We now consider the Gross-Pitaevskii equations

$$i\partial_t\psi - \Delta\psi + \frac{1}{\varepsilon^2}(|\psi|^2 - 1)\psi = 0, \qquad 0 < \varepsilon \ll 1$$

for $\psi : \mathcal{D} \times [0, T) \to \mathbb{C}$, where $\mathcal{D}$ is bounded open sunset of $\mathbb{R}^2$ with $\partial\mathcal{D}$ smooth.

Notation: For $v, w \in \mathbb{C}$, we will write

$$(v, w) := Re(v\bar{w}) = \frac{1}{2}(v\bar{w} + \bar{v}w).$$

Given $v \in \mathbb{C}$, we will sometimes write $v_1 = Re(v)$ and $v_2 = Im(v)$. Note that

$$(iv, w) = (w, iv) = -(iw, v) = -(v, iw) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}^{\perp} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}.$$

Physical quantities:

- *energy density*

$$e_\varepsilon(\psi) := \frac{1}{2}|\nabla\psi|^2 + \frac{(|\psi|^2 - 1)^2}{4\varepsilon^2}.$$

- mass density $|\psi|^2$.
- current, or momentum density

$$j(\psi) := (i\psi, \nabla\psi) = \text{ vector with } k\text{th component } (i\psi, \partial_k\psi).$$

  If one writes $\psi = \rho e^{i\phi}$, then $j(\psi) = \rho^2\nabla\phi$.
- The vorticity

$$\omega(\psi) := \frac{1}{2}\nabla^{\perp} \cdot j(\psi) = \det \nabla\psi = \frac{\partial(\psi_1, \psi_2)}{\partial(x_1, x_2)}$$

The quantities introduced above all satisfy conservation laws. We will write these in differential form.

$$
\begin{aligned}
\frac{\partial}{\partial_t} e_\varepsilon(\psi) &= \nabla \cdot (\partial_t \psi, \nabla \psi). \\
\frac{d}{dt} \frac{|\psi|^2}{2} &= \nabla \cdot j(\psi). \\
\frac{d}{dt} j(\psi) &= 2\nabla \cdot (\nabla \psi \otimes \nabla \psi) - \nabla \big[2e_\varepsilon(\psi) + (i\psi_t, \psi) \big] \\
\frac{d}{dt} \omega(\psi) &= \nabla^\perp (\nabla \cdot (\nabla \psi \otimes \nabla \psi))
\end{aligned}
$$

Here $\nabla\psi \otimes \nabla\psi$ is the 2×2 matrix whose (k, ℓ) entry is $(\partial_k \psi, \partial_\ell \psi)$.
We will write

$$
E_\varepsilon(\psi) := \int_{\mathcal{D}} e_\varepsilon(\psi).
$$

We will always consider situations where $0 < \varepsilon \ll 1$ and

$$
E_\varepsilon(\psi) \leqslant C|\log \epsilon| \qquad \text{and thus} \qquad \| \, |\psi|^2 - 1 \|_{L^2} \leqslant \varepsilon |\log \epsilon|^{1/2} \ll 1.
$$

Our goal is to prove a theorem decribing dynamocs of point vortices in solutions of $(GP)_\varepsilon$.

Main tools: theorems that describe connections between energy and vorticity.

A first such theorem states that vorticity concentration can be deduced from control over energy.

Theorem

Assume that $\psi \in H^1(\Omega;\mathbb{C})$, *that* $a = (a_1, \ldots, a_M) \in \Omega^M$ *satisfies*

$$\rho_a := \min\left(\{|a_i - a_j|, i \neq j\} \cup \{\mathrm{dist}(a_i, \partial\mathcal{D}\}\right) > \rho_0 > 0$$

and that $d_i = \pm 1$ *for* $i = 1, \ldots, M$. *Suppose further that*

$$\|\omega(\psi) - \sum_{i=1}^{M} d_i \delta_{a_i}\|_{\mathcal{F}} \leqslant \frac{1}{10}\rho_a,$$

$$E_\varepsilon(\psi) \leqslant M\pi \log\frac{1}{\varepsilon} + C.$$

Then there exists $\varepsilon_0, \alpha > 0$ *(depending on* M, ρ_0, C*) and points* $\xi_1, \ldots, \xi_M \in \Omega$ *such that*

$$\|\omega(\psi) - \sum_{i=1}^{M} d_i \delta_{\xi_i}\|_{\mathcal{F}} \leqslant \varepsilon^\alpha$$

