) \mid \operatorname{spt}(\omega) \text{ is cpt} \}$$

Here, $\operatorname{spt}(\omega) := \overline{\{ p \mid \omega_p \neq 0 \}} \subseteq M$

For $\omega \in \Omega^n_c(M)$ cover $\operatorname{spt}(\omega)$ by finitely many pos. oriented charts (U_i, φ_i), and let ρ_i be a subordinate smooth partition of unity.

So $\omega = \sum_i \rho_i \omega$, where $\operatorname{spt}(\rho_i \omega) \subset U_i$,

and $U_i \xrightarrow[\cong]{\varphi_i} \varphi_i(U_i) \subset \mathbb{R}^n$.

<u>Def:</u> $$\int_M \omega := \sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* (\rho_i \omega)$$

<u>Prop:</u> $\int_M \omega$ is well-defined, i.e. independent of the above choices of cover & partition of unity.

*) with or without boundary

<u>Proof</u> Suppose $(\tilde{U}_j, \tilde{\varphi}_j)$, $\tilde{\rho}_j$ is some other choice.

Note that $\operatorname{spt}(\rho_i \tilde{\rho}_j \omega) \subset U_i \cap \tilde{U}_j$,

so thanks to $(*)$ we get

$$\int_{\varphi_i(U_i \cap \tilde{U}_j)} (\varphi_i^{-1})^* (\rho_i \tilde{\rho}_j \omega) = \int_{\tilde{\varphi}_j(U_i \cap \tilde{U}_j)} (\tilde{\varphi}_j^{-1})^* (\rho_i \tilde{\rho}_j \omega) . \quad (**)$$

Now, compute

$$\sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* (\rho_i \omega)$$

$$= \sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* \left(\left(\sum_j \tilde{\rho}_j\right) \rho_i \omega \right)$$

$$= \sum_{i,j} \int_{\varphi_i(U_i \cap \tilde{U}_j)} (\varphi_i^{-1})^* (\rho_i \tilde{\rho}_j \omega)$$

$$\underset{\substack{\text{use}(**) \\ \text{and undo steps}}}{=} \sum_j \int_{\tilde{\varphi}_j(\tilde{U}_j)} (\tilde{\varphi}_j^{-1})^* (\tilde{\rho}_j \omega) \qquad \square$$

Note: By construction the integral has the following properties:

- $\int_M a\omega + b\eta = a\int_M \omega + b\int_M \eta$ (linearity)

- $\int_{-M} \omega = -\int_M \omega$ (orientation reversal)

- $\omega > 0 \Rightarrow \int_M \omega > 0$ (positivity)

- $F: N \to M$ orientation preserving diffeo

$$\Rightarrow \int_M \omega = \int_N F^*\omega \quad \text{(diffeo invariance)}$$

If $\omega \in \Omega^k(M)$ and $i_S : S \hookrightarrow M$ is an oriented k-dim submfd, we set

$$\int_S \omega := \int_S i_S^*\omega \quad \left(\begin{array}{l}\text{think of } i_S^*\omega \\ \text{as } \underline{\text{restriction}} \text{ of } \omega \text{ to } S\end{array}\right)$$

In particular, for $\omega \in \Omega^{n-1}(M)$ this defines

$$\int_{\partial M} \omega := \int_{\partial M} i_{\partial M}^*\omega \quad \left(\begin{array}{l}\text{here } \partial M \subset M \\ \text{has the} \\ \text{induced orientation}\end{array}\right)$$

Ex $\omega = x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy$

$$\int_{S^2}\omega = ?$$

Discarding a null set, we can work with a single parametrization $F:(0,\pi)\times(0,2\pi)\to S^2$,

$$F(\varphi,\theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

Compute: $F^*dx = \cos\varphi\cos\theta\,d\varphi - \sin\varphi\sin\theta\,d\theta$
$F^*dy = \cos\varphi\sin\theta\,d\varphi + \sin\varphi\cos\theta\,d\theta$
$F^*dz = -\sin\varphi\,d\varphi$

$$\Rightarrow \int_{S^2}\omega = \int_{(0,\pi)\times(0,2\pi)} F^*\omega = \int_0^{2\pi}\int_0^{\pi}\sin\varphi\,d\varphi\,d\theta = 4\pi.$$

Application: Can average over compact Lie groups:

Prop: G oriented compact Lie group. Then $\exists!$ left-invariant n-form ω_G with $\int_G\omega_G = 1$.

Proof $e_1,\dots,e_n\in\Gamma(TG)$ left-invariant orthonormal frame.

Then $e^i\in\Gamma(T^*G)$ is also left-invariant

$\Rightarrow \eta := e^1\wedge\dots\wedge e^n$ is left-invariant pos. n-form.

$\Rightarrow \omega_G := \eta/\int_G\eta$ does the job. (uniqueness clear) $\boxed{\checkmark}$

Rmk The map $f\mapsto\int_G f\omega_G$ is called the Haar integral.