Here

$$\|\mu\|_{\mathcal{F}} := \sup\left\{\int_{\mathcal{D}} \phi\mu \;:\; \phi \in W_0^{1,\infty}(\mathcal{D}), \quad Lip(\phi) \leqslant 1\right\}.$$

First: if $\psi \in C^1(\mathcal{D};\mathbb{C})$ and O is a subset of $\mathcal{D}$ with ∂O Lipschitz, and if in addition $|\psi| > 0$ on ∂O, then we define

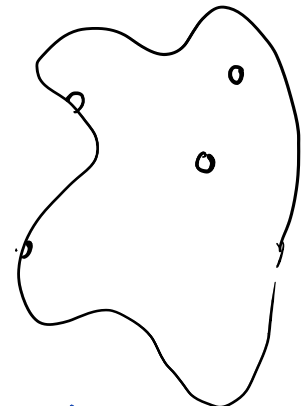
$$deg(\psi; \partial O) := \frac{1}{2\pi} \int_{\partial O} \frac{j(\psi)}{|\psi|^2} \cdot \tau$$

where τ is the unit tangent to ∂O, oriented according to the standard left(?)-hand rule, that is, counterclockwise if O is simply connected.

Lemma

Assume that $\psi \in C^1(\mathcal{D};\mathbb{C})$ and that $B_1, \ldots, B_K$ are balls such that

$$S := \{x \in \mathcal{D} : |\psi(x)| \leqslant \frac{1}{2}\} \subset \bigcup_k B_k.$$



Then

$$\|\omega(\psi) - \pi \sum_{k=1}^{K} d_k \delta_{\xi_k}\|_{\mathcal{F}} \leqslant C(\varepsilon + \sum r_k) E_\varepsilon(\psi)$$

where

(we can set $d_k = 0$ if $B_k \cap \partial\mathcal{D} \neq \emptyset$.)

$$r_k \text{ is the radius of } B_k, \qquad d_k = \deg(\psi; \partial(B_k \cap \mathcal{D})),$$

and ξ_k is any point in $B_k \cap \mathcal{D}$ (for example the center, if it belongs to $\mathcal{D}$).

Since for us, it will always be the case that $E_\varepsilon(\psi) \approx |\log \epsilon|$, the lemmaprovides a good estimate if $\sum r_k \ll |\log \epsilon|^{-1}$.

Lemma

Assume that $\psi \in C^1(\mathcal{D};\mathbb{C})$ *and that* $B_r(x) \subset \mathcal{D}$ *is a ball such that* $r \geqslant \varepsilon$ *and*

$$d = \deg(\psi; \partial B_r) \neq 0.$$

$|\psi| > 0$ on $\partial B_r(x)$

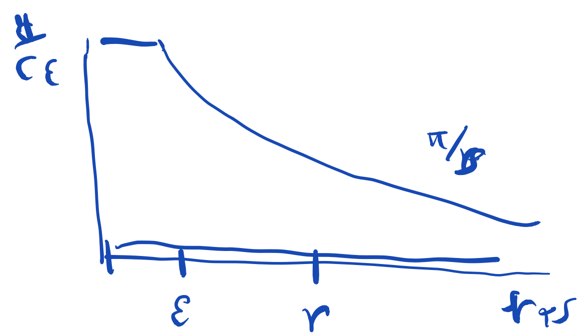
Then

$$\int_{\partial B_r} e_\varepsilon(\psi)\, d\mathcal{H}^1 \geqslant \lambda_\varepsilon\left(\frac{r}{|d|}\right),$$

where

$$\lambda_\varepsilon(s) := \min_{m\in[0,1]} \left(\frac{m^2 \pi}{s} + \frac{(1-m^2)^2}{C\varepsilon} \right) = \begin{cases} \frac{\pi}{s} - \frac{C\varepsilon\pi^2}{4s^2} & \text{if } s \geqslant \frac{C\pi\varepsilon}{2} \\ \frac{1}{C\varepsilon} & \text{if } s \leqslant \frac{C\pi\varepsilon}{2} \end{cases}$$

for a suitable constant C.



Lemma

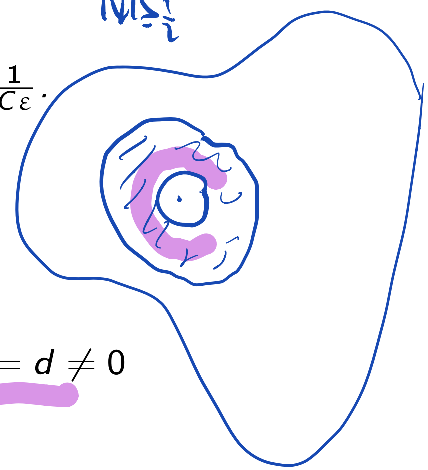
Define

$$\Lambda_\varepsilon(s) := \int_0^s \lambda_\varepsilon(r)\,dr$$

$$\frac{1}{\sigma}\Lambda_\varepsilon(\sigma) = \frac{1}{\sigma}\int_0^\sigma \lambda_\varepsilon(s)\,ds$$

Then the following hold.

1. $\Lambda_\varepsilon(r+s) \leqslant \Lambda_\varepsilon(r) + \Lambda_\varepsilon(s)$
2. $\sigma \mapsto \frac{1}{\sigma}\Lambda_\varepsilon(\sigma)$ *is nonincreasing, and is always bounded by* $\frac{1}{C\varepsilon}$.
3. $\Lambda_\varepsilon(r) \geqslant \pi \log \frac{r}{\varepsilon} - C$ *for all* $r \geqslant 0$.
4. *If* $\psi \in H^1(\mathcal{D};\mathbb{C})$ *and*



$r \geq \varepsilon$

$$|\psi| \geqslant \frac{1}{2} \text{ in } B_R \setminus B_r(a), \qquad \deg(\psi; \partial B_r(a)) = d \neq 0$$

then

$$\int_{B_R\setminus B_r(a)} e_\varepsilon(\psi)\,dx \geqslant d\left(\Lambda_\varepsilon\left(\frac{R}{|d|}\right) - \Lambda_\varepsilon\left(\frac{r}{|d|}\right)\right).$$

$$\int_{B_R\setminus B_r} e_\varepsilon(\psi)\,dx = \int_r^R \left(\int_{\partial B_s(a)} e_\varepsilon(u)\,d\mathcal{H}^1\right) ds \geq d\int_{r/d}^{R/d} \lambda_\varepsilon(\sigma)\,d\sigma$$

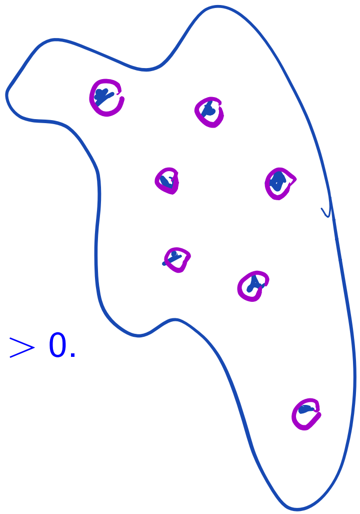
$$= d\left[\Lambda_\varepsilon\left(\tfrac{R}{\delta}\right) - \Lambda_\varepsilon\left(\tfrac{r}{\delta}\right)\right]$$

At this point, to convey the spirit of the proof, I am sweeping certain technical details under the carpet.

"Lemma" (not 100% correct as stated!)

Assume that $\psi \in C^1(\mathcal{D};\mathbb{C})$. Then there exist closed balls $B_k^0, k = 1, \ldots, K$ such that

$$S := \{x \in \mathcal{D} : |\psi(x)| \leqslant \frac{1}{2}\} \subset \bigcup_k B_k^0$$



$$r_k^0 := \text{ radius of } k\text{th ball } \geqslant \varepsilon \qquad \text{for all } k$$

$$B_k^0 \cap S \neq \emptyset \qquad \text{for all } k$$

$$\int_{B_k^0 \cap \mathcal{D}} e_\varepsilon(\psi) \geqslant \frac{r_k^0}{C\varepsilon} \geqslant \frac{r_k^0}{\sigma}\Lambda_\varepsilon(\sigma) \qquad \text{for all } k \text{ and all } \sigma > 0.$$

Let's pretend the lemma is correct, then see what one can do with it.

Ideas 1) If $\deg(\psi, B) \neq 0 \Rightarrow \int_B e_\varepsilon(\psi) \geq \pi/4$

2) If $r > \varepsilon$, then

$$\int_{B_r} \cdots = \int_{B_r \setminus B_\varepsilon} + \int_{B_\varepsilon}$$

$\geq \int_\varepsilon^r \int_{\partial B_r} e_\varepsilon \, d\mathcal{H}^1 \, dr + \frac{\pi}{4}$

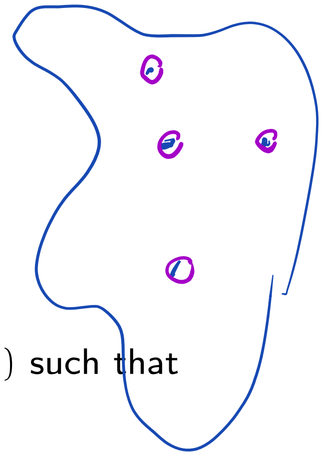
$\geq c/\varepsilon$

Proposition

Assume that $\psi \in C^1(\mathcal{D};\mathbb{C})$, and for the balls B_k^0 found above, let

$1 \gg \varepsilon(\log\frac{1}{\varepsilon}) \approx$

$$\sigma^0 := \min\left\{\frac{r_k^0}{|d_k^0|} : B_k^0 \subset \mathcal{D}\}\right\} \qquad d_k^0 := deg(\psi; \partial B_k^0\}$$



Then for every $\sigma \geqslant \sigma_0$, there exist closed balls $B_k(\sigma), k = 1, \ldots, K(\sigma)$ such that

$$S := \{x \in \mathcal{D} : |\psi(x)| \leqslant \frac{1}{2}\} \subset \bigcup_k B_k(\sigma)$$

$\pi d \log(\frac{r}{d\varepsilon}) - c$

$\approx$

$$- \; r_k(\sigma) \geqslant \sigma d_k(\sigma) \qquad \text{for all } k \text{ such that } B_k(\sigma) \subset \mathcal{D}.$$

$$\int_{B_k(\sigma)\cap\mathcal{D}} e_\varepsilon(\psi) \geqslant \frac{r_k(\sigma)}{\sigma}\Lambda_\varepsilon(\sigma) \; = \; d_k(\sigma)\,\Lambda_\varepsilon\left(\frac{r_k(\sigma)}{d_k(\sigma)}\right) \qquad \text{for all } k.$$

$\&\ r_k(\sigma) = \sigma d_k(\sigma)$

and

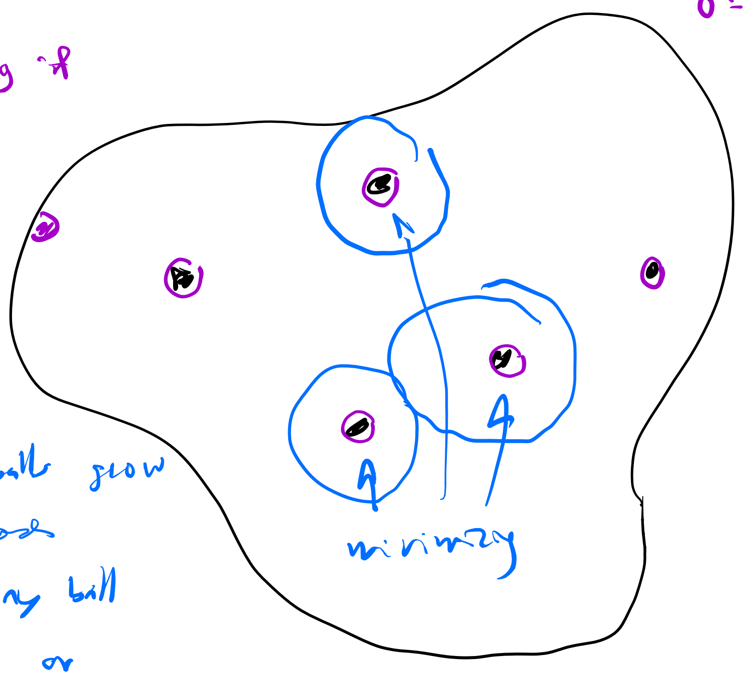
$$\sigma \mapsto R(\sigma) := \sum_k r_k(\sigma) \qquad \text{is a continuous nondecreasing function}$$

Proof of Proposition:

$$\int_{B_k} e_\varepsilon \ge \frac{r_k}{\sigma_k}\Lambda_\varepsilon(\sigma_k)$$

$$\sigma = \min_k \frac{r_k}{|d_k|}$$

Ball is minimizing if $\frac{r_k}{|d_k|} = \sigma$



- increase σ, letting all minimizing balls grow as σ increases
- stop if any ball hits ∂D or if 2 ball hit each other

$$\int_{B_{r_k(\sigma_2)}} e_\varepsilon(\psi)\,dx = \int_{B_{r_k(\sigma_2)}\setminus B_{r_k(\sigma_1)}} e_\varepsilon + \int_{B_{r_k(\sigma_1)}} e_\varepsilon$$

$$\ge \cancel{d_k(\sigma)}\,\frac{r_k(\sigma_2)}{\sigma_2}\left[\Lambda_\varepsilon\left(\frac{r_k(\sigma_2)}{\sigma_2}\right) - \Lambda_\varepsilon\left(\cancel{\frac{r_k(\sigma_1)}{\sigma}}\right)\right] + \frac{r_k(\sigma_1)}{\sigma_1}\Lambda_\varepsilon(\sigma_1)$$

$$= \frac{r_k(\sigma_2)}{\sigma_2}\Lambda_\varepsilon(\sigma_2)$$

B_1, B_2

If 2 balls touch replace by $\tilde{B}$ s.t.

$$B_1 \cup B_2 \subseteq \tilde{B}$$

$$r_1 + r_2 = \tilde{r}$$

Then

$$\int_{\tilde{B}} e_\varepsilon(\psi) \ge \int_{B_1} + \int_{B_2} e_\varepsilon(\psi)$$

$$\ge (r_1 + r_2)\left(\frac{1}{8}\Lambda_\varepsilon(\sigma)\right)$$

$$\geq \frac{\tilde{r}}{\sigma} \lambda_t(\sigma)$$

$$\tilde{d} = d_1 + d_2$$

$$\Rightarrow \quad |\tilde{d}| \leq |d_1| + |d_2|$$

$$\Rightarrow \quad \frac{\tilde{r}}{|\tilde{d}|} \geq \sigma = \frac{r_1}{d_1} = \frac{r_2}{d_2}$$